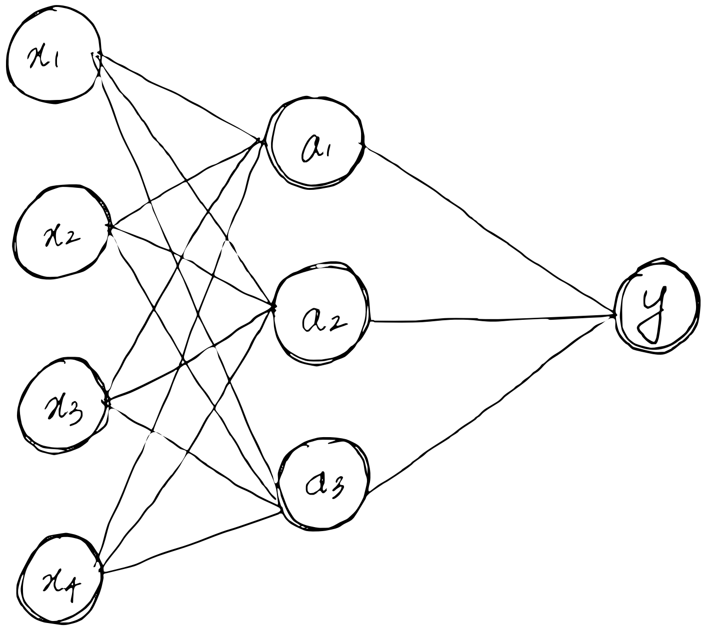
1)



Objective :

Return 1 if x_1, x_2, x_3, x_4 are sorted
else return 0

$$\approx \quad f(x) = \begin{cases} 1 & \text{if } x_1 < x_2 < x_3 < x_4 \\ 0 & \end{cases}$$

$W^1 \in \mathbb{R}^{3\times 4}$ $\qquad$ $b^1 \in \mathbb{R}^{3\times 1}$

$W^2 \in \mathbb{R}^{1\times 3}$ $\qquad$ $b^2 \in \mathbb{R}^{1\times 1}$

Finding weight matrix and bias for 1^{st} layer

$$w^{1} = \begin{bmatrix} w_{11}^{(1)} & w_{12}^{(1)} & w_{13}^{(1)} & w_{14}^{(1)} \\ w_{21}^{(1)} & w_{22}^{(1)} & w_{23}^{(1)} & w_{24}^{(1)} \\ w_{31}^{(1)} & w_{32}^{(1)} & w_{33}^{(1)} & w_{34}^{(1)} \end{bmatrix} \qquad b^{1} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \\ b_3^{(1)} \end{bmatrix}$$

$$w^{(1)}x = \begin{matrix} w_{11}x_1 + w_{12}x_2 + w_{13}x_3 + w_{14}x_4 \\ w_{21}x_1 + w_{22}x_2 + w_{23}x_3 + w_{24}x_4 \\ w_{31}x_1 + w_{32}x_2 + w_{33}x_3 + w_{34}x_4 \end{matrix}$$

- Defining inequalities for the hidden nodes

let
$$a_1 = \phi(x_2 - x_1)$$
$$a_2 = \phi(x_3 - x_2)$$
$$a_3 = \phi(x_4 - x_3)$$

where $\phi(x) = 1$ if x is positive else 0

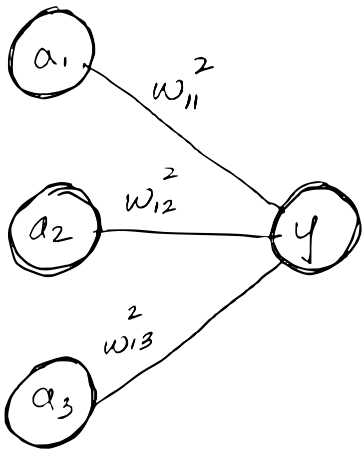
- Finding weight matrix & bias for 2nd layer.

$$\begin{bmatrix} w_{11}^{(2)} & w_{12}^{(2)} & w_{13}^{(2)} \end{bmatrix} \quad \begin{bmatrix} b^{(2)} \end{bmatrix}$$

