Matrix Methods Homework 9
APPM 3310-003

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Section 5.3

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(a)

Find the QR factorization of the following: $A = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$

$$u_1 = \frac{v_1 - ||v_1||e_1}{||v_1 - ||v_1||e_1||} = \frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \sqrt{5}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{2.35} = \begin{bmatrix} \frac{20-20\sqrt{5}}{47} \\ \frac{40}{47} \end{bmatrix}$$

$$H_1 = I - 2u_1u_1^T = I - 2\begin{bmatrix} \frac{20-20\sqrt{5}}{47} \\ \frac{40}{47} \end{bmatrix}\begin{bmatrix} \frac{20-20\sqrt{5}}{47} & \frac{40}{47} \end{bmatrix} = I - \begin{bmatrix} .54 & -.88 \\ -.88 & 1.44 \end{bmatrix}$$

$$H_1 = \boxed{\begin{bmatrix} .46 & -.88 \\ -.88 & -.44 \end{bmatrix} = Q}$$

$$\boxed{R = HA = \begin{bmatrix} -2.24 & .447 \\ 0 & 3.13 \end{bmatrix}}$$

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(a)

Find the QR factorization of the following coefficient matrix

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

$$u_1 = \frac{v_1 - ||v_1||e_1}{||v_1 - ||v_1||e_1||} = \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \sqrt{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{1.0824} = \begin{bmatrix} \frac{1-\sqrt{2}}{1.0824} \\ \frac{-1}{1.0824} \end{bmatrix}$$

$$H_1 = I - 2u_1u_1^T = I - 2\begin{bmatrix} \frac{1-\sqrt{2}}{1.0824} \\ \frac{-1}{1.0824} \end{bmatrix} \begin{bmatrix} \frac{1-\sqrt{2}}{1.0824} & \frac{-1}{1.0824} \end{bmatrix} = I - \begin{bmatrix} .2929 & .7071 \\ .7071 & 1.7071 \end{bmatrix}$$

$$H_1 = \boxed{\begin{bmatrix} .7071 & -.7071 \\ -.7071 & .7071 \end{bmatrix} = Q}$$

$$\boxed{R = H_1A = \begin{bmatrix} -1.414 & .7071 \\ 0 & 3.535 \end{bmatrix}}$$

(b)

Solve the above system using $Rx = Q^Tb$

$$\begin{bmatrix} -1.414 & .7071 \\ 0 & 3.535 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .7071 & -.7071 \\ -.7071 & .7071 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$-1.414x + .7071y = 2.1213$$

$$3.535y = .7071$$

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1.4 \\ .2 \end{bmatrix}}$$

Section 5.5

2

Find the projection of the vector $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ into the following

(a)

The vector $v_1 = \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

$$w = \frac{v \cdot v_1}{||v_1||^2} v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \boxed{\begin{bmatrix} \frac{-1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}}$$

(b)

The vector $v_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

$$w = \frac{v \cdot v_1}{||v_1||^2} v_1 = \frac{4}{14} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{4}{7} \\ \frac{-2}{7} \\ \frac{6}{7} \end{bmatrix}}$$

(c)

Plane spanned by $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$

$$w = \frac{v \cdot v_1}{||v_1||^2} v_1 + \frac{v \cdot v_2}{||v_2||^2} v_2 = \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{7}{9} \\ \frac{11}{9} \\ \frac{1}{9} \end{bmatrix}}$$

(d)

Plane spanned by $v_1 = \begin{bmatrix} -\frac{3}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} \frac{4}{13} \\ \frac{3}{13} \\ -\frac{12}{13} \end{bmatrix}$

$$w = \frac{v \cdot v_1}{||v_1||^2} v_1 + \frac{v \cdot v_2}{||v_2||^2} v_2 = \frac{\frac{1}{5}}{1} \begin{bmatrix} -\frac{3}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix} + \frac{\frac{-5}{13}}{1} \begin{bmatrix} \frac{4}{13} \\ \frac{3}{13} \\ -\frac{12}{13} \end{bmatrix} = \boxed{\begin{bmatrix} \frac{-1007}{4225} \\ \frac{301}{4225} \\ \frac{12}{5} \end{bmatrix}}$$

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(a)

Proof. **Seek to prove that the set of all vectors orthogonal to a given subspace** $V \subset \mathbb{R}^m$ **forms a subspace.**

Have to check closed under addition, scalar multiplication, and contains 0. All other subspace axioms follow from the fact that this is a smaller subspace of a valid existing subspace.

