Matrix Methods Homework 8
APPM 3310-003

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Section 5.1

1

Let $\mathbb{R}^2$ have the standard dot product. Classify the following pairs as:

$$i.)Basis \qquad ii.)Orthogonal\ Basis \qquad iii.)Orthonormal\ Basis$$

(a)

$$v_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow v_1 \cdot v_2 = 0, ||v_1|| \neq 1, \boxed{\text{orthogonal basis}}$$

(b)

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow v_1 \cdot v_2 = 0, ||v_1|| = ||v_2|| = 1, \boxed{\text{orthonormal basis}}$$

(c)

$$v_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow v_1 \cdot v_2 = -4, \boxed{\text{Not a basis}}$$

(d)

$$v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -6 \end{bmatrix} \Rightarrow v_1 \cdot v_2 = -16, , \boxed{\text{basis}}$$

(e)

$$v_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Rightarrow v_1 \cdot v_2 = 0, ||v_2|| \neq 1, \boxed{\text{orthogonal basis}}$$

(f)

$$v_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} \Rightarrow v_1 \cdot v_2 = 0, ||v_1|| = ||v_2|| = 1, \boxed{\text{orthonormal basis}}$$

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Repeat *5.1.1* but use $\langle v, w \rangle = v_1 w_1 + \frac{1}{9} v_2 w_2$ as the inner product instead.

(d)

$$v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -6 \end{bmatrix} \Rightarrow v_1 \cdot v_2 = 2 + \frac{1}{9}(-18) = 0, \boxed{\text{orthogonal basis}}$$

(e)

$$v_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Rightarrow v_1 \cdot v_2 = 0, ||v_1|| = ||v_2|| = 1, \boxed{\text{orthonormal basis}}$$

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Find all values of a such that the vectors $\begin{bmatrix} a \\ 1 \end{bmatrix}, \begin{bmatrix} -a \\ 1 \end{bmatrix}$ form an orthogonal basis of $\mathbb{R}^2$ under...

(c)

The inner product prescribed by the positive definite matrix $K = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$

$$\begin{bmatrix} a \\ 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2a - 1 \\ -a + 3 \end{bmatrix} \begin{bmatrix} -a \\ 1 \end{bmatrix} = -2a^2 + a - a + 3$$

Vectors are orthogonal if their inner product equals zero, so set the above result equal to 0:

$$3 - 2a^2 = 0, \boxed{a = \sqrt{\frac{2}{3}}}$$

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True or False: If v_1, v_2, v_3 are a basis for $\mathbb{R}^3$, then they form an orthogonal basis under some appropriately weighted inner product $\rangle v, w\rangle = av_1w_1 + bv_2w_2 + cv_3w_3$

Three vectors are linearly independent if and only if $0 \neq c_1v_1 + c_2v_2 + c_3v_3 \,\forall\, c_i$ not all 0.

$\boxed{\text{False}}$. It's not possible to guarantee this property will hold, especially for vectors with all positive elements. Since a, b, c are the diagonals of a positive definite matrix, they must be positive as well.

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Proof. **Seek to prove that every orthogonal basis of** $mathbbR^2$ **under the standard dot product has the form** $u_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, u_2 = \begin{bmatrix} \sin\theta \\ \pm\cos\theta \end{bmatrix}$

Every orthonormal basis using the standard dot product must satisfy the following properties:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1y_1 + x_2y_2 = 0$$

and

$$\left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\| = \sqrt{x_1^2 + x_2^2} = 1, \ \left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\| = \sqrt{y_1^2 + y_2^2} = 1$$

The unit length condition is trivial, since

$$\sqrt{\sin^2\theta + \cos^2\theta} = 1 \,\forall\, \theta$$

For the other condition, we get

$$\begin{aligned} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \cdot \begin{bmatrix} \sin\theta \\ \pm\cos\theta \end{bmatrix} &= \cos\theta\sin\theta \pm \sin\theta\cos\theta = 0 \\ &= \frac{1}{2}\big[\sin(\theta+\theta) - \sin(\theta-\theta)\big] \pm \frac{1}{2}\big[\sin(\theta+\theta) - \sin(\theta-\theta)\big] \\ &= \frac{1}{2}\sin(2\theta) \pm \frac{1}{2}\sin(2\theta) \end{aligned}$$

