Introduction to Reinforcement Learning

References:

- https://github.com/dalmia/David-Silver-Reinforcement-learning
- https://deeplearningindaba.com/2020/
- http://introtodeeplearning.com

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4. Value Functions and Bellman Equation
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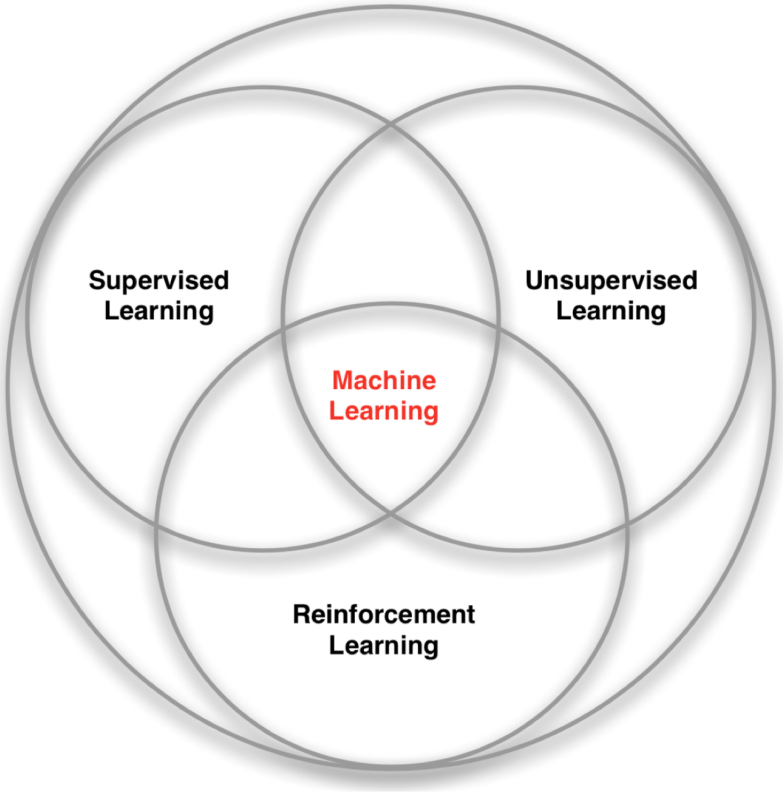
1. What is reinforcement Learning



Classes of Learning Problems

Supervised Learning

Data: (x, y)
x is data, y is label

Goal: Learn function to map

$$x \rightarrow y$$

Apple example:

This thing is an apple.

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Unsupervised Learning

Data: x
x is data, no labels!

Goal: Learn underlying structure

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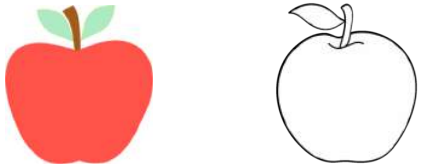

This thing is like the other thing.

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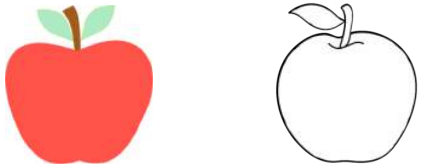


This thing is like the other thing.

Reinforcement Learning

Data: state-action pairs

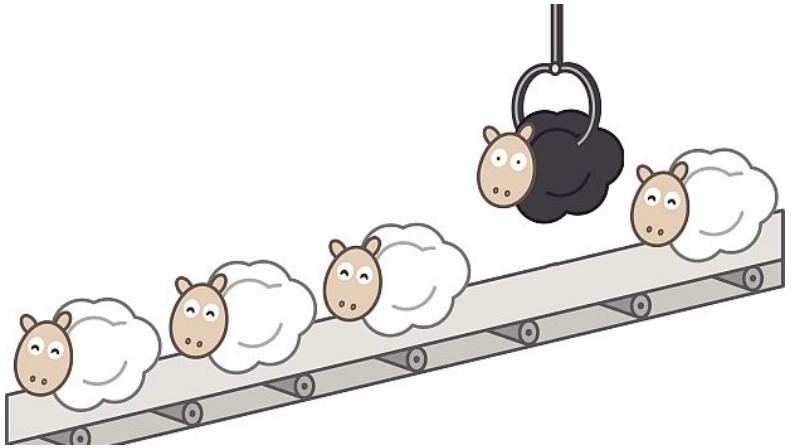
Goal: Maximize future rewards over many time steps

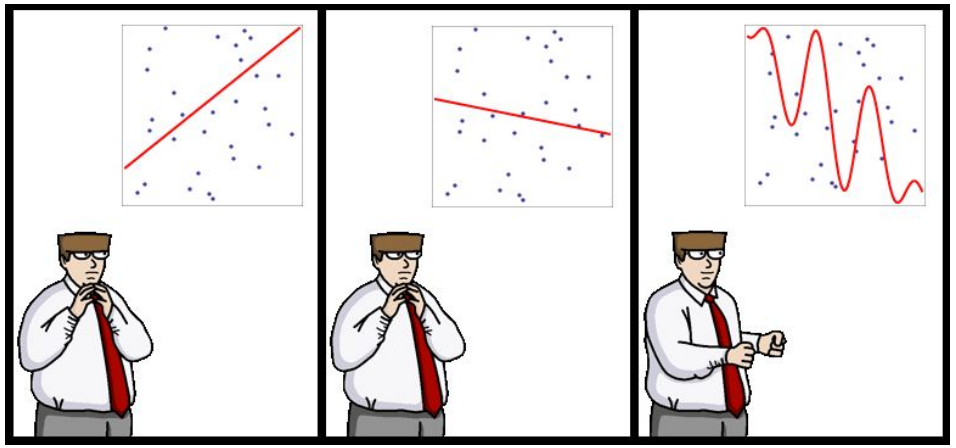
Apple example:

Eat this thing because it will keep you alive.

What is reinforcement learning?

We've seen how to solve many cool problems around supervised and unsupervised learning





But a major component of intelligence is decision making

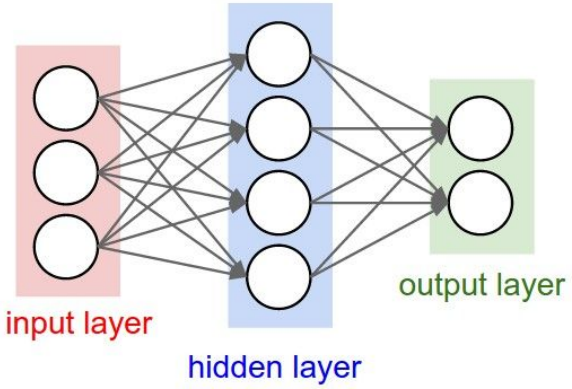
What is reinforcement learning?

Reinforcement learning is the branch of machine learning relating to learning in **sequential decision making settings**

Behaviour learning

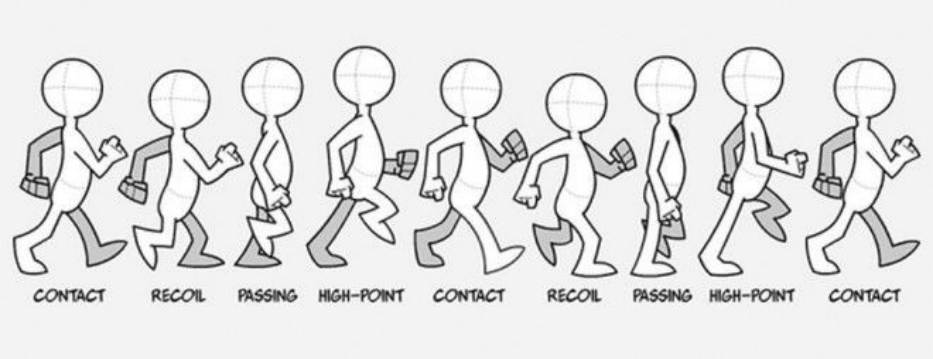
From supervised to reinforcement

Supervised learning, single decision point



Multiple decision points

- How do I know if I'm doing the right thing?
- How do my decisions now impact the future?
- Actions affect the environment!



Interacting with an environment

Decision maker (agent) exists within an environment

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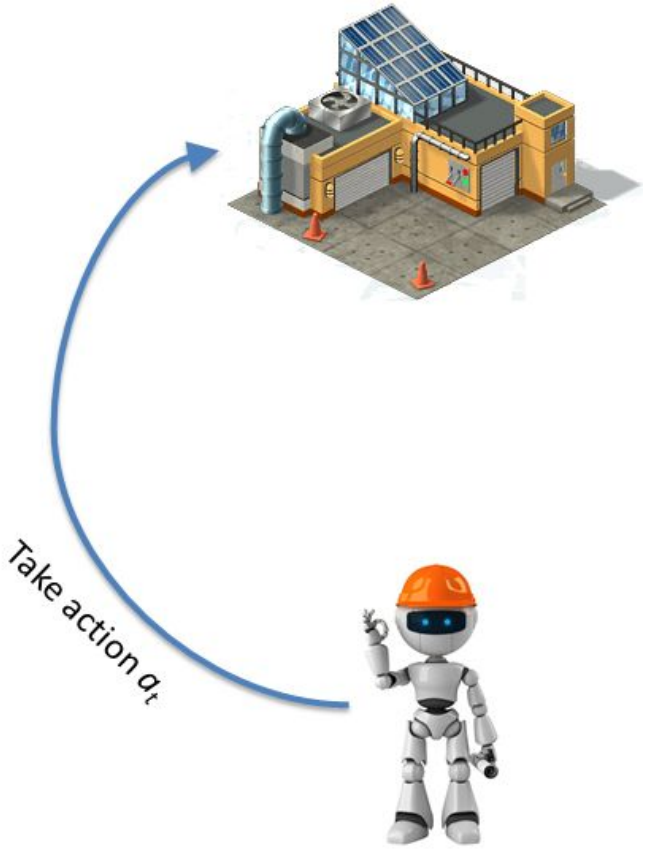
Decision maker (agent) exists within an environment

Agent takes **actions** based on the environment **state**



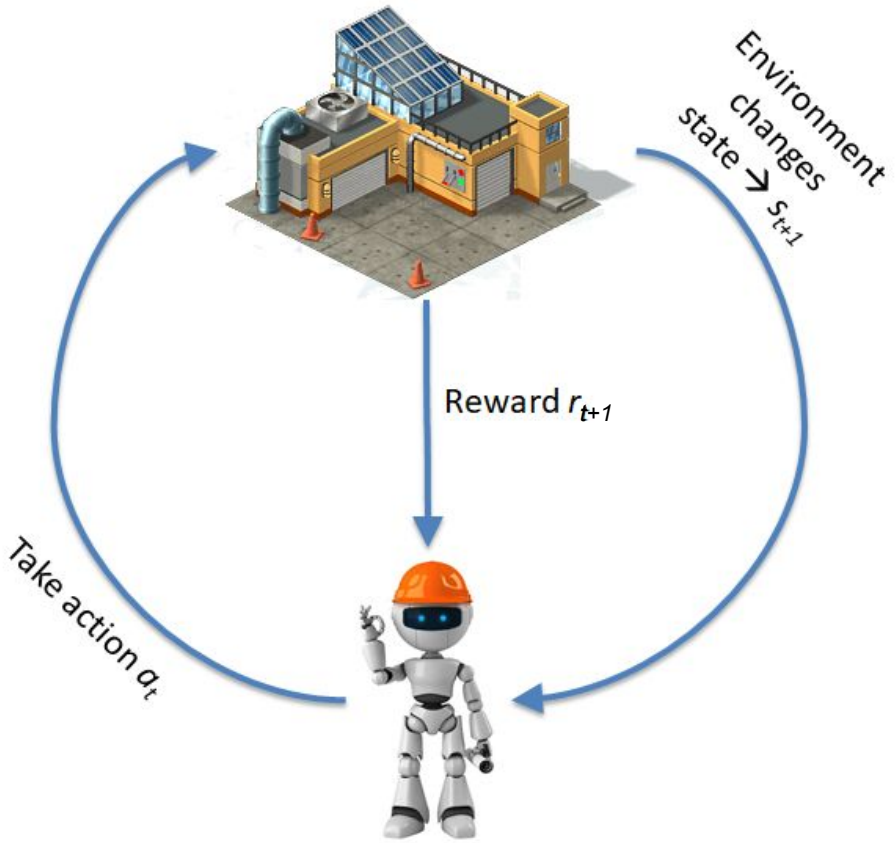
Interacting with an environment

Decision maker (agent) exists within an environment

Agent takes **actions** based on the environment **state**

Environment **state** updates

Agent receives feedback as **rewards**



2. Markov Decision Process (MDP)

A model for decision making

Markov Decision Process (MDP)

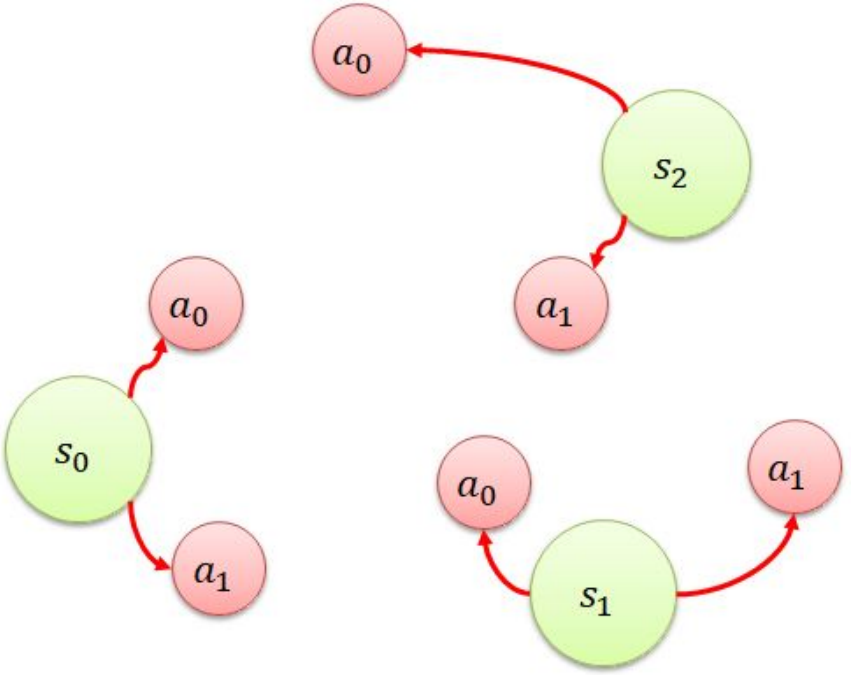
$M = \langle S, A, T, R, \gamma \rangle$

A model for decision making

Markov Decision Process (MDP)

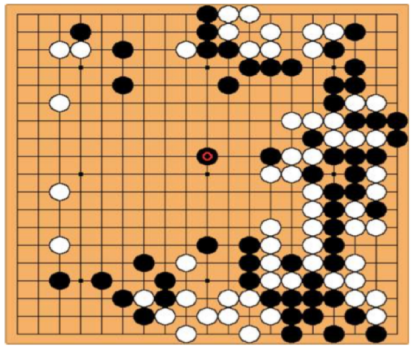
M = $\langle S, A, T, R, \gamma \rangle$

- States: encode world configurations
- Actions: choices made by agent

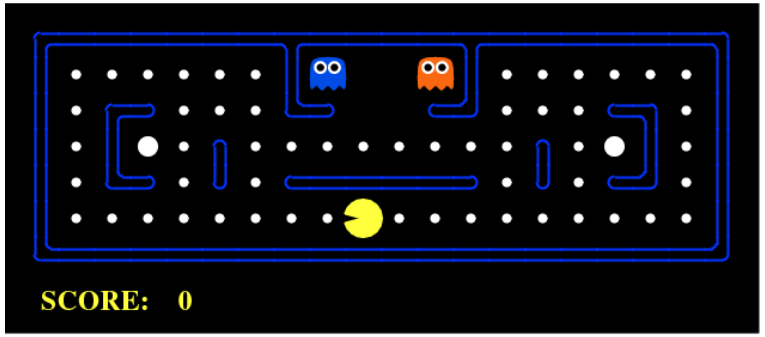


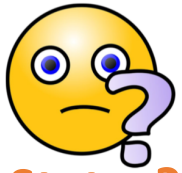
Questions:
- What are the states and actions in the following examples?
- Are they discrete or continuous?

Board Game



PacMan





States?

Actions?

Inventory Replenishment

Marketing

A model for decision making

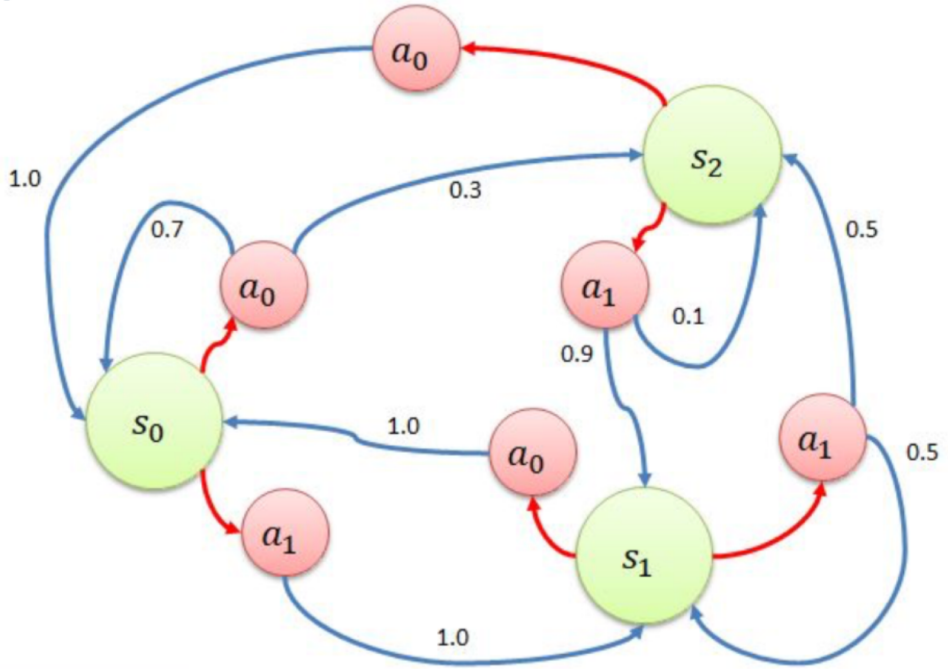
Markov Decision Process (MDP)

M = $\langle S, A, T, R, \gamma \rangle$

Markov: ***Future is independent of the past, given the present***

$$\mathcal{P}_{ss'} = P(S_{t+1} = s' \mid S_t = s)$$

$$= P(S_{t+1} = s' \mid S_0, S_{t-1}, \dots, S_t = s)$$



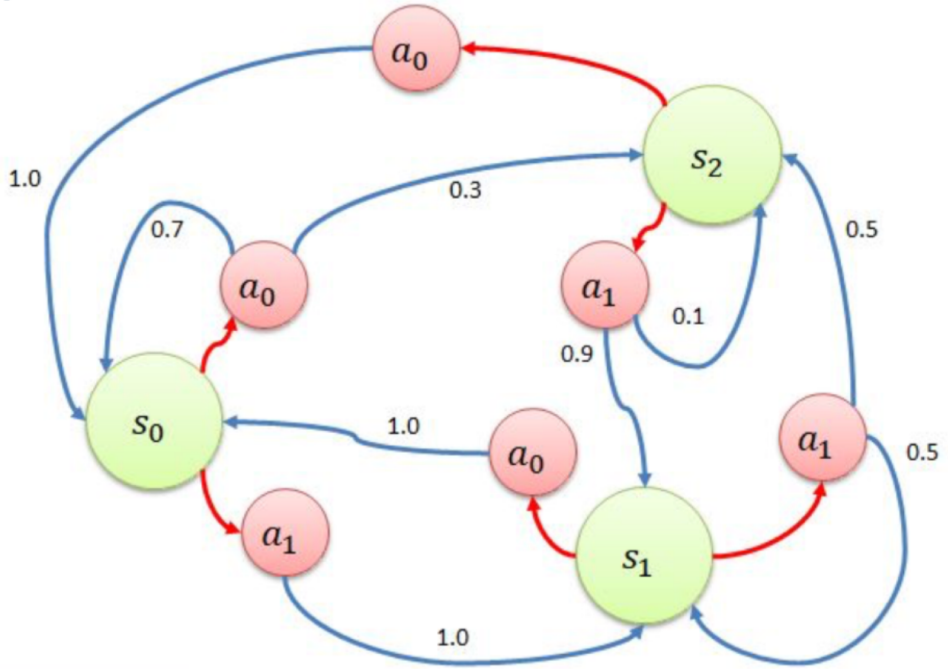
A model for decision making

Markov Decision Process (MDP)

M = $\langle S, A, T, R, \gamma \rangle$

Transition function: how the world evolves under actions

$$\mathcal{P}_{ss'}^{a} = P(S_{t+1} = s' \mid S_t = s, A_t = a)$$

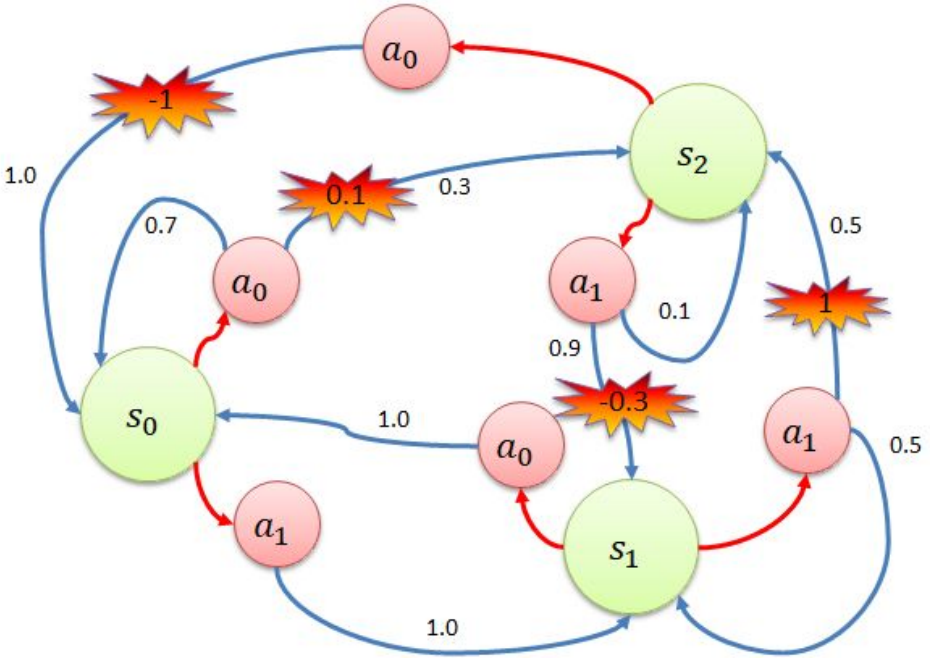


A model for decision making

Markov Decision Process (MDP)

M = ⟨S, A, T, R, γ⟩

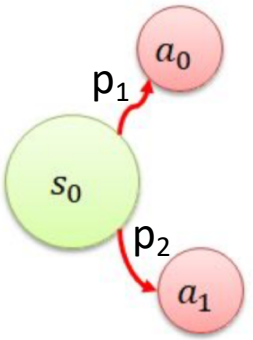
Rewards: feedback signal to agent



Questions:
- What are the rewards received by each action/state in our previous examples?
- Does every action receive a reward?
- Immediate or long-term reward? Positive or negative?

