

Quantum Computation

Quantum phase estimation and the quantum Fourier transform

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Encoding information in phases

In several quantum algorithms **information** is encoded in the **relative phases** of a quantum state.

The effect of Hadamard (once again)

$$H|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle) = \frac{1}{\sqrt{2}} \sum_{y \in 2} (-1)^{xy} |y\rangle$$
$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in 2^n} (-1)^{x \cdot y} |y\rangle$$

is to encode information about the value of x into the phases $(-1)^{x \cdot y}$ of basis states $|y\rangle$.

Encoding information in phases

Of course, as a reversible gate, the Hadamard gate also **decodes** information from phases:

$$\begin{aligned} H^{\otimes n} \frac{1}{\sqrt{2^n}} \sum_{y \in 2^n} (-1)^{x \cdot y} |y\rangle &= H^{\otimes n} (H^{\otimes n} |x\rangle) \\ &= (H^{\otimes n} H^{\otimes n}) |x\rangle \\ &= I |x\rangle \\ &= |x\rangle \end{aligned}$$

Encoding information in phases

In general, phases are complex numbers

$$e^{2\pi i w}$$

for any real $w \in [0, 1[$.

Of course, $H^{\otimes n}$ cannot encode/decode information over such generic phases. The **general situation** can be described as follows:

The phase estimation problem

Determine a good estimation of the phase parameter w given a general quantum state

$$\frac{1}{\sqrt{2^n}} \sum_{y \in 2^n} e^{2\pi i w y} |y\rangle$$

An algorithm for phase estimation

Notation

$$w = 0.x_1x_2\cdots$$

is written in base 2 (i.e. $w = x_12^{-1} + x_22^{-2} + \cdots$); thus

$$2^k w = x_1x_2\cdots x_k.x_{k+1}x_{k+2}\cdots$$

and

$$\begin{aligned} e^{2\pi i(2^k w)} &= e^{2\pi i(x_1x_2\cdots x_k.x_{k+1}x_{k+2}\cdots)} \\ &= e^{2\pi i(x_1x_2\cdots x_k)} e^{2\pi i(0.x_{k+1}x_{k+2}\cdots)} \\ &= e^{2\pi i(0.x_{k+1}x_{k+2}\cdots)} \end{aligned}$$

because $e^{2\pi iz} = 1$ for any integer z .

Case A: 1-qubit state and $w = 0.x_1$

$$\begin{aligned}\frac{1}{\sqrt{2}} \sum_{y \in 2} e^{2\pi i (0.x_1)y} |y\rangle &= \frac{1}{\sqrt{2}} \sum_{y \in 2} e^{2\pi i (\frac{x_1}{2})y} |y\rangle \\ &= \frac{1}{\sqrt{2}} \sum_{y \in 2} e^{\pi i (x_1)y} |y\rangle \\ &= \frac{1}{\sqrt{2}} \sum_{y \in 2} (-1)^{x_1 y} |y\rangle \\ &= \frac{1}{\sqrt{2}} (|0\rangle + (-1)^{x_1} |1\rangle)\end{aligned}$$

Clearly H will decode and retrieve x_1 because

$$H \left(\frac{1}{\sqrt{2}} (|0\rangle + (-1)^{x_1} |1\rangle) \right) = |x_1\rangle$$

Case B: 2-qubit state and $w = 0.x_1x_2$

Observe that

$$\frac{1}{\sqrt{2^2}} \sum_{y \in 2^2} e^{2\pi i(0.x_1x_2)y} |y\rangle = \left(\frac{|0\rangle + e^{2\pi i(0.x_2)}|1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + e^{2\pi i(0.x_1x_2)}|1\rangle}{\sqrt{2}} \right)$$

which means that x_2 , but not x_1 , can be retrieved from the first qubit through an application of H .

The phase rotator

$$R_2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i(0.01)} \end{bmatrix}$$

where 0.01 is in base 2 (thus, equal to 2^{-2}).

