Tandon Bridge Program - Homework 7

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Question 7

a) Exercise 6.1.5: A 5-card hand is dealt from a perfectly shuffled deck of playing cards.

b. Probability that the hand is three of a kind (Three cards with the same rank, other two cards do not have the same rank as each other and do not have the same rank as the three with the same rank.)
Sample space or total ways of choosing 5 cards hand from a deck of cards is $\binom{52}{5}$.
The event of a three of a kind can be calculated the following way: The three of a kind ranks can be chosen in $\binom{13}{1}$ ways and within each it, 3 cards can be chosen in $\binom{4}{3}$ ways. The rank of the remaining two cards can be chosen in $\binom{12}{2}$ ways. Suits within each of these cards can be chosen in $\binom{4}{1}$ ways each.
Therefore probability of a three of a kind is $\frac{\binom{13}{1}*\binom{4}{3}*\binom{12}{2}*\binom{4}{1}*\binom{4}{1}}{\binom{52}{5}}$

c. Probability that all 5 cards have the same suit
Sample space or total ways of choosing 5 cards hand from a deck of cards is $\binom{52}{5}$.
The event where all 5 cards belong to the same suit is calculated the following way. The suit can be selected in $\binom{4}{1}$ ways and the five ranks within that suit can be selected in $\binom{13}{5}$ ways.
Therefore the probability that all 5 cards have the same suit is $\frac{\binom{4}{1}*\binom{13}{5}}{\binom{52}{5}}$

d. Probability that the hand is a two of a kind
Sample space or total ways of choosing 5 cards hand from a deck of cards is $\binom{52}{5}$.
The event where the hand is a two of a kind can be calculated in the following way:
The rank for the pair can be chosen in $\binom{13}{1}$ ways. The suits for this pair can be chosen in $\binom{4}{2}$ ways. The remaining three cards need to be of distinct ranks. These can be chosen in $\binom{12}{3}$ ways. We reduced one rank from our pool of choices to account for the rank that has been used in the pair. Within each rank, the suit for each of these three cards can be selected in $\binom{4}{1}$ or 4 ways.
Sample space or total ways of choosing 5 cards hand from a deck of cards is $\binom{52}{5}$. Therefore the probability of a two of a kind hand is $\frac{\binom{13}{1}*\binom{4}{2}*\binom{12}{3}*4*4*4}{\binom{52}{5}}$

b) Exercise 6.2.4: A 5-card hand is dealt from a perfectly shuffled deck of playing cards. Find the probability that:

a. The hand has at least one club

This can be found by first calculating the probability of there being no club in the hand and then subtracting this result from 1.
If there are no clubs in the 5 card hand, all 5 cards will be chosen from a deck of 39 cards (52 - 13 club cards). This can be done in $\binom{39}{5}$ ways. That makes the probability of a 5 card hand having no clubs $\frac{\binom{39}{5}}{\binom{52}{5}}$. Therefore the probability of having a 5-card hand with at least one club is 1 - $\frac{\binom{39}{5}}{\binom{52}{5}}$

b. The hand has at least two cards with the same rank
This can be found by first calculating the probability of there being no no two cards with the same rank and then subtracting this result from 1.
The five ranks for the cards can be chosen in $\binom{13}{5}$ ways. Within each rank, the suits for the five cards can be chosen in $4*4*4*4*4$ ways.
Sample space or total ways of choosing 5 cards hand from a deck of cards is $\binom{52}{5}$.
Therefore the probability of having a 5-card hand with no two cards of the same rank is 1 - $\frac{\binom{13}{5}*4*4*4*4*4}{\binom{52}{5}}$.

c. The hand has exactly one club or one spade.
Exactly one club can be chosen in $\binom{13}{1} * \binom{39}{4}$ ways. Similarly, exactly one spade can be chosen in $\binom{13}{1} * \binom{39}{4}$ ways. The intersection of the two events: a hand containing exactly one club and exactly one spade can occur in $\binom{13}{1} * \binom{13}{1} * \binom{26}{3}$ ways. With the number of ways of choosing 5-hand cards being $\binom{52}{5}$, the probability of a hand containing exactly one club or exactly one spade is $\frac{2*(\binom{13}{1}*\binom{39}{4})}{\binom{52}{5}}$ - $\frac{\binom{13}{1}*\binom{13}{1}*\binom{26}{3}}{\binom{52}{5}}$

d. The hand has at least one club or at least one spade.
The event of a card having at least one club or one spade can be calculated by taking 1 minus the probability of a five-card hand with neither spades nor clubs. One can make the assessment that if a hand is not part of the group of hands with neither spades nor clubs, then there is at least one spade or one club.