Reward

- Reward may be associated with states or state-action pairs
  - Reward received when arriving at a state: $\mathcal{R}_s = E[R_{t+1}|S_t = s]$
  - Reward received when taking an action at a state: $\mathcal{R}_s^a = E[R_{t+1}|S_t = s, A_t = a]$
- Note some notations used here:
  - Given state S_t and action A_t, reward is received at time t+1, i.e. R_{t+1}
  - An episode can be shown as (_, S_1 A_1), (R_2, S_2, A_2), …, (R_t, S_t, A_t), …



$$\mathcal{R}_{s_0} = p_1\mathcal{R}_{s_0}^{a_0} + p_2\mathcal{R}_{s_0}^{a_1}$$

- Reward R_{t+1} is a scalar feedback signal
- Markov Reward Process (MRP): A Markov reward process is a Markov chain with values.
- It indicates how well agent is doing at step t
- The agent's job is to ***maximize cumulative reward***
- Reinforcement learning is based on the reward hypothesis:

Definition (Reward Hypothesis)

All goals can be described by the **maximization of expected cumulative reward**

Examples of Rewards

- Play Atari games
  - + increase score
  - – decrease score
- Play chess or board game Go
  - + win a game
  - – lose a game
- Make a humanoid robot walk
  - + reward for forward motion
  - – reward for failing over
- Manage an investment portfolio
  - + each $ gained in bank
  - – each $ lost
- Manage inventory:
  - – Order loss due to out-of-stock, storage cost
- Questions:
  - Does every action receive a reward in each of these examples?
  - Immediate or long-term reward?

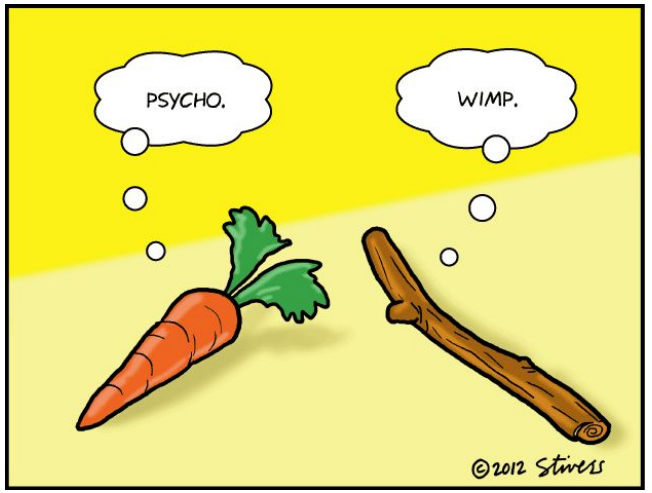
Rewards

Scalar feedback signal

Encode (un)desirable features of behaviours:

Winning/losing, collisions, taking expensive actions, ...

- Sparse
- Delayed
- Only have relative value



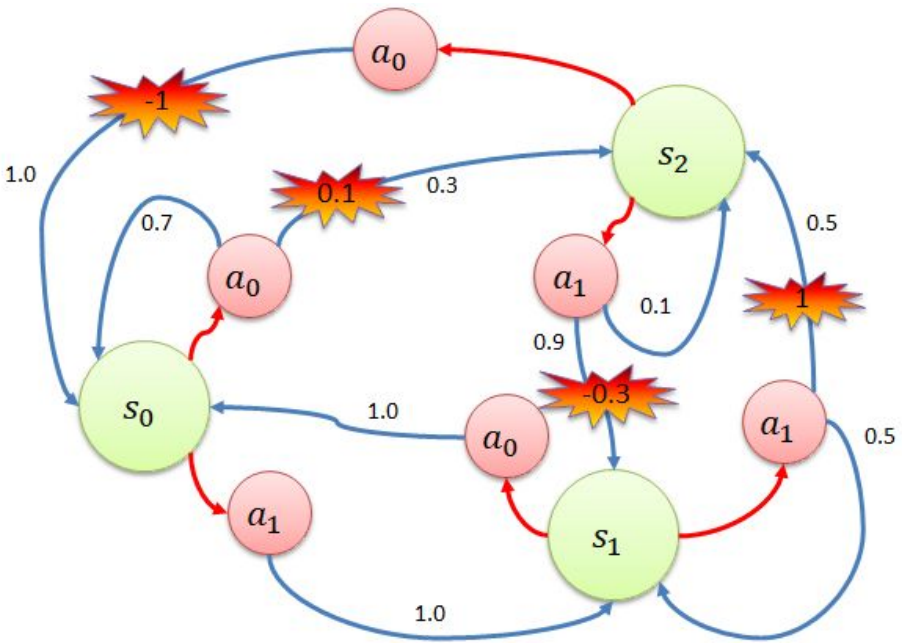
A model for decision making

Markov Decision Process (MDP)

M = ⟨S, A, T, R, γ⟩

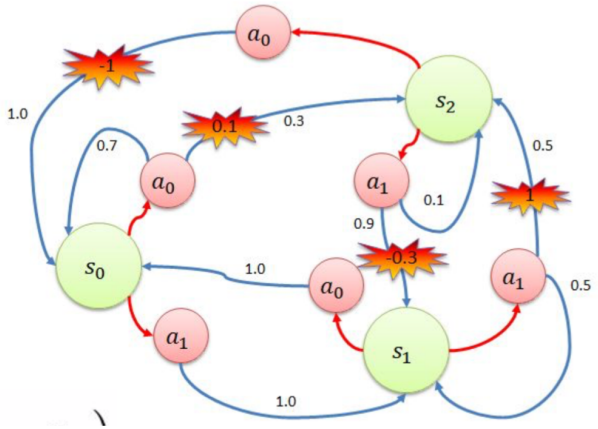
$\gamma \in [0,1]$ discounting for future rewards

Question: Why reward dicounts?



Reward Discounts

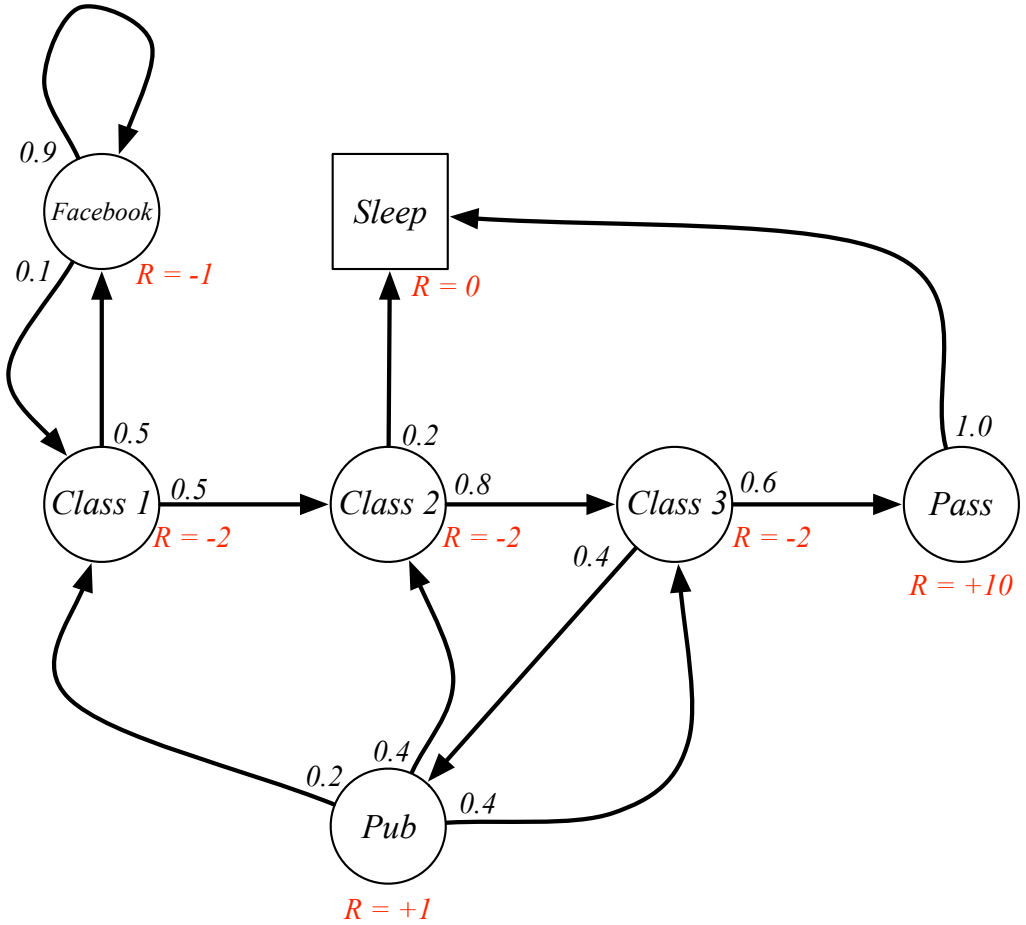
- Uncertainty about the future may not be fully represented
- If the reward is financial, immediate rewards may earn more interest than delayed rewards
- Animal/human behavior shows preference for immediate reward
  - $\gamma \rightarrow 0$: "myopic" evaluation
  - $\gamma \rightarrow 1$: "far–sighted" evaluation
- If there is no discount, what would happen in a cyclic Markov process?



Sequential Decision Making

- Goal: *select actions to maximise total future reward*
- Actions may have long term consequences
- Reward may be delayed
- It may be better to sacrifice immediate reward to gain more long-term reward
- Examples:
  - A financial investment (may take months to mature)
  - Refuelling a helicopter (might prevent a crash in several hours)
  - Blocking opponent moves (might help winning chances many moves from now)

Example: Student MRP



Return

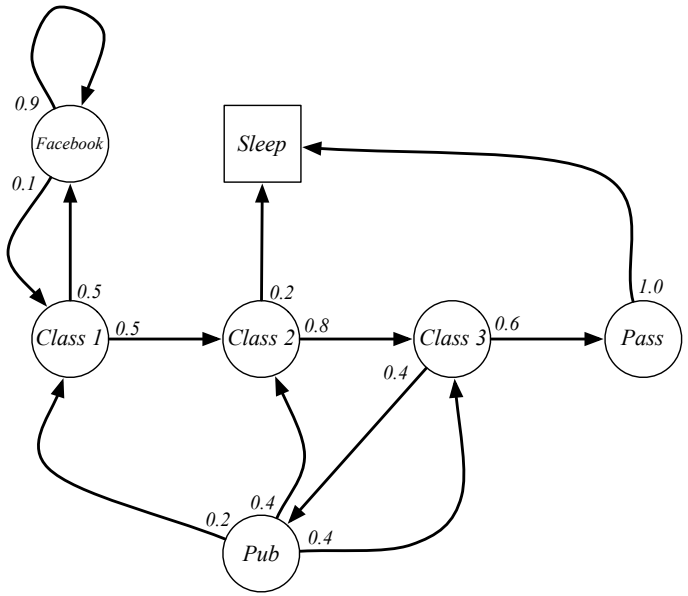
Definition

The *return* G_t is the total discounted reward from time-step t.

$$G_t = R_{t+1} + \gamma R_{t+2} + ... = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

- The *discount* $\gamma \in [0, 1]$ is the present value of future rewards
- The value of receiving reward R after $k + 1$ time-steps is $\gamma^k R$.
- This values immediate reward above delayed reward.
  - γ close to 0 leads to "myopic" evaluation
  - γ close to 1 leads to "far-sighted" evaluation

Example: Student Markov Chain Episodes



Sample episodes for Student Markov Chain starting from $S_1 = \text{C1}$

$$S_1, S_2, ..., S_T$$

How to determine an episode?

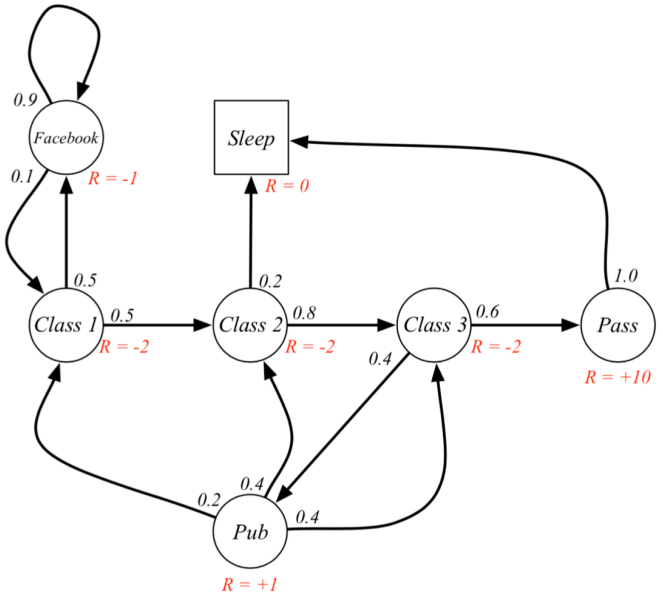
- C1 C2 C3 Pass Sleep
- C1 FB FB C1 C2 Sleep
- C1 C2 C3 Pub C2 C3 Pass Sleep
- C1 FB FB C1 C2 C3 Pub C1 FB FB FB C1 C2 C3 Pub C2 Sleep

Example: Student MRP Returns

Sample **episodes** for Student Markov Chain

Starting from $s_1 = \text{C1}$ with $\gamma = \frac{1}{2}$

$G_1 = R_2 + \gamma R_3 + \ldots + \gamma^{T-2} R_T$

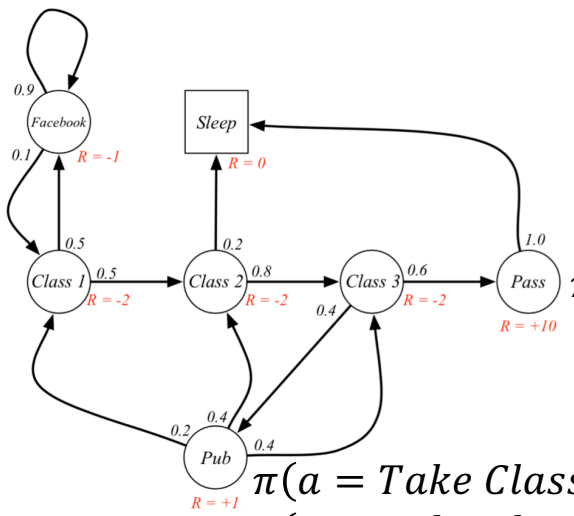


C1 C2 C3 Pass Sleep	$v_1 = -2 - 2 * \frac{1}{2} - 2 * \frac{1}{4} + 10 * \frac{1}{8}$	$=$	-2.25
C1 FB FB C1 C2 Sleep	$v_1 = -2 - 1 * \frac{1}{2} - 1 * \frac{1}{4} - 2 * \frac{1}{8} - 2 * \frac{1}{16}$	$=$	-3.125
C1 C2 C3 Pub C2 C3 Pass Sleep	$v_1 = -2 - 2 * \frac{1}{2} - 2 * \frac{1}{4} + 1 * \frac{1}{8} - 2 * \frac{1}{16} \ldots$	$=$	-3.41
C1 FB FB C1 C2 C3 Pub C1 … FB FB FB C1 C2 C3 Pub C2 Sleep	$v_1 = -2 - 1 * \frac{1}{2} - 1 * \frac{1}{4} - 2 * \frac{1}{8} - 2 * \frac{1}{16} \ldots$	$=$	-3.20

3. Policy

Policy

- A policy is the agent's behavior
- It is a map from states to actions
- Two types of Policy:
  - **Deterministic** policy: $a = \pi(s)$
  - **Stochastic** policy:
    $\pi(a|s) = \mathrm{P}(A_t = a|S_t = s)$
- **Optimal Policy** (π^*): A policy return the maximal accumulated rewards
  $G_1 = R_2 + \gamma R_3 + \gamma^2 R_4 + \ldots$
- This is what we want to learn!



$\pi(Pass)$ = Go to sleep

$\pi(a = Take\ Class\ 1|s = Pub)$ = 0.2

$\pi(a = Take\ Class\ 2|s = Pub)$ = 0.4

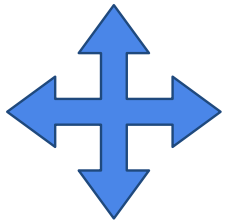
…

Start from "Class 1", what is the optimal policy? What if we don't know the transition probability, i.e. $\pi(a = Pass|s = Class\ 3)$? How to find it?

An example

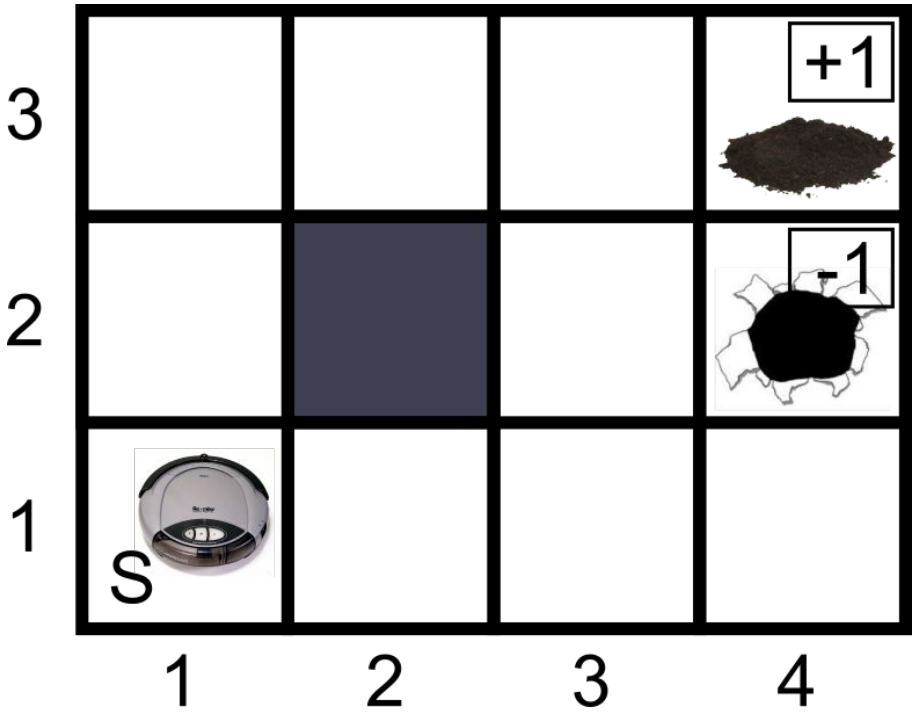
Cleaning Robot

Actions:



Reward:

- +1 for finding dirt
- -1 for falling into hole
- -0.001 for every move

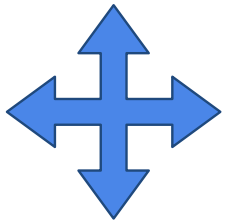


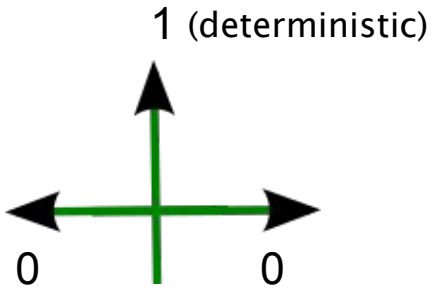
An example

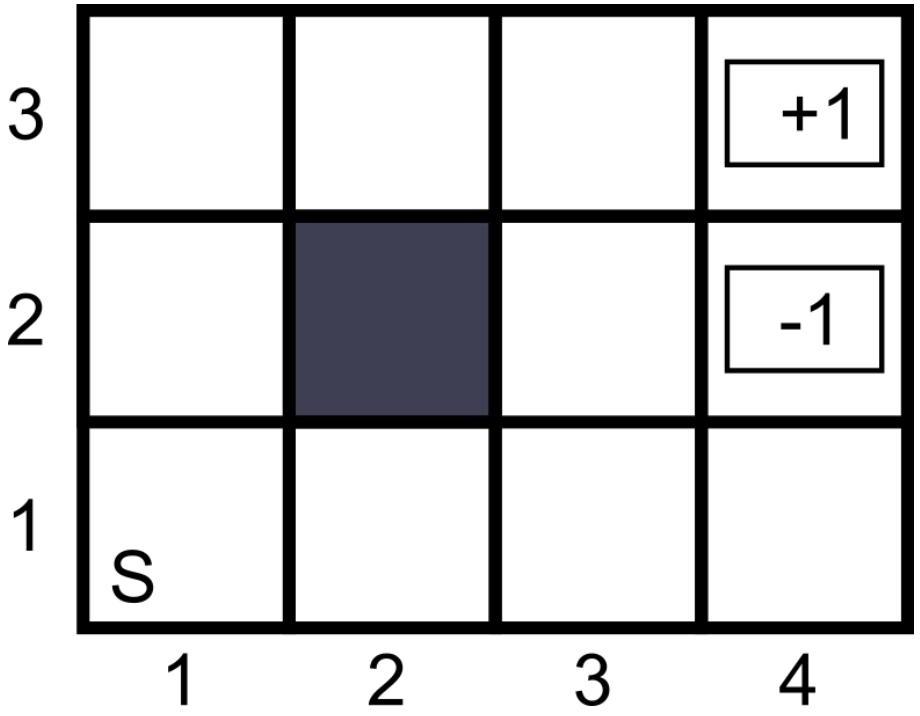
States:

- Position on grid e.g.
  - S is (1,1), goal (4,3)

Actions:







Reward:

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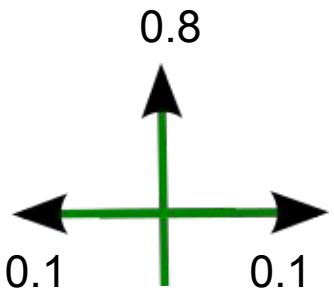
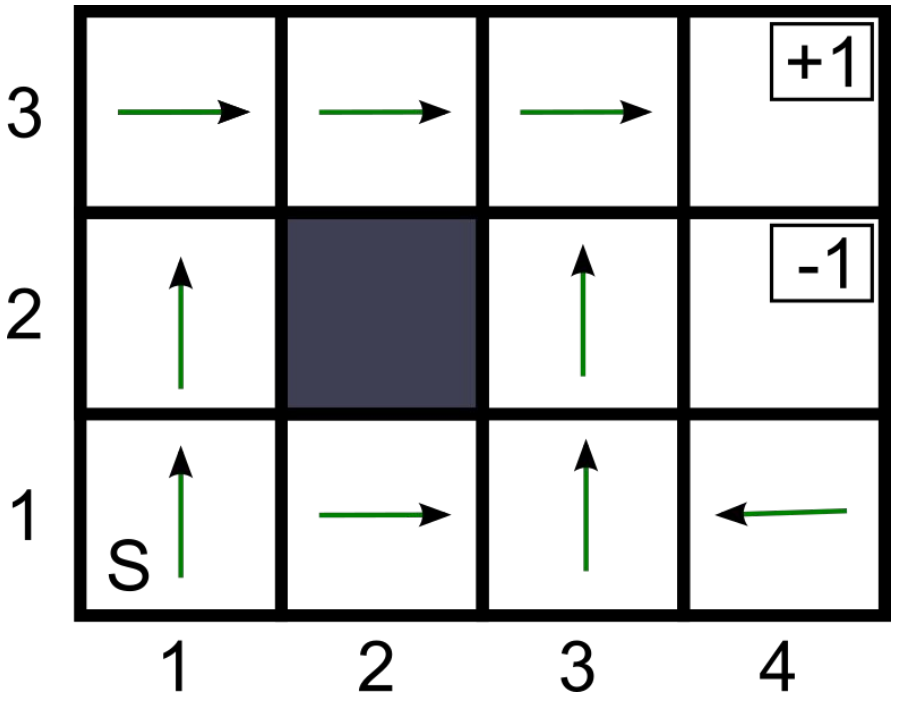
What is the optimal policy?