Case B: 2-qubit state and $w = 0.x_1x_2$

Taking $x_2 = 1$ and applying the inverse of the **phase rotator** to the second qubit, yields

$$\begin{aligned} R_2^{-1} \left(\frac{|0\rangle + e^{2\pi i(0.x_11)}|1\rangle}{\sqrt{2}} \right) &= \begin{bmatrix} 1 & 0 \\ 0 & e^{-2\pi i(0.01)} \end{bmatrix} \left(\frac{|0\rangle + e^{2\pi i(0.x_11)}|1\rangle}{\sqrt{2}} \right) \\ &= \frac{|0\rangle + e^{2\pi i(0.x_11-0.01)}|1\rangle}{\sqrt{2}} \\ &= \frac{|0\rangle + e^{2\pi i(0.x_1)}|1\rangle}{\sqrt{2}} \end{aligned}$$

Concluding

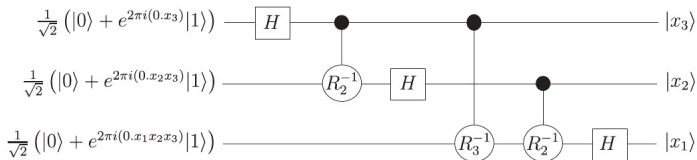
- x_1 can now be determined by an application of H , as before.
- Moreover, the decision to apply R before the application of H depends on x_2 being 1 or 0, respectively.
- Thus, to find $w = 0.x_1x_2$ it is enough to apply a **controlled** version of R , precisely controlled by the state of the first qubit.

Case C: 3-qubit state and $w = 0.x_1x_2x_3$

The state is now

$$\begin{aligned} \frac{1}{\sqrt{2^3}} \sum_{y \in \mathbb{Z}^3} e^{2\pi i (0.x_1x_2x_3)y} |y\rangle &= \\ &= \left(\frac{|0\rangle + e^{2\pi i (0.x_3)} |1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + e^{2\pi i (0.x_2x_3)} |1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + e^{2\pi i (0.x_1x_2x_3)} |1\rangle}{\sqrt{2}} \right) \end{aligned}$$

In this case the third qubit has to **conditionally rotate** both x_2 and x_3 , leading to the following circuit



Going generic

Gate R_3 in the circuit is an instance of a 1-qubit phase rotator

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^k}} \end{bmatrix}$$

whose inverse acts as

$$\begin{aligned} R_k^{-1}|0\rangle &= |0\rangle \\ R_k^{-1}|1\rangle &= e^{-2\pi i(0.0\dots 1)}|1\rangle \end{aligned}$$

with 1 in $0.0\dots 1$ appearing in position k .

Going generic

The output state of the circuit is

$$|x_3 x_2 x_1\rangle$$

Thus, relabelling the qubits in **reverse** order, this provides an efficient circuit to estimate the phase (actually, to give a totally accurate estimation ...), by computing

$$\frac{1}{\sqrt{2^n}} \sum_{y \in 2^n} e^{2\pi i (\frac{x}{2^n}) y} |y\rangle \rightsquigarrow |x\rangle$$

Inverting ...

The inverse of the [phase estimation transformation](#) computes

$$|\mathbf{x}\rangle \rightsquigarrow \frac{1}{\sqrt{2^n}} \sum_{y \in 2^n} e^{2\pi i (\frac{\mathbf{x}}{2^n})y} |y\rangle$$

which is obtained by taking the inverses of each gate and building the circuit in reverse order.

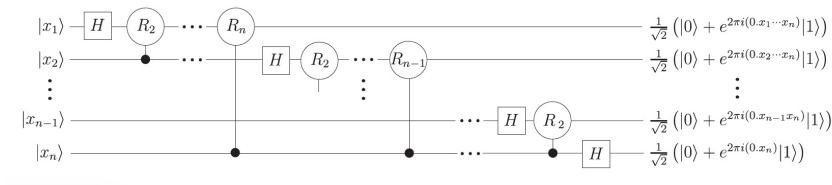
The result is formally identical to the [discrete Fourier transform](#).

