

Quantum Computation

(Lecture 8)

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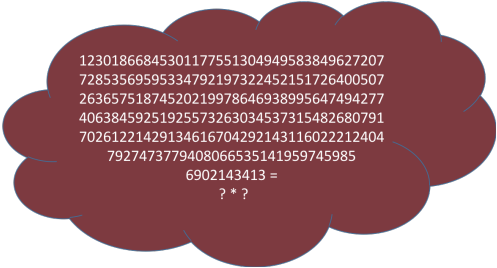
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Shor's algorithm

Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer

Proc. 35th Annual Symp. on Foundations of Computer Science, IEEE Computer Society Press, pp. 124-134 (1994)

was a turning point in quantum computing for its spectacular decrease of the **time complexity** of factoring from $\mathcal{O}(e^{\sqrt[3]{n}})$ to $\mathcal{O}(n^3 \log n)$, with potential impact in cryptography.



12301866845301177551304949583849627207
72853569595334792197322452151726400507
26365751874520219978646938995647494277
40638459251925573263034537315482680791
70261221429134616704292143116022212404
7927473779408066535141959745985
6902143413 =
? * ?

Factorization

In this famous 1994 paper, Peter Shor proved that it is possible to factor a n -bit number in time that is **polynomial** to n .

The factorization problem

Given an integer n , find positive integers $p_1, p_2, \dots, p_m, r_1, r_2, \dots, r_m$ such that

- Integers p_1, p_2, \dots, p_m are distinct **primes**;
- and, $n = p_1^{r_1} \times p_2^{r_2} \times \dots \times p_m^{r_m}$.

Note that one may assume n to be odd and contain at least two distinct odd prime factors (why?)

Factorization

Since the **test for primality** can be done **classically** in polynomial time, the **factoring problem** can be **reduced** to a $\mathcal{O}(\log n)$ instances of the following problem:

The odd non-prime-power integer splitting problem

Given an odd integer n , with at least two distinct prime factors, compute two integers

$$1 < n_1 < n \quad \text{and} \quad 1 < n_2 < n$$

$$\text{st } n = n_1 \times n_2$$

Factorization

Miller proved in 1975 that this problem **reduces probabilistically** to another problem whose solution resorts to the **eigenvalue estimation problem**, already studied.

The order-finding problem

Given two coprime integers a and n (i.e. st $\gcd(a, n) = 1$), find the **order of a modulo n** .

Preliminaries

Order of an element in a group

The order of an element a in a group $G = (A, \theta, e, {}^{-1})$ is the **least positive integer r such that $a^r = e$** , if any such r exists

Examples

- Every element of the **permutation group of degree 4**

(bijections onto $\{1, 2, 3, 4\}, \cdot, id, {}^{-1}$)

has order 4. For example, consider element

$(1, 2, 3, 4) = \{1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 1\}$

$$(1, 2, 3, 4)^1 = (1, 2, 3, 4) \neq id$$

$$(1, 2, 3, 4)^2 = (1, 3)(2, 4) \neq id$$

$$(1, 2, 3, 4)^3 = (1, 4, 3, 2) \neq id$$

$$(1, 2, 3, 4)^4 = id$$

Preliminaries

- In $\mathcal{Z} = (\mathbb{Z}, \times, 1, {}^{-1})$ every element but 0 has order ∞ .
- Consider the group of **integers modulo n** ,

$$\mathcal{Z}_n = (\{0, 1, 2, \dots, n-1\}, \times_n, 1, {}^{-1})$$

Note then when defining the order of a as the smallest positive integer r such that $a^r = 1$, the exponentiation is taken **modulo n** , and therefore the equality can be written as

$$a^r = 1 \pmod{n}$$

where $x = y \pmod{n}$ abbreviates $\text{rem}(x - y)n = 0$, i.e. $\text{rem}(x)n = \text{rem}(y)n$, which is the **equality** in \mathcal{Z}_n

So, e.g. the order of 4 in \mathcal{Z}_5 is **2** because

$$4^1 = 4 \pmod{5}$$

$$4^2 = 1 \pmod{5}$$

The problem

Note that any integer a st $\gcd(a, n) = 1$ the number 1 will appear somewhere in the sequence

$$\text{rem}(a, n), \text{rem}(a^2, n), \text{rem}(a^3, n), \dots$$

after what the sequence repeats itself in a periodic way.