Is this an optimal policy?

How to search for the optimal policy?

4. Value Functions and Bellman Equations

Value Function

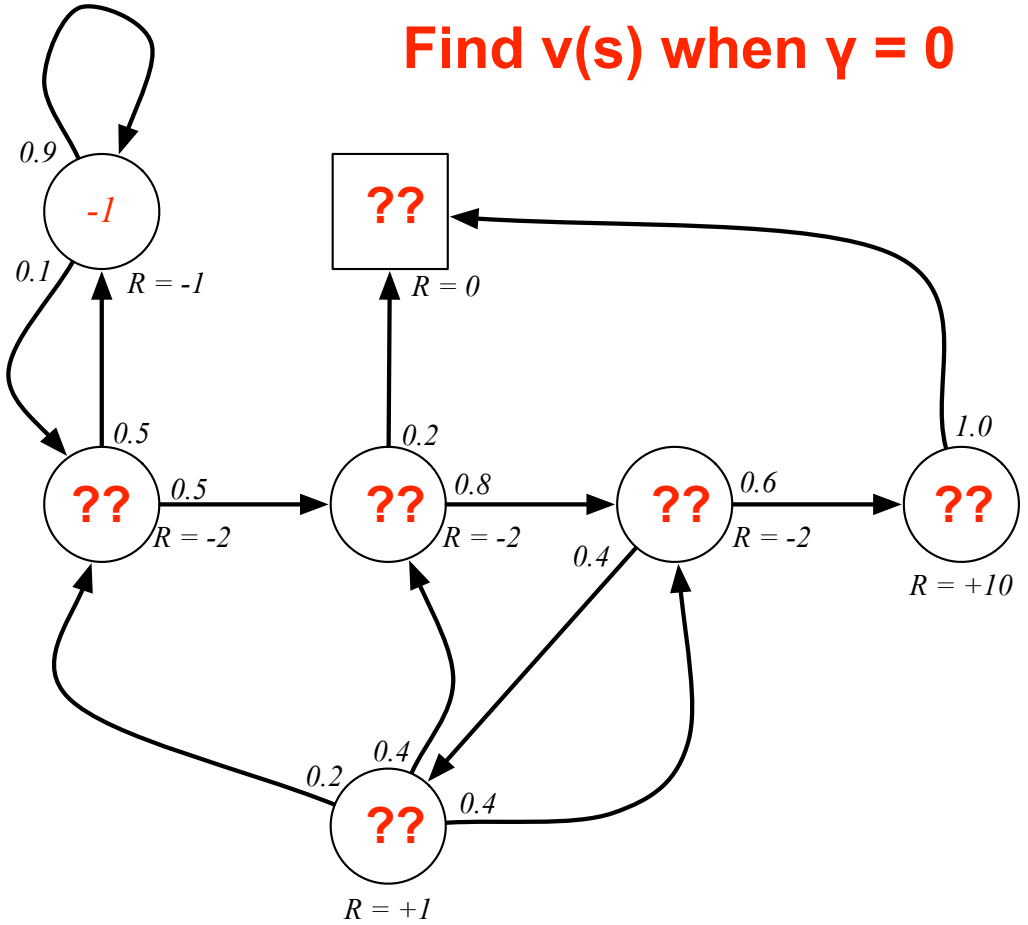
The value function $v(s)$ gives the long-term value of state s

Definition

The *state value function* $v(s)$ of an MRP is the expected return starting from state s

$$v(s) = \mathbb{E}\left[G_t \mid S_t = s\right]$$

Example: State-Value Function for Student MRP



What if γ = 0.9?

Bellman Equation for MRPs

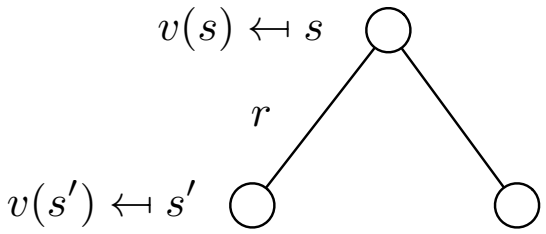
The value function can be decomposed into two parts:

- immediate reward R_{t+1}
- discounted value of successor state $\gamma v(S_{t+1})$

$$
\begin{aligned}
v(s) &= \mathbb{E}\left[G_t \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma\left(R_{t+2} + \gamma R_{t+3} + \ldots\right) \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma G_{t+1} \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma v(S_{t+1}) \mid S_t = s\right]
\end{aligned}
$$

Bellman Equation for MRPs (2)

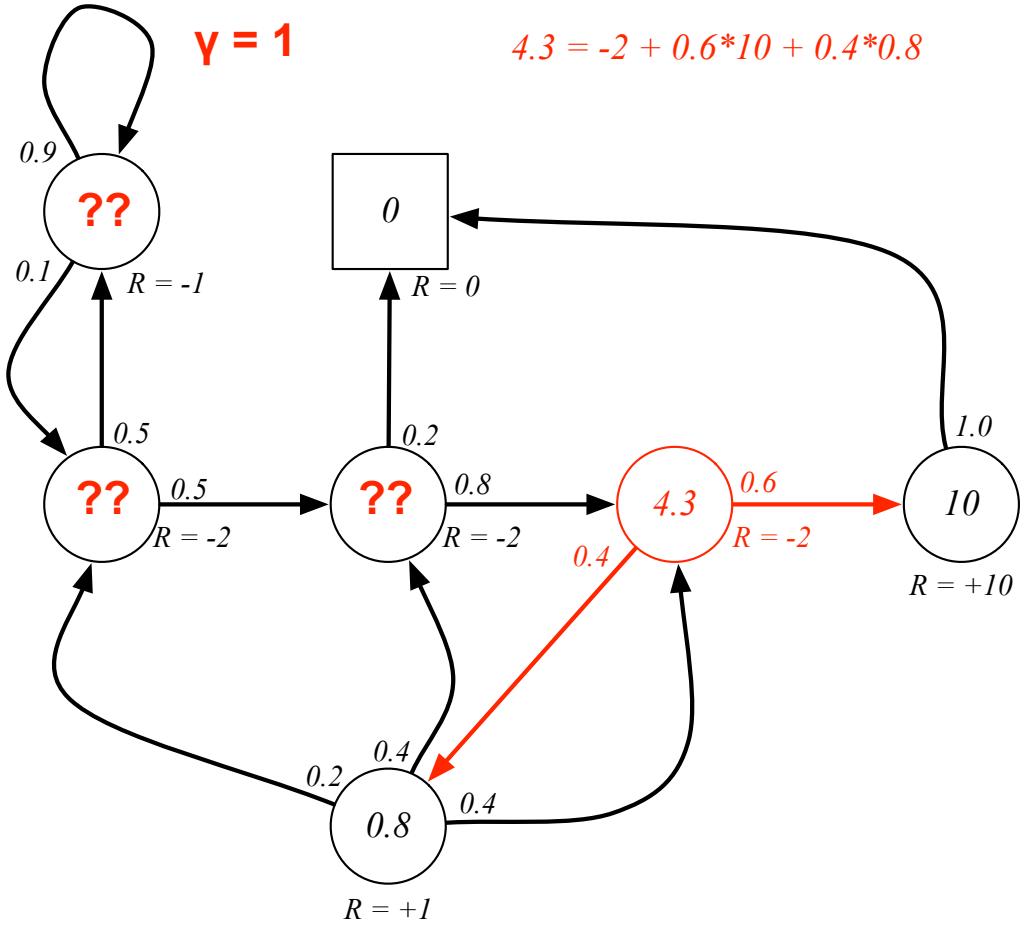
$$v(s) = \mathbb{E}\left[R_{t+1} + \gamma v(S_{t+1}) \mid S_t = s\right]$$



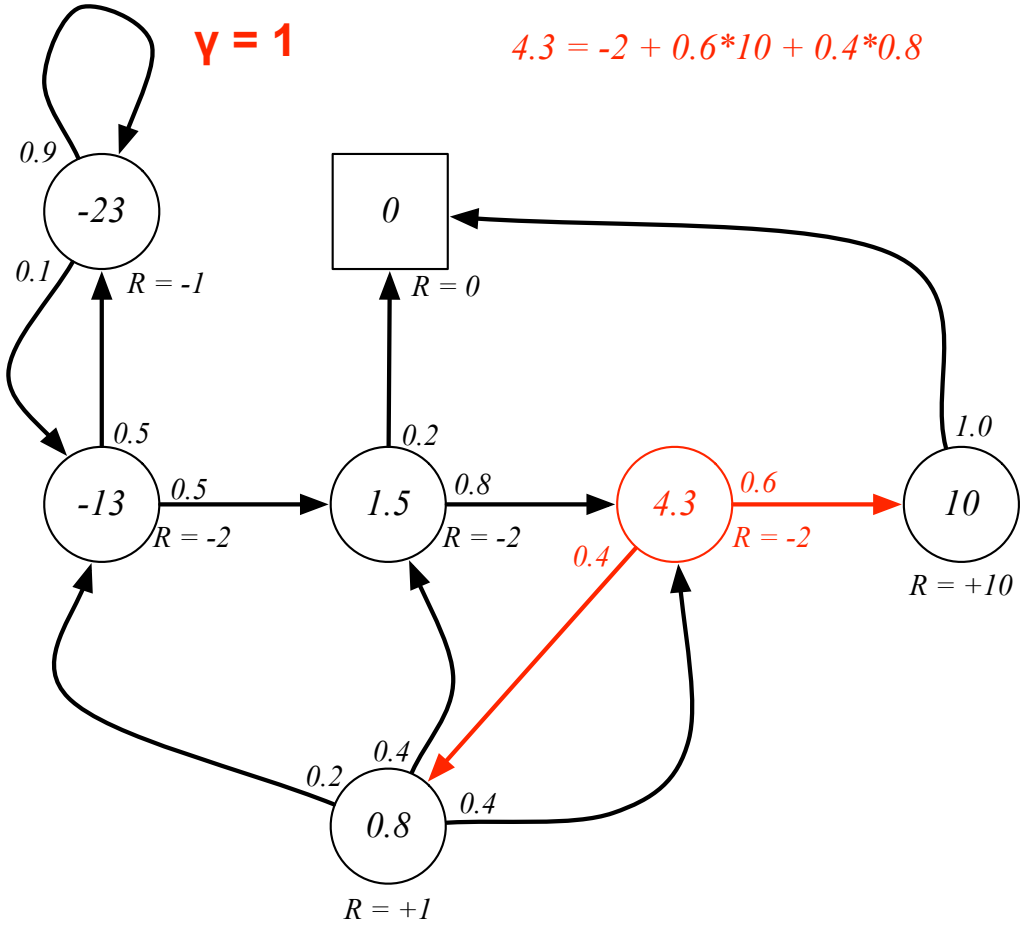
$$v(s) = \mathcal{R}_s + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'} v(s')$$

Transition probability from s to s'

Example: Bellman Equation for Student MRP



Example: Bellman Equation for Student MRP



- What if γ = 0.9?

- What if transition probabilities between states (i.e. policy) are changed? Will state values change?

- Which policy is optimal?

Value Function under a Policy

Definition

The *state-value function* $v_\pi(s)$ of an MDP is the expected return starting from state s, and then following policy π

$$v_\pi(s) = \mathbb{E}_\pi \left[G_t \mid S_t = s \right]$$

Definition

The *action-value function* $q_\pi(s, a)$ is the expected return starting from state s, taking action a, and then following policy π

$$q_\pi(s, a) = \mathbb{E}_\pi \left[G_t \mid S_t = s, A_t = a \right]$$

Bellman Expectation Equation

The state-value function can again be decomposed into immediate reward plus discounted value of successor state,

$$v_\pi(s) = \mathbb{E}_\pi \left[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s \right]$$

The action-value function can similarly be decomposed,

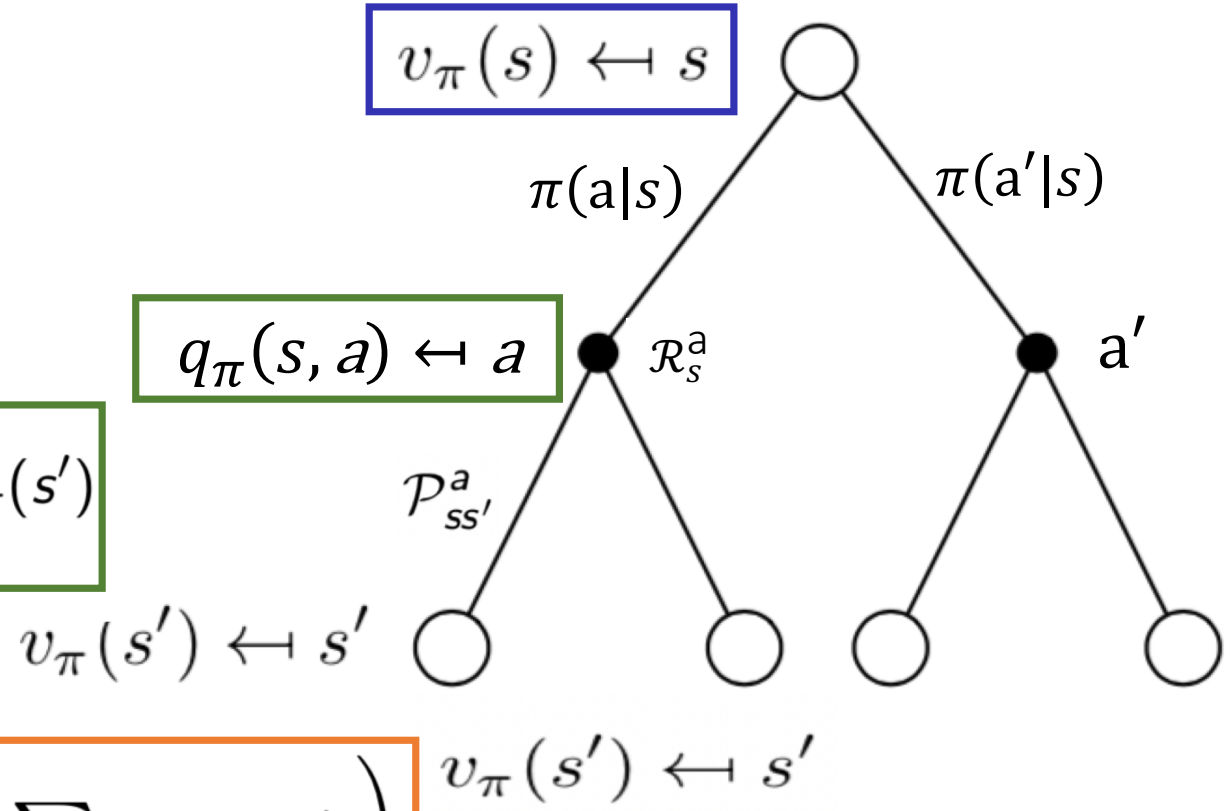
$$q_\pi(s, a) = \mathbb{E}_\pi \left[R_{t+1} + \underbrace{\gamma q_\pi(S_{t+1}, A_{t+1})}_{\text{discounted value of successor action}} \mid S_t = s, A_t = a \right]$$

Relationship between $V_\pi(s)$ and $q_\pi(s,a)$

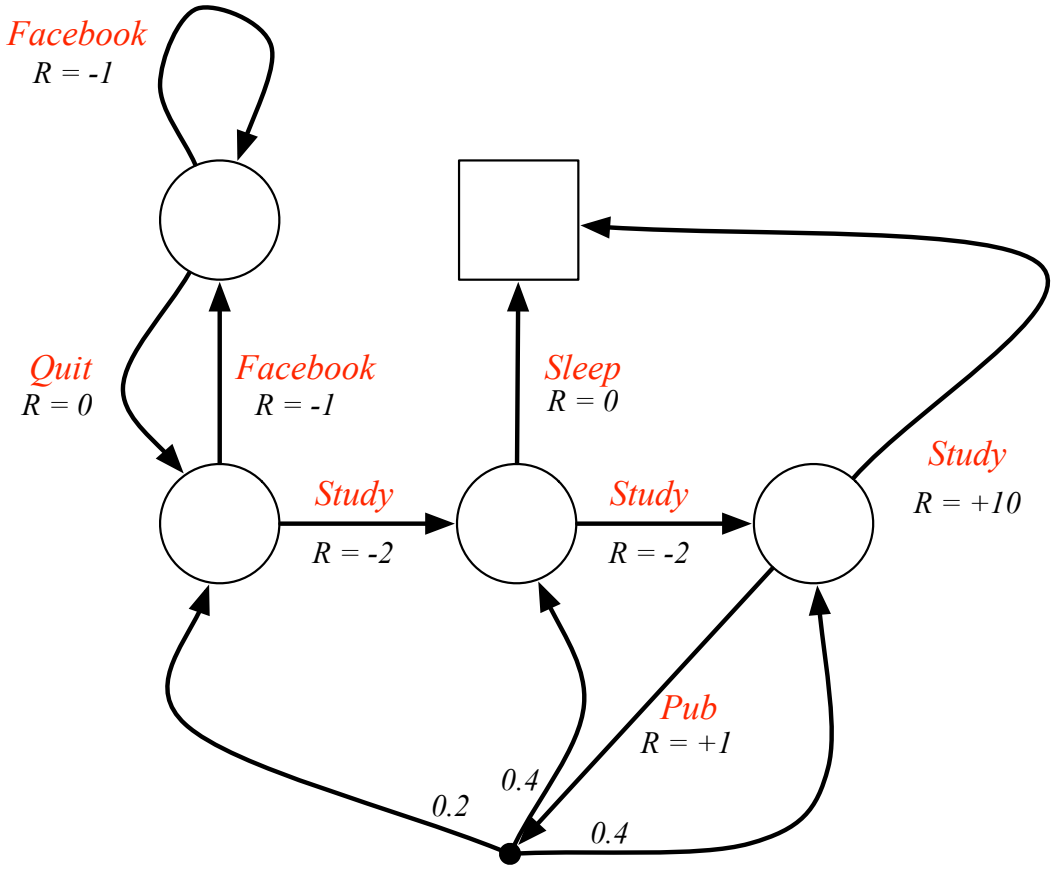
1 $$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) q_\pi(s,a)$$