The quantum Fourier transform

QFT on basis states $|0\rangle, |1\rangle \dots |k-1\rangle$

$$QFT_k(|x\rangle) = \frac{1}{\sqrt{k}} \sum_{y=0}^{k-1} e^{2\pi i (\frac{x}{k})y} |y\rangle$$

The circuit



The quantum Fourier transform

Complexity (number of gates)

- one H plus $n - 1$ conditional rotations on the first qubit
- one H plus $n - 2$ conditional rotations on the second qubit
- ...

$$n + (n - 1) + (n - 2) + \dots + 1 = \frac{n(n - 1)}{2}$$

- plus $\frac{n}{2}$ swaps (each implemented by 3 CNOT gates)

Thus

$$\frac{n(n - 1)}{2} + 3 \times \frac{n}{2} = \frac{n^2 + 2n}{2} \approx \mathcal{O}(n^2)$$

The quantum Fourier transform

Complexity (number of gates)

$$\frac{n(n-1)}{2} + 3 \times \frac{n}{2} = \frac{n^2 + 2n}{2} \approx \mathcal{O}(n^2)$$

which compares to the **classical** case for the **Fast FT**: $\mathcal{O}(n2^n)$

The result is **impressive**: the quantum version requires **exponentially** less operations to compute the Fourier transform than the (best) classical one.

- However, typical uses (e.g. in speech recognition) are **limited** by the impossibility of directly measuring the Fourier transformed amplitudes of the original state.
- This requires a **subtler** use of QFT in practice: the phase estimation procedure, underlying many quantum algorithms, is one of them.

Are we done?

- The circuit for QFT_k computes the QFT for k a power of 2, i.e. $k = 2^n$
- The phase estimation algorithm works only when the phase is of the form $w = 0.x_1x_2 \cdots x_n$, i.e. $\frac{x}{2^n}$ for some integer x

However, it can be shown that, for an arbitrary w , the algorithm will compute x such that $\frac{x}{2^n}$ is closest to w with high probability.

The question

What is the error emerging when w is not an integer multiple of $\frac{1}{2^n}$?

Are we done?

QFT^{-1} computes some superposition

$$\sum_x \alpha_x(\textcolor{red}{w})|x\rangle$$

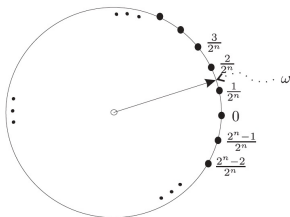
which represents the values of x that once measured gives a good estimate of $\textcolor{red}{w}$, outputting x with probability $|\alpha_x(\textcolor{red}{w})|^2$.

This output $\textcolor{blue}{x}$ corresponds to an estimate

$$\tilde{w} = \frac{\textcolor{blue}{x}}{2^n}$$

Are we done?

Consider w an integer **not** multiple of $\frac{1}{2^n}$, and let \hat{w} be the nearest integer multiple of $\frac{1}{2^n}$ to w , i.e. $\hat{w} = \frac{\hat{x}}{2^n}$ is the closest number of this form to w .



Theorem

The phase estimation algorithm returns \hat{x} with probability at least $\frac{4}{\pi^2}$, i.e. the algorithm outputs an estimate \hat{x} with the given probability such that

$$\left| \frac{\hat{x}}{2^n} - w \right| \leq \frac{1}{2^{n+1}}$$

Are we done?

Theorem

$$\text{If } \frac{x}{2^n} \leq w \leq \frac{x+1}{2^n}$$

The phase estimation algorithm returns either x or $x+1$ with probability at least $\frac{8}{\pi^2}$ i.e. the algorithm outputs an estimate \hat{x} with the given probability such that

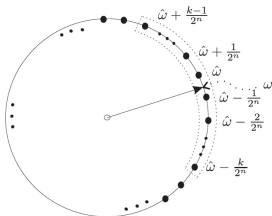
$$\left| \frac{\hat{x}}{2^n} - w \right| = \frac{1}{2^n}$$

The reverse question

How many qubits are required to get w accurate to n bits, with a probability p below a certain level?

