Lecture 1: Categories

Summary.
(1) The concept of a category: motivation and formal definition.

(2) Examples. The category of sets and functions as a prototypical category for classical computation. The categories of sets and binary relations, and of Hilbert spaces and linear maps, as a step towards quantum physics and quantum computation. Algebraic structures as categories and categories of algebraic structures.

(3) Special arrows in a category: (purely categorical) definitions of isomorphism, monomorphism and epimorphism. Sub-objects.

(4) New categories from old: product of categories, arrow and slice categories. Subcategories.

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Opening.

Roughly speaking, categories deal with *arrows* and their composition in the same sense that sets deal with *elements*, their aggregation and membership. An *arrow* is an abstraction of the familiar notion of a function in set theory, a homomorphism in algebra, or a linear map between Hilbert spaces. Depicted as $f : X \longrightarrow Y$, it may be thought of as a transformation, or, simply, a connection, between two *objects* X and Y, called its *source* (or domain) and *target* (or codomain), respectively. The sources and targets of all the arrows in a category, form the class of its *objects*. If the same object is both the target of an arrow f and the source of another arrow g, f and g are said to be composable. Arrow composition is thus a partial operation and what the axioms for a category say is that arrows and arrow composition form a sort of generalised monoid.

Arrows are everywhere. In Physics, they typically represent *processes* which turn a state of one physical system into a state of another one; objects standing obviously for such physical systems. In Computer Science, arrows are *programs*, or algorithms, whose input and output are typed by data types — the objects. Types play an essential role in any programming discipline, as a technology which fits well with the categorial principle that *arrows do not stand in the void*. In Logic objects are logical *sentences* (propositions) and an arrow represents a formal derivation of a sentence (the conclusion) from another one (the hypothesis). In other words, arrows in Logic stand for *proofs*.

The essentially categorial structure underlying Logic and Computation motivated the study, since the early 1970's, of the so-called *Curry-Howard correspondence* between proofs and programs, after Haskell Curry and William Howard who explored an idea coming from the operational interpretation of intuitionistic logic. Later this correspondence, connecting essentially 'syntactical' entities, became tripartite bringing 'semantics' (*models*) into the picture. The crucial structure of the semantic universes for Computation are, not surprisingly, captured by suitable categories. The broader *Curry-Howard-Lambek* correspondence, takes the form of a triangle, depicted in Figure 1, whose links to Categories were initiated by the wok of Joachim Lambek.

$$\begin{tikzcd} \text{Computation} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{Categories} \arrow[dl, leftrightarrow] \\ & \text{Logic} & \end{tikzcd}$$

Figure 1: The *Curry-Howard-Lambek* correspondence.

The study of this correspondence, for both classical and quantum computation, will be a major theme in this course. Classical computation has a set-theoretic 'flavour' — think of functional programs as arrows between types modelled as sets, a slightly *naive* view of functional programming ... When Physics come into the picture a sort of similar description of a system in terms of a set of states and functions between them, is clearly distorted. For example, a quantum state is given by an Hilbert space, but the latter is not the set of states underlying the system — a state is indeed a ray in a Hilbert Space! Similarly, a linear map is not a function from states to states. The notion of a category, however, also captures this case perfectly well. As J. Baez and M. Stay remark [4],

> *One of the virtues of category theory is that it frees usfrom the 'Set-centric' view of traditional mathematics and let us view quantum physics on its own terms.*

Indeed category theory provides a very general language to describe, organise and understand many different mathematical structures. It does so by neglecting what objects are, and focusing entirely on how they relate to each other. This includes the usual algebraic structures (e.g. preorders, groups or rings), but also data and algorithms in Computer Science, assertions and implications in Logic, systems and processes in Physics. Actually, as Colin McLarty put it [9]:

> *The spread of applications led to a general theory, and what had been a tool for handling structures became more and more a means of defining them. (...) In the 1960s, Lawvere began to give purely categorical descriptions of new and old structures, and developed several styles of categorical foundations for mathematics. This lead to new applications, notably in logic and computer science.*

Categories find striking application not only in Mathematics, but also in Computer Science, providing a semantic framework for programming concepts like parametrization, abstraction and compositionality, in Logic, with a syntax-independent version of the underlying structures, and even in Physics, as witnessed by numerous application to quantum information and quantum-based computational processes.

There are a number of good introductions to Category Theory. Reference [3] would be my choice for a comprehensive, accessible textbook. Abramsky and Tzevelekos introductory chapter [1] in the *New Structures for Physics* sort of handbook [6], is also an excellent starting point. Both can be complemented by more extensive introductions such as [2] or [10], both excellent, even if more oriented to a 'traditional' Mathematics graduate student. Shorter, sharp introductions are offered in [9], which includes a smooth introduction to toposes, and T. Leinster's recent book [8]. To read on the beach, as pleasant motivations for further study, references [7] and [5] should be mentioned.