The probability of a five-card hand with at least one club or one spade can be represented as below:

$$1 - \frac{\binom{26}{5}}{\binom{52}{5}}$$

Question 8

a) Exercise 6.3.2: The letters (a, b, c, d, e, f, g) are put in a random order. Each permutation is equally likely. Define the following events:
A: The letter b falls in the middle (with three before it and three after it)
B: c appears to the right of b, althrough not necessarily immediately to the right of b.
C: The letters def occur together in that order.

a. p(A) $= \frac{6!}{7!} = \frac{1}{7}$(ways of arranging six letters on the two sides of b divided by the ways of arranging all seven letters)
p(B) $= \frac{(6+5+4+3+2+1)5!}{7!} = \frac{1}{2}$ because b can only take 6 positions because in the last position, we cannot have an outcome where c is to the right of b. For i^{th} position b takes, c can take 7 - i positions. For example, if b is in the 1^{st} position, c can take 6 positions, if b is in the 2^{nd} position, c can take 5 positions and so on. For each of these, the remaining letters can be arranged in 5! ways. Therefore total ways of arranging the letters such that c appears to the right of b is (6+5+4+3+2+1)5! ways.
p(C) $= \frac{5!}{7!} = \frac{1}{42}$. Here 'def' is treated as a unit in that order. Along the remaining 4 letters, we have 5! ways of arranging a formation where 'def' occurs together.

b. $p(A \mid C) = \frac{p(A \cap C)}{p(C)}$
$p(A \cap C)$ is the event where the letter b falls in the middle and the letters 'def' occur together. As b has a fixed position and 'def' can only be arranged as a single unit, there are only 2 ways of positioning 'def' - either before b or after b. The remaining three letters can be arranged in 3! ways on the remaining side. Therefore $p(A \cap C) = \frac{2*3!}{7!}$.
$p(A \mid C) = \frac{\frac{2*3!}{7!}}{\frac{5!}{7!}}$
$p(A \mid C) = \frac{1}{10}$

c. $p(B \mid C) = \frac{p(B \cap C)}{p(C)}$
$p(B \cap C)$ is the event where 'def' occur together and c is on the right of b. 'def' is considered a single unit here, which along with the rest of the letters can be arranged into 5 positions. Within these 5, b can only take 4 positions as at the 5th position, c cannot be placed to its right. For each of b's position, c has one less way of being arranged. Therefore b and c can be arranged in (4+3+2+1) ways and the rest of the letters - a, g, and def can be arranged in 3! ways. Therefore $p(B \cap C) = \frac{(4+3+2+1)*3!}{7!}$.
$p(A \mid C) = \frac{\frac{(4+3+2+1)*3!}{7!}}{\frac{5!}{7!}} = \frac{1}{2}$

d. $p(A \mid B) = \frac{p(A \cap B)}{p(B)}$

$p(A \cap B)$ is the event where the letter b is fixed in the middle and the letter c appears to the right of b. As b is fixed, c only has 3 positions it can take to the right of b. The remaining letters can be arranged in 5! ways. Therefore $p(A \cap B) = \frac{3*5!}{7!}$.
$p(A \mid B) = \frac{\frac{3*5!}{7!}}{\frac{1}{2}} = \frac{1}{7}$

e. We will prove that A and B are independent by satisfying the condition $p(A \cap B) =$ p(A)*p(B).
$p(A \cap B) = \frac{1}{14}$
p(A)*p(B) $= \frac{1}{7} * \frac{1}{2} = \frac{1}{14}$
A and B are independent events.

Proving $p(A \mid C)$ are not independent:

1. $p(A \cap C) = \frac{2*3!}{7!} = \frac{1}{420}$
p(A)*p(C) $= \frac{1}{7} * \frac{1}{42} = \frac{1}{294}$
2. $p(A \mid C) = \frac{p(A \cap C)}{p(C)} =$ p(A)
$\frac{1}{2} = \frac{p(A \cap C)}{p(C)} = \frac{1}{7}$
We don't have to simplify further as we've already seen $p(A \mid C) \neq p(A)$
3. $p(C \mid A) = \frac{p(A \cap C)}{p(A)} =$ p(C)
$\frac{1}{2} = \frac{p(A \cap C)}{p(A)} = \frac{1}{42}$
We don't have to simplify further as we've already seen $p(C \mid A) \neq p(C)$

We can conclude that A and C are not independent events, given that none of the three conditions held true.

