Matrix Methods Homework 8
APPM 3310-003

Dawson Beatty

March 9, 2016

Section 5.1

1

Let $\mathbb{R}^2$ have the standard dot product. Classify the following pairs as:

$$i.)Basis \qquad ii.)Orthogonal\ Basis \qquad iii.)Orthonormal\ Basis$$

(a)

$$v_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow v_1 \cdot v_2 = 0, ||v_1|| \neq 1, \boxed{\text{orthogonal basis}}$$

(b)

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow v_1 \cdot v_2 = 0, ||v_1|| = ||v_2|| = 1, \boxed{\text{orthonormal basis}}$$

(c)

$$v_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow v_1 \cdot v_2 = -4, \boxed{\text{Not a basis}}$$

(d)

$$v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -6 \end{bmatrix} \Rightarrow v_1 \cdot v_2 = -16, , \boxed{\text{basis}}$$

(e)

$$v_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Rightarrow v_1 \cdot v_2 = 0, ||v_2|| \neq 1, \boxed{\text{orthogonal basis}}$$

(f)

$$v_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} \Rightarrow v_1 \cdot v_2 = 0, ||v_1|| = ||v_2|| = 1, \boxed{\text{orthonormal basis}}$$

3

Repeat *5.1.1* but use $\langle v, w \rangle = v_1 w_1 + \frac{1}{9} v_2 w_2$ as the inner product instead.

(d)

$$v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -6 \end{bmatrix} \Rightarrow v_1 \cdot v_2 = 2 + \frac{1}{9}(-18) = 0, \boxed{\text{orthogonal basis}}$$

(e)

$$v_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Rightarrow v_1 \cdot v_2 = 0, ||v_1|| = ||v_2|| = 1, \boxed{\text{orthonormal basis}}$$

5

Find all values of a such that the vectors $\begin{bmatrix} a \\ 1 \end{bmatrix}, \begin{bmatrix} -a \\ 1 \end{bmatrix}$ form an orthogonal basis of $\mathbb{R}^2$ under...

(c)

The inner product prescribed by the positive definite matrix $K = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$

$$\begin{bmatrix} a \\ 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2a - 1 \\ -a + 3 \end{bmatrix} \begin{bmatrix} -a \\ 1 \end{bmatrix} = -2a^2 + a - a + 3$$

Vectors are orthogonal if their inner product equals zero, so set the above result equal to 0:

$3 - 2a^2 = 0, \boxed{a = \sqrt{\frac{2}{3}}}$

9

True or False: If v_1, v_2, v_3 are a basis for $\mathbb{R}^3$, then they form an orthogonal basis under some appropriately weighted inner product $\rangle v, w\rangle = av_1w_1 + bv_2w_2 + cv_3w_3$

Three vectors are linearly independent if and only if $0 \neq c_1v_1 + c_2v_2 + c_3v_3 \,\forall\, c_i$ not all 0.

$\boxed{\text{False}}$. It's not possible to guarantee this property will hold, especially for vectors with all positive elements. Since a, b, c are the diagonals of a positive definite matrix, they must be positive as well.

11

Proof. **Seek to prove that every orthogonal basis of** $mathbbR^2$ **under the standard dot product has the form** $u_1 = \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}, u_2 = \begin{bmatrix}\sin\theta \\ \pm\cos\theta\end{bmatrix}$

Every orthonormal basis using the standard dot product must satisfy the following properties:

$$\begin{bmatrix}x_1 \\ x_2\end{bmatrix} \cdot \begin{bmatrix}y_1 \\ y_2\end{bmatrix} = x_1y_1 + x_2y_2 = 0$$

and

$$\left\|\begin{bmatrix}x_1 \\ x_2\end{bmatrix}\right\| = \sqrt{x_1^2 + x_2^2} = 1, \quad \left\|\begin{bmatrix}y_1 \\ y_2\end{bmatrix}\right\| = \sqrt{y_1^2 + y_2^2} = 1$$