Actually, the **crucial choice** is the value of n (number of qubits used) to ensure the estimation is close enough.

For $p = 1 - \frac{1}{2^{(k-1)}}$, the algorithm returns one of the $2k$ closest integer multiples of $\frac{1}{2^n}$, i.e.



which means that $|w - \hat{w}| \leq \frac{k}{2^n}$.

The reverse question

Thus, to estimate \hat{w} such that $|w - \hat{w}| \leq \frac{1}{2^r}$ with probability at least

$$1 - \frac{1}{2^m}$$

the maximum number of qubits required is

$$n = r + m + 1$$

- In practice a much smaller error is obtained: for example, with probability at least $\frac{8}{\pi^2}$, the error will be at most

$$\frac{1}{2^{r+m}}$$

Exercises

Recall the definition of QFT on K basis states:

$$QFT_K(|x\rangle) = \frac{1}{\sqrt{K}} \sum_{y=0}^{K-1} e^{2\pi i (\frac{x}{K})y} |y\rangle$$

Exercise 1

Compute $QFT_K(|00 \cdots 0\rangle)$.

Exercise 2

Verify the following equality, used in the slides but not proved.

$$QFT_K(|x_1 \cdots x_n\rangle) = \left(\frac{|0\rangle + e^{2\pi i (0 \cdot x_n)} |1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + e^{2\pi i (0 \cdot x_n x_{n-1})} |1\rangle}{\sqrt{2}} \right) \cdots \otimes \cdots \left(\frac{|0\rangle + e^{2\pi i (0 \cdot x_1 x_2 \cdots x_n)} |1\rangle}{\sqrt{2}} \right)$$

Exercises

Hint to Exercise 2: The case of QFT_4 applied to $|x\rangle = |x_1x_2\rangle$

$$\begin{aligned} QFT_4(|x\rangle) &= \frac{1}{2} \sum_{y=0}^3 e^{2\pi ixy2^{-2}} |y\rangle \\ &= \frac{1}{2} \sum_{y_1, y_2=0}^1 e^{2\pi ix(y_12^{-1}+y_22^{-2})} |y_1y_2\rangle \end{aligned}$$

because, for $|y\rangle = |y_1y_2\rangle$,

$$\frac{y}{2^n} = \sum_{j=1}^n y_j 2^{-j}$$

Exercises

Hint to Exercise 2: The case of QFT_4 applied to $|x\rangle = |x_1x_2\rangle$

$$\begin{aligned}\dots &= \frac{1}{2} \sum_{y_1, y_2=0}^1 (e^{2\pi i x y_1 2^{-1}} |y_1\rangle \otimes e^{2\pi i x y_2 2^{-2}} |y_2\rangle) \\&= \frac{1}{2} \sum_{y_1=0}^1 (e^{2\pi i x y_1 2^{-1}} |y_1\rangle \otimes \sum_{y_2=0}^1 e^{2\pi i x y_2 2^{-2}} |y_2\rangle) \\&= \frac{(|0\rangle + e^{2\pi i x 2^{-1}} |1\rangle)}{\sqrt{2}} \otimes \frac{(|0\rangle + e^{2\pi i x 2^{-2}} |1\rangle)}{\sqrt{2}} \\&= \frac{(|0\rangle + e^{2\pi i (x_1 \cdot x_2)} |1\rangle)}{\sqrt{2}} \otimes \frac{(|0\rangle + e^{2\pi i (0 \cdot x_1 x_2)} |1\rangle)}{\sqrt{2}} \\&= \frac{(|0\rangle + e^{2\pi i (0 \cdot x_2)} |1\rangle)}{\sqrt{2}} \otimes \frac{(|0\rangle + e^{2\pi i (0 \cdot x_1 x_2)} |1\rangle)}{\sqrt{2}}\end{aligned}$$

because, $e^{2\pi i(a.b)} = e^{2\pi i a} e^{2\pi i(0.b)} = e^{2\pi i(0.b)}$