The order-finding problem

Given two coprime integers a and n (i.e. st $\gcd(a, n) = 1$), find the **order of a modulo n** , i.e. the smallest positive integer r such that

$$a^r = 1 \pmod{n}$$

Strategy: The eigenvalue approach

Consider the following operator:

$$U_a(|q\rangle) = |\text{rem}(qa, n)\rangle \quad \text{for } 0 \leq q < n$$

Clearly, U_a is unitary: being a coprime with n , a has an inverse modulo n and, thus, is reversible.

Note that U_a can be extended reversibly to an implementation in a circuit over m qubits ($2^m > n$) making

$$U_a(|q\rangle) = |\text{rem}(qa, n)\rangle \quad \text{for } 0 \leq q < n$$

$$U_a(|q\rangle) = |q\rangle \quad \text{for } q \geq n$$

In any case, let us focus on the action of U_a restricted to the state space spanned by $\{|0\rangle, |1\rangle, \dots, |n-1\rangle\}$.

Strategy: The eigenvalue approach

Since $a^r = 1 \pmod{n}$,

$$U_a^r(|q\rangle) = |\text{rem}(qa^r, n)\rangle = |q\rangle$$

i.e. U_a is the r th root of the identity operator I , i.e. $U_a^r = I$.

It can be shown that the eigenvalues λ of such an operator satisfy $\lambda^r = 1$, i.e. they are the r th root of 1, which means they take the form $e^{2\pi i \frac{k}{r}}$, for some integer k .

Thus, suppose one is able to prepare the state

$$|u_k\rangle = \frac{1}{\sqrt{r}} \sum_{q=0}^{r-1} e^{-2\pi i \frac{k}{r} q} |\text{rem}(a^q, n)\rangle$$

Strategy: The eigenvalue approach

$$\begin{aligned}U_a|u_k\rangle &= \frac{1}{\sqrt{r}} \sum_{q=0}^{r-1} e^{-2\pi i \frac{k}{r} q} U_a |\text{rem}(a^q, n)\rangle \\&= \frac{1}{\sqrt{r}} \sum_{q=0}^{r-1} e^{-2\pi i \frac{k}{r} q} |\text{rem}(a^{q+1}, n)\rangle \\&= e^{2\pi i \frac{k}{r}} \frac{1}{\sqrt{r}} \sum_{q=0}^{r-1} e^{-2\pi i \frac{k}{r} (q+1)} |\text{rem}(a^{q+1}, n)\rangle \\&= e^{2\pi i \frac{k}{r}} |u_k\rangle\end{aligned}$$

Observing that, for the last step, we have

$$e^{2\pi i \frac{k}{r} r} |\text{rem}(a^{q+1}, n)\rangle = e^{2\pi i \frac{k}{r} 0} |\text{rem}(a^0, n)\rangle$$

Strategy: The eigenvalue approach

... concluding that

$$|u_k\rangle \text{ is an eigenstate for } U_a \text{ with eigenvalue } e^{2\pi i \frac{k}{r}}$$

Thus, for any value $0 \leq k \leq r-1$, the eigenvalue estimation algorithm will compute an approximation $\widetilde{k/r}$ to $\frac{k}{r}$ mapping

$$|0\rangle|u_k\rangle \mapsto |\widetilde{k/r}\rangle|u_k\rangle$$

However ...

Without knowing r we do not know how to prepare $|u_k\rangle$.

Fortunately, it is not necessary!

Strategy: The eigenvalue approach

Instead of preparing an eigenstate corresponding to an eigenvalue $e^{2\pi i \frac{k}{r}}$ for a randomly selected $k \in \{0, 1, \dots, r-1\}$, it suffices to prepare a uniform superposition of the eigenstates

Then the eigenvalue estimation algorithm will compute a superposition of these eigenstates entangled with estimates of their eigenvalues.

Thus, when a measurement is performed, the result is an estimate of a random eigenvalue.

Question

How to prepare such a superposition without knowing r ?