The unit length condition is trivial, since

$$\sqrt{\sin^2\theta + \cos^2\theta} = 1 \,\forall\, \theta$$

For the other condition, we get

$$\begin{aligned}\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \cdot \begin{bmatrix}\sin\theta \\ \pm\cos\theta\end{bmatrix} &= \cos\theta\sin\theta \pm \sin\theta\cos\theta = 0 \\ &= \frac{1}{2}\big[\sin(\theta+\theta) - \sin(\theta-\theta)\big] \pm \frac{1}{2}\big[\sin(\theta+\theta) - \sin(\theta-\theta)\big] \\ &= \frac{1}{2}\sin(2\theta) \pm \frac{1}{2}\sin(2\theta)\end{aligned}$$

At the point shown above, we split into two cases. If a minus is chosen in the $\pm$, then the result of the whole expression will be zero, regardless of the choice of θ. On the other hand, if plus is chosen, we get that $\sin(2\theta) = 0$. Which is only true when $\theta = n \cdot \frac{\pi}{2}, n \in \mathbb{Z}$. $\square$

22

(a)

Prove the vectors $v_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\1\\-2\end{bmatrix}, v_3 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$ **form a basis of** $\mathbb{R}^3$ **with the standard dot product.**

It's trivial to show the following.

$$\boxed{v_1 \cdot v_2 = 0, v_1 \cdot v_3 = 0, v_2 \cdot v_3 = 0}$$

(b)

Use orthogonality to write $v = \begin{bmatrix}1\\2\\3\end{bmatrix}$ **as a combination of the three vectors from part** (a)**.**

$$v = av_1 + bv_2 + cv_3$$
$$a = \frac{\langle v, v_1\rangle}{||v_1||} = \frac{6}{3} = 2$$
$$b = \frac{\langle v, v_2\rangle}{||v_2||} = \frac{-3}{6} = \frac{-1}{2}$$
$$c = \frac{\langle v, v_3\rangle}{||v_3||} = \frac{1}{2}$$
$$\begin{bmatrix}1\\2\\3\end{bmatrix} = 2\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\-2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix}$$
$$= \begin{bmatrix}2\\2\\2\end{bmatrix} + \begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix} + \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{bmatrix} \boxed{= \begin{bmatrix}1\\2\\3\end{bmatrix}}$$

Section 5.2

1

(a)

Use Gram-Schmidt to find an orthonormal basis for $\mathbb{R}^3$ **starting with the following vectors:** $w_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, w_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}, w_1 = \begin{bmatrix}-1\\2\\1\end{bmatrix}$

Arbitrarily choose $v_1 = w_1$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{||v_1||^2} v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{||v_1||^2} v_1 - \frac{\langle w_3, v_2 \rangle}{||v_2||^2} v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \frac{0}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Now to normalize...

$$\boxed{u_1 = \frac{v_1}{||v_1||} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}}$$

3

Try Gram-Schmidt on the following vectors

$$w_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, w_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}, w_4 = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 1 \end{bmatrix}$$

Arbitrarily set $v_1 = w_1$.

$$v_2 = w_2 - \frac{w_2 \cdot v_1}{||v_1||^2} v_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = v_2$$

$$v_3 = w_3 - \frac{w_3 \cdot v_1}{||v_1||^2} v_1 - \frac{w_3 \cdot v_2}{||v_2||^2} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{-3}{\sqrt{3}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - \sqrt{3} \\ 0 \\ -1 + \sqrt{3} \\ -1 + \sqrt{3} \end{bmatrix}$$

Now we can see that v_2 and v_3 are parallel, which obviously means they're not orthogonal. $v_3(\frac{1+\sqrt{3}}{2}) = v_2$

4

Use the Gram-Schmidt process to construct an orthonormal basis for the following subspaces of $\mathbb{R}^3$

(a)

Plane spanned by $w_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ Can arbitrarily use $v_1 = w_1$.

