Tandon Bridge Program - Homework 7

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Question 5

a. **Theorem:** For any positive integer n, 3 divides $n^3 + 2n$ leaving no remainder.
Proof.
By induction on n.
Base case: n = 1.
Plug in 1 for n: $1^3 + 2 * (1) = 3$.
Since 3 evenly divides 3, the theorem holds true for $n = 1$.
Inductive step: Suppose that for positive integer k, 3 evenly divides $k^3 + 2k$, then we will prove that 3 evenly divides $(k+1)^3 + 2(k+1)$.
By the inductive hypothesis, 3 evenly divides $k^3 + 2k$, meaning $k^3 + 2k = 3m$ for some integer m.
We now show that $(k+1)^3 + 2(k+1)$ can be expressed as 3 times an integer.

$$
\begin{aligned}
(k+1)^3 + 2(k+1) &= k^3 + 1 + 3 * k^2 + 3k + 2k + 2 \\
&\quad \text{(Using the expansion of } (a+b)^3\text{)} \\
&= k^3 + 3 + 3k^2 + 5k \\
&= k^3 + 2k + 3k^2 + 3k + 3 \\
&= 3m + 3k^2 + 3k + 3 \\
&\quad \text{(By the inductive hypothesis)} \\
&= 3(m + k^2 + k + 1)
\end{aligned} \tag{1}
$$

Since m and k are integers, $(m + k^2 + k + 1)$ is also an integer. Therefore $(k+1)^3 + 2(k+1)$ is equal to 3 times an integer, proving that it is perfectly divisible by 3. ■

b. **Theorem:** A positive integer n, such that $n \geq 2$, can be written as a product of primes.
Proof.
By strong induction on n.
Inductive step: Assume that for $k \geq 2$, any integer j in the range 2 through k can be expressed as a product of prime numbers. We will prove that $k+1$ can also be expressed as a product of primes.
There are two cases to consider:

   i. $k+1$ is prime
   In this case, $k+1$ is already a product of the prime $k+1$

   ii. $k+1$ is not a prime
   In this case $k+1$ can be expressed as a product of two integers a and b that are each at least 2.
   For strong induction, we need to prove that both a and b fall within the range from 2 to k, that is $2 \leq a \leq k$ and $2 \leq b \leq k$.
   Since $k+1 = a * b$ by assumption, $a = \frac{k+1}{b}$. Since $b \geq 2$, $a = \frac{k+1}{b} < k+1$. Since a is an integer and is strictly less than $k+1$, $a \leq k$.
   Using the same logic, $b \leq k$.

Since both a and b are less than or equal to k and, by the inductive hypothesis, 2 through k can be written as a product of primes:

$$a = p_1.p_2....p_l$$
$$b = q_1.q_2....q_m$$

$k + 1$ can be expressed as the product of primes:

$$k + 1 = a.b = p_1.p_2....p_l.q_1.q_2....q_m$$

Question 6

a. Exercise 7.4.1

a. P(n) is the assertion that $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2*n+1)}{6}$
P(3) is $\sum_{j=1}^{3} j^2 = \frac{3(3+1)(2*3+1)}{6}$
$1^2 + 2^2 + 3^2 = \frac{3(3+1)(2*3+1)}{6}$
$1 + 4 + 9 = \frac{3(4)(7)}{6}$
$14 = 14$

b. P(k) is the expression $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2*k+1)}{6}$

c. P(k+1) is the expression $\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2*(k+1)+1)}{6}$
$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

d. In the base case, we prove that P(n) is true for the first relevant value of n. Here that value is 1, so we prove that P(1) or $\sum_{j=1}^{1} j^2 = \frac{1(1+1)(2*1+1)}{6}$ is true.

e. In the inductive step, we prove that if P(k) is true, then P(k+1) is true.
Here, that would be the same as proving:

$$P(k+1), \text{ which is } \sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2*k+3)}{6}$$

using the inductive hypothesis P(k) = $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2*k+1)}{6}$.