Strategy: The eigenvalue approach

The uniform superposition is

$$\frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} |u_k\rangle = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} \frac{1}{\sqrt{r}} \sum_{q=0}^{r-1} e^{-2\pi i \frac{k}{r} q} |\text{rem}(a^q, n)\rangle$$

Note that

$$|\text{rem}(a^q, n)\rangle = |1\rangle \text{ iff } \text{rem}(q, r) = 0$$

Thus, the amplitude of $|1\rangle$ in the above state is the sum over the terms for which $q = 0$

(because q takes values in $[0, r-1]$ and must be a multiple of r)

$$\frac{1}{\sqrt{r}} \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-2\pi i \frac{k}{r} 0} = \frac{1}{r} \sum_{k=0}^{r-1} 1 = 1$$

Strategy: The eigenvalue approach

If the amplitude of $|1\rangle$ is 1, this means that the amplitudes of all other basis states are 0, yielding

$$\frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} |u_k\rangle = |1\rangle$$

Thus, the eigenvalue estimation algorithm maps

$$|0\rangle|1\rangle = |0\rangle \left(\frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} |u_k\rangle \right) = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} |0\rangle |u_k\rangle \mapsto \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} \widetilde{|k/r\rangle} |u_k\rangle$$

Strategy: The eigenvalue approach

Thus, after executing the eigenvalue estimation algorithm the first register contains a **uniform superposition** of states $\widetilde{|k/r\rangle}$ for $k \in \{0, 1, \dots, r-1\}$.

Measuring this register yields an integer x st $\frac{x}{2^n}$ is an estimate of $\frac{k}{r}$ for **some** k selected uniformly at random.

Finally, to estimate r one resorts to the following result in **number theory**:

Estimating r

Theorem: Let r be a positive integer, and take integers k_1 to k_2 selected independently and uniformly at random from $\{0, 1, \dots, r-1\}$. Let c_1, c_2, r_1, r_2 be integers st $\gcd(r, c_1) = \gcd(r, c_2) = 1$ and

$$\frac{k_1}{r} = \frac{c_1}{r_1} \quad \text{and} \quad \frac{k_2}{r} = \frac{c_2}{r_2}$$

Then, $r = \text{lcm}(r_1, r_2)$ with probability at least $\frac{6}{\pi^2}$.

Thus

- To obtain $\frac{c_1}{r_1}$ from $\widetilde{k/r}$, i.e. the nearest fraction approximating $\frac{k}{r}$ up to some precision dependent on the number of qubits used, one resorts to the **continued fractions** method.
- As a second pair (c_2, r_2) is needed, the whole algorithm is repeated.

The order-finding algorithm

1. Prepare a n -qubit register, identified as the control register, for an integer n st $2^n > 2r^2$, with $|0\rangle$ over n qubits.
2. Prepare a n -qubit register, identified as the target register, with $|1\rangle$.
3. Apply QFT to the control register, followed by cU_a^x to the target and control registers, and then QFT^{-1} to the control register.
4. Measure the control register to retrieve an estimate $\frac{x_1}{2^n}$ of a random integer multiple of $\frac{1}{r}$.
5. With the continued fractions method obtain integers c_1, r_1 such that

$$\left| \frac{x_1}{2^n} - \frac{c_1}{r_1} \right| \leq \frac{1}{2^{\frac{n-1}{2}}}$$

Fail otherwise.

6. Repeat steps 1. to 6. to find another integer x_2 , and a second pair (c_2, r_2) st $\left| \frac{x_2}{2^n} - \frac{c_2}{r_2} \right| \leq \frac{1}{2^{\frac{n-1}{2}}}$. Fail otherwise.
7. Compute $r = \text{lcm}(r_1, r_2)$. If $\text{rem}(a^r, n) = 1$ output r ; fail otherwise.

Afterthoughts

How can the algorithm fail?

- The eigenvalue estimation algorithm produces a **bad** estimate of $\frac{k}{r}$. This occurs with a **bounded** probability that can be made smaller by an increase in the size of the circuit.
- The value found is not r itself, but a **factor** of r , which will be the case if the computed c_1, c_2 have **common factors**, eventually requiring additional repetitions of the algorithm

Recall

Like all quantum algorithms, this one is **probabilistic**: it gives the correct answer with high probability, and the probability of failure can be decreased by repeating the algorithm.

Afterthoughts

Cost

$\mathcal{O}((\log n)^3)$, the major cost coming from the modular exponentiation:

- The critical computation is the $cU_a^{2^j}$ operations, for $j \in \{0, 1, 2, \dots, 2^{n-1}\}$, which constitutes cU_a^x and requires 2^j applications of operator U_a .
- However, $cU_a^{2^j} = cU_{a^{2^j}}$ — multiplying by $\text{rem}(a, n)$ for 2^j times is equivalent to multiplying by $\text{rem}(a^{2^j}, n)$ only once.
- $\text{rem}(a^{2^j}, n)$ can be computed with j multiplications modulo n (exponential improvement over multiplying $\text{rem}(a, n)$ for 2^j times).
- *QFT* requires $\mathcal{O}(\log n)^2$ gates.