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{||v_1||^2} v_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} - \frac{-5}{5} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Now we find the norms of each of those, since we need an orthonormal basis.

$$\boxed{u_1 = \frac{v_1}{||v_1||} = \begin{bmatrix} 0 \\ \frac{2}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \; u_2 = \frac{v_2}{||v_2||} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

(b)

The plane $2x - y + 3z = 0$ We pick two arbitrary vectors in the plane, then use the Gram-Schmidt process to find an orthonormal basis.

We'll pick $w_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Again, arbitrarily pick $v_1 = w_1$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{||v_1||^2} v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{5}{3} \\ \frac{-1}{3} \end{bmatrix}$$

Now we just need to normalize each of our vectors.

$$\boxed{u_1 = \frac{v_1}{||v_1||} = \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, u_2 = \frac{v_2}{||v_2||} = \begin{bmatrix} 4\sqrt{42} \\ 5\sqrt{42} \\ -\sqrt{42} \end{bmatrix}}$$

(c)

Set of all vectors orthogonal to $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ Choose two vectors perpendicular to the vector above, we'll pick $w_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ Set $v_1 = w_2$ because it's easy and we don't like change.

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{||v_1||^2} v_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} - \frac{-1}{5} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ 2 \\ \frac{-4}{5} \end{bmatrix}$$

Now we normalize:

$$\boxed{u_1 = \frac{v_1}{||v_1||} = \begin{bmatrix} \frac{2}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix}, u_2 = \frac{v_2}{||v_2||} = \begin{bmatrix} 4\sqrt{30} \\ \frac{4\sqrt{30}}{5} \\ -8\sqrt{30} \end{bmatrix}}$$

11

(a)

How many orthonormal bases does $\mathbb{R}$ have? Only 2, since only 1 and -1 have magnitude 1, and nothing can be 'orthogonal' to a Real except 0.

(b)

What about $\mathbb{R}^2$? By *Theorem 5.15*, any inner product space with dimension greater than one has infinitely many orthonormal bases.

(c)

Does the answer change for a different inner product? No. *Theorem 5.15* applies to *any* inner product space, so our answer still holds.

15

True or False: Reordering the original basis before starting Gram-Schmidt leads to the same orthogonal basis.

$\boxed{\text{FALSE}}$. Counter example: Given the vectors $w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

First we choose $v_1 = w_1$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{||v_1||^2} v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Which gives us the basis of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} \\ \frac{1}{2} \end{bmatrix}$

On the other hand, if we initially choose $v_1 = w_2$ (Really mixing things up here):

$$v_2 = w_1 - \frac{\langle w_1, v_1 \rangle}{||v_1||^2} v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Which gives us the basis $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Both of the above are valid orthogonal bases, but they're not the same.

Section 5.3

1

Determine which of the following matrices are orthogonal

(b)

The matrix $b = \begin{bmatrix} \frac{12}{13} & \frac{5}{13} \\ \frac{-5}{13} & \frac{12}{13} \end{bmatrix}$ Any orthogonal matrix Q must satisfy the property $Q^T Q = I$

$$b^T b = \begin{bmatrix} \frac{12}{13} & \frac{-5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{bmatrix} \cdot \begin{bmatrix} \frac{12}{13} & \frac{5}{13} \\ \frac{-5}{13} & \frac{12}{13} \end{bmatrix} = \begin{bmatrix} \frac{144}{13} + \frac{25}{13} & \frac{60}{13} - \frac{60}{13} \\ \frac{60}{13} - \frac{60}{13} & \frac{25}{13} + \frac{144}{13} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

(e)

$$e = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix}$$

$$e^T \cdot e = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{61}{144} & \frac{3}{10} & \frac{7}{30} \\ \frac{3}{10} & \frac{769}{3600} & \frac{1}{6} \\ \frac{7}{30} & \frac{1}{6} & \frac{469}{3600} \end{bmatrix}$$

Not orthogonal.

