Diagrams and monoidal categories

Summary.
(1) Diagrams, processes and categories. Examples of process theories.
(2) Diagram composition and equivalence. Circuits.
(3) States, effects and scalars.

Luís Soares Barbosa,
Universidade do Minho (Informatics Department) & INESC TEC

Introduction.

We've studied categories as a general theory of mathematical structures, and emphasised the role of *morphisms*: entities are characterised in terms of how they are composed and interact, rather than of what they actually are. In computer science, as in many other application domains, it is also useful to think of a category as a *process theory*, i.e. collection of system *types*, a collection of *processes* linking them, and a discipline of *wiring* processes together, i.e. a form of *composition*. Diagrams provide a way to represent such theories in a pictorial, two-dimensional way (dually one may say a process theory as providing an interpretation of diagrams).

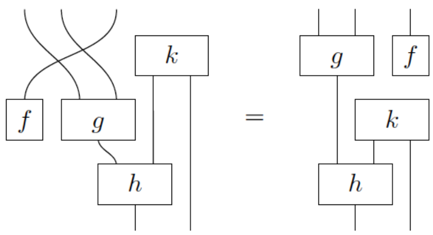
This set of lectures explores the application of such a perspective to understand, and reason within quantum informatics. Exposition is based on the so-called *quantum picturalism*, an approach to quantum foundations initiated by Samson Abramsky and Bob Coecke which gives prominence to diagramatic reasoning in the specification and transformation of quantum systems. Reference [1] is the basic textbook[1], while C. Heunen and J. Vicary [3] provides a more concise introduction.

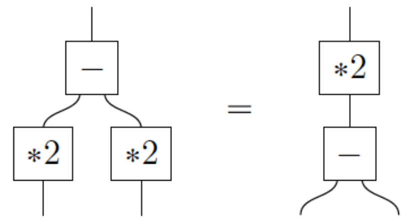
In this lecture we will introduce basic diagrams and examples of process theories embodied in the familiar categories Set and Rel. Two forms of composition — *vertical* (i.e. *sequential*) and *horizontal* (i.e. *parallel*) will be discussed. This leads to the theory of *monoidal categories* as the mathematical framework to capture the common structure of process theories. Graphical languages for monoidal categories are discussed in detail in [4] (which appears as chapter 4 in reference [2]).

[1] from which almost all pictures reproduced in the sequel were taken.

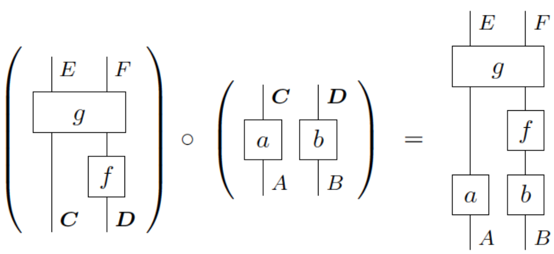
Diagrams.

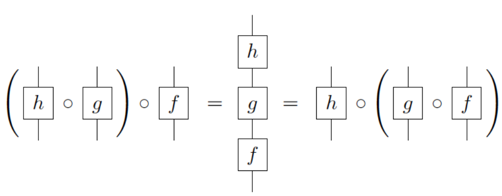
- Wires (system types), boxes (processes), and a slogan: *only connectivity matters.*
- Diagram equations (always hold regardless how boxes and diagrams are interpreted) vs process equations[2]:

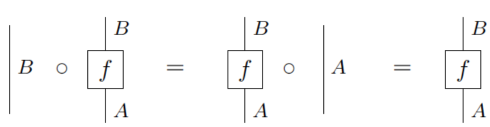




- Diagram equality: two diagrams are equal if they can be deformed into each other.
- Vertical (sequential) diagram composition: ·