The classical algorithm is **exponential** on n : the best known one uses $e^{\mathcal{O}(\sqrt{\log n} \sqrt{(\log \log(n))})}$ classical gates.

Reducing to order-finding

The odd non-prime-power integer splitting problem

Given an odd integer n , with at least two distinct prime factors, compute two integers

$$1 < n_1 < n \quad \text{and} \quad 1 < n_2 < n$$

$$\text{st } n = n_1 \times n_2$$

Miller proved in 1975 that this problem **reduces probabilistically** to the **order-finding problem**, all reductions being **classical**: only the **sampling estimates problem** is quantum.

Reduction to order-finding

- To split n , choose randomly, with uniform probability, an integer a and compute its order r such that a and n are coprime (test a from $\{2, 3, \dots, n-2\}$). If they are not coprime, their greatest common divisor is already a non-trivial factor of n .
- If r is even (it will be with at least a probability of 0.5), $a^r - 1$ can be factorized as

$$a^r - 1 = (a^{\frac{r}{2}} - 1)(a^{\frac{r}{2}} + 1)$$

- As r is the order of a , n divides $a^r - 1$, which means n must share a factor with $(a^{\frac{r}{2}} - 1)$, or $(a^{\frac{r}{2}} + 1)$, or both.

This factor can be extracted by the Euclidean algorithm which efficiently returns $\gcd(a^{\frac{r}{2}} - 1, n)$.

Question

But how can we be sure such a factor is non-trivial?

Reduction to order-finding

- Clearly n does not divide $(a^{\frac{r}{2}} - 1)$.

Actually, if $\text{rem}(a^{\frac{r}{2}} - 1, n) = 0$, $\frac{r}{2}$, rather than r , would be the order of a .

- However, n may divide $(a^{\frac{r}{2}} + 1)$, i.e. $a^{\frac{r}{2}} = 1 \pmod{n}$ and not share any factor with $(a^{\frac{r}{2}} - 1)$.

Thus, the reduction is probabilistic according to the following

Theorem: Let $n = p_1^{r_1} \times p_2^{r_2} \times \cdots \times p_m^{r_m}$ be the prime factorization of an odd number with $m \geq 2$. Then for a random a , chosen uniformly as before, the probability that its order is even and $a^{\frac{r}{2}} \neq -1 \pmod{n}$ is at least $(1 - \frac{1}{2^m}) \geq \frac{9}{16}$.

For number theoretic results see N. Koblitz. *A Course in Number Theory and Cryptography*, Springer, 1994.

Shor's algorithm

1. Choose $1 \leq a \leq n - 1$ randomly.
2. If $\gcd(a, n) > 1$, then return $\gcd(a, n)$.
3. If $\gcd(a, n) = 1$, then use the **order-finding** algorithm to compute r — the order of a wrt n .
4. If r is odd or $a^{\frac{r}{2}} = -1 \pmod{n}$ then return to 1.
else return $\gcd(a^{\frac{r}{2}} - 1, n)$ and $\gcd(a^{\frac{r}{2}} + 1, n)$.

Shor's algorithm

Shor's approach to [estimate a random integer multiple of \$\frac{1}{r}\$](#) in his original paper was different from the one discussed in this lecture, as an application of the [eigenvalue estimation algorithm](#).

Shor's approach (based on period finding)

- Create a state

$$\sum_{x=0}^{2^n-1} \frac{1}{\sqrt{2^n}} |x\rangle |\text{rem}(a^x, n)\rangle$$

which is shown to be re-written as

$$\sum_{b=0}^{r-1} \left(\frac{1}{\sqrt{2^n}} \sum_{z=0}^{m_b-1} |zr + b\rangle \right) |\text{rem}(a^x, n)\rangle$$

where m_b is the largest integer st $(m_b-1)r + b \leq 2^n - 1$.