- Possible combinations of $a_1\ a_2\ a_3$ are:-

a_1	a_2	a_3
1	1	1
1	1	0
1	0	1
0	1	1
0	0	1
1	0	0
0	1	0
0	0	0

2nd layer:-



$$y = w_{11}^{(2)} a_1 + w_{12}^{(2)} a_2 + w_{13}^{(2)} a_3 + b^{(2)}$$

- 3X4 weight matrix $w^{(1)}$ for hidden layer:-

$$w^{(1)} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

- A 3-dimensional vector of biases $b^{(1)}$ for the hidden layer:

$$b^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- A 3-dimensional weight vector $w^{(2)}$ for the output layer.

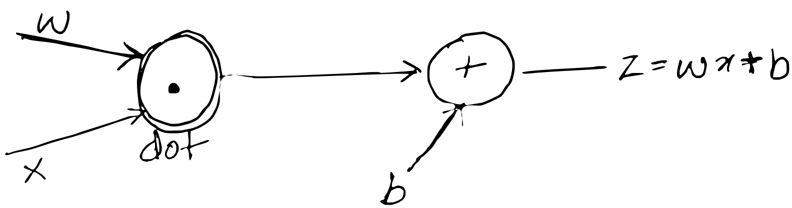
$$w^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- A scalar $b^{(2)}$ for output layer: $b^{(2)} = 3$

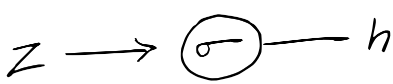
2. Back prop:

