Tandon Bridge Program - Extended - Homework 3

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Question 7

a) Exercise 3.1.1, sections a-g
$A = \{x \in \mathbb{Z} : x \text{ is an integer multiple of } 3\}$
$B = \{x \in \mathbb{Z} : x \text{ is a perfect square}\}$
$C = \{4, 5, 9, 10\}$
$D = \{2, 4, 11, 14\}$
$E = \{3, 6, 9\}$
$F = \{4, 6, 16\}$

(a.) $27 \in A$
True, since A is the set of all integer multiples of 3 and 27 is an integer multiple of 3 (3*9 = 27).

(b.) $27 \in B$
False, since B is the set of all integer perfect squares, and 27 is not a perfect square.

(c.) $100 \in B$
True, since B is the set of all integer perfect squares and 100 is a perfect square of 10, which is an integer.

(d.) $E \subseteq C$ or $C \subseteq E$
False. E is not the subset of C, because it contains 3 and 6, which are not in C. C is not a subset of E, because it contains 4, 5, 10, which are not in E.

(e.) $E \subseteq A$
True. A is the set of all integer multiples of three, and 3, 6, 9 are all multiples of three.

(f.) $A \subseteq E$
False, since A includes all integers multiples of 3 while E contains only 3 integer multiples of 3 (3, 6 and 9).

(g.) $E \in A$
False. While the elements in E are included in A, the set E is not.

b) Exercise 3.1.2, sections a-e
$A = \{x \in \mathbb{Z} : x \text{ is an integer multiple of } 3\}$
$B = \{x \in \mathbb{Z} : x \text{ is a perfect square}\}$
$C = \{4, 5, 9, 10\}$
$D = \{2, 4, 11, 14\}$
$E = \{3, 6, 9\}$
$F = \{4, 6, 16\}$

(a.) $15 \subset A$
False. $15 \in A$, but is not a subset of A, as it is not a set.

(b.) $\{15\} \subset A$
True. For A to be a subset of B, all elements in A should be contained in B. The set in question here has a single element 15 and 15 is also an element in A. We also know it's a perfect subset, because A contains every other integer multiple of 3 in addition to 15.

(c.) $\emptyset \subset C$
True. $\emptyset$ is a subset of all sets. We also learned that $A \subset B$ if B contains an element that is not in A. An example of an element in C not in $\emptyset$ is 4.

(d.) $D \subseteq D$
True. A set can be a subset of itself. It is, however, not a perfect subset, because they contain the exact same elements.

(e.) $\emptyset \in B$
False. $\emptyset \notin B$, but it is a subset of B.

c) Exercise 3.1.5, sections b, d
Express each set using set builder notation. Then if the set is finite, give its cardinality. Otherwise, indicate that the set is infinite

(b.) $\{3, 6, 9, 12, ...\}$
Builder notation: $A = \{x \in \mathbb{Z}^+ : x \text{ is an integer multiple of } 3\}$
Infinite set, so not including cardinality.

(d.) $\{0, 10, 20, 30, ...1000\}$
Builder notation: $A = \{x \in \mathbb{Z} : 0 \leq x \leq 1000 \text{ and } x \text{ is an integer multiple of } 10\}$
Finite set. $|A| = 101$

d) Exercise 3.2.1, sections a-k
Let X = {1, {1}, {1, 2}, 2, {3}, 4}. Which statements are true?

(a.) $2 \in X$
True. 2 is an element in X.

(b.) $\{2\} \subseteq X$
True. All elements in {2}, that is 2 is also an element in X, satisfying the condition for a subset.

(c.) $\{2\} \in X$
False. 2 is an element of X, but the set {2} is not an element of X.

(d.) $3 \in X$
False. 3 is not an element of X, the set {3} is.

(e.) $\{1, 2\} \in X$
True. The set $\{1, 2\}$ is an element of X.

(f.) $\{1, 2\} \subseteq X$
True. Both 1 and 2 are elements of X, so they would make up one subset of X.