2 $$q_\pi(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s')$$

3 $$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s') \right)$$



Example: Student MDP

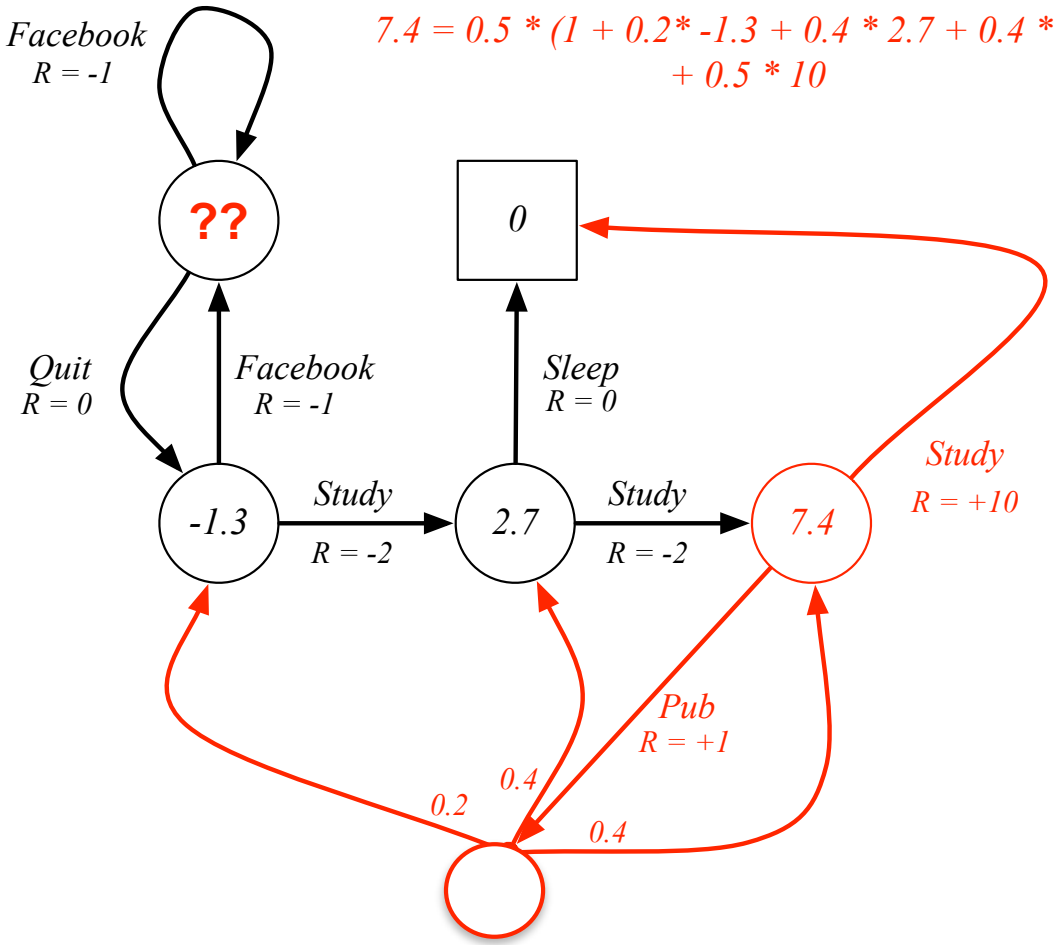


Note, rewards are associated with actions

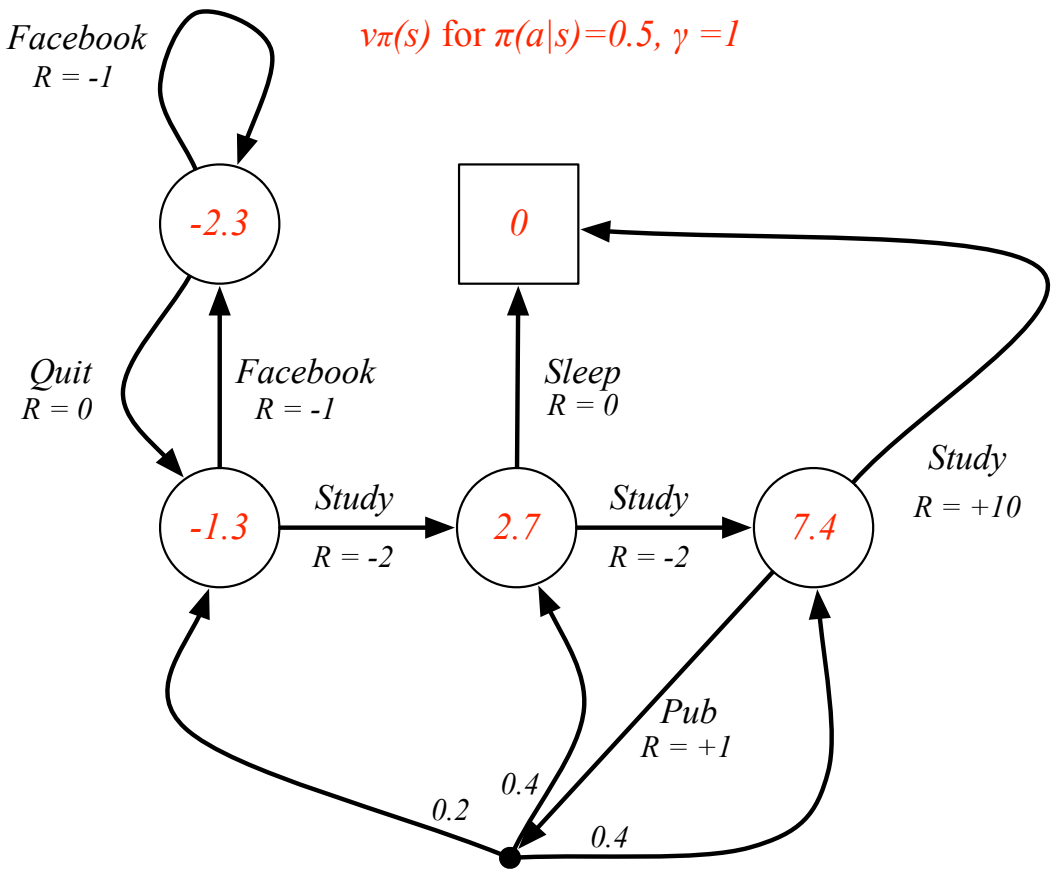
Example: Bellman Expectation Equation in Student MDP

Assume π(a|s) = 0.5, γ = 1

$7.4 = 0.5 * (1 + 0.2 * -1.3 + 0.4 * 2.7 + 0.4 * 7.4) + 0.5 * 10$



Example: State-Value Function for Student MDP



GridWorld Demo:
http://www.cs.toronto.edu/~lcharlin/courses/80-629/reinforcejs/gridworld_dp.html

Optimal Value Function

Definition

The *optimal state-value function* $v_*(s)$ is the maximum value function over all policies

$$v_*(s) = \max_{\pi} v_\pi(s)$$

The *optimal action-value function* $q_*(s, a)$ is the maximum action-value function over all policies

$$q_*(s, a) = \max_{\pi} q_\pi(s, a)$$

- The optimal value function specifies the best possible performance in the MDP.
- An MDP is "solved" when we know the optimal value fn.

Finding an Optimal Policy

An optimal policy can be found by maximising over $q_*(s, a)$,

$$\pi_*(a|s) = \begin{cases} 1 & \text{if } a = \underset{a \in \mathcal{A}}{\operatorname{argmax}}\, q_*(s, a) \\ 0 & \textit{otherwise} \end{cases}$$

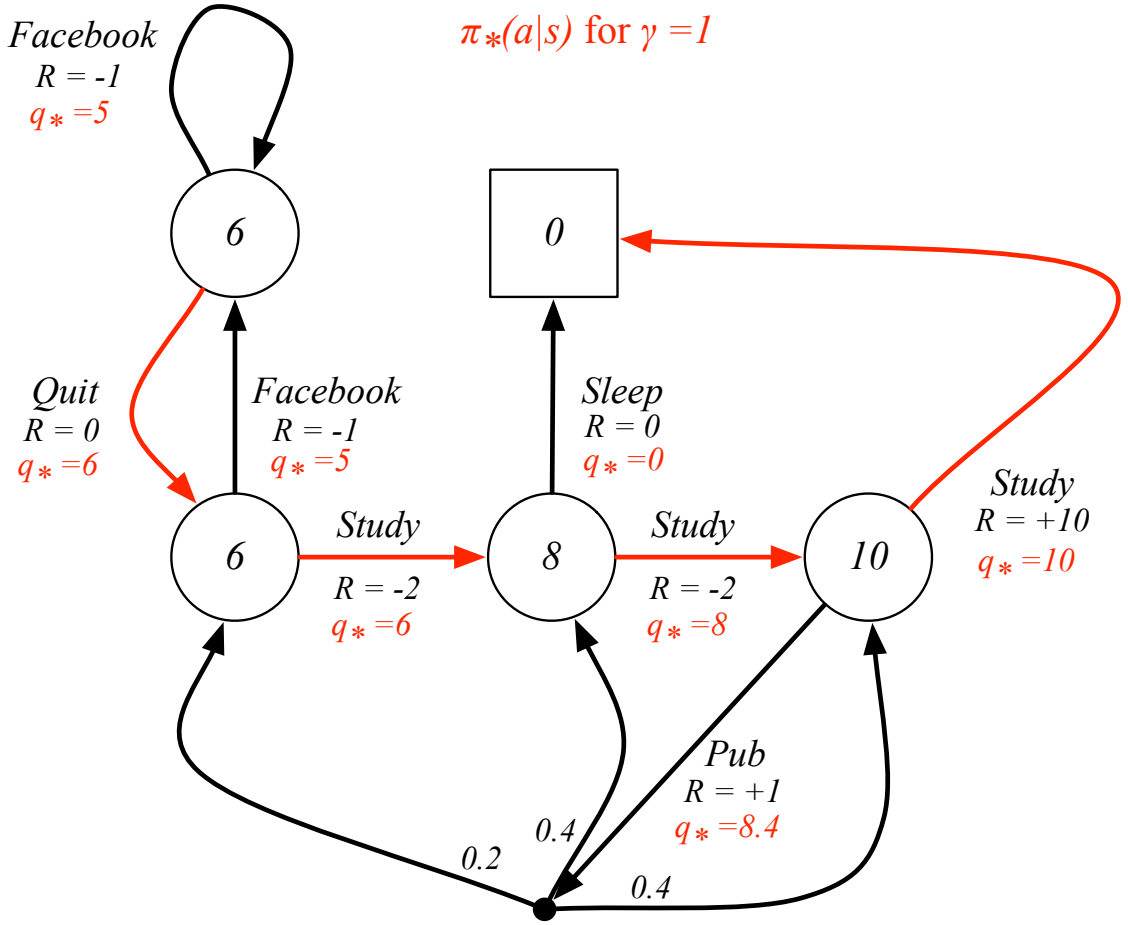
Move in direction of greatest value

- There is always a deterministic optimal policy for any MDP
- If we know $q_*(s, a)$, we immediately have the optimal policy

Finding Q* (or V*) is equivalent to finding π*

GridWorld demo: http://www.cs.toronto.edu/~lcharlin/courses/80-629/reinforcejs/gridworld_dp.html

Example: Optimal Policy for Student MDP

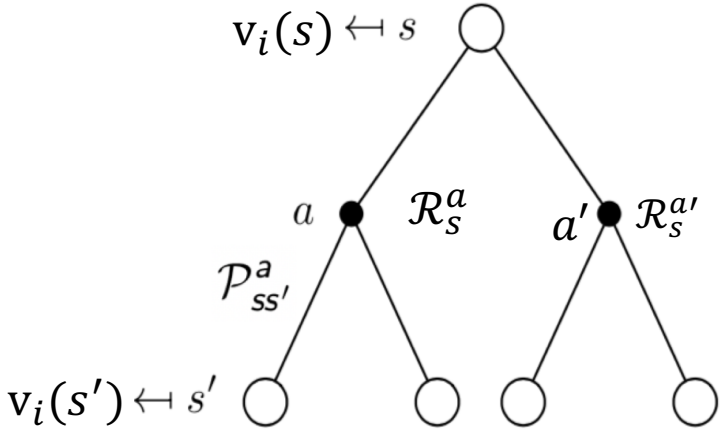


What if we don't know V* or q*? Can we find the optimal poilcy incrementally?

The Goal of Reinforcement Learning

How to learn an optimal policy based on Bellman Optimality Equation?

$$\mathrm{v}^*(s) = \max_a \left(\mathcal{R}_s^a + \gamma \sum_{s' \in S} \mathcal{P}_{ss'}^a \, \mathrm{v}^*(s') \right)$$



Value Iteration: Iteratively update V

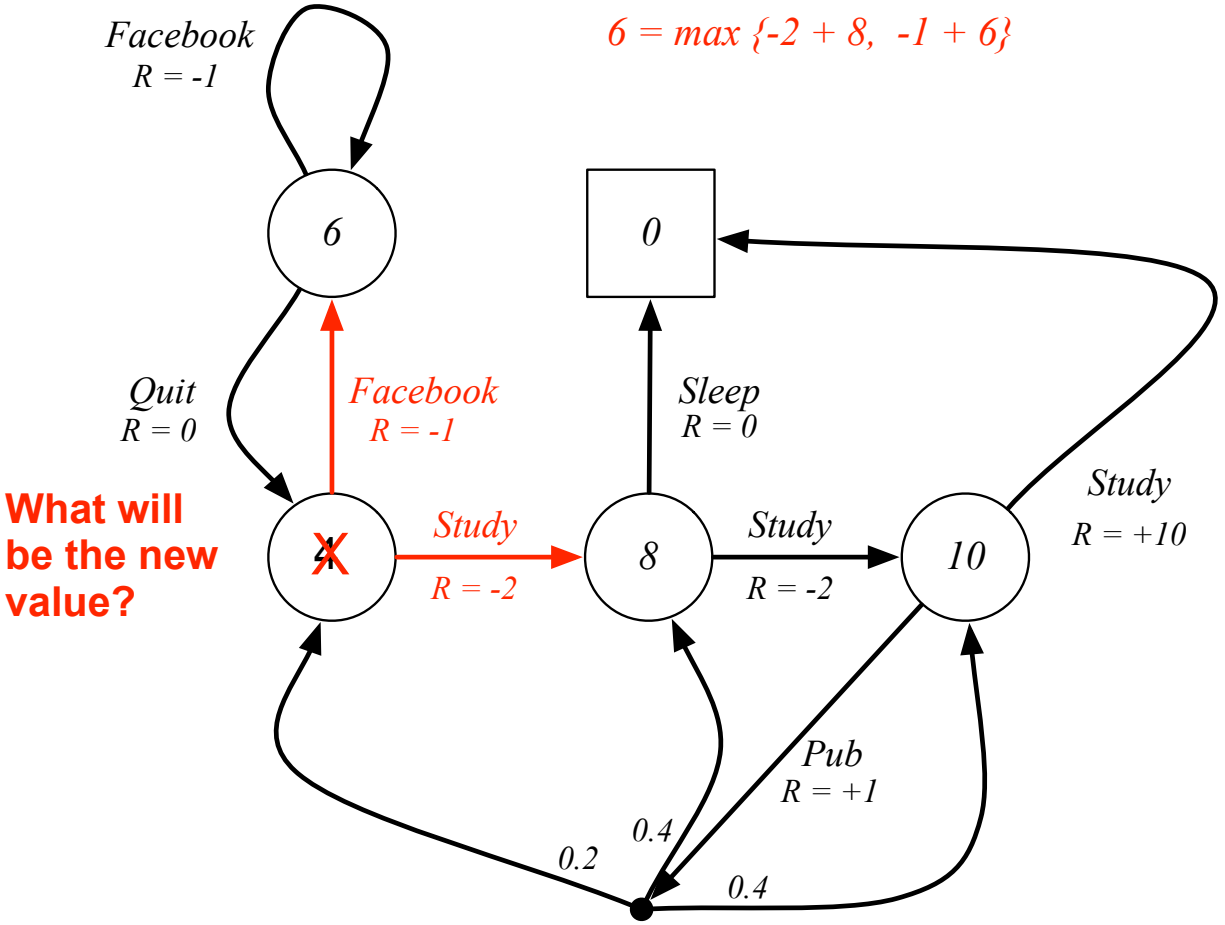
- At each iteration i :
  - For all states s in S :
    - Update $v_{i+1}(s) := \max_a (\mathcal{R}_s^a + \gamma \sum_{s' \in S} \mathcal{P}_{ss'}^a v_i(s'))$

Try the GridWorld demo!

Challenges in searching $\mathrm{v}^*(s)$:

- Often $\mathcal{P}_{ss'}^a$ and sometimes $\mathcal{R}_s^a$ are unknown in advance
- There may be a large number of possible states and actions
- Exhaustive search is often impossible and ineffective

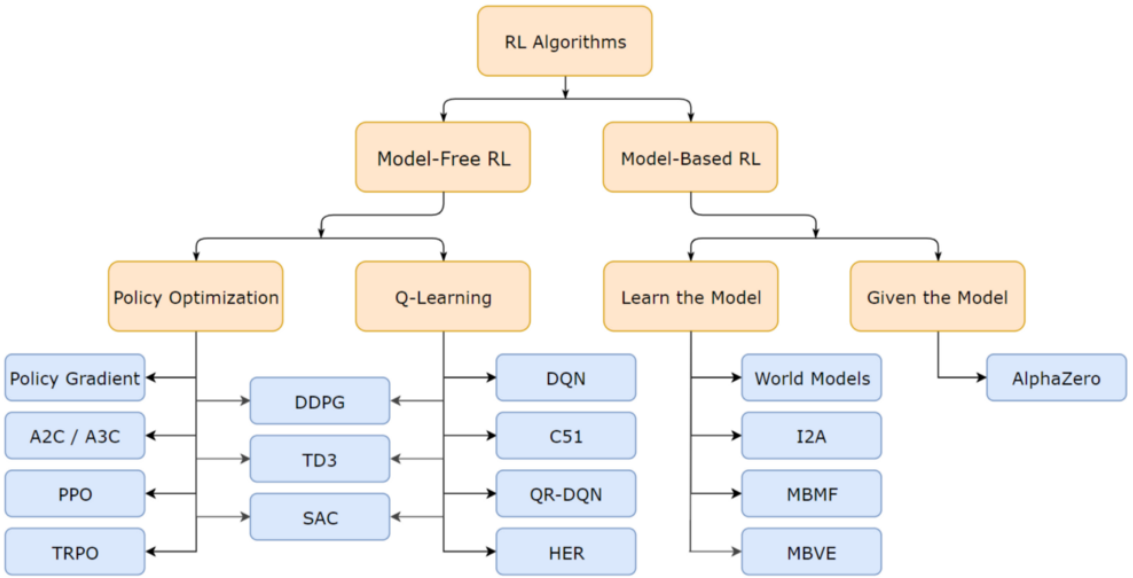
Example: Bellman Optimality Equation in Student MDP



5. Approaches to Reinforcement Learning

Taxonomy of RL Algorithms

- Value-based vs. Policy-based RL
  - Value-based: Estimate the optimal value function $v^*(s)$ or $q^*(s,a)$, the maximum value achievable under any policy
  - Policy-based: Search directly for the optimal policy π^*, the policy achieving maximum future reward
- Model-based vs. Model-free RL
  - Model-based RL build a model of the environment and plan (e.g. by lookahead) using model

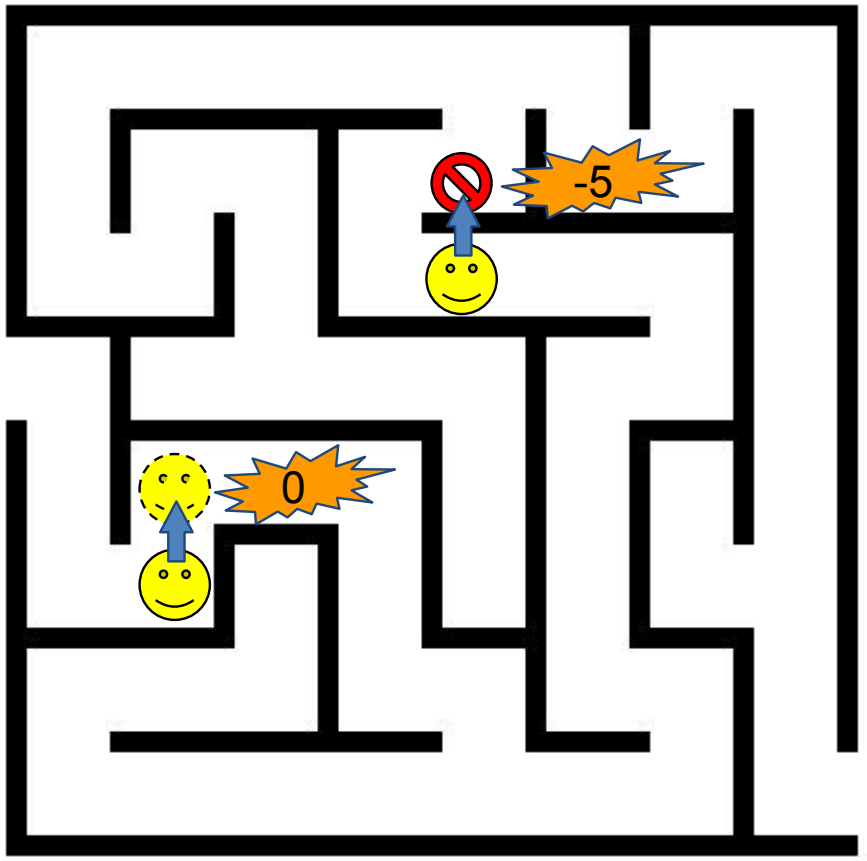


https://spinningup.openai.com/en/latest/spinningup/rl_intro2.html

Data generation

Transition probabilities and rewards are often unknown in advance

Instead, generate samples of training data ***(s,a,r,s')*** from environment

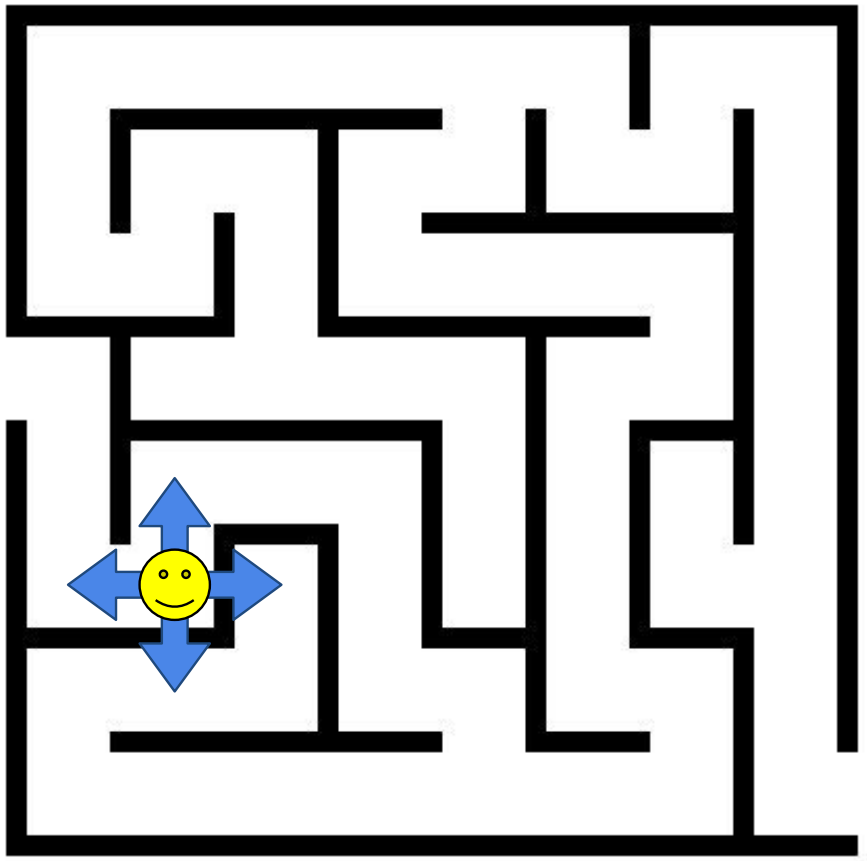


Learning from Experience

We need

- **A method to choose actions**

- Some model to keep track of and learn
  - *Value Function*



The Bandit Problem

Consider a row of one-arm bandit machines in a casino

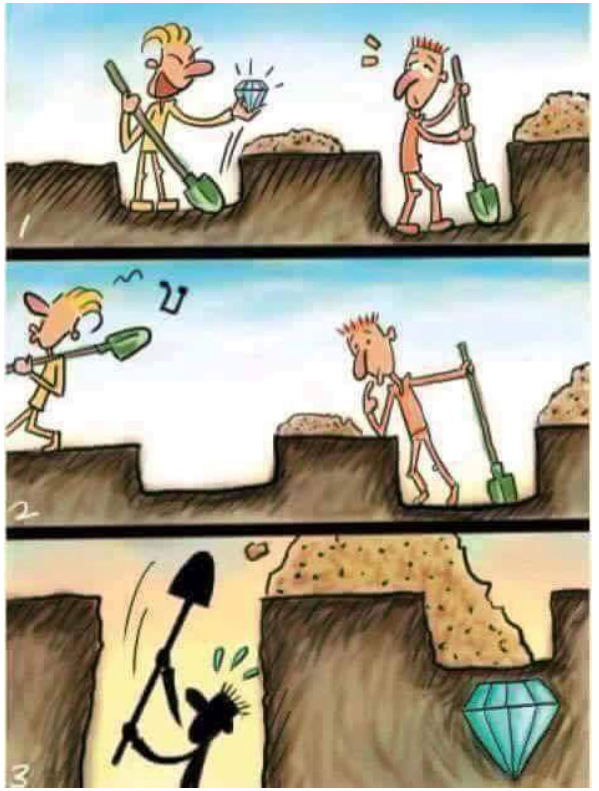
Set of “arms” (actions) that each generate rewards from different distributions