8

(a)

Proof. **Seek to prove that the transpose of an orthogonal matrix is also orthogonal.**

For any orthogonal matrix Q, we know that $Q^T Q = I$, by the definition of an orthogonal matrix. We also know that $Q^{-1} = Q^T$

Let F be an orthogonal matrix, which means that $F^T F = I$. We can take the transpose of F, F^T and seek to show that $(F^T)^T F^T = I$. We know that $(F^T)^T$ is just F, and we can use the second property from above to show that $F^T = F^{-1}$. Shown symbolically:

$$(F^T)^T F^T = F F^{-1} = I$$

Showing that the transpose of an orthogonal matrix is itself orthogonal. *QED* □

(b)

Explain why the rows od an $n \times n$ orthogonal matrix also form an orthonormal basis of $\mathbb{R}^n$ To be a basis for $\mathbb{R}^n$, we need n linearly independent elements of $\mathbb{R}^n$.

Let $A_{n\times n}$ be an orthogonal matrix composed of column vectors $\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$

We know that A is orthogonal, so $A^TA = I$, or

$$\begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} \cdot \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1^T \cdot v_1 & v_1^T \cdot v_2 & \cdots & v_1^T \cdot v_n \\ v_2^T \cdot v_1 & v_2^T \cdot v_2 & \cdots & v_2^T \cdot v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n^T \cdot v_1 & v_n^T \cdot v_2 & \cdots & v_n^T \cdot v_n \end{bmatrix} = I$$

We get zero on each of the non-diagonal entries, showing that each of the n vectors making up the rows of A are orthogonal to all the others, which shows that we have n orthogonal basis vectors.

We can also see that this basis is orthonormal, because on each of the diagonals, the magnitude of $v_1v_1 = ||v_1||^2 = 1$.

10

True or False

(a)

A matrix whose columns form an orthogonal basis of $\mathbb{R}^n$ is an orthogonal basis.

$\boxed{\text{False}}$. Counter example:

$$\text{Let } A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \neq I$$

The columns must be normalized for this to work.

(b)

A matrix whose rows form an orthonormal basis of $mathbbR^n$ is an orthogonal basis.

$\boxed{\text{True}}$. The only reason (a) didn't work is that the columns weren't normalized, but the property holds in this case.

(c)

An orthogonal matrix is symmetric if and only if it's a diagonal matrix. $\boxed{\text{False}}$. Counter example: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is an symmetric orthogonal matrix.

16

(a)

Proof. **Seek to prove that if Q is an orthogonal matrix, then $||Qx|| = ||x|| \; \forall \, x \, \epsilon \, \mathbb{R}^n$, where $|| \cdot ||$ denotes the Euclidean norm.**

Proposition 5.19 states that a matrix Q is orthogonal if and only if its columns form an orthonormal basis with respect to the Euclidean dot product on $\mathbb{R}^n$

Let Q be composed of $\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ vectors, which we know form an orthonormal basis with respect to the dot product. Also let each vector $v_1, v_2, \cdots, v_n$ be composed of elements such that $v_1 = \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{bmatrix}$

$$Q \cdot x = \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{n1} \\ v_{21} & v_{22} & \cdots & v_{n2} \\ v_{13} & v_{23} & \ddots & \vdots \\ \vdots & \vdots & \cdots & v_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} v_{11}x_1 + v_{21}x_2 + \cdots + v_{n1}x_n \\ v_{12}x_1 + v_{22}x_2 + \cdots + v_{n2}x_n \\ \vdots \\ v_{1n}x_1 + v_{2n}x_2 + \cdots + v_{nn}x_n \end{bmatrix}$$

Taking the norm of the above, we get

$$\sqrt{(v_{11}x_1 + v_{21}x_2 + \cdots + v_{n1}x_n)^2 + (v_{12}x_1 + v_{22}x_2 + \cdots + v_{n2}x_n)^2 + \cdots (v_{1n}x_1 + v_{2n}x_2 + \cdots + v_{nn}x_n)^2}$$

We proved in *5.3.8* that the transpose of an orthogonal matrix is itself orthogonal. From this and *Proposition 5.19* we can see that the rows of Q are orthonormal. This means that each of the squared terms above (which each contain an orthonormal vector times the vector x) must exactly equal the magnitude of x, since multiplying by an orthonormal vector cannot change the magnitude. □

(b)