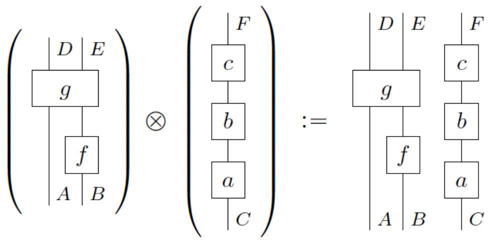


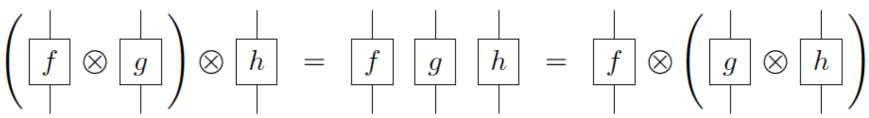


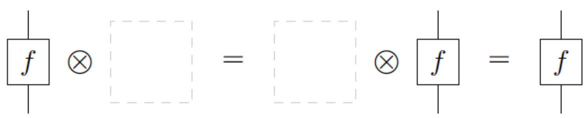


[2]Theories where diagram equations and process equations coincide are called *free* — recall the notion of a free group, free monoid, etc.

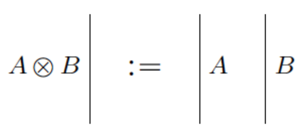
- Horizontal (parallel) diagram composition: $\otimes$ represented by juxtaposition.







Clearly, $\otimes$ is functorial:



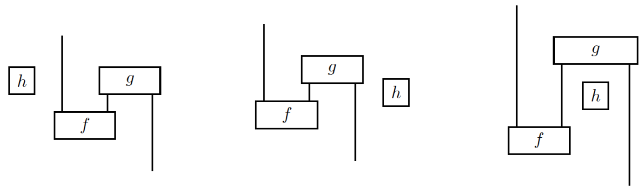
Diagramatic reasoning: algebraic laws are built-in!

$\boxed{\text{Example}}$ the interchange law $(\mathsf{h}_1 \otimes \mathsf{h}_2) \cdot (\mathsf{l}_1 \otimes \mathsf{l}_2) \;=\; (\mathsf{h}_1 \cdot \mathsf{l}_1) \otimes (\mathsf{h}_2 \cdot \mathsf{l}_2)$.

Moreover, note that a decomposition of a diagram does not uniquely determine its assembly process, which makes the non-diagrammatic treatment of processes especially boring.

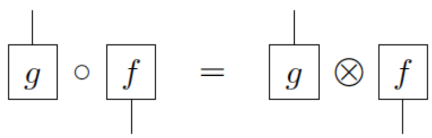
$\boxed{\textbf{Exercise } 1}$

Discuss whether the following circuits are equivalent.



Exercise 2

Prove that



Monoidal categories.

A *strict monoidal category* C is a category equipped with a parallel composition operator $\otimes$ on objects and arrows, i.e. a functor $\otimes : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$, and a unit object I such that

- Is associative on both objects and arrows:

$$(\mathsf{A} \otimes \mathsf{B}) \otimes \mathsf{C} \;=\; \mathsf{A} \otimes (\mathsf{B} \otimes \mathsf{C}) \quad \text{and} \quad (\mathsf{f} \otimes \mathsf{g}) \otimes \mathsf{h} \;=\; \mathsf{f} \otimes (\mathsf{g} \otimes \mathsf{h})$$

- Is unital on both objects and arrows:

$$\mathsf{A} \otimes \mathrm{I} \;=\; \mathrm{I} \otimes \mathsf{A} \;=\; \mathsf{A} \quad \text{and} \quad \mathsf{f} \otimes \mathsf{id}_{\mathrm{I}} \;=\; \mathsf{id}_{\mathrm{I}} \otimes \mathsf{f} \;=\; \mathsf{f}$$

- $\otimes$ and $\cdot$ verify the interchange law.

Exercise 3

A monoidal category is not strict if equalities in the definition of the strict case are relaxed to natural isomorphisms, making $\otimes$ associative and unital only *up to* an isomorphism. Draw the corresponding diagrams.

Actually, non strict monoidal categories must also assume a number of so-called *coherence* conditions that somehow guarantee such isomorphisms behave as expected. In practice, it can be shown that every monoidal has a strict equivalent, which is of course easier to work with.

A main result on diagrammatic reasoning: *Diagrams are sound and complete for monoidal categories*, i.e. given two morphisms f and g such that the equation $\mathsf{f} = \mathsf{g}$ is *well-typed*, then,

- *soundness:* If the equation holds under the axioms of a monoidal category, then the graphical representations of f and g are planar isotopic (i.e. one can be deformed continuously into the other).
- *completeness:* the reverse implication holds.