Addition: By definition, the set of all vectors orthogonal to V are in the range of at most $m-1$ basis vectors. If any vectors (or the sum of any vectors in the set) were able to leave the range of these basis vectors, it would not have been an orthogonal set to begin with. This is because we would need another basis vector to express something outside the existing range.

Scalar Multiplication: As with above, all of the vectors in our orthogonal set lie in the range of at least $m-1$ basis vectors. The range is composed of all possible linear combinations of these basis vectors, which already includes any possible scalar multiples of vectors in the subspace.

Contains $\vec{0}$: The zero vector is orthogonal to all vectors, so it must be in the set of all vectors orthogonal to a given subspace. $\square$

(b)

Find basis for the set of all vectors in $\mathbb{R}^4$ that are orthogonal to the subspace spanned by $v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \end{bmatrix}$

For a vector $\vec{w} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ to be orthogonal to the above vectors, both $w \cdot v_1$ and $w \cdot v_2$ must be zero. In other words,

$$a + 2b + d = 0 \qquad\qquad 2a + 3c + d = 0$$

From this point it's trivial to find a basis, setting first a then b equal to zero, arriving at a basis of

$$\boxed{\begin{bmatrix} 0 \\ 3 \\ 2 \\ -6 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}}$$

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Find the closest point to the vector $b = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ **belonging to the plane spanned by** $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

Vectors are already orthogonal, so we can jump right into orthogonal projection without Gram-Schmidt

$$w = \frac{b \cdot v_1}{||v_1||^2} v_1 + \frac{b \cdot v_2}{||v_2||^2} v_2 = \frac{3}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 2 \end{bmatrix}}$$

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Find the least squares solution to the following linear systems.

(a)

$$\begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

The columns of the coefficient matrix are already orthogonal, so we can skip Gram-Schmidt. b is not in the range of A, so we'll find the closest vector within A to b, which we'll call w. We'll then find the least squares solution to that system.

$$w = \frac{b \cdot v_1}{||v_1||^2} v_1 + \frac{b \cdot v_2}{||v_2||^2} v_2 = \frac{-2}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{0}{6} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} \\ -\frac{2}{7} \\ -\frac{3}{7} \end{bmatrix}$$

Now that we have the closest vector in the range of the coefficient matrix, now we solve $Ax = w$

$$\begin{bmatrix} 1 & -1 & | & -\frac{1}{7} \\ 2 & 2 & | & -\frac{2}{7} \\ 3 & -1 & | & -\frac{3}{7} \end{bmatrix} \quad -\frac{3}{2} R_2 \quad = \begin{bmatrix} 1 & -1 & | & -\frac{1}{7} \\ 2 & 2 & | & -\frac{2}{7} \\ 0 & 4 & | & 0 \end{bmatrix} \quad -2R_1 .$$

$$\begin{bmatrix} 1 & -1 & | & -\frac{1}{7} \\ 0 & 4 & | & 0 \\ 0 & -4 & | & 0 \end{bmatrix} \quad +R_2 \quad = \begin{bmatrix} 1 & -1 & | & -\frac{1}{7} \\ 0 & 4 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} \\ 0 \end{bmatrix}}$$

(b)

23

Apply the method from problem *5.5.22* to find the least squares solution to the systems in excercise *4.3.14*.