(g.) $\{2, 4\} \subseteq X$
True. Both 2 and 4 are elements of X, so they would make up one subset of X.

(h.) $\{2, 4\} \in X$
False. The set $\{2, 4\}$ does not belong to X.

(i.) $\{2, 3\} \subseteq X$
False. The set $\{2, 3\}$ is not a subset of X because 3 is not an element in X.

(j.) $\{2, 3\} \in X$
False. The set $\{2, 3\}$ does not belong to X

(k.) $|X| = 7$
False. The cardinality of X is 6, as there are 6 elements in the set.

Question 8 Solve Exercise 3.2.4, section b

b . Let $A = \{1, 2, 3\}$. What is $\{X \in P(A) : 2 \in X\}$?
Answer: $\{X \in P(A) : 2 \in X\} = \{\{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$

Rationale: For $\{X \in P(A) : 2 \in X\}$, we need to find those elements in P(A) that contain the element 2.
P(A) $= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. We identify those elements of $P(A)$ that contain 2, resulting in the answer above.

Question 9

a) Exercise 3.3.1, sections c-e
$A = \{-3, 0, 1, 4, 17\}$
$B = \{-12, -5, 1, 4, 6\}$
$C = \{x \in \mathbb{Z} : x \text{ is odd}\}$
$D = \{x \in \mathbb{Z} : x \text{ is positive}\}$

(c.) $A \cap C$

C contains all odd integers while A = $\{-3, 0, 1, 4, 17\}$.
$A \cap C$ contains all elements common to both A and C, essentially all odd integers in A.
$A \cap C = \{-3, 1, 17\}$

(d.) $A \cup (B \cap C)$

Here $(B \cap C)$ is tabulated using the same logic as the previous question: $(B \cap C) = \{-5, 1\}$
$A \cup (B \cap C)$ contains all the elements present in either A or $\{B \cap C\}$.
$\therefore A \cup (B \cap C) = \{-3, -5, 0, 1, 4, 17\}$

(e.) $A \cap B \cap C$

$A \cap B \cap C$ or the intersection of A, B and C contains all the elements common to A, B and C. There is only one element that is common to the three sets: 1
$\therefore A \cap B \cap C = \{1\}$

b) Exercise 3.3.3, sections a, b, e, f

$A_i = \{i^0, i^1, i^2\}$

$C_i = \{x \in \mathbb{R} : \frac{-1}{i} \leqslant x \leqslant \frac{1}{i}\}$

(a.) $\bigcap_{i=2}^{5} A_i$

$A_2 = \{2^0, 2^1, 2^2\} = \{1, 2, 4\}$
$A_3 = \{3^0, 3^1, 3^2\} = \{1, 3, 9\}$
$A_4 = \{4^0, 4^1, 4^2\} = \{1, 4, 16\}$
$A_5 = \{5^0, 5^1, 5^2\} = \{1, 5, 25\}$

$\bigcap_{i=2}^{5} A_i = A_2 \cap A_3 \cap A_4 \cap A_5$
$\bigcap_{i=2}^{5} A_i = \{1\}$

(b.) $\bigcup_{i=2}^{5} A_i$

We use A_2, A_3, A_4, A_5 from the previous question

$\bigcup_{i=2}^{5} A_i = A_2 \cup A_3 \cup A_4 \cup A_5$

$\bigcup_{i=2}^{5} A_i = \{1, 2, 3, 4, 5, 9, 16, 25\}$

(e.) $\bigcap_{i=1}^{100} C_i$

$C_1 = \{x \in \mathbb{R} : \frac{-1}{1} \leqslant x \leqslant \frac{1}{1}\}$

$C_2 = \{x \in \mathbb{R} : \frac{-1}{2} \leqslant x \leqslant \frac{1}{2}\}$

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.
$C_{99} = \{x \in \mathbb{R} : \frac{-1}{99} \leqslant x \leqslant \frac{1}{99}\}$

$C_{100} = \{x \in \mathbb{R} : \frac{-1}{100} \leqslant x \leqslant \frac{1}{100}\}$

The range of x reduces as we go from C_1 to C_{100}. It starts with all the elements where $x \in [-1, 1]$ and reduces all the way to only those elements where $x \in [\frac{-1}{100}, \frac{1}{100}]$.