Circuits.

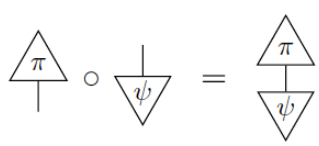
A circuit is a diagram that can be built by composing boxes, including identities and swaps, by means of vertical and horizontal composition.

Exercise 4

Show that a circuit is a diagram that contains no directed loops.

States, effects and scalars.

- States are processes without any inputs, representing preparation procedures.
- Effects are processes without any outputs, representing tests. A test consists of a *question* about a system, a verification *procedure*, and the *event of obtaining yes* as the answer, i.e. the assertion that effect occured.
- Composing a state with an effect yields a *scalar*:



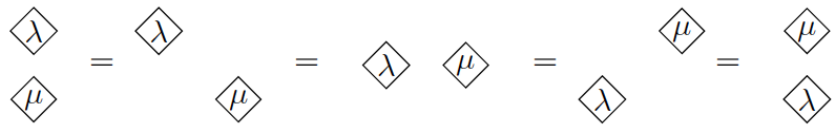
All numbers (naturals, complexes, probabilities, ...) emerge this way; cf the *generalised Born rule* returning the probability that the effect happens, given the system is in a particular state. Actually this interpretation makes sense in any process theory.

Exercise 5

How do scalars compose?

Exercise 6

Interpret the following equalities (known as the Eckmann-Hilton lemma)



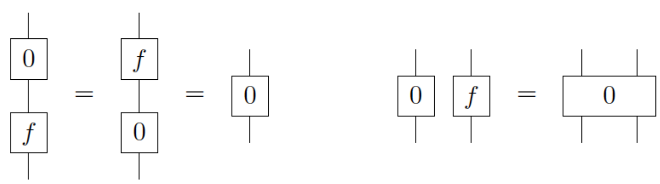
Why do they hold and what do they tell us about scalars?

Exercise 7

Consider the process theories represented in categories $\mathtt{Set}$ and $\mathtt{Rel}$. Characterise states, effects and scalars in both of them.

Exercise 8

A *zero* process is a process that verifies the following two *absorption* laws:



Show that if it exists in a process theory, it is unique. Due to its uniqueness, it can be written just a 0, ignoring input and output wires.

Symmetric monoidal categories.

A monoidal category is called *braided* if it comes equipped with a natural isomorphism (known as the *swap* arrow)

$$\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A$$

such that

- $\sigma_{A,I} \;=\; \mathtt{id}_A$
- $(\mathtt{id}_B \otimes \sigma_{A,C}) \cdot (\sigma_{A,B} \otimes \mathtt{id}_C) \;=\; \sigma_{A,B\otimes C}$

It is called *symmetric* if

$$\sigma_{B,A} \cdot \sigma_{A,B} = \mathrm{id}_{A \otimes B}$$

for all objects A and B.

Exercise 9

Express the naturality of σ through a diagram. Draw the commutative diagrams corresponding to the properties defining a symmetric monoidal category.

The soundness/completeness result discussed above for diagrams and monoidal cetegories can be rephrased in this setting: *circuit diagrams are sound and complete for symmetric monoidal categories*, i.e.

- *soundness:* two arrows are provably equal using the axioms of a symmetric monoidal category, if they can be expressed by the same circuit,
- *completeness:* two arrows are expressed by the same circuit, if can be proved equal using the axioms of a symmetric monoidal category.

References

[1] B. Coecke and A. Kissinger. *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning.* Cambridge University Press, 2017.

[2] B. Coecke (ed). *New Structures for Physics.* Springer Lecture Notes on Physics (813), 2011.

[3] C. Heunen and J. Vicary. *Categories for Quantum Theory.* Oxford Graduate texts in Mathematics. Oxford University Press, 2019.

[4] P. Selinger. A survey of graphical languages for monoidal categories. In B. Coecke, editor, *New Structures for Physics*, pages 289–355. Springer Lecture Notes on Physics (813), 2011.