Lecture 4: Naturality

Summary.
(1) Natural transformation: motivation and formal definition. Naturality as a source of genericity. Examples: parametric operators in programming.

(2) Vertical and horizontal composition of natural transformations. Functor categories.

(3) Natural isomorphisms and equivalence of categories.

(4) Small case study: revisiting Hom-functors and natural transformation between them. Brief mention to the Yoneda lemma.

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Opening.

If functors are arrows between categories, natural transformations can be regarded as arrows between functors. Historically, the concept seems to predate those of a functor or a category, to describe structural transformations which are canonical in the sense of being built without resorting to any sort of arbitrary choices. As T. Leinster puts it [2]:

> *In fact, it was the desire to formalize the notion of natural transformation that led to the birth of category theory. By the early 1940s, researchers in algebraic topology had started to use the phrase 'natural transformation', but only in an informal way. Two mathematicians, Samuel Eilenberg and Saunders Mac Lane, saw that a precise definition was needed. But before they could define natural transformation, they had to define functor; and before they could define functor, they had to define category. And so the subject was born.*

In programming natural transformations model parametric operators, i. e. operations defined in a way which does not depend on the argument basic types but only on the 'shape' used to organise them. For example, function $\mathsf{elems} : X^* \longrightarrow \mathcal{P}(X)$ which maps a sequence into the set of its elements does not depend on what X actually is. In the language of categories, elems can be regarded as a family of arrows $(\mathsf{elems}_A)_{A \,\mathrm{in}\, \mathsf{Set}}$ such that $\mathsf{elems}_A : A^* \longrightarrow \mathcal{P}(A)$ is uniformly defined, i.e. the following diagram commutes for any set A and function $f : A \longrightarrow B$:

$$\begin{tikzcd} A^* \arrow[r, "\mathsf{elems}_A"] \arrow[d, "f^*"'] & \mathcal{P}(A) \arrow[d, "\mathcal{P}(f)"] \\ B^* \arrow[r, "\mathsf{elems}_B"] & \mathcal{P}(B) \end{tikzcd}$$

The corresponding equation — $\mathcal{P}(f) \cdot \mathsf{elems}_A \;=\; \mathsf{elems}_B \cdot f^*$ — is the *naturality* condition.

Exercise 1

Recall from the Functional Programming course, functions elems, mentioned above, and $\frown : X^* \times X^* \longrightarrow X^*$ which merges two sequences. Prove their naturality.

Exercise 2

Show that $\eta : \mathrm{Id}_{\mathsf{Set}} \Longrightarrow \mathcal{P}$ mapping x to $\{x\}$ is a natural transformation. Draw the corresponding diagram.

Exercise 3

Let Mon be the category of monoids and monoid homomorphisms, and consider a functor $F : \mathsf{Set} \longrightarrow \mathsf{Mon}$ which builds a monoid freely from a set S, i. e.

$$S \;\mapsto\; (S^*, \frown, \mathsf{nil})$$

where $\frown$ is word (sequence) concatenation and nil denotes the empty word. What is the action of F on functions?

Let $U : \mathsf{Mon} \longrightarrow \mathsf{Set}$ be the forgetful functor (which, as the name says, 'forgets' the monoid structure). Show that $\epsilon : FU \Longrightarrow \mathrm{Id}_{\mathsf{Mon}}$ and $\eta : \mathrm{Id}_{\mathsf{Set}} \Longrightarrow UF$ defined by

$$\begin{tikzcd}
FU(M,\times,1) = (M^*,\frown,\mathsf{nil}) \arrow[r, "\epsilon_{(M,\times,1)}"] \arrow[d, "h^*"'] & (M,\times,1) \arrow[d, "h"] \\
FU(N,+,0) = (N^*,\frown,\mathsf{nil}) \arrow[r, "\epsilon_{(N,+,0)}"] & (N,+,0)
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, "\eta_S"] \arrow[d, "f"'] & UF(S) = S^* \arrow[d, "f^*"] \\
T \arrow[r, "\eta_T"] & UF(T) = T^*
\end{tikzcd}$$

$$\epsilon_{(P,\theta,u)}(p_1 p_2 \cdots p_n) \;\widehat{=}\; p_1 \theta p_2 \theta \cdots \theta p_n \qquad\qquad \eta_S(s) \;\widehat{=}\; s$$

are indeed natural transformations.

