Tandon Bridge Program - Homework 5

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Question 3

1. Exercise 4.1.3, sections b, c

   b) Not a well-defined function from $\mathbb{R} \to \mathbb{R}$.
   For x = -2 and x = 2, the denominator of the function is 0. In these cases, the functional value is undefined.
   Hence the function is not well-defined.

   c) Well-defined function from $\mathbb{R} \to \mathbb{R}$.
   Each real number x in the domain maps to a valid real number in the co-domain. More specifically, it maps to a positive real number in the codomain.
   The range is $\mathbb{R}^+$.

2. Exercise 4.1.5, sections b, d, h, i, l

   b) $\{4, 9, 16, 25\}$

   d) $\{0, 1, 2, 3, 4, 5\}$

   h) $\{(1,1),(2,1),(3,1),(1,2),(2,2),(3,2),(1,3),(2,3),(3,3)\}$

   i) $\{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$

   l) $\{\emptyset, \{2\}, \{3\}, \{2,3\}\}$

Question 4

I Solve the following questions from the Discrete Math zyBook:

(a) Exercise 4.2.2, sections c, g, k

c) The function is one to one, but not onto.

One-one: There are no two elements x_1 and x_2 in the domain such that $h(x_1) = h(x_2)$.

Not onto: The range for the function only comprises of perfect cubes of integers. Whereas the co-domain contains all integers. Therefore the range is smaller than the target (i.e., there are values in the target that don't have values mapped from the domain to them), so the function is not onto.

Counterexample for onto: 5 is in the co-domain - there is no number in the domain that maps to 5. We know this by taking the cube root of 5 and seeing it is not an integer; we know the domain is only integers.

g) The function is one to one, but not onto.

One-to-one: There are no two ordered pairs such that both of them map to the same co-domain value.

Not onto: Each ordered pair maps to an ordered pair in the target, but not every ordered pair in the target has a mapping from the domain.

Counterexample for onto: (1,1) is part of the co-domain, but does not form the range of the function because (1,1) cannot be written in the form (x+1, 2y) where both x and y belong to the domain of integers.

k) The function is neither one to one nor onto.

Counterexample for one-to-one: both (1,3) and (2,1) in the domain map to 5 in the co-domain.

Counterexample for onto: no value in the domain maps to 1 and 2 in the target.

(b) Exercise 4.2.4, sections b, c, d, g

b) The function is neither one-to-one nor onto.

Counterexample for one-to-one: f(010) and f(110) map to 110 in the co-domain.

Counterexample for onto: no value from the domain maps to 000 in the target.

c) The function is a bijection, both one-to-one and onto.

There are no two values in the domain that map to the same value in the co-domain and the co-domain equals the range of the function.

d) The function is one to one but not onto.

Not onto: The codomain $\{0,1\}^4$ has 16 elements while the domain $\{0,1\}^3$ only has 8 elements. Which means there are elements in the co-domain which are not mapped to any element in the domain. Range does not equal co-domain, hence it is not onto.

Counterexample for onto: The value 0001 in the co-domain has no mapping from the domain.

g) The function is neither one-to-one nor onto.

Counterexample for one-to-one: Both $\emptyset$ and 1 map to $\emptyset$ in the co-domain.

Counterexample for onto: No value in the domain maps to 1 in the target.

II Give an example of a function from the set of integers ($\mathbb{Z}$) to the set of positive integers ($\mathbb{Z}^+$) that is:

(a) one-to-one, but not onto
g: $\mathbb{Z} \to \mathbb{Z}^+$, where g(x) $= |x^3| + 1$ when $x \leq 0$ and g(x) $= x^3 + 3$ when $x > 0$

(b) onto, but not one-to-one
g: $\mathbb{Z} \to \mathbb{Z}^+$, where g(x) $= |x| + 1$

(c) one-to-one and onto
g: $\mathbb{Z} \to \mathbb{Z}^+$, where g(x) $= |2x| + 1$ when $x \leq 0$ and g(x) $= 2x$ when $x > 0$

(d) neither one-to-one nor onto
g: $\mathbb{Z} \to \mathbb{Z}^+$, where g(x) $= |x| + 2$

Question 5

a) Exercise 4.3.2, sections c, d, g, i

c) The function is both one-to-one and onto and therefore has a well-defined inverse function.
$f^{-1}(x) = \frac{x-3}{2}$

d) The function does not have a well-defined inverse, because it's not a bijection; specifically, the function is not one-to-one.
For example, $\{1\}$ and $\{2\}$ both have a cardinality of 1, and map to 1 in the co-domain. This would ensure the inverse function would not be well-defined.

g) The function has a well defined inverse.
$f^{-1}(x)$ is found by taking the input string and reversing the bits.

i) The function has a well-defined inverse as it is bijective.
$f^{-1}(x, y) =$ (x-5, y+2)

b) Exercise 4.4.8, sections c, d

c) $f \circ h : f(h(x)) = f(x^2 + 1) = 2(x^2 + 1) + 3 = 2x^2 + 5$
Answer: $2x^2 + 5$

d) $h \circ f : h(f(x)) = h(2x + 3) = (2x + 3)^2 + 1 = 4x^2 + 12x + 10$
Answer: $4x^2 + 12x + 10$

c) Exercise 4.4.2, sections b-d

b) $(f \circ h)(52)$: $f(h(52)) = f(\lceil \frac{52}{5} \rceil) = f(11) = 11^2$
Answer: 121

c) $(g \circ h \circ f)(4)$: $g(h(f(4))) = g(h(4^2)) = g(h(4^2)) = g(h(16)) = g(\lceil \frac{16}{5} \rceil) = g(4) = 2^4$
Answer: 16

d) $h \circ f$: h(f(x)) = h(x^2) = $\lceil \frac{x^2}{5} \rceil$
Answer: $\lceil \frac{x^2}{5} \rceil$

d) Exercise 4.4.6, sections c-e

c) $(h \circ f)(010)$: h(f(010)) = h(110)
Answer: 111

d) Range of $h \circ f$:
Range of f: {111, 110, 100, 101}
Range of $h \circ f$: {111, 101}

e) Range of $g \circ f$:
Range of f: $\{111, 110, 100, 101\}$
Range of $g \circ f$: $\{111, 011, 001, 101\}$

Extra credit: Exercise 4.4.4, sections c, d

c) No, this is not possible.
   If f is not one-one, there exists elements m_1 and m_2 in the domain of f such that $f(m_1) = f(m_2)$.
   When calculating g ∘ f, there will always exist $f(m_1)$ and $f(m_2)$ in the domain of $g \circ f$ such that $g(f(m_1)) = g(f(m_2))$, hence g cannot be one-one.

d) Yes, this is possible.
   If g is not 1-1, that means there exists two elements m_1 and m_2 in Y such that both of them map to the same element in the codomain of g, Z. that is $g(m_1) = g(m_2)$.
   However if f is not onto, it could be the case that one of either m_1 or m_2 is not in the range of f. Which means that when calculating g ∘ f, only one of them could possibly be in the domain of g ∘ f, Y. This would negate the scenario where g ∘ f cannot be one-one.
   Therefore, there does exist a possibility where g is not one-one but g ∘ f is. And this is the case when f is not onto.