Exploration vs exploitation

Action Selection: Exploration vs. Exploitation

Maximizing expected returns means balancing between:

- **Exploitation**: Exploiting gained knowledge (greedy)
  – Take the best known action
- **Exploration**: Exploring new actions/states (random)
  – Try something new



https://medium.com/deep-math-machine-learning-ai/ch-12-1-model-free-reinforcement-learning-algorithms-monte-carlo-sarsa-q-learning-65267cb8d1b4

Action selection strategies

ε-Greedy ($0 < \varepsilon \leq 1$):

- With probability $1-\varepsilon$ exploit
  - Choose the best action for a state
- With probability ε explore
  - Randomly choose action

ε usually higher at beginning of learning, decay later

Softmax

- Sample action given softmax

$$P_t(a) = \frac{\exp(Q_t(a)/\tau)}{\sum_{i=1}^{n} \exp(Q_t(i)/\tau)},$$

ϵ-Greedy Exploration

- Simplest idea for ensuring continual exploration
- All m actions are tried with non-zero probability
- With probability $1 - \epsilon$ choose the greedy action
- With probability ϵ choose an action at random

$$\pi(a|s) = \begin{cases} \epsilon/m + 1 - \epsilon & \text{if } a^* = \underset{a \in \mathcal{A}}{\operatorname{argmax}}\, Q(s, a) \\ \epsilon/m & \text{otherwise} \end{cases}$$

5 (A). Value-based RL: SARSA and Q-Learning

Basic Idea: Bellman Equations

Under a policy π, at time t, if **Q-value** ($\boldsymbol{q_\pi(S_t, A_t)}$) estimates are accurate, the following must hold:

$$q_\pi(S_t, A_t) = \mathbb{E}_\pi \left[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) \mid S_t = s, A_t = a\right]$$

If not, then there is an error (a.k.a. **Temporal Difference** (**TD**)):

$$\delta = q_\pi(S_t, A_t) - \mathbb{E}_\pi \left[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) \mid S_t = s, A_t = a\right]$$

Then we learn better Q-values to minimize the δ

- **SARSA**: Follow a policy π to determine the next action A_t(**On Policy**)
- **Q-Learning**: Reuse experience from other policies to determine the next action A_t (**Off Policy**)

SARSA Algorithm for On-Policy Control

Initialize $Q(s,a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
  Initialize S
  Choose A from S using policy derived from Q (e.g., ε-greedy)
  Repeat (for each step of episode):
    Take action A, observe R, S'
    $\boxed{\text{Choose } A' \text{ from } S' \text{ using policy derived from } Q \text{ (e.g., } \varepsilon\text{-greedy)}}$
    $Q(S,A) \leftarrow Q(S,A) + \alpha\underbrace{\left[R + \gamma Q(S',A') - Q(S,A)\right]}_{\text{learned value}}$
    $S \leftarrow S'$; $A \leftarrow A'$;
  until S is terminal

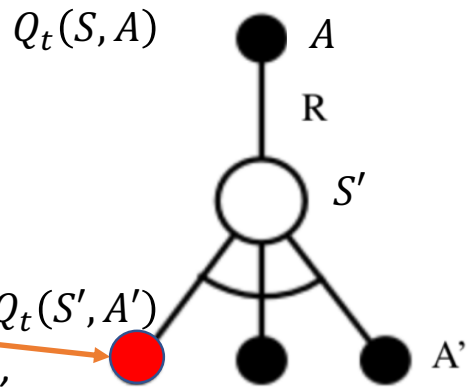
new value

old value

learning rate

expected target by Bellman Equation

old value



Pick A'

A' may not be the action with max $Q_t(S',A')$ over all possible A's because of exploration (ε)

Q-Learning Algorithm for Off-Policy Control

Initialize $Q(s,a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$

Repeat (for each episode):

  Initialize S

  Repeat (for each step of episode):

    Choose A from S using policy derived from Q (e.g., ε-greedy)

    Take action A, observe R, S'

    $$\boxed{Q(S,A) \leftarrow Q(S,A) + \alpha\left[R + \gamma \boxed{\max_a Q(S',a)} - Q(S,A)\right]}$$

    $S \leftarrow S'$;

  until S is terminal

new value

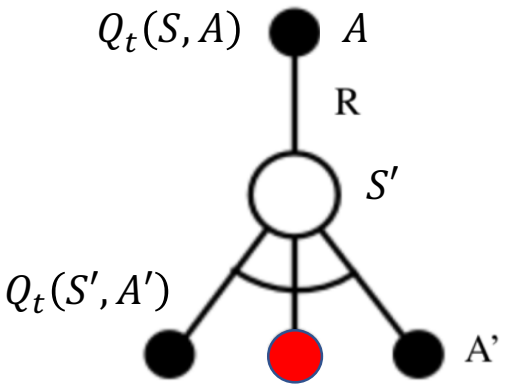
old value

learning rate

the most optimistic estimate so far

learned value

old value



- Estimate $Q_t(S,A)$ using $\max_{A'} Q_t(S,A')$, which is learned from past experience, but do not pick A' at Step t.
- Directly go to Step $t+1$ and then pick A based on updated Q values.

SARSA vs. Q-Learning

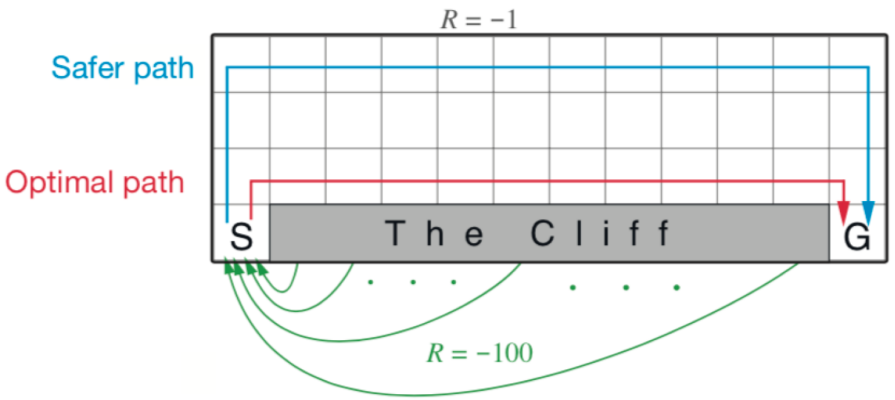
Initialize $Q(s,a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
  Initialize S
  Choose A from S using policy derived from Q (e.g., ε-greedy)
  Repeat (for each step of episode):
    Take action A, observe R, S'
    Choose A' from S' using policy derived from Q (e.g., ε-greedy)
    $Q(S,A) \leftarrow Q(S,A) + \alpha[R + \gamma Q(S',A') - Q(S,A)]$
    $S \leftarrow S'$; $A \leftarrow A'$;
  until S is terminal

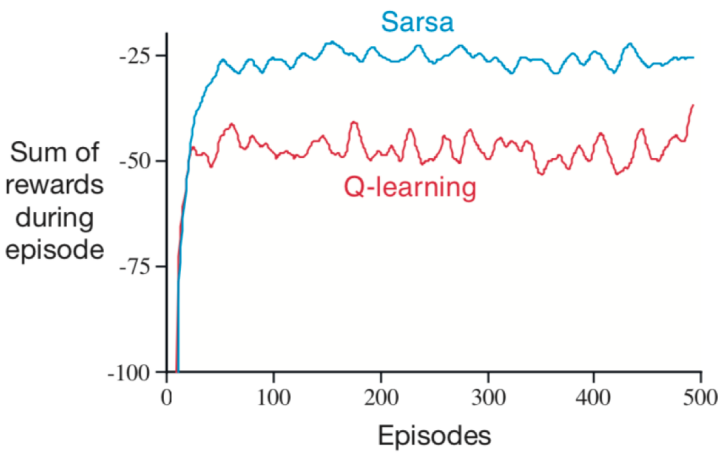
Figure 6.9: Sarsa: An on-policy TD control algorithm.

Initialize $Q(s,a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
  Initialize S
  Repeat (for each step of episode):
    Choose A from S using policy derived from Q (e.g., ε-greedy)
    Take action A, observe R, S'
    $Q(S,A) \leftarrow Q(S,A) + \alpha[R + \gamma \max_a Q(S',a) - Q(S,A)]$
    $S \leftarrow S'$; what about A'?
  until S is terminal

this is like following a greedy policy (e.g. ε=0, NO exploration)

Figure 6.12: Q-learning: An off-policy TD control algorithm.





Sutton & Barto, 2018

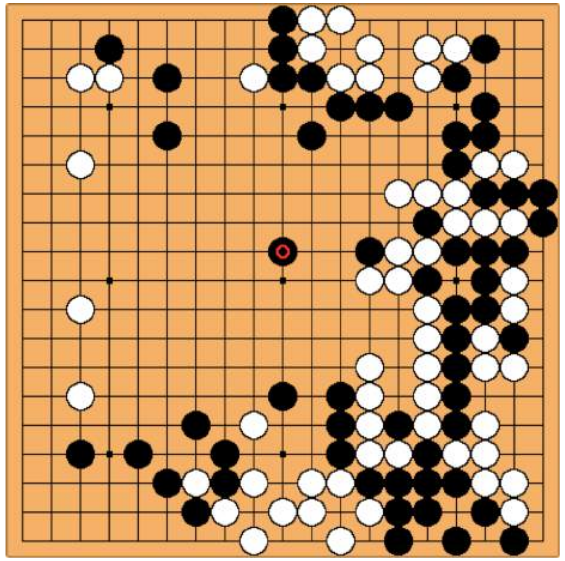
Q-Learning demo

(*T*,*R*,*S*,*A*)

Shreyas Skandan: https://www.youtube.com/watch?v=RTu7G0y4Os4

The Game of Go

Aim: Get more board territory than your opponent.



Board Size $n \times n$	Positions 3^{n^2}	% Legal	Legal Positions
1×1	3	33.33%	1
2×2	81	70.37%	57
3×3	19,683	64.40%	12,675
4×4	43,046,721	56.49%	24,318,165
5×5	847,288,609,443	48.90%	414,295,148,741
9×9	$4.434264882 \times 10^{38}$	23.44%	$1.03919148791 \times 10^{38}$
13×13	$4.300233593 \times 10^{80}$	8.66%	$3.72497923077 \times 10^{79}$
19×19	$1.740896506 \times 10^{172}$	1.20%	$2.08168199382 \times 10^{170}$

Greater number of legal board positions than atoms in the universe.

It is impossible to store legal positions (states) in memory

Source: Wikipedia.

Large-Scale Reinforcement Learning

Reinforcement learning can be used to solve *large* problems, e.g.

- Backgammon: 10^{20} states
- Computer Go: 10^{170} states
- Helicopter: continuous state space

Large-Scale Reinforcement Learning

Reinforcement learning can be used to solve *large* problems, e.g.

- Backgammon: 10^{20} states
- Computer Go: 10^{170} states
- Helicopter: continuous state space

How can we scale up the model-free methods for *prediction* and *control* ? **Use models to approximate v(s) and q(s, a)!**

5 (B). Model-Based Methods

Deep Reinforcement Learning Algorithms

Value Learning

Find $Q(s, a)$

$$a = \underset{a}{\operatorname{argmax}}\, Q(s, a)$$

Policy Learning

Find $\pi(s)$

Sample $a \sim \pi(s)$

Deep Reinforcement Learning Algorithms

Value Learning

Find $Q(s, a)$

$$a = \underset{a}{\operatorname{argmax}}\, Q(s, a)$$

Policy Learning

Find $\pi(s)$

Sample $a \sim \pi(s)$

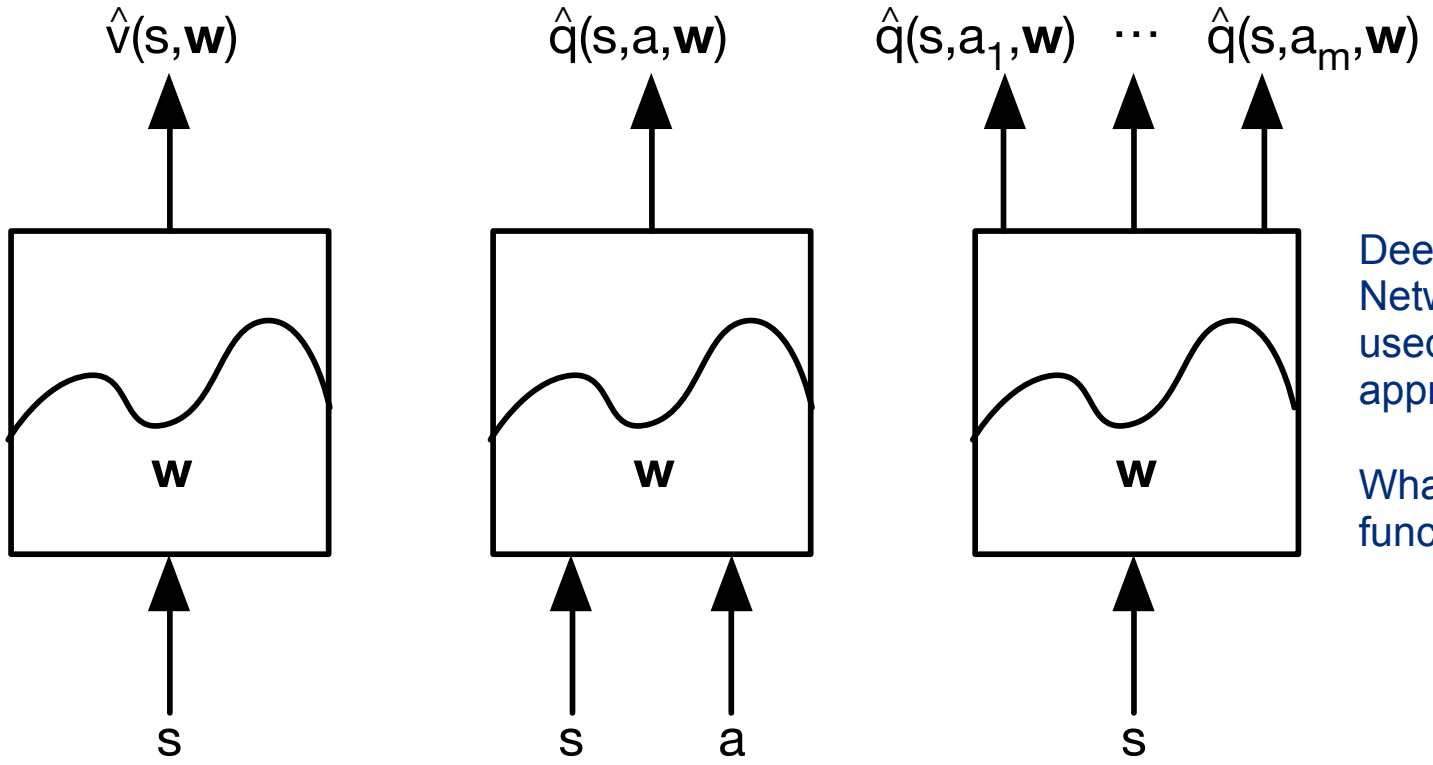
Value Function Approximation

- So far we have represented value function by a *lookup table*
  - Every state s has an entry $V(s)$
  - Or every state-action pair s, a has an entry $Q(s, a)$
- Problem with large MDPs:
  - There are too many states and/or actions to store in memory
  - It is too slow to learn the value of each state individually
- Solution for large MDPs:
  - Estimate value function with *function approximation*

$$\hat{v}(s, \mathbf{w}) \approx v_\pi(s)$$
$$\text{or } \hat{q}(s, a, \mathbf{w}) \approx q_\pi(s, a)$$

  - *Generalise* from seen states to unseen states
  - *Update* parameter $\mathbf{w}$ using gradient descent

Types of Value Function Approximation



Deep neural Networks can be used as function approximators

What about cost function?

Least Squares Prediction

- Given value function approximation $\hat{v}(s, \mathbf{w}) \approx v_\pi(s)$
- And *experience* $\mathcal{D}$ consisting of $\langle state, value \rangle$ pairs

$$\mathcal{D} = \{\langle s_1, v_1^\pi \rangle, \langle s_2, v_2^\pi \rangle, ..., \langle s_T, v_T^\pi \rangle\}$$

- Which parameters $\mathbf{w}$ give the *best fitting* value fn $\hat{v}(s, \mathbf{w})$?
- Least squares algorithms find parameter vector $\mathbf{w}$ minimising sum-squared error between $\hat{v}(s_t, \mathbf{w})$ and target values v_t^π,

$$\begin{aligned} LS(\mathbf{w}) &= \sum_{t=1}^{T} (v_t^\pi - \hat{v}(s_t, \mathbf{w}))^2 \\ &= \mathbb{E}_\mathcal{D} \left[(v^\pi - \hat{v}(s, \mathbf{w}))^2 \right] \end{aligned}$$

Stochastic Gradient Descent with Experience Replay

Given experience consisting of $\langle state, value \rangle$ pairs

$$\mathcal{D} = \{\langle s_1, v_1^\pi \rangle, \langle s_2, v_2^\pi \rangle, ..., \langle s_T, v_T^\pi \rangle\}$$

Repeat:

1. Sample state, value from experience

$$\langle s, v^\pi \rangle \sim \mathcal{D}$$

2. Apply stochastic gradient descent update

$$\Delta \mathbf{w} = \alpha(v^\pi - \hat{v}(s, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(s, \mathbf{w})$$

Stochastic Gradient Descent with Experience Replay

Given experience consisting of $\langle state, value \rangle$ pairs

$$\mathcal{D} = \{\langle s_1, v_1^\pi \rangle, \langle s_2, v_2^\pi \rangle, ..., \langle s_T, v_T^\pi \rangle\}$$

Repeat:

1. Sample state, value from experience

$$\langle s, v^\pi \rangle \sim \mathcal{D}$$

2. Apply stochastic gradient descent update

$$\Delta \mathbf{w} = \alpha(v^\pi - \hat{v}(s, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(s, \mathbf{w})$$

Converges to least squares solution

The question is how to get V^π

$$\mathbf{w}^\pi = \underset{\mathbf{w}}{\text{argmin}}\ LS(\mathbf{w})$$

Action-Value Function Approximation

- Approximate the action-value function

$$\hat{q}(S, A, \mathbf{w}) \approx q_\pi(S, A)$$

- Minimise mean-squared error between approximate action-value fn $\hat{q}(S, A, \mathbf{w})$ and true action-value fn $q_\pi(S, A)$

$$J(\mathbf{w}) = \mathbb{E}_\pi \left[(q_\pi(S, A) - \hat{q}(S, A, \mathbf{w}))^2 \right]$$

- Use stochastic gradient descent to find a local minimum

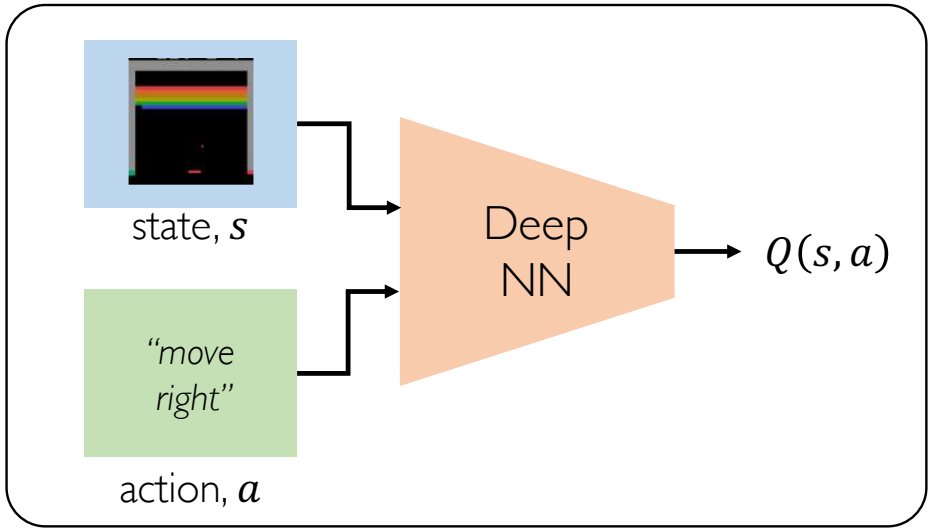
$$-\frac{1}{2}\nabla_{\mathbf{w}} J(\mathbf{w}) = (q_\pi(S, A) - \hat{q}(S, A, \mathbf{w}))\nabla_{\mathbf{w}} \hat{q}(S, A, \mathbf{w})$$

$$\Delta \mathbf{w} = \alpha(q_\pi(S, A) - \hat{q}(S, A, \mathbf{w}))\nabla_{\mathbf{w}} \hat{q}(S, A, \mathbf{w})$$

Similarly, how to calculate $q_\pi(S, A)$?

