Tandon Bridge Program - Extended - Homework 2

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Question 5

a) Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.12.2: Proving arguments are valid using rules of inference

   b.

$$\begin{array}{l} p \to (q \land r) \\ \neg q \\ \hline \therefore \neg p \end{array}$$

1	$\neg q$	Hypothesis
2	$\neg q \lor \neg r$	Addition, 1
3	$\neg(q \land r)$	De Morgan's, 2
4	$p \to (q \land r)$	Hypothesis
5	$\neg p$	Modus tollens 3,4

   e.

$$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \neg q \\ \hline \therefore r \end{array}$$

1	$p \lor q$	Hypothesis
2	$\neg p \lor r$	Hypothesis
3	$q \lor r$	Resolution 1, 2
4	$\neg q$	Hypothesis
5	r	Disjunctive syllogism 3,4

2. Exercise 1.12.3: Proving the rules of inference using other rules.

   c.

$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$

1	$p \lor q$	Hypothesis
2	$\neg p \to q$	Conditional identity, 1
3	$\neg p$	Hypothesis
4	q	Modus ponens 2,3

3. Exercise 1.12.5: Proving arguments in English are valid or invalid

c.

I will buy a new car and a new house only if I get a job
I am not going to get a job

$\therefore$ I will not buy a new car

c: I will buy a new car
h: I will buy a new house
j: I will get a job

$(c \wedge h) \to j$
$\neg j$
$\therefore \neg c$

c	h	j	$(c \wedge h)$	$(c \wedge h) \to j$	$\neg j$	$\neg c$
T	T	T	T	T	F	F
T	T	F	T	F	T	F
T	F	T	F	T	F	F
T	F	F	F	T	T	F
F	T	T	F	T	F	T
F	T	F	F	T	T	T
F	F	T	F	T	F	T
F	F	F	F	T	T	T

The argument is invalid because of the truth values corresponding to the red labeled row in the table above. In this case, both hypothesis $(c \wedge h) \to j$ and $\neg j$ are true, but the conclusion $\neg c$ is false.

d.

I will buy a new car and a new house only if I get a job
I am not going to get a job
I will buy a new house

$\therefore$ I will not buy a new car

c: I will buy a new car
h: I will buy a new house

j: I will get a new job

$(c \wedge h) \rightarrow j$
$\neg j$
h
$\therefore \neg c$

1	$(c \wedge h) \rightarrow j$	Hypothesis
2	$\neg j$	Hypothesis
3	$\neg(c \wedge h)$	Modus tollens 1, 2
4	$\neg c \vee \neg h$	De Morgan's law 3
5	$\neg h \vee \neg c)$	Commutative law 4
6	h	Hypothesis
7	$\neg(\neg h)$	Double negation law 6
8	$\neg c$	Disjunctive syllogism 5,7

Following rules of inference, the above argument is valid.

b) Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.13.3: Show that the given argument is invalid by giving values for the predicates P and Q over the domain a,b

b.

$$\begin{array}{l} \exists x(P(x) \vee Q(x)) \\ \exists x \neg Q(x) \\ \hline \therefore \exists x P(x) \end{array}$$

	P	Q
a	F	T
b	F	F

At x = a, $\exists(P(x) \vee Q(x)$ is true
At x = b, $\exists \neg Q(x)$ is true
But there is no value of x at which $\exists P(x)$ is true
$\therefore$ The argument is invalid

2. Exercise 1.13.5: Determine and prove whether an argument in English is valid or invalid.

d.

Every student who missed class got a detention
Penelope is a student in the class
Penelope did not miss class

$\therefore$ Penelope did not get a detention

M(x): x missed a class
D(x): x got a detention

$\forall x(M(x) \rightarrow D(x))$
Penelope, a student in the class
$\neg M(Penelope)$
$\therefore \neg D(Penelope)$

M	D	$M \rightarrow D$	$\neg M$	$\neg D$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

The argument is invalid because of the truth values corresponding to the red labeled row in the table above. In this case, both hypothesis $(M(x) \rightarrow D(x))$ and $\neg M(Penelope)$ are true, but the conclusion $\neg D(Penelope)$ is false.

e.