i) computation graph relating x, z, h, y, R, S, E

$$z = w^{(1)} x + b^{(1)}$$

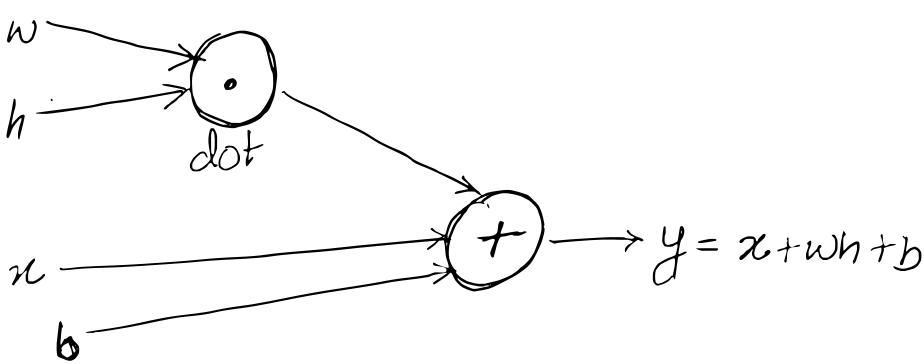


$$h = \sigma(z)$$

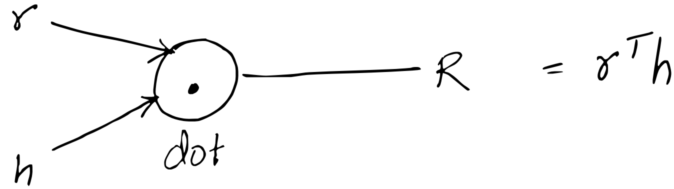


$h = \sigma(z)$

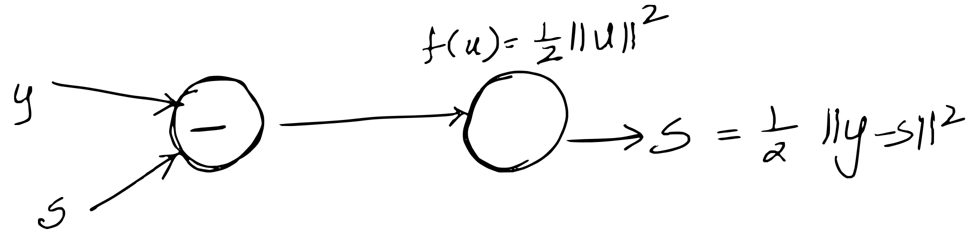
$$y = x + w^{(2)} h + b^{(2)}$$

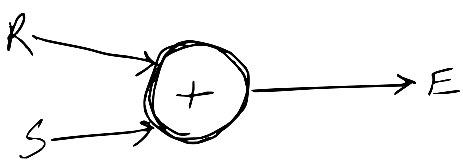


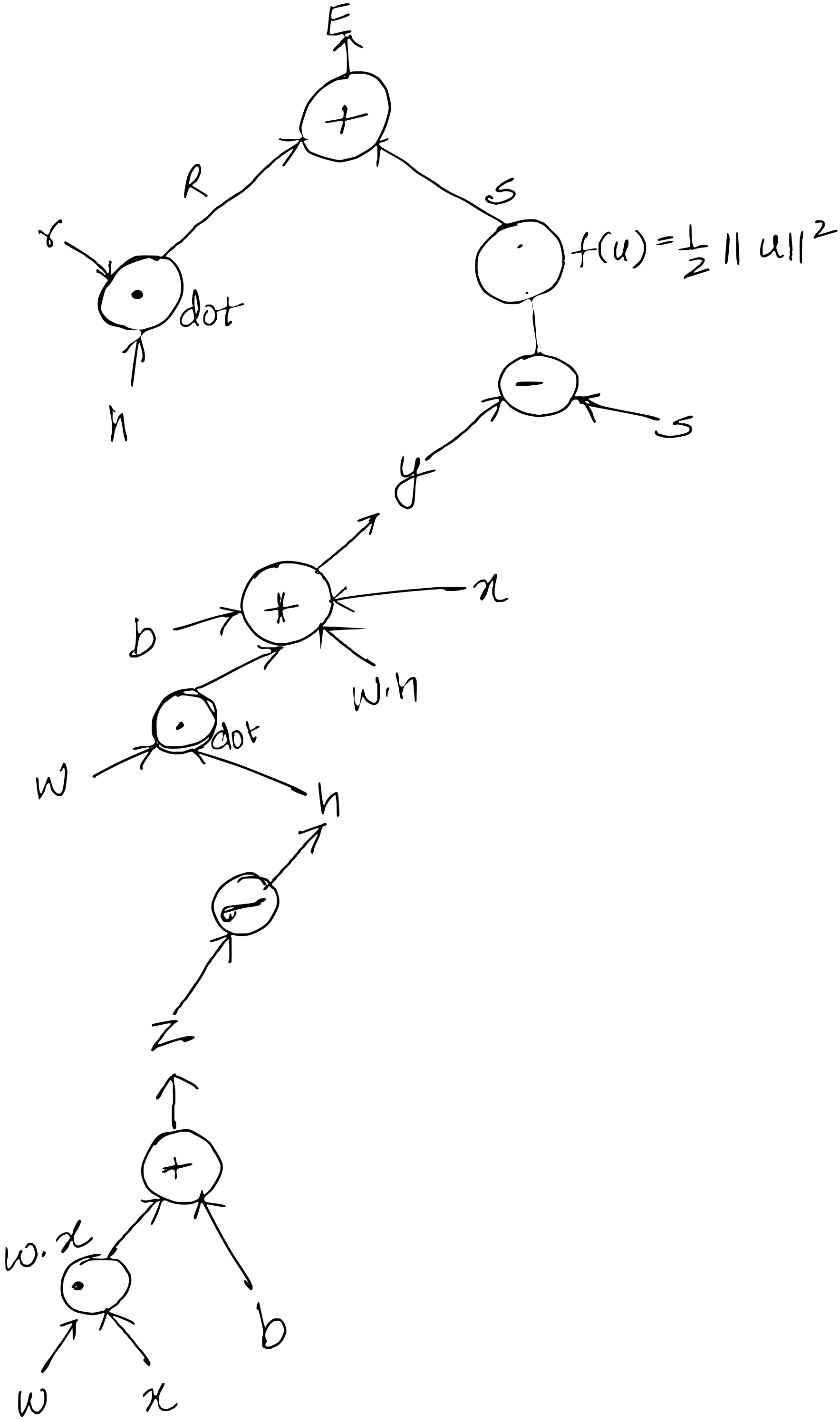
$$R = r^T h$$



$$S = \frac{1}{2} \|y - s\|^2$$



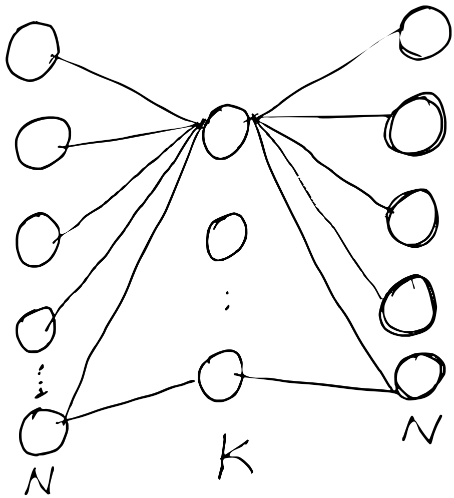




E
+
R
s
r
dot
h
$f(u) = \frac{1}{2} \|u\|^2$
−
s
y
+
b
x
w.h
dot
w
h
σ
z
+
w.x
b
w
x

· Derive the backprop equations for computing

$$x = \frac{dE}{dx}$$



$$z = w^{(1)}x + b$$

$$h = \delta(z)$$

$$h' = \delta'(z)$$

$$y = x + w^{(2)}h + b^{(2)}$$

$$w^{(1)} = \underset{K \times N}{\begin{bmatrix} w'_{11} & w'_{12} & \cdots & w'_{1N} \\ w'_{21} & w'_{22} & \cdots & w'_{2N} \\ \vdots & & & \\ w'_{K1} & w'_{K2} & \cdots & w'_{KN} \end{bmatrix}}$$

$$b' = \underset{K \times 1}{\begin{bmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_K \end{bmatrix}}$$

$$w^2 = \begin{bmatrix} w_{11}^{(2)} & w_{12}^{(2)} & \cdots & w_{1K}^{(2)} \\ w_{21}^{(2)} & w_{22}^{(2)} & \cdots & w_{2K}^{(2)} \\ \vdots & & & \\ w_{N1}^{(2)} & w_{N2}^{(2)} & \cdots & w_{NK}^{(2)} \end{bmatrix} \qquad b^{(2)} = \begin{bmatrix} b_1^{(2)} \\ b_2^{(2)} \\ \vdots \\ b_K^{(2)} \end{bmatrix}$$

