Bivariate Gaussian Distribution

⇒ Transforming 1-D normal distribution to bivariate distribution

$$z_1 , z_2 \sim N(0, 1)$$

(ρ - correlation coefficient)

→ Params : $\mu_x , \mu_y , \sigma_x , \sigma_y , \rho$

$$X = \mu_x + \sigma_x z_1$$
$$Y = \mu_y + \sigma_y [\rho z_1 + \sqrt{1-\rho^2} z_2]$$

→ X would be $\mathcal{N}(\mu_x, \sigma_x^2)$
Y would be $\mathcal{N}(\mu_y, \sigma_y^2)$
$Cov(X, Y) = \rho \cdot \sigma_x \cdot \sigma_y$
$\rho(X, Y) = \rho$

⇒ $$f(x, y) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} \exp\left[\frac{-1}{2(1-\rho^2)} \left(\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - 2\rho \frac{(x-\mu_x)(y-\mu_y)}{\sigma_x \cdot \sigma_y} \right) \right]$$

$$f(\vec{x}) = \frac{1}{{}^{k}\!\sqrt{2\pi} \sqrt{|\Sigma|}} \exp\left[-\frac{1}{2} (\vec{x} - \vec{\mu})^T \Sigma^{-1} (\vec{x} - \vec{\mu}) \right]$$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \vec{\mu} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} \quad \Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_x \cdot \sigma_y \cdot \rho \\ \sigma_x \cdot \sigma_y \cdot \rho & \sigma_y^2 \end{pmatrix}$$

- Σ will alway be symmetric.
- Note $(2\pi)^{\cancel{k}/2}$

Example: Suppose heights of married couple can be explained by a bivariate normal distribution.

wife's height :- $\mu_w = 66.8$ $\quad \sigma_w = 2$

husband's height :- $\mu_h = 70$ $\quad \sigma_h = 2$

correlation $\rho = 0.68$

What is the probability that for a randomly selected couple the wife is taller than husband ?

My Approach

$$\text{ans} = \int_{-\infty}^{\infty} \int_{w}^{\infty} f(w, h)\, dh \cdot dw$$

Another easier Approach

→ we need to find $Pr(\overbrace{w-h} > 0)$

→ Since h, w have bivariant normal distribution $(w-h)$ will be normal distribution.

$$E(w-h) = 66.8 - 70$$
$$= -3.2$$

$$Var(w-h) = Var(w) + Var(h) - 2\,cov(x, y)$$
$$= \sigma_w^2 + \sigma_h^2 - 2\rho \cdot \sigma_w \cdot \sigma_h$$
$$= 4 + 4 - 2(0.68)(2)(2)$$
$$= 2.56$$

⇒ For a 1-D distribution with $\mu = -3.2$ and $\sigma^2 = 2.56$

$$Pr(x > 0) = 0.0227$$

Ans: $Pr(\text{wife} > \text{husband}) = 0.0227$