Observe that each η component, $\eta_S : S \longrightarrow S^*$, is universal in the following sense: for any function $f : S \longrightarrow M$, from S to the carrier M of a monoid $(M,\times,1)$, there is a unique function f' making the following diagram to commute:

$$\begin{tikzcd}
S \arrow[r, "\eta_S"] \arrow[dr, "f"'] & S^* \arrow[d, "f'"] \\
& M
\end{tikzcd}$$

Moreover, $f' \;=\; U(\;\; (S^*,\frown,\mathsf{nil}) \xrightarrow{f^*} (M^*,\frown,\mathsf{nil}) \xrightarrow{\epsilon_{(M,\times,1)}} (M,\times,1) \;\;)$.

$\boxed{\textbf{Exercise } 4}$

There is a functor $(-)^* : \mathsf{Vect}_K^{op} \longrightarrow \mathsf{Vect}_K$ that caries a K-vector space to its *dual* vector space

$$V^* = \mathrm{Hom}(V, K)$$

A vector in the dual space V^* is a linear transformation from V to K (i.e. a linear functional on V). Convince yourself that this functor is contravariant: it sends a linear map $h : V \longrightarrow W$ to $h^* : W^* \longrightarrow V^*$ which *pre-composes* any linear map $f : W \longrightarrow K$ with h to obtain $f \cdot h : V \longrightarrow K$.

Clearly, for the finite-dimensional case, vector spaces V and V^* are isomorphic because both have the same dimension. This can proved through the explicit construction a basis for V^* starting from a basis $\{b_i\}_{i \le n}$ for V. Indeed each element b_i^* of the dual basis corresponds to the Dirac function δ_i. The map realting both bases extends by linearity to an isomorphism between V and its dual space. However, the identity functor on Vect_K and the dual functor are not naturallly isomorphic. The failure of naturallity is precisely related to the need for a concrete (non 'parametric') choice of a basis[1].

On the other hand, for vector spaces of any dimension, the map

$$\mathsf{ev}_V : V \longrightarrow V^{**}$$

that sends a vector $v \in V$ to the linear function $f \mapsto f(v) : V^* \longrightarrow K$, defines a natural transformation from the identity functor on Vect_K to the double dual functor $(-)^{**}$. Verify this statement by checking the commutativity of the following diagram:

$$\begin{tikzcd} V \arrow[r, "\mathsf{ev}_V"] \arrow[d, "h"'] & V^{**} \arrow[d, "h^{**}"] \\ W \arrow[r, "\mathsf{ev}_W"] & W^{**} \end{tikzcd}$$

$\boxed{\textbf{Exercise } 5}$

Define an endofunctor $\Delta_\otimes$ in the category Vect_K of vector spaces over a field K mapping a vector space U to $U \otimes U$, for $\otimes$ denoting the tensor product. Show that there is a natural transformation $\zeta : \mathrm{Id}_{\mathsf{Vect}_K} \Longrightarrow \Delta_\otimes$ whose components map each vector to 0 (the additive identity).

Observe this is the only natural transformation that can be defined between the two functors: actually, there is no basis-independent way to define a linear map from U to $U \otimes U$[2].

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[1] A proof can be found in one of the initial , seminal papers in category theory by Eilenberg and Mac Lane [1].

[2] This observation, which also holds in the category of Hilbert spaces, the classical semantic universe for quantum computing, is related to what is called the no-cloning theorem in that setting.

$\boxed{\textbf{Exercise } 6}$

Let $\mathbf{2}$ be the discrete category with two objects. Observe that a functor from $\mathbf{2}$ to a category C is a pair of objects of C, and a natural transformation is a pair of maps. Show that the functor category $C^{\mathbf{2}}$ is therefore isomorphic to the product category $C \times C$.

$\boxed{\textbf{Exercise } 7}$

Functors between posets regarded as categories are monotone functions. Show that between two such functions f and g, between posets P and Q, there exists at most a natural transformation iff, for all $x \in P$, $f(x) \leq g(x)$.