f. The inductive hypothesis would be that P(k) = $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2*k+1)}{6}$ is true

g. **Theorem:** $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2*n+1)}{6}$
Proof.
By induction on n.
Base case: $n = 1$
$1^2 = \frac{1(2)(3)}{6}$
$1 = 1$
Therefore for $n = 1$, $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2*n+1)}{6}$
Inductive step: Suppose that for any integer $k \geq 1$, $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2*k+1)}{6}$, then we will show that

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2*k+3)}{6}$$

$$
\begin{aligned}
\sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k+1)^2 \\
&\text{(Separating out the last term)} \\
&= \frac{k(k+1)(2*k+1)}{6} + (k+1)^2 \\
&\text{(By the inductive hypothesis)} \\
&= (k+1)(\frac{k(2k+1)+6(k+1)}{6}) \\
&= \frac{(k+1)(2*k^2+k+6k+6)}{6} \\
&= \frac{(k+1)(2*k^2+7k+6)}{6} \\
&= \frac{(k+1)(2*k^2+4k+3k+6)}{6} \\
&= \frac{(k+1)(2k(k+2)+3(k+2))}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned} \tag{2}
$$

Therefore $\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2*k+3)}{6}$ ■

b. Exercise 7.4.3

c. **Theorem:** For $n \geq 1$, $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$
Proof.
By induction on n.
Base case: $n = 1$
$\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$ is $\sum_{j=1}^{1} \frac{1}{j^2} \leq 2 - \frac{1}{1}$.
This is equivalent to $\frac{1}{1^2} \leq 2 - \frac{1}{1}$
$1 \leq 1$.
Therefore, for $n = 1$, $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$.
Inductive step: Suppose that for any integer $k \geq 1$, $\sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$, we will prove that $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$.

$$\begin{aligned}\sum_{j=1}^{k+1}\frac{1}{j^2} &= \sum_{j=1}^{k}\frac{1}{j^2}+\frac{1}{(j+1)^2}\\ &\text{(Separating out the last term)}\\ &\leq 2-\frac{1}{k}+\frac{1}{(k+1)^2}\\ &\text{(By the inductive hypothesis)}\\ &\leq 2-\frac{1}{k}+\frac{1}{(k+1)k}\\ &\leq 2+\frac{1}{k}(\frac{1-(k+1)}{k+1})\\ &\leq 2-\frac{1}{k+1}\end{aligned} \tag{3}$$

Therefore, $\sum_{j=1}^{k+1}\frac{1}{j^2} \leq 2-\frac{1}{k+1}$. ■

c. Exercise 7.5.1

a. **Theorem:** For any positive integer n, 4 evenly divides $3^{2n}-1$.
Proof.
By induction on n.
Base case: n = 1.
$3^{2*1}-1 = 3^2-1 = 8$. Since 4 evenly divides 8 into 4 and 2, the theorem holds for $n = 1$.
Inductive step: Suppose that for positive integer k, 4 evenly divides $3^{2k}-1$. Then we will show that 4 evenly divides $3^{2(k+1)}-1$. By the inductive hypothesis, 4 evenly divides $3^{2k}-1$, which means $3^{2k}-1 = 4m$ for some integer m. This is equivalent to $3^{2k} = 4m+1$. We will show that $3^{2(k+1)}-1$ can be expressed as 4 times an integer.

$$\begin{aligned}3^{2(k+1)}-1 &= 3^{2k+2}-1\\ &= 3^{2k}*3^2-1\\ &= (4m+1)*3^2-1\\ &\text{(By the inductive hypothesis)}\\ &= (4m+1)*9-1\\ &= 36m+9-1\\ &= 36m+8\\ &= 4(9m+2)\end{aligned} \tag{4}$$

Since m is an integer, $(9m+2)$ is also an integer. Therefore, $3^{2(k+1)}-1$ can be perfectly divided by 4.