At the point shown above, we split into two cases. If a minus is chosen in the $\pm$, then the result of the whole expression will be zero, regardless of the choice of θ. On the other hand, if plus is chosen, we get that $\sin(2\theta) = 0$. Which is only true when $\theta = n \cdot \frac{\pi}{2}, n \,\epsilon\, \mathbb{Z}$. □

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(a)

Prove the vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ **form a basis of** $\mathbb{R}^3$ **with the standard dot product.**

It's trivial to show the following.

$$\boxed{v_1 \cdot v_2 = 0, v_1 \cdot v_3 = 0, v_2 \cdot v_3 = 0}$$

(b)

Use orthogonality to write $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ **as a combination of the three vectors from part** (a)**.**

$$\begin{aligned}
&v = av_1 + bv_2 + cv_3 \\
&a = \frac{\langle v, v_1 \rangle}{||v_1||} = \frac{6}{3} = 2 \\
&b = \frac{\langle v, v_2 \rangle}{||v_2||} = \frac{-3}{6} = \frac{-1}{2} \\
&c = \frac{\langle v, v_3 \rangle}{||v_3||} = \frac{1}{2} \\
&\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \boxed{= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}
\end{aligned}$$

Section 5.2

1

(a)

Use Gram-Schmidt to find an orthonormal basis for $\mathbb{R}^3$ **starting with the following vectors:** $w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, w_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

Arbitrarily choose $v_1 = w_1$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{||v_1||^2}v_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{2}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{||v_1||^2}v_1 - \frac{\langle w_3, v_2\rangle}{||v_2||^2}v_2 = \begin{bmatrix}-1\\2\\1\end{bmatrix} - \frac{0}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{2}{1}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$$

Now to normalize...

$$\boxed{u_1 = \frac{v_1}{||v_1||} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}, u_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, u_3 = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}}$$

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Use the Gram-Schmidt process to construct an orthonormal basis for the following subspaces of $\mathbb{R}^3$

(a)

Plane spanned by $w_1 = \begin{bmatrix}0\\2\\1\end{bmatrix}, w_2 = \begin{bmatrix}1\\-2\\-1\end{bmatrix}$ Can arbitrarily use $v_1 = w_1$.

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{||v_1||^2}v_1 = \begin{bmatrix}1\\-2\\-1\end{bmatrix} - \frac{-5}{5}\begin{bmatrix}0\\2\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

Now we find the norms of each of those, since we need an orthonormal basis.

$$\boxed{u_1 = \frac{v_1}{||v_1||} = \begin{bmatrix}0\\\frac{2}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix},\ u_2 = \frac{v_2}{||v_2||} = \begin{bmatrix}1\\0\\0\end{bmatrix}}$$

(b)

The plane $2x - y + 3z = 0$ We pick two arbitrary vectors in the plane, then use the Gram-Schmidt process to find an orthonormal basis.

We'll pick $w_1 = \begin{bmatrix}-1\\1\\1\end{bmatrix}, w_2 = \begin{bmatrix}1\\2\\0\end{bmatrix}$. Again, arbitrarily pick $v_1 = w_1$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{||v_1||^2}v_1 = \begin{bmatrix}1\\2\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{4}{3}\\\frac{5}{3}\\\frac{-1}{3}\end{bmatrix}$$

Now we just need to normalize each of our vectors.

$$\boxed{u_1 = \frac{v_1}{||v_1||} = \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, u_2 = \frac{v_2}{||v_2||} = \begin{bmatrix} 4\sqrt{42} \\ 5\sqrt{42} \\ -\sqrt{42} \end{bmatrix}}$$

(c)

Set of all vectors orthogonal to $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ Choose two vectors perpendicular to the vector above, we'll pick $w_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ Set $v_1 = w_2$ because it's easy and we don't like change.

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{||v_1||^2} v_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} - \frac{-1}{5} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ 2 \\ \frac{-4}{5} \end{bmatrix}$$

Now we normalize:

$$\boxed{u_1 = \frac{v_1}{||v_1||} = \begin{bmatrix} \frac{2}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix}, u_2 = \frac{v_2}{||v_2||} = \begin{bmatrix} 4\sqrt{30} \\ \frac{4\sqrt{30}}{5} \\ -8\sqrt{30} \end{bmatrix}}$$

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(a)

How many orthonormal bases does $\mathbb{R}$ have? Only 2, since only 1 and -1 have magnitude 1, and nothing can be 'orthogonal' to a Real except 0.

(b)

What about $\mathbb{R}^2$? By *Theorem 5.15*, any inner product space with dimension greater than one has infinitely many orthonormal bases.