(a)

$$Ax = b \Rightarrow \begin{bmatrix} w_1 & w_2 \end{bmatrix} x = b \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

First we apply Gram-Schmidt:

$$v_1 = w_1 \qquad\qquad A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$$

$$v_2 = w_2 - \frac{w_2 \cdot v_1}{||v_1||^2} v_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} - \frac{-3}{11} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{25}{11} \\ \frac{-2}{11} \\ \frac{19}{11} \end{bmatrix}$$

$$e_1 = \frac{v_1}{||v_1||} = \begin{bmatrix} \frac{379}{1257} \\ \frac{379}{419} \\ \frac{-379}{1257} \end{bmatrix} \qquad\qquad e_2 = \frac{v_2}{||v_2||} = \begin{bmatrix} \frac{700}{881} \\ \frac{-56}{881} \\ \frac{532}{881} \end{bmatrix}$$

$$Q = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \qquad\qquad R = \begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 \\ 0 & a_2 \cdot e_2 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{379}{1257} & \frac{700}{881} \\ \frac{379}{419} & \frac{-56}{881} \\ \frac{-379}{1257} & \frac{532}{881} \end{bmatrix} \qquad R = \begin{bmatrix} \frac{1456}{439} & \frac{-379}{419} \\ 0 & \frac{2520}{881} \end{bmatrix}$$

Now solve using $Rx = Q^T b$

$$\begin{bmatrix} \frac{1456}{439} & \frac{-379}{419} \\ 0 & \frac{2520}{881} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{379}{1257} & \frac{379}{419} & \frac{-379}{1257} \\ \frac{700}{881} & \frac{-56}{881} & \frac{532}{881} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\frac{1456}{439} x - \frac{379}{419} y = -\frac{758}{1257} \qquad\qquad \frac{2520}{881} = \frac{1191}{457}$$

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .067 \\ .911 \end{bmatrix}}$$

(b)

$$A = \begin{bmatrix} 4 & -2 \\ 2 & 3 \\ 1 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$$

$$e_1 = \frac{a_1}{||a_1||} = \begin{bmatrix} \frac{4}{5} \\ \frac{2}{5} \\ \frac{1}{5} \\ \frac{2}{5} \end{bmatrix} \qquad e_2 = \frac{a_2}{||a_2||} = \begin{bmatrix} \frac{-769}{1762} \\ \frac{2089}{3191} \\ \frac{-769}{1762} \\ \frac{769}{1762} \end{bmatrix}$$

$$Q = \begin{bmatrix} e_1 & e_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{-769}{1762} \\ \frac{2}{5} & \frac{2089}{3191} \\ \frac{1}{5} & \frac{-769}{1762} \\ \frac{2}{5} & \frac{769}{1762} \end{bmatrix} \qquad R = \begin{bmatrix} a_1 \cdot e_1 & a_1 \cdot e_2 \\ 0 & a_2 \cdot e_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & \frac{3524}{769} \end{bmatrix}$$

Now we can solve $Rx = Q^T b$.

$$\begin{bmatrix} 5 & 0 \\ 0 & \frac{3524}{769} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{-769}{1762} & \frac{2089}{3191} & \frac{-769}{1762} & \frac{769}{1762} \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ -1 \\ 2 \end{bmatrix}$$

$$5x = -\frac{1}{5} \qquad \frac{3524}{769} y = -\frac{1538}{881}$$

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -.381 \end{bmatrix}}$$

Section 5.6

2

Find a basis for the orthogonal complement to the following subspaces of $\mathbb{R}^3$

(a)

The plane $3x + 4y - 5z = 0$

Need $\begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0, \qquad 3a + 4b - 5c = 0$

We'll arbitrarily use $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ because we like whole numbers. We can use this as the basis:

$$\boxed{\text{span}\left\{ \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \right\}}$$

(b)

The line in the direction $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

Need $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

Arbitrarily choose $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Now we need another basis vector that's perpendicular to the other two– we can use our old friend cross product.

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$$

And so a basis is the following

$$\boxed{\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} \right\}}$$

(c)

(d)

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15

17

(c)

22

(a)

(f)