Tandon Bridge Program - Homework 6

Archana Balachandran (ab12864) and Rachel Butler (rb3028)

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Question 5
Use the definition of Θ in order to show the following:

a $5n^3 + 2n^2 + 3n = \Theta(n^3)$

Proof.
We'll prove $5n^3 + 2n^2 + 3n = \Theta(n^3)$ by proving first that $5n^3 + 2n^2 + 3n = O(n^3)$ and then proving $5n^3 + 2n^2 + 3n = \Omega(n^3)$

Proving $5n^3 + 2n^2 + 3n = O(n^3)$: Let $n_o = 1$ and $c_1 = 10$. We will show that for any $n \geq 1$, $f(n) \leq 10 * g(n)$, where $f(n) = 5n^3 + 2n^2 + 3n$ and $g(n) = n^3$.

$$5n^3 + 2n^2 + 3n \leq 5n^3 + 2n^3 + 3n^3$$
$$\text{Simplified: } 5n^3 + 2n^2 + 3n \leq 10n^3$$

Finally, $5n^3 + 2n^2 + 3n \leq 10n^3 = 10 * g(n)$. Putting the inequalities together, we get that for any $n \geq 1$,

$$f(n) = 5n^3 + 2n^2 + 3n \leq 5n^3 + 2n^3 + 3n^3 = 10n^3 = 10 * g(n).$$

and therefore, $f(n) \leq 10 * g(n)$, which means $f = O(g)$ or in this case $5n^3 + 2n^2 + 3n = O(n^3)$.

Proving $5n^3 + 2n^2 + 3n = \Omega(n^3)$: Let $n_o = 1$ and $c_2 = 5$. We will show that for any $n \geq 1$, $5n^3 + 2n^2 + 3n \geq 5n^3$.
Since $n \geq 1$, we know $2n^2 \geq 0$ and $3n \geq 0$. Putting the inequalities together gives:

$$2n^2 + 3n \geq 0$$

Adding $5n^3$ to both sides, we get:

$$5n^3 + 2n^2 + 3n \geq 5n^3$$

Thus, for $n \geq 1$, $5n^3 + 2n^2 + 3n = \Omega(n^3)$.

Now we have a proof for big O and a proof for Ω with the following values:
1) For n ≥ 1, $f(n) \leq 10 * g(n)$ where $c_1 = 10$ and $n_0 = 1$
2) For n ≥ 1, $f(n) \geq 5 * g(n)$ where $c_2 = 5$ and $n_0 = 1$

Hence we finally have a proof in the form $c_2.g(n) \leq f(n) \leq c_1.g(n)$ for $n \geq n_0$ with the values $5n^3 \leq 5n^3 + 2n^2 + 3n \leq 10n^3$ and $c_1 = 10$, $c_2 = 5$ and $n_0 = 1$ proving $5n^3 + 2n^2 + 3n = \Theta(n^3)$. ■

b $\sqrt{7n^2+2n-8} = \Theta(n)$

Proof.
We'll prove $\sqrt{7n^2+2n-8} = \Theta(n)$ by proving first that $\sqrt{7n^2+2n-8}$ $= O(n)$ and then proving $\sqrt{7n^2+2n-8} = \Omega(n)$

Proving $\sqrt{7n^2+2n-8} = O(n)$: Let $n_o = 1$ and $c_1 = 3$. We will show that for any $n \geq 1$, $f(n) \leq 3 * g(n)$, where $f(n) = \sqrt{7n^2+2n-8}$ and $g(n) = n$.

$$f(n) = \sqrt{7n^2+2n-8} \leq \sqrt{7n^2+2n}$$

$n \leq n^2$, so:

$$\sqrt{7n^2+2n} \leq \sqrt{7n^2+2n^2}$$

Putting the inequalities together, we get:

$$f(n) = \sqrt{7n^2+2n-8} \leq \sqrt{7n^2+2n} \leq \sqrt{7n^2+2n^2}$$
$$\text{Simplifying: } f(n) = \sqrt{7n^2+2n-8} \leq \sqrt{7n^2+2n} \leq \sqrt{9n^2} = 3n$$

Finally, $3n = 3 * g(n)$. Putting this in with the other inequalities we get:

$$f(n) = \sqrt{7n^2+2n-8} \leq \sqrt{7n^2+2n} \leq \sqrt{9n^2} = 3n = 3 * g(n)$$

This shows that $f(n) \leq 3 {*} g(n)$, which means $f = O(g)$ or $\sqrt{7n^2+2n-8} = O(n)$

Proving $\sqrt{7n^2+2n-8} = \Omega(n)$:
Let $n_o = 4$ and $c_2 = \sqrt{7}$.

We will show that for any $n \geq 4$, $f(n) \geq \sqrt{7} * g(n)$, where $f(n) = \sqrt{7n^2+2n-8}$ and $g(n) = n$.

We begin with $n \geq 4$
Multiplying both sides by 2, we have $2n \geq 8$
Subtracting both sides by 8, we have $2n - 8 \geq 0$
Adding $7n^2$ to both sides, we have $7n^2 + 2n - 8 \geq 7n^2$
Since $n \geq 4$, we know that both sides of the equation would be greater than 1. Hence without affecting the inequality, we can take the square root of both sides of the equation.
We then have $\sqrt{7n^2+2n-8} \geq \sqrt{7n^2}$
On simplification, this would be $\sqrt{7n^2+2n-8} \geq \sqrt{7}n$
This equation corresponds to $f(n) \geq \sqrt{7}*g(n)$ where $f(n) = \sqrt{7n^2+2n-8}$ and $g(n) = n$.
Hence proving $\sqrt{7n^2+2n-8} = \Omega(n)$

Now we have a proof for big O and a proof for Ω with the following values:
1) For n ≥ 1, $f(n) \leq 3 * g(n)$ where $c_1 = 3$ and $n_0 = 1$
2) For n ≥ 4, $f(n) \geq \sqrt{7} * g(n)$ where $c_2 = \sqrt{7}$ and $n_0 = 4$

To standardize and find a common n for the two proofs, we can modify O to $n \geq 4$ as it is true for all $n \geq 1$ so it will remain true for all values above 1, including 4. This allows us to rewrite our O and Ω as
1) For n ≥ 4, $f(n) \leq 3 * g(n)$ where $c_1 = 3$ and $n_0 = 4$
2) For n ≥ 4, $f(n) \geq \sqrt{7} * g(n)$ where $c_2 = \sqrt{7}$ and $n_0 = 4$

Hence we finally have a proof in the form $c_2.g(n) \leq f(n) \leq c_1.g(n)$ for $n \geq n_0$ with the values $\sqrt{7}n \leq \sqrt{7n^2+2n-8} \leq 3n$ for $n \geq 4$ and $c_1 = 3$, $c_2 = \sqrt{7}$ and $n_0 = 4$ proving $\sqrt{7n^2+2n-8} = \Theta(n)$. $\blacksquare$