Shor's algorithm

Shor's approach (based on period finding)

- Measuring the target register yields $\text{rem}(a^b, n)$ for b chosen uniformly at random from $\{0, 1, 2, \dots, r-1\}$, and leaves the control register in

$$\frac{1}{\sqrt{m_b}} \sum_{z=0}^{m_b-1} |zr + b\rangle$$

- Apply $QFT_{2^n}^{-1}$ to the control register

Note that, if r, m_b were known (!), applying $QFT_{m_b r}^{-1}$ would lead to

$$\sum_{j=0}^{r-1} e^{-2\pi i \frac{b}{r} j} |m_b j\rangle$$

i.e. only values x such that $\frac{x}{rm_b} = \frac{j}{r}$ would be measured.

- Measure x and output $\frac{x}{2^n}$.

Shor's algorithm

Note that in both approaches the circuit is the [same](#).

The only difference is the [basis](#) in which the state of the system is analysed:

- the eigenvector basis
- the computational basis in Shor's original algorithm.

Shor's original algorithm is based on the [period finding algorithm](#), which is another application of phase estimation (see [Nielsen & Chuang, 2010] for a complete account)

In all cases, the underlying quantum component is, of course, the [QFT](#).

Quantum algorithms

Recall the overall idea:

engineering quantum effects as computational resources

Classes of algorithms

- Algorithms with superpolynomial speed-up, typically based on the quantum Fourier transform, include Shor's algorithm for prime factorization. The level of resources (qubits) required is not yet currently available.
- Algorithms with quadratic speed-up, typically based on amplitude amplification, as in the variants of Grover's algorithm for unstructured search. Easier to implement in current NISQ technology, typical component of other algorithms.
- Quantum simulation

... and we are done!

Where to look further

- Quantum computation is an extremely **young and challenging** area, looking for young people either with a **theoretical** or **experimental** profile.
Get in touch if you are interested in pursuing studies/research in the area at UMinho, INESC TEC and INL.
- Follow-up courses next semester on
 - **Quantum Logic** (**calculi and logics for quantum programs**)
 - **Quantum Data Science** (**algorithms and exciting applications**)



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... and we are done!

Where to look further

Two research directions at INL

(dissertation themes coming around May)

- **Quantum Software Engineering (INESC TEC & INL)**: oriented towards the development of foundations and mathematical methods for Quantum Computer Science and Software Engineering and its application to strategic problem-areas.
- **Quantum and Linear-Optical Computation (INL)**: to explore the features of quantum theory that enable advantage in quantum information processing tasks, in particular those present in photonic implementations of quantum computers.



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Continued Fractions

Method to approximate any real number t with a sequence of rational numbers of the form

$$[a_0, a_1, \dots, a_p] \text{ defined by } a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_p}}}}$$

computed inductively as follows

$$\begin{aligned} a_0 &= \lfloor t \rfloor & r_0 &= t - a_0 \\ a_j &= \left\lfloor \frac{1}{r_{j-1}} \right\rfloor & r_j &= \frac{1}{r_{j-1}} - \left\lfloor \frac{1}{r_{j-1}} \right\rfloor \end{aligned}$$

The sequence $[a_0, a_1, \dots, a_p]$ is called the **p -convergent** of t .

If $r_p = 0$ the continued fraction terminates with a_p and

$$t = [a_0, a_1, \dots, a_p],$$

Continued Fractions

Example: $\frac{47}{13} = [3, 1, 1, 1, 1, 2]$

$$\begin{aligned}
 \frac{47}{13} &= 3 + \frac{8}{13} = 3 + \frac{1}{\frac{13}{8}} \\
 &= 3 + \frac{1}{1 + \frac{5}{8}} = 3 + \frac{1}{1 + \frac{1}{\frac{8}{5}}} \\
 &= 3 + \frac{1}{1 + \frac{1}{1 + \frac{3}{5}}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\frac{5}{3}}}} \\
 &= 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{2}{3}}}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}}}
 \end{aligned}$$

Continued Fractions

Theorem: The expansion **terminates** iff t is a **rational** number.
[which makes continued fractions the *right*, finite expansion for rational numbers, differently from decimal expansion]

Theorem: $[a_0, a_1, \dots, a_p] = \frac{p_j}{q_j}$ where

$$p_0 = a_0, q_0 = 1$$

$$p_1 = 1 + a_0 a_1$$

$$p_j = a_j p_{j-1} + p_{j-2}, \quad q_j = a_j q_{j-1} + q_{j-2}$$

Theorem: Let x and $\frac{p}{q}$ be rationals st

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{2q^2}.$$

Then, $\frac{p}{q}$ is a convergent of the continued fraction for x .