Every student who missed a class or got a detention did not get an A
Penelope is a student in the class
Penelope got an A

$\therefore$ Penelope did not get a detention

D(x): x got a detention
M(x): x missed a class
A(x): x got an A

$\forall x((M(x) \vee D(x)) \rightarrow \neg A(x))$
Penelope, a student in the class
A (Penelope)

$\therefore \neg D(Penelope)$

1	$\forall x((M(x) \vee D(x)) \to \neg A(x))$	Hypothesis
2	Penelope, a student in the class	Hypothesis
3	$M(Penelope) \vee D(Penelope) \to \neg A(Penelope)$	Universal instantiation 1,2
4	A (Penelope)	Hypothesis
5	$\neg(\neg A(Penelope))$	Double negation law, 4
6	$\neg(M(Penelope) \vee D(Penelope))$	Modus tollens 3,5
7	$\neg M(Penelope) \wedge \neg D(Penelope)$	De Morgan's law 6
8	$\neg D(Penelope)$	Simplification, 7

Following rules of inference, the above argument is valid.

Question 6

a. Solve exercise 2.4.1: Prove the statement using a direct proof

   d. The product of two odd integers is an odd integer

      Proof:
      Let x and y be two odd integers. We will show that xy is an odd integer.
      Since x is odd, there is an integer m such that x = 2m + 1. Similarly since y is an odd integer, there exists an integer n such that y = 2n + 1.

      Plugging this into xy, we have:
      xy = (2m+1)(2n+1)
      xy = 4mn + 2m + 2n + 1
      xy = 2(2mn+m+n) + 1

      Since m and n are integers, 2mn+m+n is also an integer.
      Then xy = 2j + 1 where j=2mn+m+n is an integer.
      Therefore, xy is an odd integer.

b. Solve Exercise 2.4.3: Prove the statement using a direct proof

   b. If x is a real number such that $x \leq 3$, then $12 - 7x + x^2 \geq 0$

      Proof:
      Let x be a real number such that $x \leq 3$. This means that x is either less than zero or equal to zero. We can prove this argument by considering each of these cases individually.

      i. x < 3
         When x < 3, subtracting 3 from both sides will give us x - 3 < 0.
         Subtracting 4 from both sides of x < 3 will give us $x-4 \leq -1$.
         If x - 4 is less than -1, x-4 is also less than 0.
         Here we have both (x-3) and (x-4) less than zero, meaning both are negative. The product of these two terms will hence

be positive.
Since $(x-3)(x-4) = 12-7x+x^2$ and the product of (x-3)(x-4) is positive, $12 - 7x + x^2$ is positive.

ii. x = 3
When x = 3, subtracting 3 from both sides will give us (x-3) = 0. When (x-3) = 0, the product of (x-3)(x-4) will always be zero since the first term equals zero. Hence in this case, $12 - 7x - x^2 = 0$

The two cases together show that for $x \leq 3$, $12 - 7x + x^2 \geq 0$

Question 7

a) Exercise 2.5.1: Prove the following statement by contrapositive

d. For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

Proof:
Let n be an integer. We assume that n is even and prove that $n^2 - 2n + 7$ is odd.

If n is even, n can be expressed as 2k for some integer k.
Plugging n = 2k into $n^2 - 2n + 7$ gives

$n^2 - 2n + 7 = (2k)^2 - 2(2k) + 7$
$n^2 - 2n + 7 = 4k^2 - 4k + 7$
$n^2 - 2n + 7 = 4k^2 - 4k + 6 + 1$
$n^2 - 2n + 7 = 2(2k^2 - 2k + 3) + 1$

Since k is an integer, $2k^2 - 2k + 3$ is also an integer.
$\therefore$ $n^2 - 2n + 7$ can be represented as 2p + 1, where p = $2k^2 - 2k + 3$ is an integer.