$\boxed{\textbf{Exercise } 8}$

Let T, S and R be functors from category C to D, and consider the following natural transformations $\sigma : T \Longrightarrow S$ and $\sigma' : S \Longrightarrow R$. Then, σ and σ' can be composed originating $\sigma' \cdot \sigma : T \Longrightarrow R$, by defining

$$(\sigma' \cdot \sigma)_X \;=\; \sigma'_X \cdot \sigma_X$$

as illustrated in the diagram

$$\begin{tikzcd} C \arrow[r, bend left=50, "T"{name=U}] \arrow[r, "S"'{name=M}] \arrow[r, bend right=50, "R"'{name=L}] & D \arrow[from=U, to=M, phantom, "\Downarrow\sigma"] \arrow[from=M, to=L, phantom, "\Downarrow\sigma'"] \end{tikzcd}$$

This is known as the *vertical* composition of natural transformations.

Show that the *functor category*, D^C, of functors from C to D and natural transformations is a category indeed. Notice that, for each functor T in D^C, the family of identity arrows on $T(X)$ in D gives rise to a trivial natural transformation denoted by $\mathtt{id}_T$, which acts as an identity in D^C.

$\boxed{\textbf{Exercise } 9}$

There is also a notion of composition for natural transformations between pairs of composable functors. It will be denoted by by ; and used in diagrammatic order. Suppose T and S are functors from C to D and T' and S' are functors from D to E. If there exist natural transformations $\sigma : T \Longrightarrow S$ and $\sigma' : T' \Longrightarrow S'$, their *horizontal* composite is $\sigma \,;\, \sigma' : T'\,T \Longrightarrow S'\,S$ whose component at X is given by

$$(\sigma \,;\, \sigma')_X \;= S'\,\sigma_X \cdot \sigma'_{T\,X} \;=\; \sigma'_{S\,X} \cdot T'\,\sigma_X$$

as, by definition of σ and σ', the following diagram commutes:

$$\begin{tikzcd}
T'T(X) \arrow[r, "\sigma'_{T(X)}"] \arrow[d, "T'(\sigma_X)"'] \arrow[dr, "(\sigma;\sigma')_X"] & S'S(X) \arrow[d, "S'(\sigma_X)"] \\
T'S(X) \arrow[r, "\sigma'_{S(X)}"'] & S'S(X)
\end{tikzcd}$$

Particular cases of this situation occur when σ or σ' are the identity $\mathtt{id}_R$ on a functor R. We may, then, pre- or post-compose σ with $\mathtt{id}_R$, yielding

$$R\sigma \stackrel{\text{abv}}{=} \sigma\,;\mathtt{id}_R : RT \longrightarrow RS \qquad \text{with} \quad (R\sigma)_X = R(\sigma_X)$$

$$\sigma R \stackrel{\text{abv}}{=} \mathtt{id}_R\,;\sigma : TR \longrightarrow SR \qquad \text{with} \quad (\sigma R)_X = \sigma_{R(X)}$$

where $T, S : C \longrightarrow D$, $\sigma : T \Longrightarrow S$ and R is a functor from D to E, in the first case, and from B to C, in the second.

Show that the horizontal composition of two natural transformations still is a natural transformation.

Observe that vertical and horizontal composition of natural transformations interact via the *interchange law*:

$$(\sigma\,;\sigma')\cdot(\gamma\,;\gamma') \;=\; (\gamma\cdot\sigma)\,;(\sigma'\cdot\gamma')$$

which gives an unambiguous meaning to the diagram

$$\begin{tikzcd}
C \arrow[r, bend left=50, "T"{name=T}] \arrow[r, "S"{name=S, description}] \arrow[r, bend right=50, "R"'{name=R}] \arrow[Rightarrow, from=T, to=S, "\gamma"] \arrow[Rightarrow, from=S, to=R, "\sigma"] & D \arrow[r, bend left=50, "T'"{name=T2}] \arrow[r, "S'"{name=S2, description}] \arrow[r, bend right=50, "R'"'{name=R2}] \arrow[Rightarrow, from=T2, to=S2, "\gamma'"] \arrow[Rightarrow, from=S2, to=R2, "\sigma'"] & E
\end{tikzcd}$$

This pattern often occurs in Computer Science, namely to relate temporal (parallel) and spatial (sequential) composition of a system's behaviour.