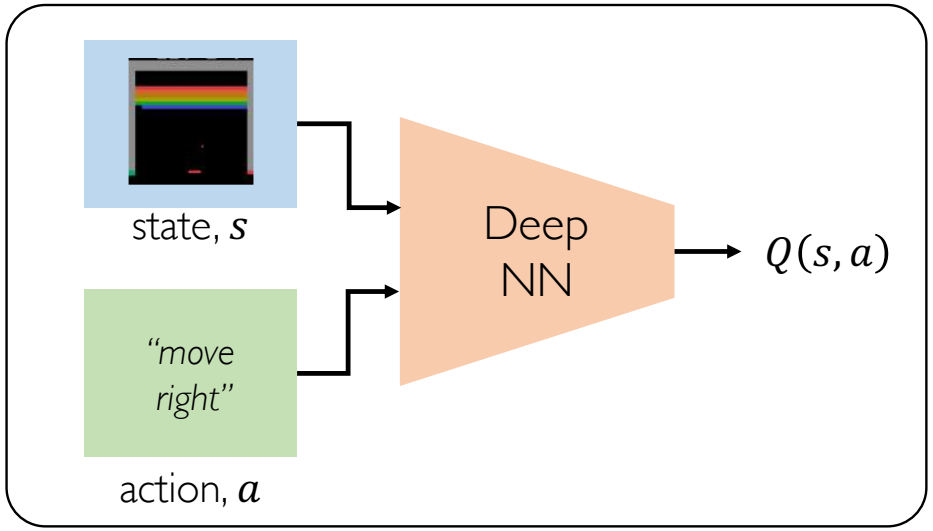
Deep Q Networks (DQN)

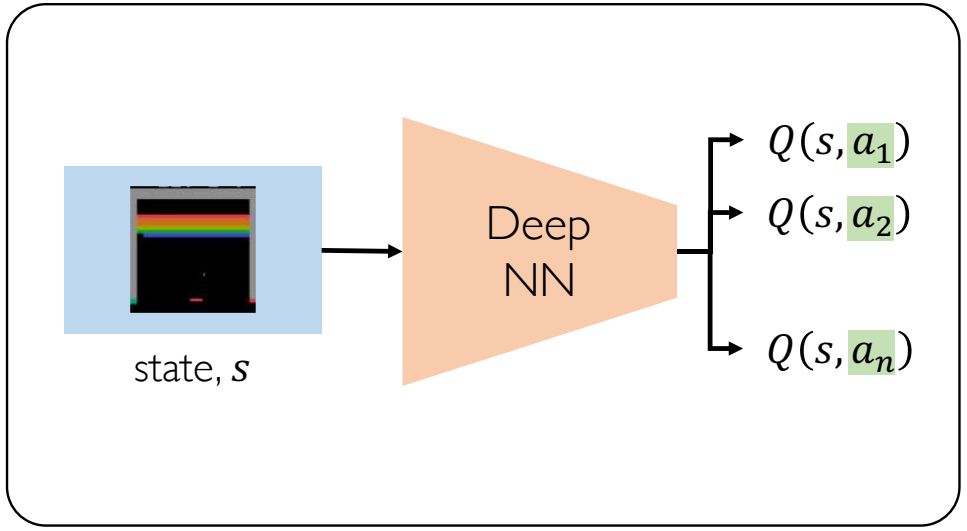
How can we use deep neural networks to model Q-functions?



Deep Q Networks (DQN)

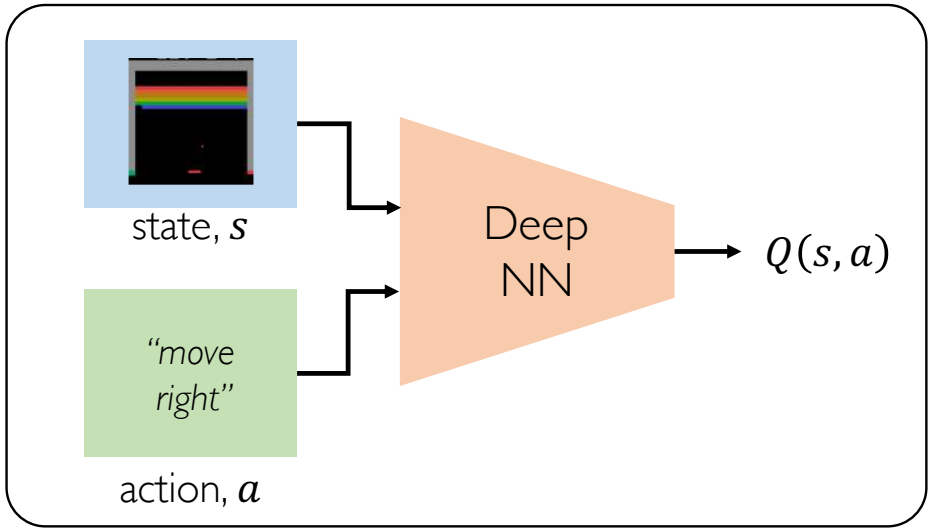
How can we use deep neural networks to model Q-functions?

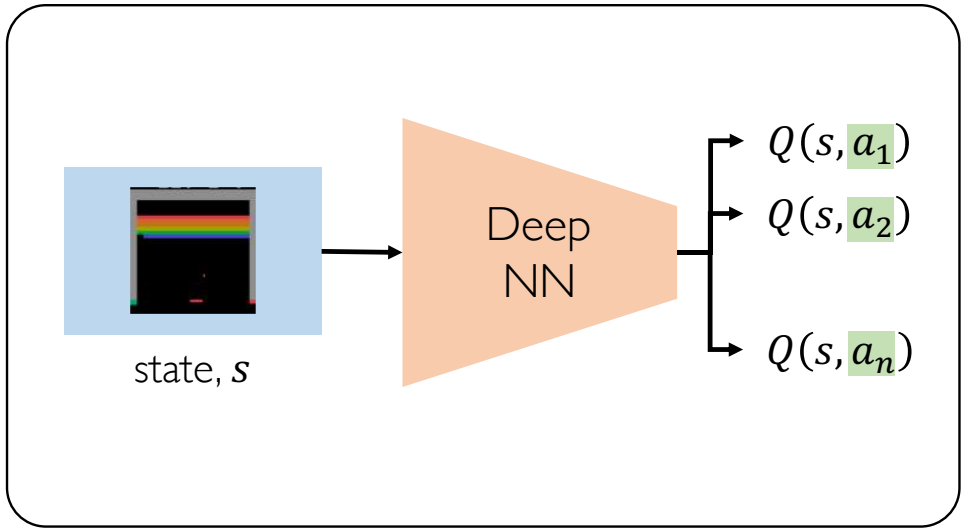




Deep Q Networks (DQN): Training

How can we use deep neural networks to model Q-functions?





Sounds familiar? This is used in Q-Learning

$$\mathcal{L} = \mathbb{E}\left[\left\|\boxed{\left(r + \gamma \max_{a'} Q(s',a')\right)} - Q(s,a)\right\|^2\right]$$

Deep Q Networks (DQN): Training

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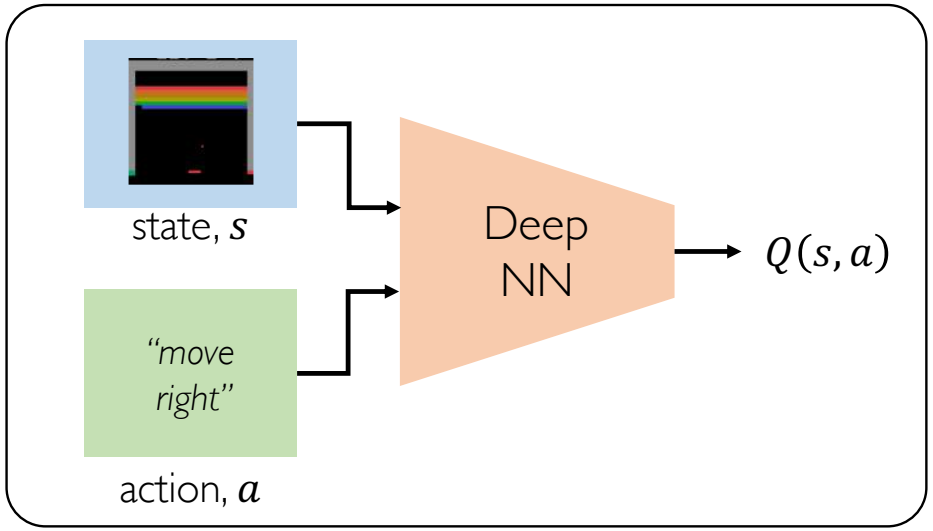
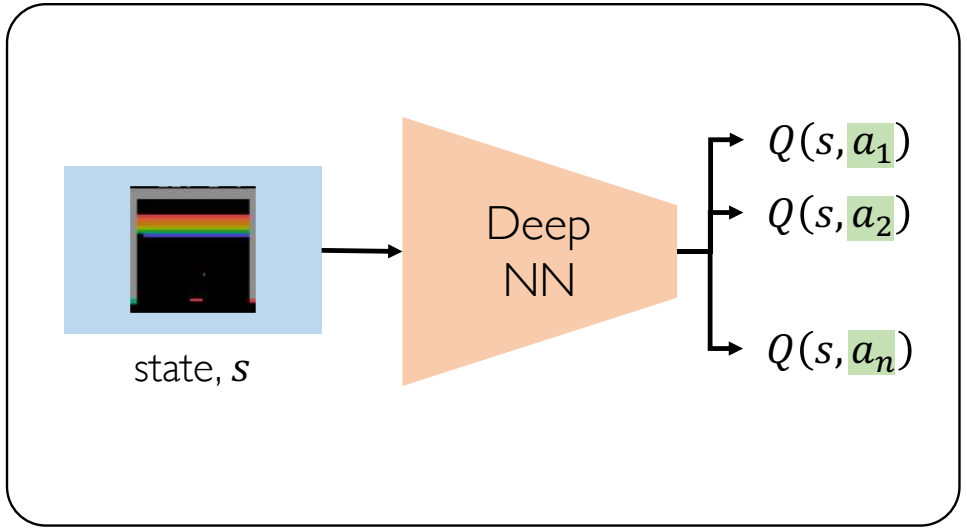
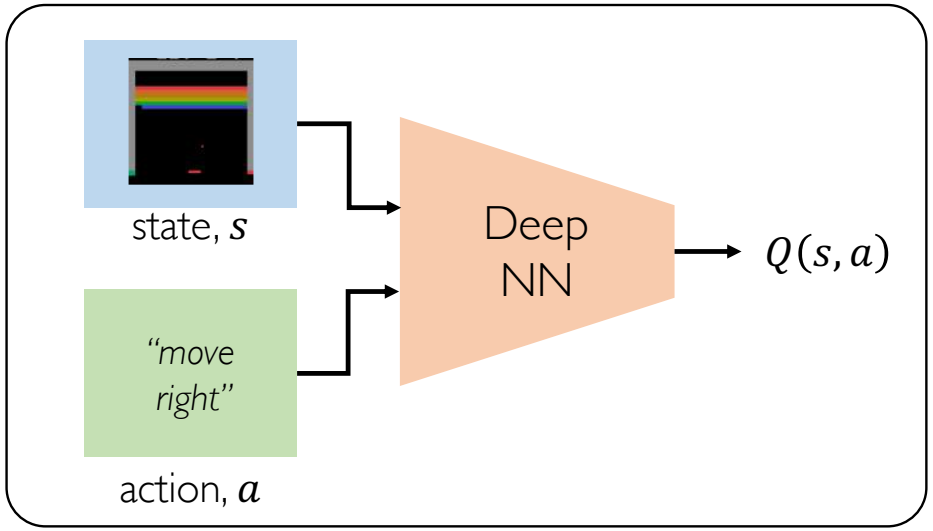


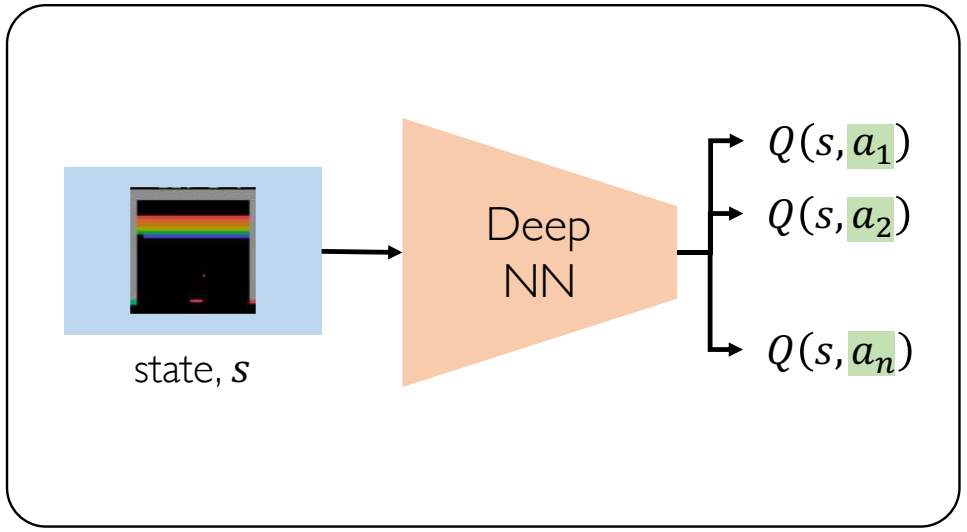


$$\mathcal{L} = \mathbb{E}\left[\left\|\overbrace{\left(r + \gamma \max_{a'} Q(s',a')\right)}^{\text{target}} - \overbrace{Q(s,a)}^{\text{predicted}}\right\|^2\right]$$

Deep Q Networks (DQN): Training

How can we use deep neural networks to model Q-functions?





$$\mathcal{L} = \mathbb{E}\left[\left\|\overbrace{\left(r + \gamma \max_{a'} Q(s',a')\right)}^{\text{target}} - \overbrace{Q(s,a)}^{\text{predicted}}\right\|^2\right]$$

Experience Replay in Deep Q-Networks (DQN)

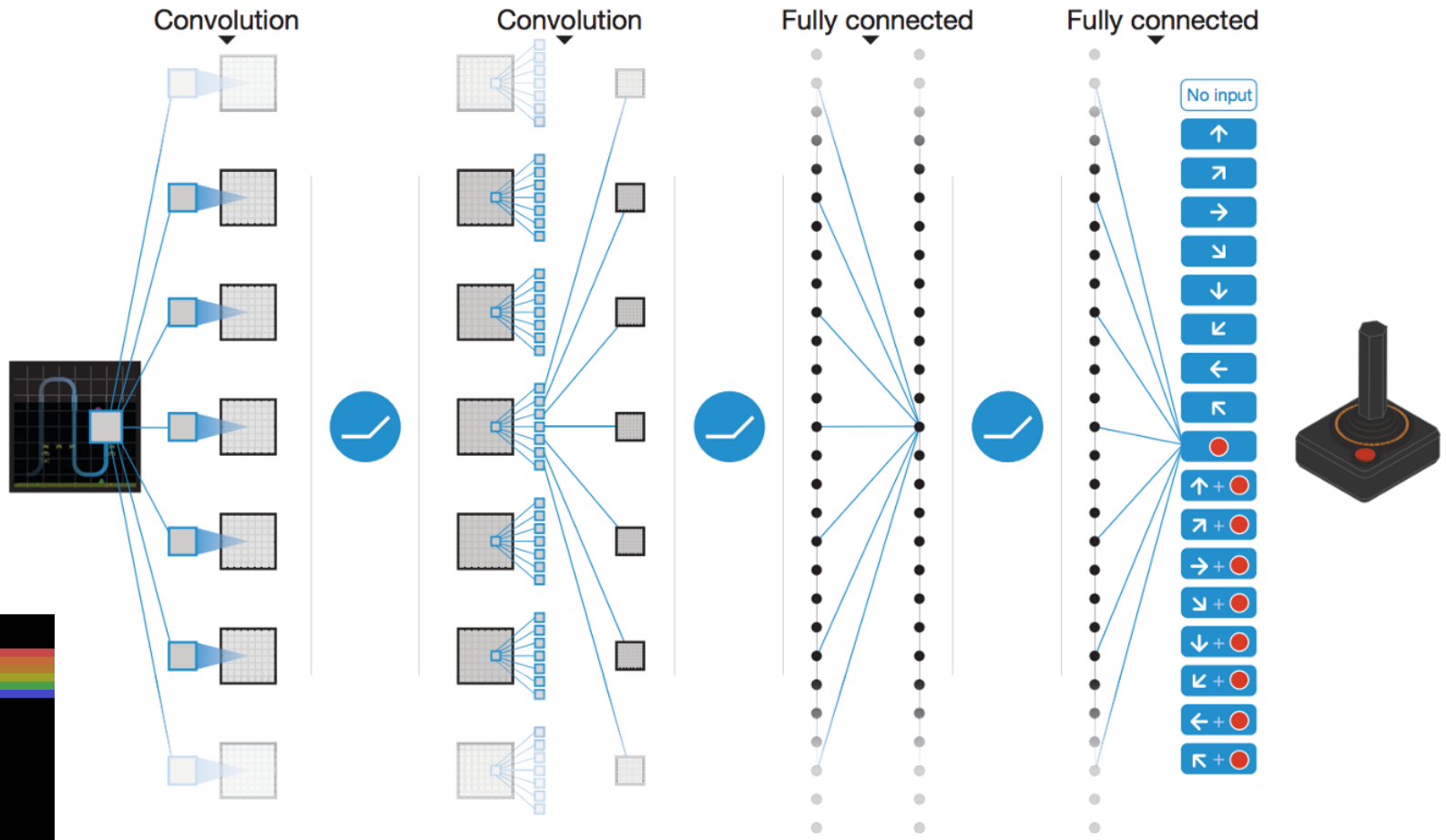
DQN uses experience replay and fixed Q-targets

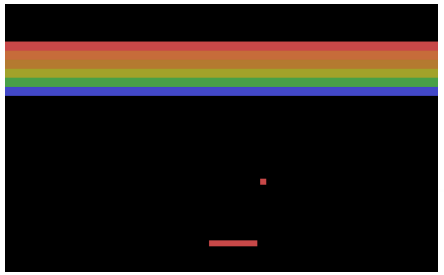
- Take action a_t according to ϵ-greedy policy
- Store transition $(s_t, a_t, r_{t+1}, s_{t+1})$ in replay memory $\mathcal{D}$
- Sample random mini-batch of transitions (s, a, r, s') from $\mathcal{D}$
- Compute Q-learning targets w.r.t. old, fixed parameters w^-
- Optimise MSE between Q-network and Q-learning targets

$$\mathcal{L}_i(w_i) = \mathbb{E}_{s,a,r,s' \sim \mathcal{D}_i} \left[\left(r + \gamma \max_{a'} Q(s', a'; w_i^-) - Q(s, a; w_i) \right)^2 \right]$$

- Using variant of stochastic gradient descent

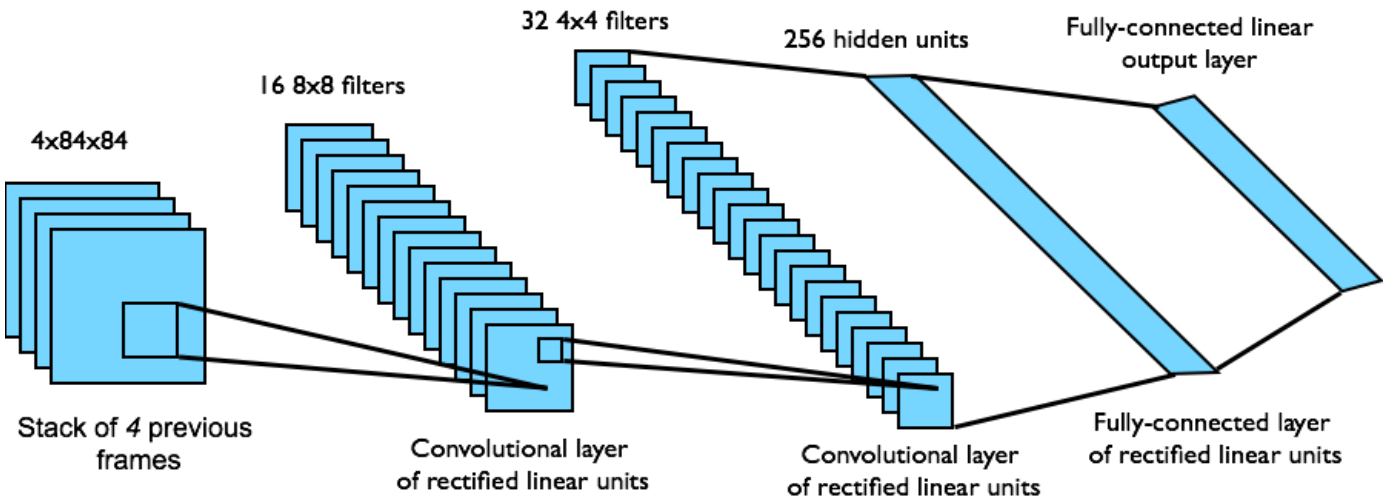
DQN: Atari


Convolution
Convolution
Fully connected
Fully connected
No input



DQN in Atari

- End-to-end learning of values $Q(s, a)$ from pixels s
- Input state s is stack of raw pixels from last 4 frames
- Output is $Q(s, a)$ for 18 joystick/button positions
- Reward is change in score for that step



Network architecture and hyperparameters fixed across all games

DQN Demo

- ConvNetJS Deep Q Learning Demo, https://cs.stanford.edu/people/karpathy/convnetjs/demo/rldemo.html

- Atari:
  - https://www.youtube.com/watch?v=MKtNv1UOaZA
  - https://becominghuman.ai/lets-build-an-atari-ai-part-1-dqn-df57e8ff3b26

- WaterWorld DQN (http://www.cs.toronto.edu/~lcharlin/courses/80-629/reinforcejs/waterworld.html)

Downsides of Q-learning

Complexity:

- Can model scenarios where the action space is discrete and small
- Cannot handle continuous action spaces

IMPORTANT:
Imagine you want to predict steering wheel angle of a car!