$\therefore$ We conclude that $n^2 - 2n + 7$ is odd.

b) Exercise 2.5.4: Prove by contrapositive

a. For each pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$

Proof
Let x and y be two real numbers. We assume x>y and prove that $x^3 + xy^2 > x^2y + y^3$.
$x^3 + xy^2$ can be rewritten such that
$x^3 + xy^2 = x(x^2 + y^2)$ ———- (a)
Similarly $x^2y + y^3$ can be rewritten as
$x^2y + y^3 = y(x^2 + y^2)$ ———- (b)

The right hand side of equations (a) and (b) have the same second terms (i.e. $(x^2 + y^2)$) . The first terms are x and y respectively

and since we have assumed that x > y, RHS of equation 1 is greater than RHS of equation 2. Consequently, LHS of equation 1 is greater than the LHS of equation 2.
$\therefore x^3 + xy^2 > x^2y + y^3$

b. For every pair of real numbers x and y, if x+y > 20, then x>10 or y>10.

Proof:
Let x and y be two real numbers. We assume $x \leq 10$ and $y \leq 10$ and prove that $x + y \leq 20$.

$$x \leq 10 \tag{1}$$

$$y \leq 10 \tag{2}$$

Without disturbing the direction of the inequalities, we can add (1) and (2).
This gives us

$$x + y \leq 20 \tag{3}$$

c. Exercise 2.5.5: Prove statement using direct proof or contrapositive.

c. For every non-zero real number x, if x is irrational, then $\frac{1}{x}$ is also irrational.

Proof:
Let x be a non-zero real number. We assume $\frac{1}{x}$ is rational and prove that x is rational.

Since $\frac{1}{x}$ is rational, it can be written in the form $\frac{p}{q}$ where $q \neq 0$.

$$\frac{1}{x} = \frac{p}{q} \tag{4}$$

Since x is non-zero by assumption, $\frac{1}{x}$ is also non-zero. This implies that $\frac{p}{q}$ is non-zero and p is non-zero. i.e, both p and q are non zero.
Rearranging 4 gives

$$x = \frac{q}{p} \tag{5}$$

$\therefore$ x can be written as a ratio of q and p where both q and p are non-zero. Hence x is also rational.

Question 8 Exercise 2.6.6: Proofs by contradiction

c. The average of three real numbers is greater than or equal to at least one of the numbers.

   Proof:
   We will assume that there exists three real numbers x, y, and z such that their average is less than each of the individual numbers.
   This can be mathematically represented as:

$$\frac{x+y+z}{3} < x \tag{6}$$
$$\frac{x+y+z}{3} < y \tag{7}$$
$$\frac{x+y+z}{3} < z \tag{8}$$

   Without disturbing the direction of the inequalities, we can add (6), (7) and (8).
   This gives us

$$\frac{x+y+z}{3} + \frac{x+y+z}{3} + \frac{x+y+z}{3} < x+y+z \tag{9}$$
$$\frac{3x+3y+3z}{3} < x+y+z \tag{10}$$
$$\frac{3(x+y+z)}{3} < x+y+z \tag{11}$$
$$x+y+z < x+y+z \tag{12}$$

   (12) is a contradiction. Hence our assumption is flawed and there does not exist three real numbers x, y, and z such that their average is less than each of the individual numbers.

d. There is no smallest integer.

   Proof:
   We will assume there exists x such that x is the smallest integer.
   Since x is an integer, x - 1 is also an integer and x - 1 is an integer smaller than x. This contradicts our assumption.
   Hence, our assumption is false and there does not exist x such that x is the smallest integer.

Question 9 Exercise 2.7.2: Proofs by cases: even/odd integers and divisibility.

b. If integers x and y have the same parity, then x+y is even.

   Proof:
   Case 1: x and y are odd.
   When x and y are odd integers, they can be represented as $x = 2m + 1$ and $y = 2n + 1$ where m and n are integers.
   Substituting these values into $x + y$ gives:
   $x + y = (2m+1) + (2n+1)$
   $x + y = 2m + 2n + 2$
   $x + y = 2(m+n+1)$
   Since m and n are integers, m+n+1 is also an integer.
   $x + y = 2a$ where $a= m+n+1$ is an integer.
   $\therefore$ We conclude that x+y is even.

   Case 2: x and y are even.
   When x and y are even integers, they can be represented as $x = 2m$ and $y = 2n$ where m and n are integers.
   Substituting these values into $x + y$ gives:
   $x + y = 2m+2n$
   $x + y = 2(m+n)$
   Since m and n are integers, m+n is also an integer.
   $x + y = 2a$ where $a= m+n$ is an integer.
   $\therefore$ We conclude that x+y is even.