Exercise 10

Two categories C and D are *equivalent* if there exists functors $F : D \longrightarrow C$ and $E : C \longrightarrow D$, and natural isomorphisms $\sigma : \mathrm{Id}_C \Longrightarrow FE$ and $\gamma : \mathrm{Id}_D \Longrightarrow EF$, called the *pseudo inverses*.

Compare with the notion of isomorphism of categories and explain why an equivalence of categories is often called an *isomorphism up to an isomorphism*.

Exercise 11

Show that the notion of equivalence of categories is in fact an equivalence relation.

Exercise 12

Consider the following two categories suitable to represent the universe of partial functions:

- Pfn is the category of partial functions: objects are sets; an arrow f from A to B is a pair $(\mathrm{dom}\, f \subseteq A, |f| : \mathrm{dom}\, f \longrightarrow B)$. The composition of two arrows f and g suitably typed is the pair $(\mathrm{dom}\,(g \cdot f) = f^{-1}(\mathrm{dom}\, g) \subseteq A, |g \cdot f| : \mathrm{dom}\,(g \cdot f) \longrightarrow C)$.
- $\mathsf{Set}_\perp$ is the category of pointed sets (A, a), with $a \in A$, whose arrows $f : (A, a) \longrightarrow (B, b)$ are functions such that $f(a) = b$. Composition and identities are as is Set.

Pfn and $\mathsf{Set}_\perp$ are related through the functors $S : \mathsf{Pfn} \longrightarrow \mathsf{Set}_\perp$ and $T : \mathsf{Set}_\perp \longrightarrow \mathsf{Pfn}$ defined by

$$
\begin{aligned}
S(A) &\mathrel{\widehat{=}} (A \cup \{\perp\}, \perp) \\
S(f)(x) &\mathrel{\widehat{=}} \begin{cases} f(x) & \Leftarrow\ x \in \mathrm{dom}\, f \\ \perp & \Leftarrow\ \text{otherwise} \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
T((A, a)) &\mathrel{\widehat{=}} A - \{a\} \\
T(f : (A, a) \longrightarrow (B, b)) &\mathrel{\widehat{=}} \begin{cases} \mathrm{dom}\, T(f) & = A - f^{-1}(b) \\ T(f)(x) & = f(x) \end{cases}
\end{aligned}
$$

Show that Pfn and $\mathsf{Set}_\perp$ are equivalent categories. Discuss why the notion of isomorphism of categories does not apply in this case.

Exercise 13

Let C be a locally small category and W an object of C. C can be *represented by* W in Set through the so-called hom-functors, defined as follows:

$$C \xrightarrow{\mathrm{Hom}_C(W,-)} \mathsf{Set} \qquad\qquad C^{op} \xrightarrow{\mathrm{Hom}_C(-,W)} \mathsf{Set}$$

$$
\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathrm{Hom}_C(W,X) \arrow[d, "\mathrm{Hom}_C(W,f)=f_*=f\cdot -"] \\
Y \arrow[r, mapsto] & \mathrm{Hom}_C(W,Y)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathrm{Hom}_C(X,W) \\
Y \arrow[r, mapsto] & \mathrm{Hom}_C(Y,W) \arrow[u, "\mathrm{Hom}_C(f,W)=f^*=-\cdot f"']
\end{tikzcd}
$$

where $f_*(g) = f \cdot g$ and $f^*(g) = g \cdot f$. Verify that both constructions are indeed functors. Explain why the restriction to locally small categories is necessary.