Here each larger set includes all smaller sets that succeed it.
$\therefore x \in [\frac{-1}{100}, \frac{1}{100}]$ which is the smallest set in the sequence of $C_1, C_2, ...C_{99}, C_{100}$ has all the elements that are common to the bigger sets that precede it.

$\therefore \bigcap_{i=1}^{100} C_i = \{x \in \mathbb{R} : \frac{-1}{100} \leqslant x \leqslant \frac{1}{100}\}$

(f.) $\bigcup_{i=1}^{100} C_i$

We use the same values for C_1 to C_{100} as the previous question. We know that C_2 to C_{100} are all proper subsets of C_1.

In this case, union of all sets will be C_1 as it contains all the elements of its subsets C_2 to C_{100} in addition to its own elements.

$\therefore \bigcup_{i=1}^{100} C_i = \{x \in \mathbb{R} : -1 \leqslant x \leqslant 1\}$

c) Exercise 3.3.4, sections b, d

A = {a, b}

B = {b, c}

(b.) $P(A \cup B)$
$A \cup B = \{a, b\} \cup \{b, c\}$
$A \cup B = \{a, b, c\}$
$P(A \cup B) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

(d.) $P(A) \cup P(B)$
$P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
$P(B) = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$
$P(A) \cup P(B) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$

Question 10

a) Exercise 3.5.1, sections b, c
$A = \{tall, grande, venti\}$
$B = \{foam, no - foam\}$
$C = \{non - fat, whole\}$

(b.) (foam, tall, non-fat)

(c.) B x C $= \{(foam, non - fat), (foam, whole), (no - foam, non - fat), (no - foam, whole)\}$

b) Exercise 3.5.3, sections b, c, e

(b.) $\mathbb{Z}^2 \subseteq \mathbb{R}^2$
True.
Rationale: $\mathbb{Z}$ is a subset of $\mathbb{R}$ and both $\mathbb{Z}^2$ and $\mathbb{R}^2$ are made up of ordered pairs.

(c.) $\mathbb{Z}^2 \cap \mathbb{Z}^3 = \emptyset$
True.
Rationale: $\mathbb{Z}^2$ contains ordered pairs whereas $\mathbb{Z}^3$ contains triples.

(e.) For any three sets, A, B, and C, if $A \subseteq B$, then $A * C \subseteq B * C$
True.
Rationale: If all elements of A are in B, then we know $x \in A$ and $x \in B$. We also know that the ordered pair in $A * C$ would be (x, y) (where $y \in C$). Thus we know (x, y) would also be in $B * C$.

c) Exercise 3.5.6, sections d, e

(d.) $\{xy :$ where $x \in \{0\} \cup \{0\}^2$ and $y \in \{1\} \cup \{1\}^2\}$
$x \in \{0\} \cup \{0\}^2; x \in \{0, 00\}$
$y \in \{1\} \cup \{1\}^2; y \in \{1, 11\}$
$\{xy :$ where $x \in \{0\} \cup \{0\}^2$ and $y \in \{1\} \cup \{1\}^2\} = \{01, 011, 001, 0011\}$

(e.) $\{xy :$ where $x \in \{aa, ab\}$ and $y \in \{a\} \cup \{a\}^2\}$
$y \in \{a\} \cup \{a\}^2; y \in \{a, aa\}$
$\{xy :$ where $x \in \{aa, ab\}$ and $y \in \{a\} \cup \{a\}^2\} = \{aaa, aaaa, aba, abaa\}$

d) Exercise 3.5.7, sections c, f, g
$A = \{a\}$
$B = \{b, c\}$
$C = \{a, b, d\}$

(c.) $(A \times B) \cup (A \times C)$
$A \times B = \{ab, ac\}$
$A \times C = \{aa, ab, ad\}$
$(A \times B) \cup (A \times C) = \{aa, ab, ac, ad\}$