Exercise 1

Show that partially ordered sets and monotone functions form a category Pos. Generalise your argument to show that other algebraic structures, and the corresponding homomorphisms, also give rise to categories. Illustrate with the case of monoids and groups.

Exercise 2

Verify that vector spaces over a field K, say the complex numbers, and linear maps, form a category Vect_K.

Exercise 3

A preorder $\langle P, \leq \rangle$ may itself be regarded as a category[1] whose objects are the elements of P and there exists a unique arrow $p \longrightarrow q$ iff $p \leq q$. Explain why transitivity (respectively, reflexivity) of relation $\leq$ implies the existence of the required composite (respectively, identity) arrow. Note that, on the other hand, a category with at most an arrow between any two objects determines a preorder on its objects with the relation $A \leq B$ iff there is an arrow from A to B.

Exercise 4

A category is *skeletal* if isomorphic objects are equal. Explain why a poset, but not a preorder, regarded as a category, is skeletal.

Exercise 5

Show that sets and binary relations $R : A \longrightarrow B$, defined as $R \subseteq A \times B$, can be organised into a category Rel, with relational composition

$$S \cdot R \;\widehat{=}\; \{\langle a, c\rangle \in A \times C \mid \exists_{b \in B} \,.\, \langle a, b\rangle \in R \wedge \langle b, c\rangle \in S\}$$

for each pair of composable relations $R : A \longrightarrow B, S : B \longrightarrow C$, and, identities

$$\mathsf{id}_A \;\widehat{=}\; \{\langle a, a\rangle \mid a \in A\}$$

[1]Compare with the first exercise in this lecture. As noted in [1], *the ability to capture mathematics both 'in the large' and 'in the small' is a first indication of the flexibility and power of categories.*

for each set A.

Exercise 6

Given a semiring $(S, +, \times)$, a category Mat_S of matrices over S, has non-negative integers as objects and $r \times c$ matrices as arrows $c \to r$ from (the number of columns) c to (the number of rows) r. Note that the way a matrix is typed provides its interface, syntactically governing the possibility of composition, and, obviously, does not bear any relationship to the structure of the matrix elements. Composition

$$\begin{tikzcd} c \arrow[r, "N"] \arrow[rr, bend right, "M \cdot N"'] & l \arrow[r, "M"] & r \end{tikzcd}$$

is matrix multiplication

$$i\,(M \cdot N)\,j \;\widehat{=}\; \sum_k (i\,M\,k) \times (k\,N\,j)$$

and diagonal matrices

$$i\,(id)\,j \;\widehat{=}\; (i = j)$$

serve as the identity arrows. Verify that Mat_S is a category indeed. Take a few minutes to explore the similarities between Mat_S and Rel. In particular, note that a binary relation can be regarded as a Boolean-valued matrix, and composition in Rel rephrased as

$$c\,(S \cdot R)\,a \;\widehat{=}\; \bigvee_k (c\,S\,b) \wedge (b\,R\,a)$$

i. e. as composition in $\mathsf{Mat}_{(2,\wedge,\vee)}$.

Exercise 7

The *dual category* C^{op} of a category C has the same objects as C and all the arrows of C reversed. Characterise C^{op} in detail, defining composition and identities in terms of the corresponding operations in C.

Exercise 8

Prove that Rel^{op} is self-dual, i.e. isomorphic to Rel. Explain why Rel is not obtained by dualizing Set.

Exercise 9

Given category C and one of its objects X, the *slice* category C/X has as objects all the arrows to X. An arrow between $f : A \longrightarrow X$ and $g : B \longrightarrow X$, is an arrow $h : A \longrightarrow B$ in C making the following diagram to commute:

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

Define identities and composition in C/X and show the axioms for a category hold. Show that if C_P is a poset $(P, \leq)$ regarded as a category, the slice C_P/x, for an element $x \in P$, is a principal ideal $x \downarrow = \{y \in P \mid y \leq x\}$.

Exercise 10

The so-called *co-slice* category X/C, for a category C and an object X of C, may be defined as

$$X/C \;\widehat{=}\; (C^{op}/X)^{op}$$

Unfold the definition in detail to arrive to a direct characterisation of X/C.

Exercise 11

If one takes the arrows of a category C and uses them as objects of a new category, the result is called an *arrow category* and denoted by $C^{\rightarrow}$. An arrow in $C^{\rightarrow}$ from f to g, is a pair of arrows (h_1, h_2) in C such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "h_1"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "h_2"] & D \end{tikzcd}$$

Define identities and composition in $C^{\rightarrow}$ and show the axioms for a category hold.

Exercise 12

Recall that an arrow $f : A \longrightarrow B$ is an *isomorphism* if it has an inverse, i.e. an arrow $h : B \longrightarrow A$ such that

$$h \cdot f = id_A \quad \text{and} \quad f \cdot h = id_B$$

Show that inverses are unique. In Set isomorphisms correspond exactly to bijective functions, but this fact does not necessarily "scale" to categories whose objects carry extra structure. Prove, by building a suitable counter-example, that in Pos a monotone bijection may not be an isomorphism.