Proof. **Seek to prove the converse of the above: if $||Qx|| = ||x||$ for all x, Q must be orthogonal.**

If we take the square of both sides of the equality we seek to prove ($||Qx||^2 = ||x||^2$), we can rewrite the left as shown below:

$$||Qx||^2 = (Qx)^T(Qx) = (x^TQ^T)(Qx) = x^TIx = ||x||^2$$

We made the assumption in the fourth step above that $Q^TQ = I$, which is only true for orthogonal matrices. Thus the only way the converse can hold is if Q is orthogonal. □

17

(a)

Show that if $u \epsilon \mathbb{R}^n$ is a unit vector, then the $n \times n$ matrix $Q = I - 2uu^T$ is an orthogonal matrix or a Householder Matrix

$$\begin{aligned} &Q^T Q = I \\ &(I - 2uu^T)^T (I - 2uu^T) = I \\ &(I^T - 2uu^T)(I - 2uu^T) = I \\ &I - 4uu^T + 4(uu^T)^2 = I \end{aligned}$$

We know that u is a unit vector, so $||uu^T|| = 1$, so we can rewrite the last line from above as $I - 4(1) + 4(1) = I \Rightarrow I = I$, QED.

(c)

Prove Q is symmetric

I is symmetric of course, and uu^T must be symmetric, since $u_i u_j = u_j u_i \forall u_i, u_j \epsilon u, i, j \epsilon \mathbb{N}$

25

(b)

A complex, square matrix U is called unitary if it satisfies $U^\dagger U = I$, where $U^\dagger = \overline{U^T}$ denotes the Hermitian transpose in which one first transposes and then takes the complex conjugates of all entries.

$$\begin{aligned} &U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\iota}{\sqrt{2}} \\ \frac{\iota}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &U^T = U \Rightarrow \overline{U^T} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-\iota}{\sqrt{2}} \\ \frac{-\iota}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = U^\dagger \\ &U^\dagger \cdot U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-\iota}{\sqrt{2}} \\ \frac{-\iota}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\iota}{\sqrt{2}} \\ \frac{\iota}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{-\iota^2}{2} & \frac{\iota}{2} + \frac{-\iota}{2} \\ \frac{-\iota}{2} + \frac{\iota}{2} & \frac{1}{2} + \frac{-\iota^2}{2} \end{bmatrix} = I \checkmark \end{aligned}$$

33

Suppose A is an $m \times n$ matrix with rank(A) $= n$

(a)

Show that applying Gram-Schmidt to the columns produces an orthonormal basis for A.

If rank$(A) = n$, columns must all be linearly independent. The Gram-Schmidt process will transform this basis for the range of A into an orthonormal basis for A.

(b)

Prove this is equivalent to $A = QR$, where Q is an $m \times n$ matrix with orthonormal columns, while R is a non-singular upper triangular matrix.

The $A = QR$ factorization turns A into an orthonormal basis Q, with a matrix of scalars, R.

For any set of vectors $w_1, w_2 ... w_n$ composing A, we can write $w_1 = r_{11}u_1, w_2 = r_{12}u_1 + r_{22}u_2$, etc. where r's come from R, and the u's are the columns of Q.

(c)

Show that the QR factorization works for rectangular $m \times n$ matrices the only difference being that row indices run from 1 to m.

A is composed of $w_1, w_2, \cdots, w_n$, all of which together form a basis for $\mathbb{R}^n$.

The process to form an orthonormal basis for $\mathbb{R}^n$ can still occur unhindered, since we'll have the same number of basis vectors.

(d)

Apply this method to the following matrix

$\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 2 \end{bmatrix}$ Set $v_1 = w_1$

$$v_2 = w_2 - \frac{w_2 \cdot v_1}{||v_1||^2} v_1 = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} - \frac{5}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 - \sqrt{5} \\ 3 - 2\sqrt{5} \\ 2 \end{bmatrix}$$

These two vectors form a basis for $\mathbb{R}^2$

(e)

Explain what happens if rank(A) ¡ n The process will fail, since we won't be able to obtain n basis vectors.