(f.) $P(A \times B)$
$A \times B = \{ab, ac\}$
$P(A \times B) = \{\emptyset, \{ab\}, \{ac\}, \{ab, ac\}\}$

(g.) $P(A) \times P(B)$
$P(A) = \{\emptyset, \{a\}\}$
$P(B) = \{\emptyset, \{b\}, \{c\}, \{bc\}\}$
$P(A)\times P(B) = \{(\emptyset, \emptyset), (\emptyset, \{b\}), (\emptyset, \{c\}), (\emptyset, \{bc\}), (\{a\}, \emptyset), (\{a\}, \{b\}), (\{a\}, \{c\}), (\{a\}, \{bc\})\}$

Question 11

a) Exercise 3.6.2, sections b, c

(b.) $(B \cup A) \cap (\overline{B} \cup A) = A$

1	$(B \cup A) \cap (\overline{B} \cup A)$	Premise
2	$(A \cup B) \cap (\overline{B} \cup A)$	Commutative law, 1
3	$(A \cup B) \cap (A \cup \overline{B})$	Commutative law, 2
4	$A \cup (B \cap \overline{B})$	Distributive law, 3
5	$A \cup \emptyset$	Complement law 4
6	A	Identity law, 5

We started from the premise and arrived at the conclusion using valid laws of logic, proving the identity.

(c.) $\overline{A \cap \overline{B}} = \overline{A} \cup B$

1	$(\overline{A \cap \overline{B}})$	Premise
2	$\overline{A} \cup \overline{\overline{B}}$	De Morgan's Law, 1
3	$\overline{A} \cup B$	Double complement law, 2

We started from the premise and arrived at the conclusion using valid laws of logic, proving the identity.

b) Exercise 3.6.3, sections b, d

(b.) $A - (B \cap A) = A$

Say $A = \{1, 2\}$ and $B = \{1\}$.
Here $B \cap A = \{1\}$, because there is only one element: 1, that is part of both A and B
Plugging $B \cap A = \{1\}$ into the left hand side of the question, you get: $\{1, 2\} - \{1\}$.
This is equal to $\{2\}$, which is not equal to $\{A\}$ or $\{1, 2\}$.

$\therefore A - (B \cap A) = A$ is not a set identity.

(d.) $(B - A) \cup A = A$

Say $A = \{1, 2\}$ and $B = \{1, 2, 3\}$.
Here $B - A = \{3\}$, because 3 is the only element only in B, not A.
Plugging $B - A = \{3\}$ and $A = \{1, 2\}$ into the left hand side of the question, you get:
$(B - A) \cup A$
$\{3\} \cup \{1, 2\}$

$\{1, 2, 3\}$.

Since $\{1, 2, 3\} \neq \{1, 2\}$, $\{1, 2, 3\} \neq A$
$\therefore (B - A) \cup A = A$ is not a set identity.

c) Exercise 3.6.4, sections b, c

(b.) $A \cap (B - A) = \emptyset$

We start with the left hand side and arrive at the right hand side.

1	$A \cap (B - A)$	Premise
2	$A \cap (B \cap \overline{A}$	Set subtraction law, 1
3	$A \cap (\overline{A} \cap B)$	Commutative law, 2
4	$(A \cap \overline{A}) \cap B$	Associative law, 3
5	$\emptyset \cap B$	Complement law, 4
6	$\emptyset$	Domination law, 5

$\therefore A \cap (B - A) = \emptyset$

(c.) $A \cup (B - A) = A \cup B$

We start with the left hand side and arrive at the right hand side.

1	$A \cup (B - A)$	Premise
2	$A \cup (B \cap \overline{A}$	Set subtraction law, 1
3	$(A \cup B) \cap (A \cup \overline{A})$	Distributive law, 2
4	$(A \cup B) \cap U)$	Complement law, 3
5	$(A \cup B)$	Identity law, 4

$\therefore A \cup (B - A) = A \cup B$