Exercise 13

Show that any group can be regarded as a category with only one object and whose arrows are isomorphisms. Conversely, show that a category of this type always determines a group.

Exercise 14

An arrow $h : A \longrightarrow B$ is *monic* (a monomorphism) if for all $f, g : C \longrightarrow A$ as in

$$\begin{tikzcd} C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[r, "h"] & B \end{tikzcd}$$

the following holds

$$h \cdot f = h \cdot g \;\Rightarrow\; f = g$$

Dually, an arrow is *epic* (or a epimorphism) in a category C if it is monic in C^{op}. Unfold this statement to arrive to a direct definition of an epimorphism. Show that in Set monic (respectively, epic) arrows correspond to injective (respectively, surjective) functions.

Exercise 15

Show that the composition of monomorphisms is a monomorphism and that, if a composition $g \cdot f$ is monic, so is f. Formulate the dual results for epimorphims. Do you need to prove them?

Exercise 16

Show that, in any category, an isomorphism is always a monic and epic arrow. The converse, however, is not true in general. Show that the inclusion of integers into the set of rational numbers, although being monic and epic in the category of monoids, fails to be an isomorphism.

Exercise 17

Consider arrows

$$\begin{tikzcd} A \arrow[r, bend right, "s"'] & B \arrow[l, bend right, "r"'] \end{tikzcd}$$

such that $r \cdot s = id_A$. Arrow s is called a *section*, or right inverse to r, whereas r is a *retraction*, or left inverse to s. Show that s is always monic and r epic. Note that, to witness these one-sided inverses, s (respectively, r) is said to be a *split* monomorphism (respectively, epimorphism). Show that in Set every epimorphism is a split one, and every monomorphism, but for the inclusion of the empty set in any other set, is a split monomorphism.

$\boxed{\textbf{Exercise } 18}$

A category S is a subcategory of another category C if it is defined by restricting to a sub-collection of objects and sub-collection of arrows such that the domain and codomain of any such arrow is in S, and S is closed for identities and composition. Show that the category of finite sets and bijections is a subcategory of Set.

$\boxed{\textbf{Exercise } 19}$

Show that in any category C monomorphisms (respectively, epimorphisms) define a subcategory of C.

$\boxed{\textbf{Exercise } 20}$

Show that isomorphisms in Rel are the graphs of bijections: a relation $S : X \longrightarrow Y$ is an isomorphism if there is some bijection $h : X \longrightarrow Y$ such that ySx iff $h(x) = y$.

$\boxed{\textbf{Exercise } 21}$

The structure of category consists of the composition operator on arrows and the identities on objects — no reference is ever made to what the individual objects really are. However, as arrows are structure-preserving maps, therefore preserving whatever structure objects carry, the latter can be recovered easily. For example i the category Set one is able to identify the elements of the objects as arrows: for a set X

$$\underline{x} : \mathbf{1} \longrightarrow X$$

where $\mathbf{1} \cong \{*\}$ is the (isomorphism class of the) one-element set, is the map $* \mapsto x$. Similarly, in the category of finite-dimension vector spaces over a field Φ, for a vector space U and some fixed vector u in U, the linear map

$$\underline{u} : \Phi \longrightarrow U$$

maps the element $1 \in \Phi$ to u. As $\underline{u}$ is linear, it is completely determined by the image of the single element 1. Show that

$$\underline{u}(\rho) = \rho \cdot u$$

and conclude that the element 1 is a basis for the one-dimensional vector space Φ.

$\boxed{\textbf{Exercise } 22}$

One way of thinking of an arrow $x : Z \longrightarrow X$ is as an 'element' of X, which is not given once and for all, but depends on Z. Such x is often called a *generalised element* of X, and Z its *stage of definition*. This suggests the alternative, set-inspired notation $x \in_Z X$ for arrow $x : Z \longrightarrow X$. The composite $f \cdot x$, for $f : X \longrightarrow Y$, can thus be written as $f(x)$. A special kind of elements of an object X consists of arrows into X whose source is the final object $\mathbf{1}$ (see Lecture 4) in the category (if it exists). These are called *points* (or *global elements*) of X.

In some categories every arrow $f : X \longrightarrow Y$ is fully determined by its effect on the points of X. Should this be the case, the category is said to be *well pointed*. Again Set is a good example: being well pointed is just a categorical version of the well known fact that 'a set is determined by its elements'. Moreover, in Set the correspondence between elements $x \in X$ and points $\underline{x} : \mathbf{1} \longrightarrow X$ is made explicit by denoting function application $f \cdot \underline{x}$ by $f(x)$.

Rephrase the definition of monic and epic arrow in an arbitrary category in the language of generalised elements.

References

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