cost fun

$$E = R + S$$

$$z = w^{(1)} x + b$$

$$h = \sigma(z)$$

$$y = x + w^{(2)} h + b^{(2)}$$

$$R = r^T h \qquad S = \frac{1}{2} \| y - s \|^2$$

$$\frac{\partial E}{\partial x} = \frac{\partial R}{\partial x} + \frac{\partial S}{\partial x}$$

we solve above derivative using chain rule.

$\longrightarrow$

$$\frac{dE}{dx} = \frac{\partial R}{\partial h}\frac{\partial h}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial s}{\partial y}\frac{\partial y}{\partial x}$$

$$R = r^T h$$

$$\therefore \frac{\partial R}{\partial h} = r^T$$

$$z = w^{(1)} x + b^{(1)}$$

$$\frac{\partial h}{\partial z} = \delta'\left(\frac{\partial h}{\partial z}\right)$$

$$\frac{\partial z}{\partial x} = w^{(1)}$$

$$\frac{\partial s}{\partial y} = \frac{1}{2} \times 2 = 1 \qquad s = \frac{1}{2}\|y - s\|^2$$

$$\frac{\partial y}{\partial x} = 1 \qquad y = x + w^{(2)} h + b^{(2)}$$

$$\frac{\partial E}{\partial x} = \frac{\partial R}{\partial x} + \frac{\partial S}{\partial x}$$

$$\frac{\partial R}{\partial h} \frac{\partial h}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial S}{\partial y} \cdot \frac{\partial y}{\partial x}$$

$$= r^T \frac{\partial h}{\partial z} w^{(1)} + 1 \times 1$$

$$\therefore \frac{dE}{dx} = r^T w^{(1)} \frac{\partial h}{\partial z}$$