We will prove that B and C are independent by satisfying the condition $p(B \cap C) =$ p(B)*p(C).
$p(B \cap C) = \frac{2*3!}{7!} = \frac{1}{84}$
p(B)*p(C) $= \frac{1}{2} * \frac{1}{42} = \frac{1}{84}$
B and C are independent events.

b) Exercise 6.3.6: p(H) $= \frac{1}{3}$ and p(T) $= \frac{2}{3}$ where outcomes of the coin flips are mutually independent.

b. p(HHHHHTTTTT) $= (\frac{1}{3})^5 * (\frac{2}{3})^5$
c. p(HTTTTTTTTT) $= (\frac{1}{3})^1 * (\frac{2}{3})^9$

c) Exercise 6.4.2

a. We assume A is the fair dice and B is the biased dice. p(A) and p(B) denote the probability of choosing each of the die and equals 0.5.

The question requires us to find the probability that the die picked is fair given the six rolls result in 4,3,6,6,5,5. This is denoted by $p(A \mid 4,3,6,6,5,5)$.
Using Baye's theorem and given probabilities of landing on each value for the fair and the biased die, we have

$$p(A \mid 4,3,6,6,5,5) = \frac{p(4{,}3{,}6{,}6{,}5{,}5|A)*p(A)}{p(4{,}3{,}6{,}6{,}5{,}5|A)*p(A)+p(4{,}3{,}6{,}6{,}5{,}5|B)*p(B)}$$

$$p(A \mid 4,3,6,6,5,5) = \frac{(\frac{1}{6})^6*\frac{1}{2}}{(\frac{1}{6})^6*\frac{1}{2}+(\frac{15}{100})^4*(\frac{25}{100})^2*\frac{1}{2}}$$

$$p(A \mid 4,3,6,6,5,5) = 0.4038$$

Question 9

a) Exercise 6.5.2

a. The range of A where A denotes the number of aces in the hand is {0,1,2,3,4}

b. The distribution over random variable A requires calculating the probability of the occurrence of each event in A.

p(A = 0) = $\frac{\binom{48}{5}}{\binom{52}{5}}$ = 0.6588

p(A = 1) = $\frac{\binom{4}{1}*\binom{48}{4}}{\binom{52}{5}}$ = 0.2995

p(A = 2) = $\frac{\binom{4}{2}*\binom{48}{3}}{\binom{52}{5}}$ = 0.0399

p(A = 3) = $\frac{\binom{4}{3}*\binom{48}{2}}{\binom{52}{5}}$ = 0.0017

p(A = 4) = $\frac{\binom{4}{4}*\binom{48}{1}}{\binom{52}{5}}$ = 0.0000185

Distribution of random variable A = {(0, 0.6588), (1, 0.2995), (2, 0.0399), (3, 0.0017), (4, 0.0000185)}

b) Exercise 6.6.1

a. $E[G] = \sum_{r \in G(S)} r.p(G = r)$
Here r can take 3 values - 0, 1, 2 corresponding to a council with no girls, one girl and two girls each.
The probability for each value of r is found using combinations and corresponds to $\frac{\binom{3}{2}}{\binom{10}{2}}$, $\frac{\binom{7}{1}*\binom{3}{1}}{\binom{10}{2}}$ and $\frac{\binom{7}{2}}{\binom{10}{2}}$.
E[G] = 0*$\frac{\binom{3}{2}}{\binom{10}{2}}$ + 1*$\frac{\binom{7}{1}*\binom{3}{1}}{\binom{10}{2}}$ + 2*$\frac{\binom{7}{2}}{\binom{10}{2}}$
$E[G] = \frac{7}{5}$

c) Exercise 6.6.4

a. When a fair die is rolled once, and X is the random variable that denotes the square of the number that shows up on the die, range of X is 1, 4, 9, 16, 25, 36. Each outcome has the same probability of $\frac{1}{6}$.
E[X] = 1*($\frac{1}{6}$) + 4*($\frac{1}{6}$) + 9*($\frac{1}{6}$) + 16*($\frac{1}{6}$)+ 25*($\frac{1}{6}$)+ 36*($\frac{1}{6}$)
E[X] = $\frac{91}{6}$