(c)

Does the answer change for a different inner product? No. *Theorem 5.15* applies to *any* inner product space, so our answer still holds.

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True or False: Reordering the original basis before starting Gram-Schmidt leads to the same orthogonal basis.

$\boxed{\text{FALSE}}$. Counter example: Given the vectors $w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

First we choose $v_1 = w_1$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{||v_1||^2} v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Which gives us the basis of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} \\ \frac{1}{2} \end{bmatrix}$

On the other hand, if we initially choose $v_1 = w_2$ (Really mixing things up here):

$$v_2 = w_1 - \frac{\langle w_1, v_1 \rangle}{||v_1||^2} v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Which gives us the basis $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Both of the above are valid orthogonal bases, but they're not the same.

Section 5.3

1

Determine which of the following matrices are orthogonal

(b)

The matrix $b = \begin{bmatrix} \frac{12}{13} & \frac{5}{13} \\ \frac{-5}{13} & \frac{12}{13} \end{bmatrix}$ Any orthogonal matrix Q must satisfy the property $Q^T Q = I$

$$b^T b = \begin{bmatrix} \frac{12}{13} & \frac{-5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{bmatrix} \cdot \begin{bmatrix} \frac{12}{13} & \frac{5}{13} \\ \frac{-5}{13} & \frac{12}{13} \end{bmatrix} = \begin{bmatrix} \frac{144}{13} + \frac{25}{13} & \frac{60}{13} - \frac{60}{13} \\ \frac{60}{13} - \frac{60}{13} & \frac{25}{13} + \frac{144}{13} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

(e)

$$e = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix}$$

$$e^T \cdot e = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{61}{144} & \frac{3}{10} & \frac{7}{30} \\ \frac{3}{10} & \frac{769}{3600} & \frac{1}{6} \\ \frac{7}{30} & \frac{1}{6} & \frac{469}{3600} \end{bmatrix}$$

Not orthogonal.

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(a)

Proof. **Seek to prove that the transpose of an orthogonal matrix is also orthogonal.**

For any orthogonal matrix Q, we know that $Q^TQ = I$, by the definition of an orthogonal matrix. We also know that $Q^{-1} = Q^T$

Let F be an orthogonal matrix, which means that $F^TF = I$. We can take the transpose of F, F^T and seek to show that $(F^T)^TF^T = I$. We know that $(F^T)^T$ is just F, and we can use the second property from above to show that $F^T = F^{-1}$. Shown symbolically:

$$(F^T)^TF^T = FF^{-1} = I$$

Showing that the transpose of an orthogonal matrix is itself orthogonal. QED

□

(b)

Explain why the rows od an $n \times n$ orthogonal matrix also form an orthonormal basis of $\mathbb{R}^n$ To be a basis for $\mathbb{R}^n$, we need n linearly independent elements of $\mathbb{R}^n$.

Let $A_{n\times n}$ be an orthogonal matrix composed of column vectors $\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$

We know that A is orthogonal, so $A^TA = I$, or

$$\begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} \cdot \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1^T \cdot v_1 & v_1^T \cdot v_2 & \cdots & v_1^T \cdot v_n \\ v_2^T \cdot v_1 & v_2^T \cdot v_2 & \cdots & v_2^T \cdot v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n^T \cdot v_1 & v_n^T \cdot v_2 & \cdots & v_n^T \cdot v_n \end{bmatrix} = I$$

We get zero on each of the non-diagonal entries, showing that each of the n vectors making up the rows of A are orthogonal to all the others, which shows that we have n orthogonal basis vectors.

We can also see that this basis is orthonormal, because on each of the diagonals, the magnitude of $v_1v_1 = ||v_1||^2 = 1$.

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True or False

(a)

A matrix whose columns form an orthogonal basis of $\mathbb{R}^n$ is an orthogonal basis.

$\boxed{\text{False}}$. Counter example:

$$\text{Let } A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \neq I$$

The columns must be normalized for this to work.

(b)

A matrix whose rows form an orthonormal basis of $mathbbR^n$ **is an orthogonal basis.**

$\boxed{\text{True}}$. The only reason (a) didn't work is that the columns weren't normalized, but the property holds in this case.

(c)

An orthogonal matrix is symmetric if and only if it's a diagonal matrix. $\boxed{\text{False}}$. Counter example: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is an symmetric orthogonal matrix.

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(a)

(c)

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(b)

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