Flexibility:

- Cannot learn stochastic policies since policy is deterministically computed from the Q function

To overcome, consider a new class of RL training algorithms: Policy gradient methods

Deep Reinforcement Learning Algorithms

Value Learning

Find $Q(s, a)$

$$a = \underset{a}{\operatorname{argmax}}\, Q(s, a)$$

Policy Learning

Find $\pi(s)$

Sample $a \sim \pi(s)$

Advantages of Policy-Based RL

Advantages:

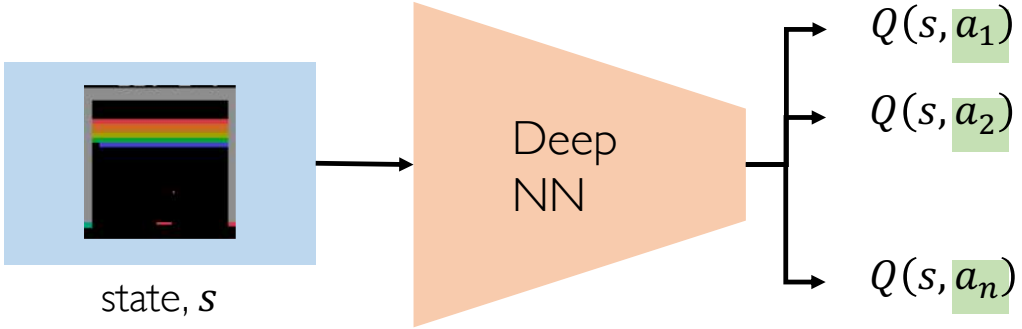
- Better convergence properties
- Effective in high-dimensional or continuous action spaces
- Can learn stochastic policies

Disadvantages:

- Typically converge to a local rather than global optimum
- Evaluating a policy is typically inefficient and high variance

Policy Gradient (PG) : Key Idea

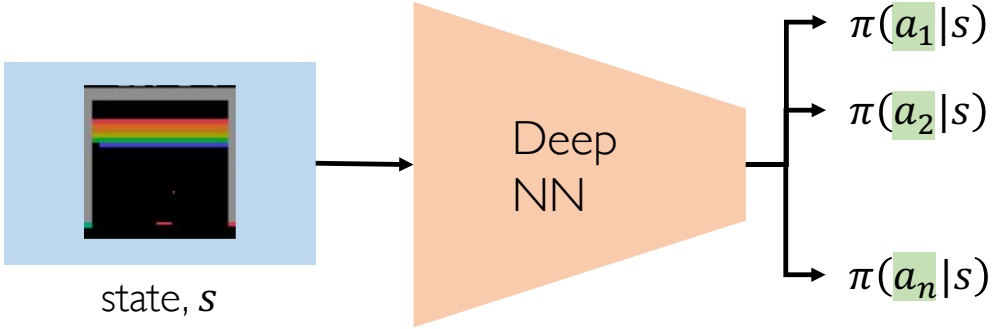
DQN (before): Approximating Q and inferring the optimal policy,



Policy Gradient (PG): Key Idea

DQN (before): Approximating Q and inferring the optimal policy,

Policy Gradient: Directly optimize the policy!



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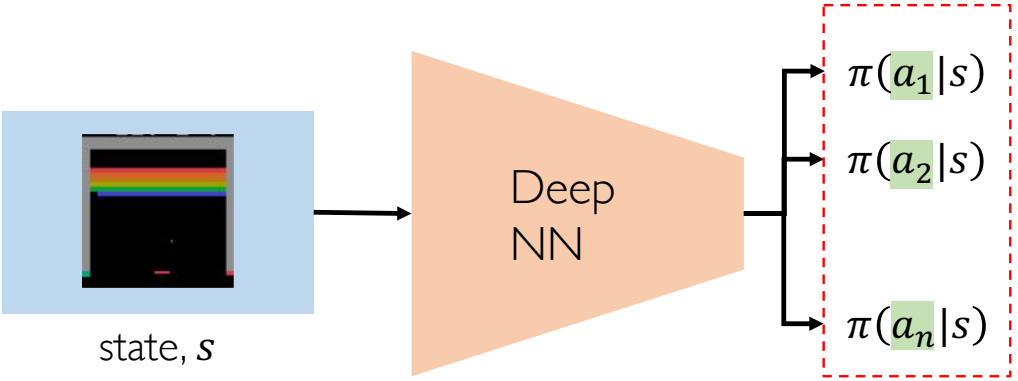

$$\sum_{a_i \in A} \pi(a_i|s) = 1$$

Policy Gradient (PG): Key Idea

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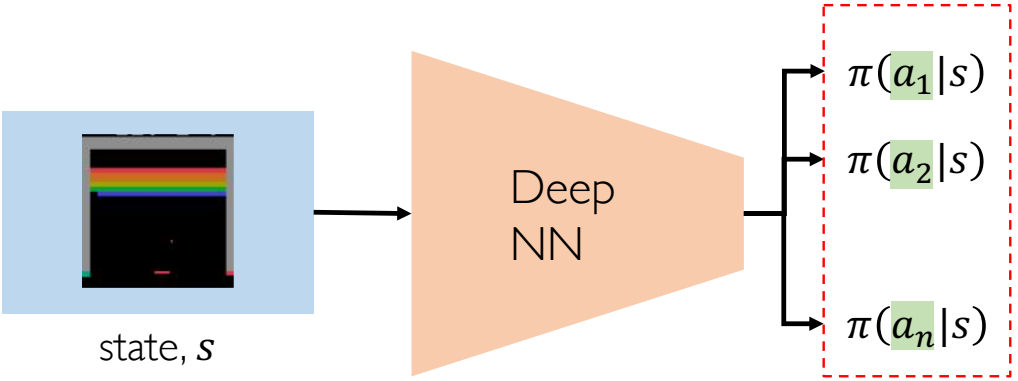

$$\sum_{a_i \in A} \pi(a_i|s) = 1$$

$$\pi(a|s) = P(action|state)$$

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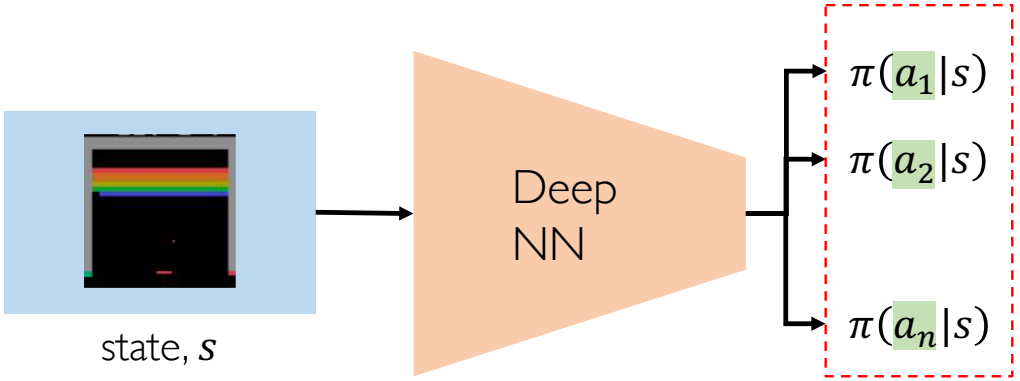


$$\sum_{a_i \in A} \pi(a_i|s) = 1$$

$$\pi(a|s) = P(action|state)$$

But how to compute gradients?

Policy Objective Functions

- Goal: given policy $\pi_\theta(s, a)$ with parameters θ, find best θ
- But how do we measure the quality of a policy π_θ?
- In episodic environments we can use the start value

$$J_1(\theta) = V^{\pi_\theta}(s_1) = \mathbb{E}_{\pi_\theta}[v_1]$$

- In continuing environments we can use the average value

$$J_{avV}(\theta) = \sum_s d^{\pi_\theta}(s) V^{\pi_\theta}(s)$$

- Or the average reward per time-step

$$J_{avR}(\theta) = \sum_s d^{\pi_\theta}(s) \sum_a \pi_\theta(s, a) \mathcal{R}_s^a$$

- where $d^{\pi_\theta}(s)$ is stationary distribution of Markov chain for π_θ

Score Function

Optimization Objective: Find θ that maximizes J(θ)

- We now compute the policy gradient *analytically*
- Assume policy π_θ is differentiable whenever it is non-zero
- and we know the gradient $\nabla_\theta \pi_\theta(s, a)$
- Likelihood ratios exploit the following identity

$$\begin{aligned}\nabla_\theta \pi_\theta(s, a) &= \pi_\theta(s, a) \frac{\nabla_\theta \pi_\theta(s, a)}{\pi_\theta(s, a)} \\ &= \pi_\theta(s, a) \nabla_\theta \log \pi_\theta(s, a)\end{aligned}$$

- The score function is $\nabla_\theta \log \pi_\theta(s, a)$

One-Step MDPs

- Consider a simple class of one-step MDPs
  - Starting in state $s \sim d(s)$
  - Terminating after one time-step with reward $r = \mathcal{R}_{s,a}$
- Use likelihood ratios to compute the policy gradient

$$
\begin{aligned}
J(\theta) &= \mathbb{E}_{\pi_\theta}[r] \\
&= \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \pi_\theta(s,a) \mathcal{R}_{s,a} \\
\nabla_\theta J(\theta) &= \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \underline{\pi_\theta(s,a) \nabla_\theta \log \pi_\theta(s,a) \mathcal{R}_{s,a}} \\
&= \mathbb{E}_{\pi_\theta}[\nabla_\theta \log \pi_\theta(s,a) r]
\end{aligned}
$$

Remember, $\pi_\theta(s, a)$ is a function of s

The gradient does not contain d(s) or transition probabilities any more

Policy Gradient Theorem

- The policy gradient theorem generalises the likelihood ratio approach to multi-step MDPs
- Replaces instantaneous reward r with long-term value $Q^{\pi}(s, a)$
- Policy gradient theorem applies to start state objective, average reward and average value objective

Theorem

For any differentiable policy $\pi_\theta(s, a)$,
for any of the policy objective functions $J = J_1, J_{avR}$, or $\frac{1}{1-\gamma} J_{avV}$,
the policy gradient is

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s, a)\ Q^{\pi_\theta}(s, a)\right]$$

Now we only need to figure out how to approximate Q values under a policy

Monte-Carlo Policy Gradient (REINFORCE)

- Update parameters by stochastic gradient ascent
- Using policy gradient theorem
- For an episode $[1, 2, \ldots, T]$, at each step t, using return:
  $G_t = \gamma^t R_t + \gamma^{t+1} R_{t+1} + \ldots + \gamma^T R_T$
  as an unbiased sample of $Q^{\pi_\theta}(s_t, a_t)$

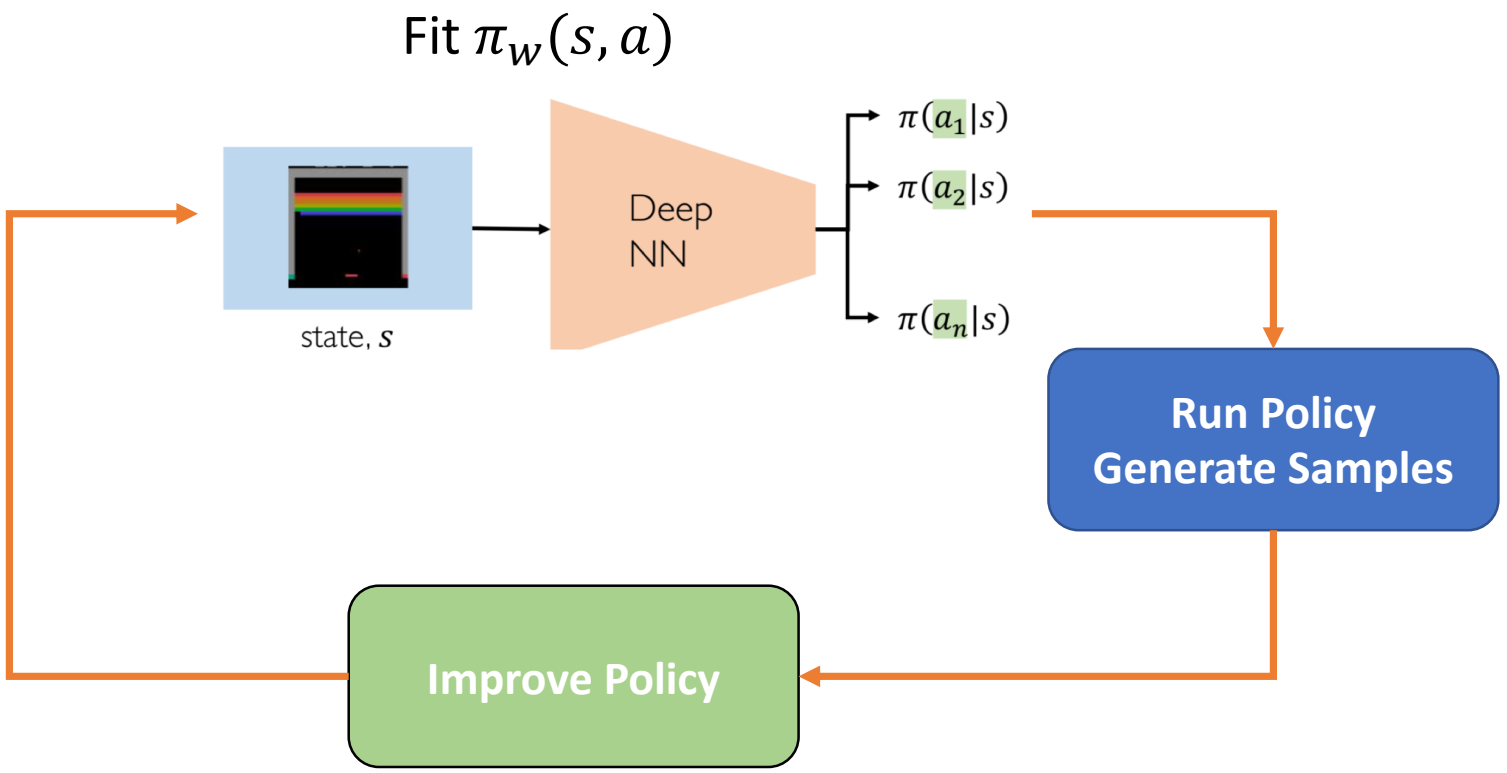
REINFORCE: Monte-Carlo Policy-Gradient Control (episodic) for π_*

Input: a differentiable policy parameterization $\pi(a|s, \boldsymbol{\theta})$
Algorithm parameter: step size $\alpha > 0$
Initialize policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ (e.g., to $\mathbf{0}$)

Loop forever (for each episode):
  Generate an episode $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$, following $\pi(\cdot|\cdot, \boldsymbol{\theta})$
  Loop for each step of the episode $t = 0, 1, \ldots, T-1$:
    $G \leftarrow \sum_{k=t+1}^{T} \gamma^{k-t-1} R_k$ $\qquad (G_t)$
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha \gamma^t G \nabla \ln \pi(A_t|S_t, \boldsymbol{\theta})$

- If G_t is high, push up the probabilities of the actions seen
- If G_t is low, push down the probabilities of the actions seen

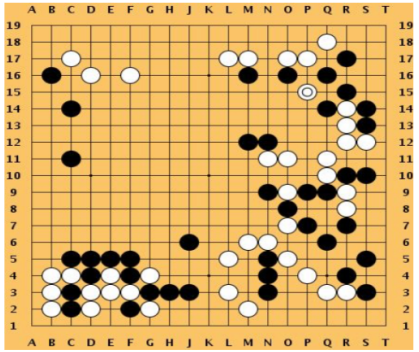
Monte-Carlo Policy Gradient (REINFORCE) Architecture



$$\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta)$$

Reducing Variance in REINFORCE

- Rewards received during trajectories may have high variance
- Different estimates of $Q^{\pi_\theta}(s_t, a_t)$ have been used, i.e., R_t, G_t, …
- Sometimes, a move without receiving a reward (or even receiving a negative reward) is not necessarily a bad move
- How to reduce variance?
  - ***Critic***: model $Q^{\pi_\theta}(s, s)$ using a function $Q^w\ (s, a)$
  - ***Advantage***: Check whether a reward is better or worse than what is expected
    - Introduce a baseline $b(s) = v^{\pi_\theta}(s)$, the expected value of s
    - Again, model $v^{\pi_\theta}(s)$ using a function $V^w(s)$
    - Advantage $A^w(s, a) = Q^w\ (s, a) - V^w(s)$



$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s, a)\, G_t\right] \qquad \text{REINFORCE}$$

$$= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s, a)\ Q^w(s, a)\right] \qquad \text{Q Actor-Critic}$$

$$= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s, a)\ A^w(s, a)\right] \qquad \text{Advantage Actor-Critic}$$

Reducing Variance Using a Critic

- Monte-Carlo policy gradient still has high variance
- We use a critic to estimate the action-value function,

$$Q_w(s, a) \approx Q^{\pi_\theta}(s, a)$$

- Actor-critic algorithms maintain *two* sets of parameters

  Critic Updates action-value function parameters w ***similar to Deep Q-Learning!***

  Actor Updates policy parameters θ, in direction suggested by critic ***similar to REINFORCE!***

- Actor-critic algorithms follow an *approximate* policy gradient

$$\nabla_\theta J(\theta) \approx \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s, a)\ Q_w(s, a)\right]$$
$$\Delta\theta = \alpha \nabla_\theta \log \pi_\theta(s, a)\ Q_w(s, a)$$

Actor-Critic Algorithm

- Critic: Estimate the value function
  - Action-value (the Q value) or state-value (the V value)
  - Updates parameter w by minimizing the temporal difference (TD) between action value $Q_w(s,a)$ and $[r_t + \gamma Q_w(s',a')]$ (updated value after a')
- Actor: Updates θ by policy gradient

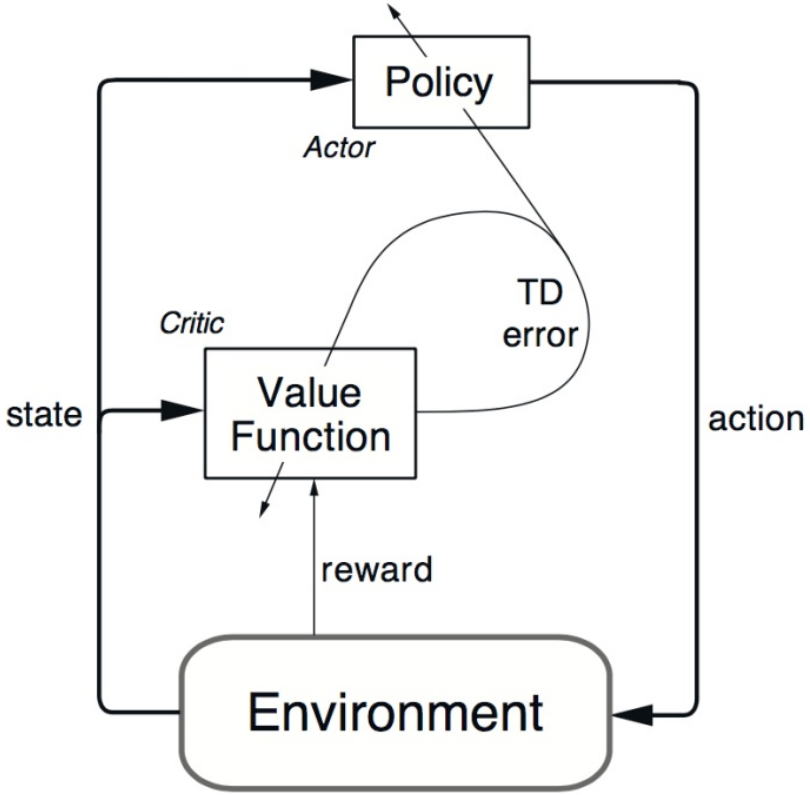
Algorithm 1 Q Actor Critic

Initialize parameters s, θ, w and learning rates α_θ, α_w; sample $a \sim \pi_\theta(a|s)$.
for $t = 1 \ldots T$: **do**
  Sample reward $r_t \sim R(s,a)$ and next state $s' \sim P(s'|s,a)$
  Then sample the next action $a' \sim \pi_\theta(a'|s')$
  Update the policy parameters: $\theta \leftarrow \theta + \alpha_\theta Q_w(s,a) \nabla_\theta \log \pi_\theta(a|s)$; Compute the correction (TD error) for action-value at time t:
  $\delta_t = r_t + \gamma Q_w(s',a') - Q_w(s,a)$
  and use it to update the parameters of Q function:
  $w \leftarrow w + \alpha_w \delta_t \nabla_w Q_w(s,a)$
  Move to a $\leftarrow a'$ and s $\leftarrow s'$
end for

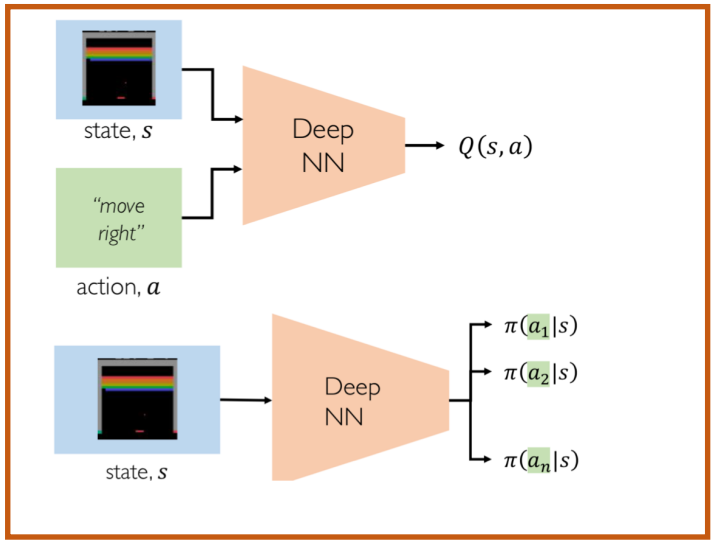
REINFORCE

Similar to Deep Q-Learning

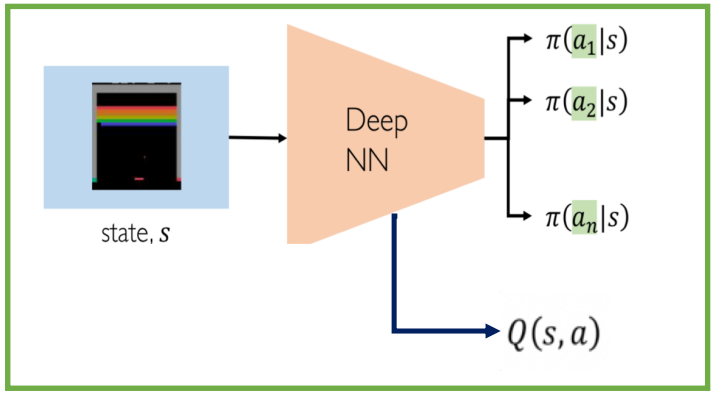
Actor-Critic Architecture Design



Separate Networks for Policy and Value



Shared Network Design



6. Reinforcement Learning Applications

AlphaGo Beats Top Human Player at Go (2016)

Silver et al., *Nature* 2016.