Similarly, one may define a bifunctor (i.e. a functor of two variables)

$$\mathrm{Hom}_C(-,-) : C^{op} \times C \longrightarrow \mathsf{Set}$$

such that any pair of arrows $f : X' \longrightarrow X$ and $g : Y \longrightarrow Y'$ is mapped to

$$(\mathrm{Hom}_c(f, g) : X' \longrightarrow Y') : \mathrm{Hom}_C(X, Y) \longrightarrow \mathrm{Hom}_C(X', Y')$$

such that $\mathrm{Hom}_c(f, g)(h : X \longrightarrow Y) \widehat{=} \; g \cdot h \cdot f$, i.e. $\mathrm{Hom}_C(f, g) = g \cdot - \cdot f$.

Exercise 14

The lemma of Yoneda is one of the most useful results in Category Theory. This exercise invites the reader to approach its core while playing with natural transformations. Informally, the message is that every natural transformation between a Hom-functor and another functor also valued in Set can be determined by a single object in the source category. More generally, its relevance comes from making explicit a representation of generic mathematical constructions through functors valued in the category of sets, and therefore reducing proofs of isomorphisms between categorial constructions to the definition of bijections between their set-theoretical analogs[3].

For any locally small category C, the lemma establishes a bijective correspondence between the set of natural transformations from $\mathrm{Hom}_C(W, -) : C \longrightarrow \mathsf{Set}$ and an arbitrary functor $F : C \longrightarrow \mathsf{Set}$, and set $F(W)$. Let us describe the two components of such a bijection.

For each element $\omega \in F(W)$ define a natural transformation $\gamma^\omega : \mathrm{Hom}_C(W, -) \Longrightarrow F$ by

$$\gamma^\omega_X\,(h : W \longrightarrow X) \;\widehat{=}\; F(h)(\omega)$$

It is easy to verify the naturality of γ^ω by showing that the diagram below commutes, for any C-arrow $f : X \longrightarrow Y$:

$$\begin{tikzcd}
\mathrm{Hom}_C(W, X) \arrow[r, "\gamma^\omega_X"] \arrow[d, "\mathrm{Hom}_C(W,f)"'] & F(X) \arrow[d, "F(f)"] \\
\mathrm{Hom}_C(W, Y) \arrow[r, "\gamma^\omega_Y"] & F(Y)
\end{tikzcd}$$

Actually,

$$\gamma^\omega_Y \cdot \mathrm{Hom}_C(W, f)\,(h : W \longrightarrow X) \;=\; \gamma^\omega_Y \cdot (f \cdot h) \;=\; F(f \cdot h)(\omega) \;=\; F(f) \cdot \gamma^\omega_X\,(h)$$

On the other hand, for each natural transformation $\eta : \mathrm{Hom}_C(W, -) \Longrightarrow F$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_C(W, W) \arrow[r, "\eta_W"] \arrow[d, "\mathrm{Hom}_C(W,f)"'] & F(W) \arrow[d, "F(f)"] \\
\mathrm{Hom}_C(W, X) \arrow[r, "\eta_X"] & F(X)
\end{tikzcd}$$

which, by naturality, commutes for any $f : W \longrightarrow X$, defines the (arbitrary) component of η at X as

$$\eta_X(f) \;=\; \eta_X(f \cdot \mathrm{id}_W) \;=\; \eta_X(\mathrm{Hom}_C(W, f)(\mathrm{id}_W) \;=\; F(f)(\eta_W(\mathrm{id}_W))$$

and, of course, $\eta_W(\mathrm{id}_W) \in F(W)$.

[3]In its essence, this is similar to the classical representation of an arbitrary abstract group by a subgroup of a permutation group.

1. Show that the correspondences $\omega \in FW \mapsto \eta^{\gamma}$ and $\eta \mapsto \eta_W(\mathrm{id}_W)$ are mutually inverse, thus establishing the isomorphism

$$\mathrm{Hom}_{\mathsf{Set}^C}(\mathrm{Hom}_C(W, -), F) \;\cong\; F(W)$$

and completing the proof.

2. The isomorphism above is natural. Write down the assertions that need be verified to establish the fact.

3. Formulate (possibly resorting to some help from a text book) the lemma of Yoneda for the contravariant Hom functor.

References

[1] S. Eilenberg and S. MacLane. General theory of natural equivalences. *Trans. Amer. Math. Soc.*, 58:231–294, 1945.

[2] T. Leinster. *Basic Category Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2014.