b. When a fair coin is tossed three times, and Y is the random variable that denotes the square of the number of heads, range of Y is 0, 1, 4, 9. This corresponds to events where zero, one, two, and three tosses result in a head.
These events have probabilities of $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$ and $\frac{1}{8}$.
E[Y] = 0*$\frac{1}{8}$ + 1*$\frac{3}{8}$ + 4*$\frac{3}{8}$ + 9*$\frac{1}{8}$
E[Y] = 3

d) Exercise 6.7.4

a. There are 10 students and a corresponding 10 number of coats for the teacher to choose from. The probability of the teacher choosing the right coat for a given student is $\frac{1}{10}$ and the probability of the teacher choosing the wrong coat is $\frac{9}{10}$.
If we choose random variable X_i such that it is equal to 1 if a student receives her own coat and 0 if they receive someone else's,
E[X] = E[X_1] + E[X_2] + E[X_3] + E[X_4] + E[X_5] + E[X_6] + E[X_7] + E[X_8] + E[X_9] + E[X_{10}]
Since each X_i has the same distribution, we can rewrite this as
E[x] = 10 E[X_1]
Where E[X_1] = 1*$\frac{1}{10}$ + 0*$\frac{9}{10}$ = $\frac{1}{10}$
E[X] = 10*$\frac{1}{10}$
E[X] = 1

Question 10

a) Exercise 6.8.1
n = 100 as we have 100 circuit boards to choose from. Probability of defect or p = 0.01 and probability of no defect = 0.99.

   a. Using Bernoulli's trial probabilities, the probability of exactly 2 out of a 100 circuit boards having defects is
   $\binom{100}{2} * 0.01^2 * 0.99^{98}$

   b. Probability that out of 100 circuit boards, at least two will have defects can be found by subtracting the probabilities of 0 and 1 defects from 1.
   p(at least 2 have defects) = 1 - $(\binom{100}{1} * 0.01 * 0.99^{99} + \binom{100}{0} * 0.01^0 * 0.99^{100})$
   p(at least 2 have defects) = 0.26425

   c. E[X] = E[X_1] + E[X_2] ..... + E[X_{100}]
   E[X] = n*E[X_1]
   E[X] = n*p
   E[X] = 100*$\frac{1}{100}$
   E[X] = 1

   d. As circuit boards are made in batches of 2, we have 50 batches or n = 50. p now denotes the probability of a batch having a defect and q denotes the probability of a batch not having a defect.
   P(atleast two circuit boards having a defect) = P(at least 1 batch having a defect)
   P(at least one batch having a defect) = 1 - P(0 batches having a defect)
   P(at least one batch having a defect) = 1 - $\binom{50}{0} * 0.01^0 * 0.99^{50}$
   P(at least one batch having a defect) = 0.3949

   Let's say A is a random variable denoting the number of defective circuit boards. To calculate the expected number of circuit boards with defects, we take $n * p$, where $n = 50$ and p = $\frac{1}{100}$. This means we'll expect that np = 0.5 batches or 1 circuit board will have defects over time (though we of course know that in individual cases, a batch is either entirely defective or neither board has defects).

   Rationale for different answers: We see that the probability of at least two circuit boards having defects is .26 when looking at 100 separate circuit boards, versus .39 when looking at 50 batches of 2 circuit boards. We see this difference because the 100 circuit boards in the first example are independent, versus the batch-structure in the second example where if one has defects the other does as well in the batch. In particular, the defect rate stays the same, 1%, but applying to one circuit board versus one entire batch.

b) Exercise 6.8.3

b. If the coin is biased, probability of heads p = 0.3 and probability of tails q = 0.7.
An incorrect conclusion in this case would to conclude that the coin is fair. We do this in case where the number of heads is at least 4.
Probability of us concluding that the coin is fair when it is actually biased can be calculated by finding the probability of the number of heads being at least 4 for the biased coin:

$$\text{P(incorrect conclusion)} = 1 - \left(\binom{10}{3} * 0.3^3 * 0.7^7 + \binom{10}{2} * 0.3^2 * 0.7^8 + \binom{10}{1} * 0.3^1 * 0.7^9 + \binom{10}{0} * 0.3^0 * 0.7^{10}\right)$$

$$\text{P(incorrect conclusion)} = 0.35$$