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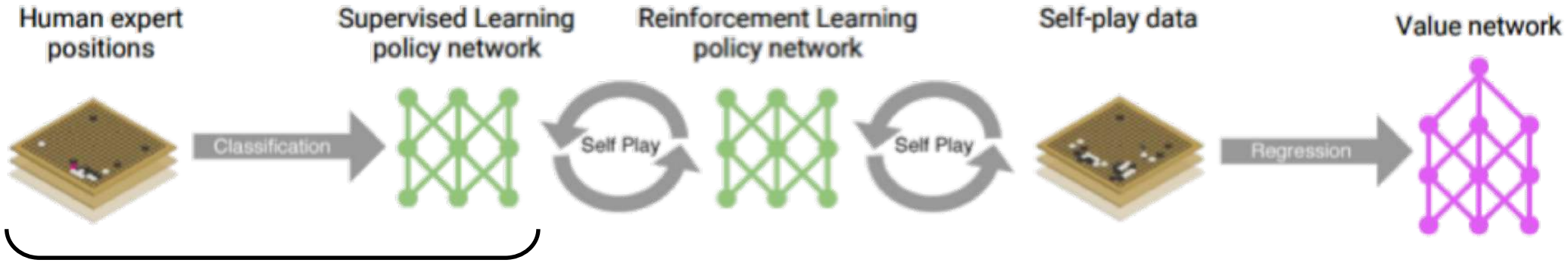

1) Initial training: human data

Silver et al., *Nature* 2016.

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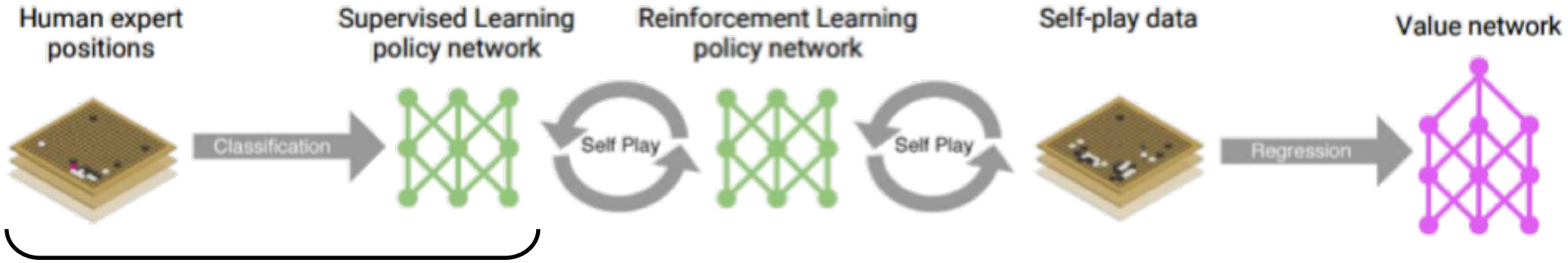

1) Initial training: human data

2) Self-play and reinforcement learning
→ super-human performance

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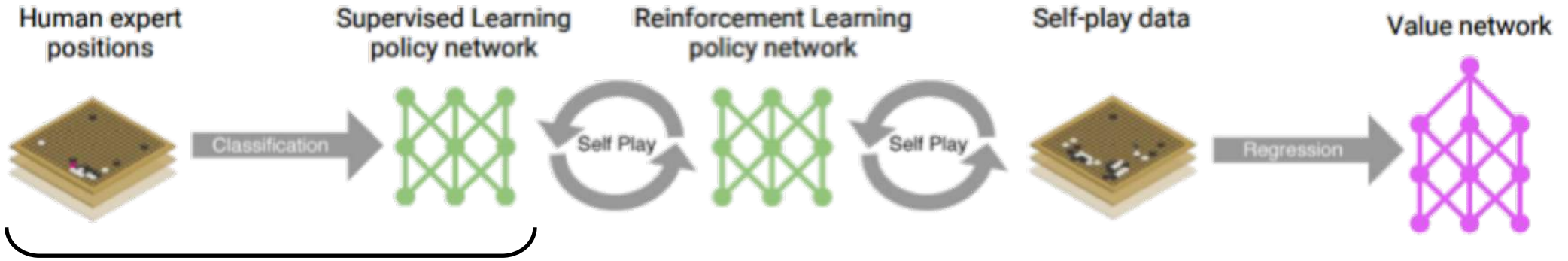


1) Initial training: human data

2) Self-play and reinforcement learning
→ super-human performance

3) "Intuition" about board state

Silver et al., *Nature* 2016.

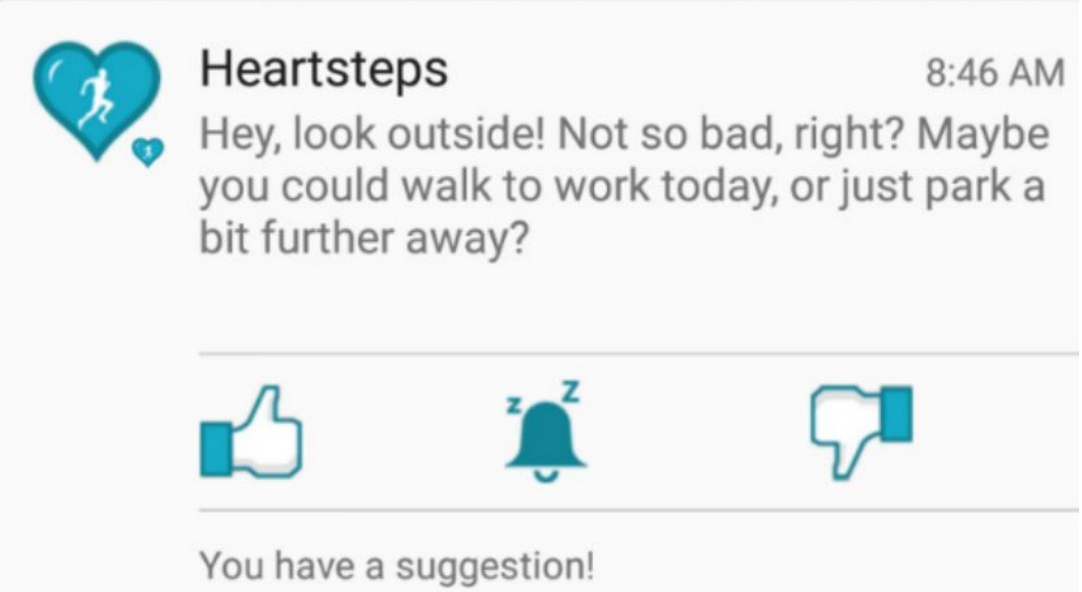
AlphaGo Beats Top Human Player at Go (2016)

Silver et al., *Nature* 2016.

Application Areas

Mobile Health Interventions

Heartsteps



Experimental Design & Machine Learning Opportunities in Mobile Health: Susan Murphy

Application Areas

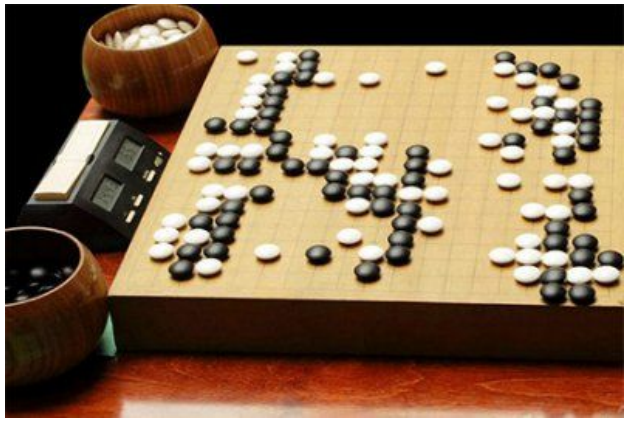
Robotics:

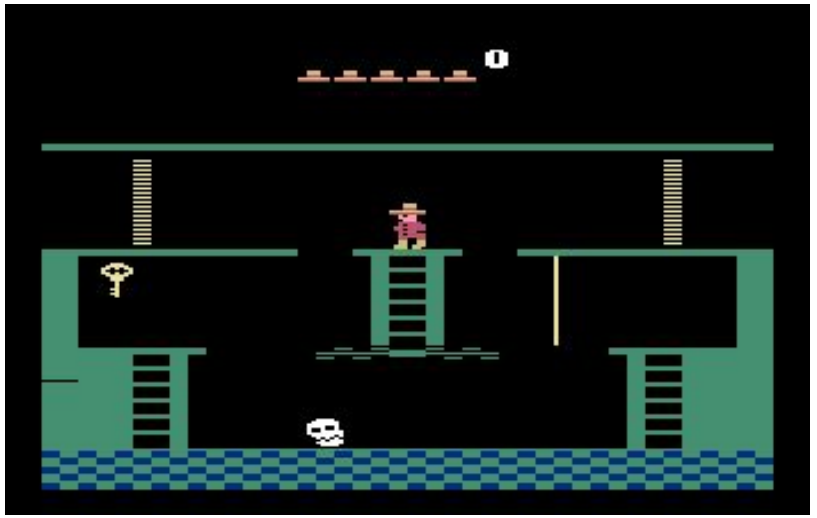
learning behaviours

RL Application Areas

Games

- Standardised testbeds
- Long decision horizons





Application Areas

Automated Trading

1:

2: ???

3:

Questions?

Principle of Optimality

Any optimal policy can be subdivided into two components:

- An optimal first action A_*
- Followed by an optimal policy from successor state S'

Theorem (Principle of Optimality)

A policy $\pi(a|s)$ achieves the optimal value from state s, $v_\pi(s) = v_*(s)$, if and only if

- *For any state s' reachable from s*
- *π achieves the optimal value from state s', $v_\pi(s') = v_*(s')$*

Bellman Optimality Equation

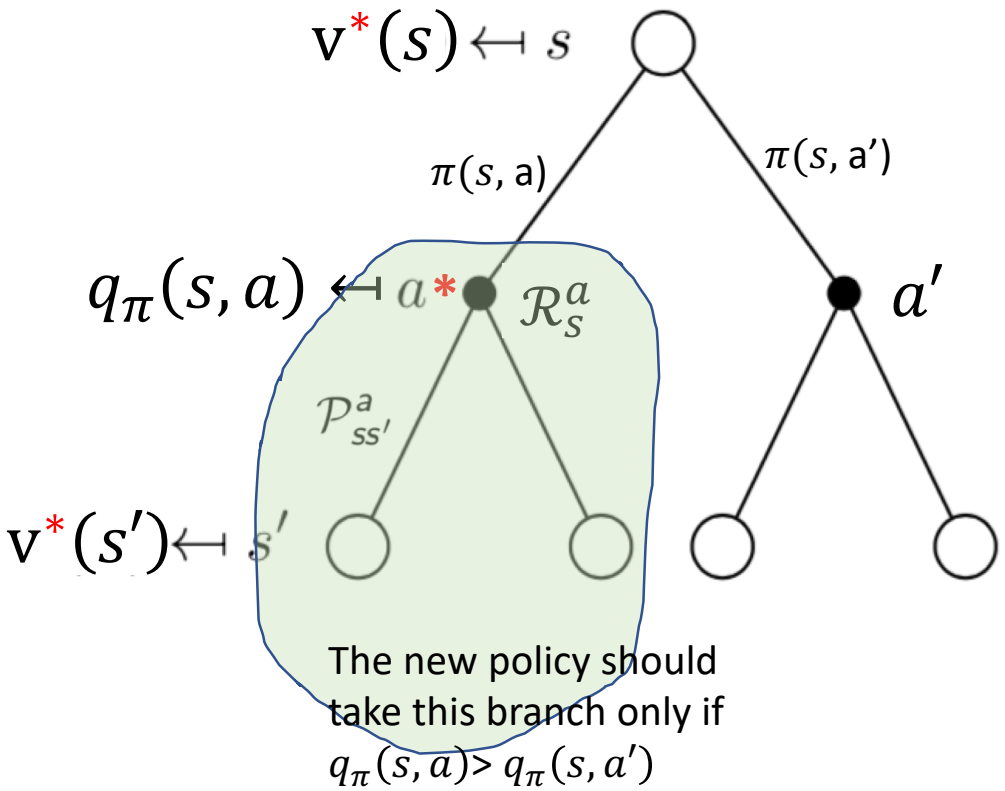
$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s') \right)$$

$$\Downarrow$$

$$\mathrm{v}^*(s) = \max_a \left(\mathcal{R}_s^a + \gamma \sum_{s' \in S} \mathcal{P}_{ss'}^a \, \mathrm{v}^*(s') \right)$$

Take the first optimal action a^* (i.e. $\max_a q_\pi(s, a)$)

followed by an optimal policy from successor state s'

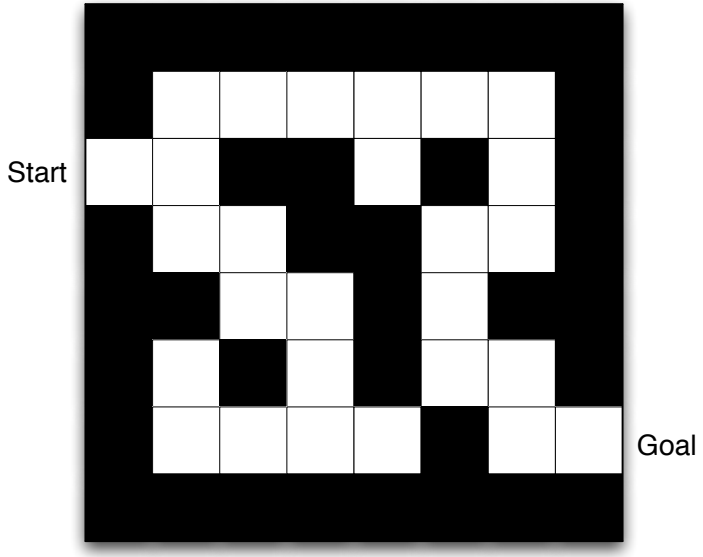


The Goal of Reinforcement Learning

How to learn an optimal policy based on Bellman Optimality Equation?

$$\mathrm{v}^*(s) = \max_a \left(\mathcal{R}_s^a + \gamma \sum_{s' \in S} \mathcal{P}_{ss'}^a \, \mathrm{v}^*(s') \right)$$

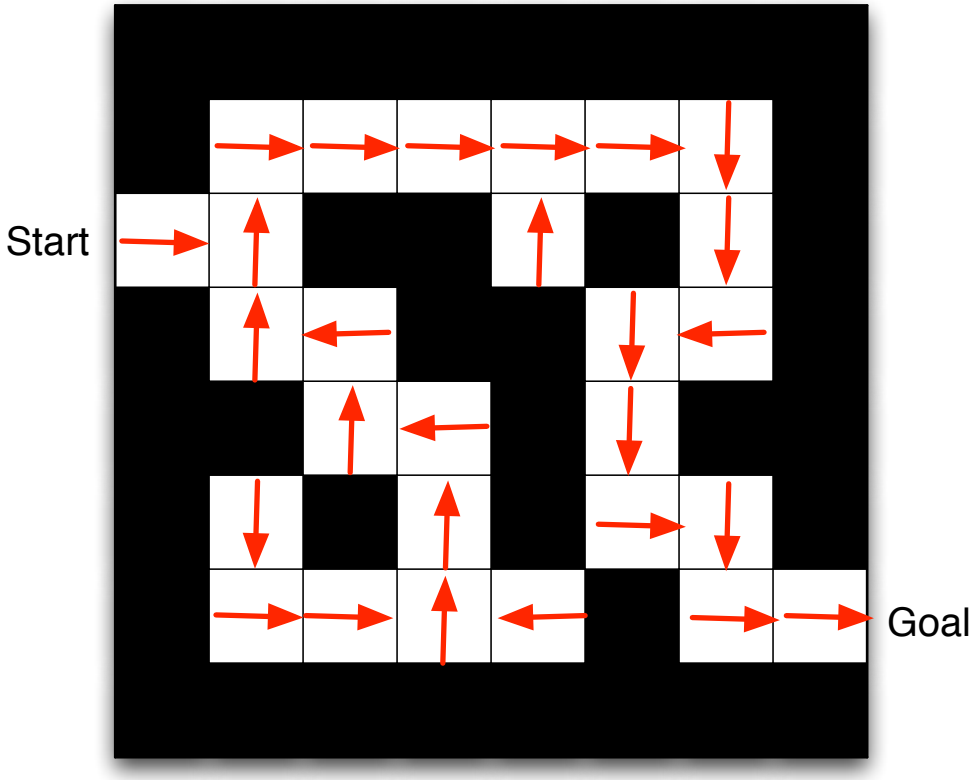
Maze Example



- Rewards: -1 per time-step
- Actions: N, E, S, W
- States: Agent's location

Can you find the optimal policy π(a|s)?

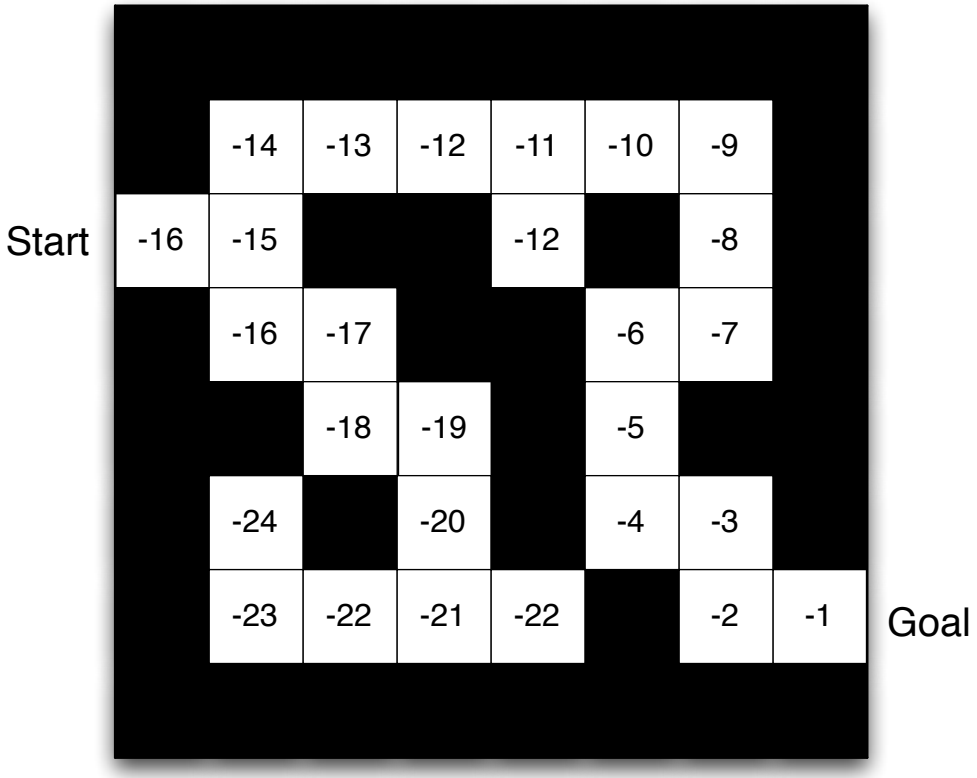
Maze Example: Policy



Can you calculate the action values under this deterministic policy?

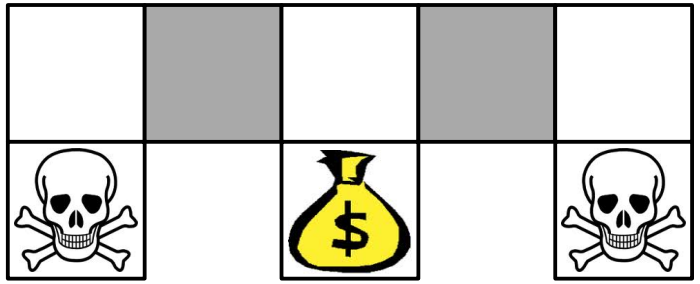
- Arrows represent policy $\pi(s)$ for each state s

Maze Example: Value Function



- Numbers represent value $v_\pi(s)$ of each state s

Example: Aliased Gridworld (1)



- The agent cannot differentiate the grey states
- Consider features of the following form (for all N, E, S, W)

$$\phi(s, a) = \mathbf{1}(\text{wall to N}, a = \text{move E})$$

i.e. Whether there is a walls to N and it can move to E

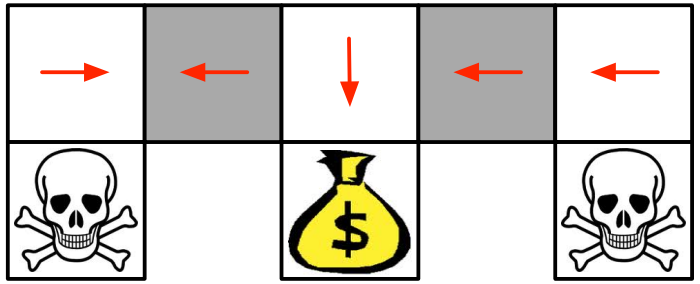
- Compare value-based RL, using an approximate value function

$$Q_\theta(s, a) = f(\phi(s, a), \theta)$$

- To policy-based RL, using a parametrised policy

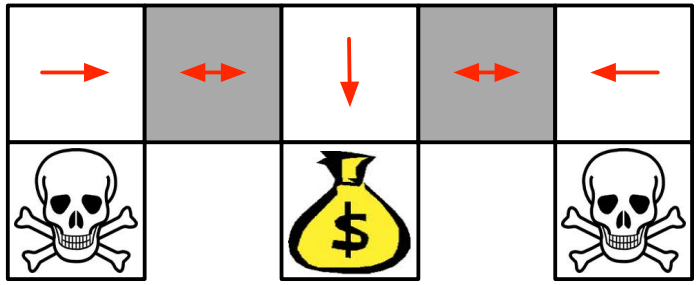
$$\pi_\theta(s, a) = g(\phi(s, a), \theta)$$

Example: Aliased Gridworld (2)



- Under aliasing, an optimal deterministic policy will either
  - move W in both grey states (shown by red arrows)
  - move E in both grey states
- Either way, it can get stuck and *never* reach the money
- Value-based RL learns a near-deterministic policy
  - e.g. greedy or ϵ-greedy
- So it will traverse the corridor for a long time

Example: Aliased Gridworld (3)



- An optimal stochastic policy will randomly move E or W in grey states

$$\pi_\theta(\text{wall to N and S, move E}) = 0.5$$
$$\pi_\theta(\text{wall to N and S, move W}) = 0.5$$

- It will reach the goal state in a few steps with high probability
- Policy-based RL can learn the optimal stochastic policy