CS-541: Artificial Intelligence
Lecture 6

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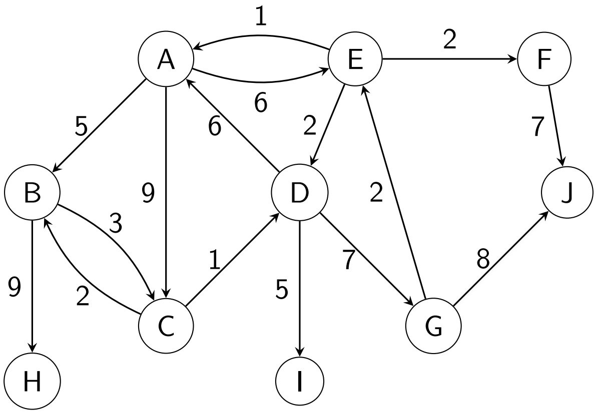
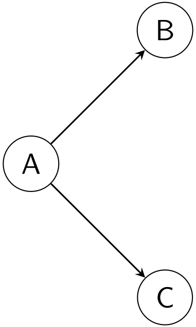
Deterministic



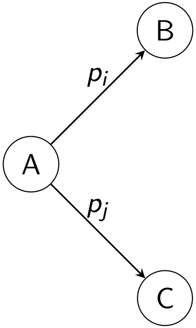
Deterministic Actions

Actions from a give state are deterministic
$Succ(s, a)$ is always the same state s'

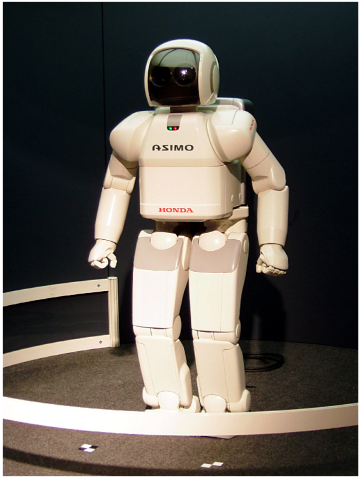


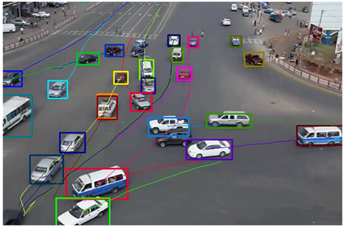
Stochastic Actions

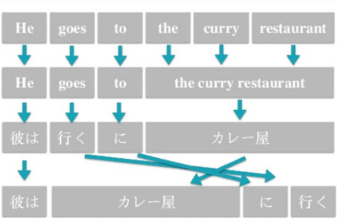
Actions from a give state are probabilistic (stochastic)
$Succ(s, a, t)$ denotes the next state given the current state s and action a taken at the time t_i
It can either be state B with a probability p_i or state C with probability p_j



Markov Decision Process







Markov Decision Process

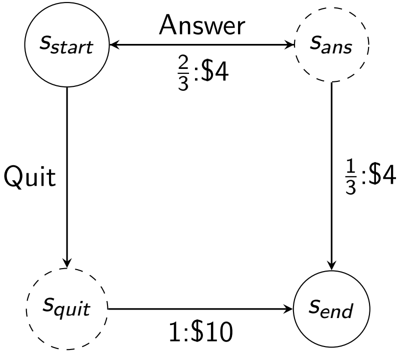
Game:

The player starts with \$0 as the prize money.

In each round, the player can take two steps:

- Quit and take \$10
- Answer a question
    - Correctly answer with a probability of $\frac{2}{3}$, get \$4 prize and move to the next round
    - Otherwise get \$4 prize and end the game

Markov Decision Process



Markov Decision Process

Gameshow:

The player starts with 0 as the prize money.

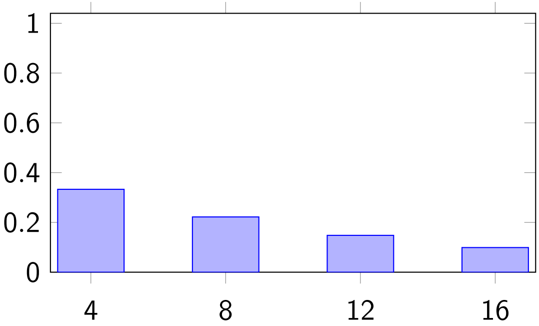
In each round, the player can take two steps:

- Quit and take \$10
- Answer a question
  - Correctly answer with a probability of $\frac{2}{3}$ and move to the next round
  - Otherwise take \$4 and end the game

What is the best strategy for the game?

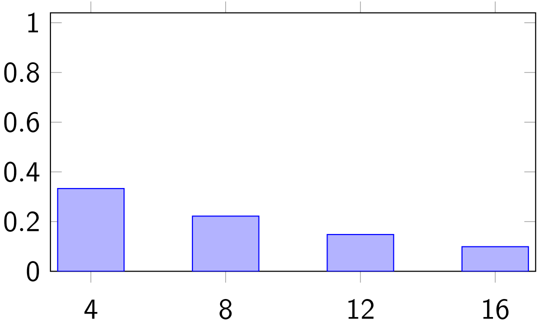
Markov Decision Process

If our policy is to 'answer':



Markov Decision Process

If our policy is to 'answer':

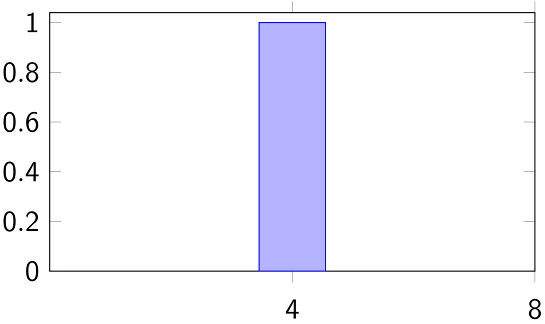


Expected Utility:

$$\frac{1}{3}(4) + \frac{2}{3} \cdot \frac{1}{3}(8) + \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3}(12) + \cdots = 12$$

Markov Decision Process

If our policy is to 'quit':



Expected Utility:

$$1(10) = 10$$

Markov Decision Process

s_{start}: start state
Actions(s**)**: all possible actions from state s
Succ(s, a**)**: next possible states given action a is taken from state s
Cost(s, a**)**: cost of transition from state s by taking action a
IsEnd(s**)**: is s a goal state

Search problem

s_{start}: start state
Actions(s): all possible actions from state s
T(s, a, s'): probability of s' if action a is taken from state s
Reward(s, a, s'): reward from the transition s to s'
IsEnd(s): is s a goal state
$0 \leq \gamma \leq 1$: discount factor (default: 1)

Markov Decision Process

Total transition probability: $\sum_{s'} T(s, a, s') = 1$
Discount factor γ is based on how much we value the future reward

Markov Decision Process

$\text{Succ}(s, a) \rightarrow \text{T}(s, a, s')$
$\text{Succ}(s, a)$ can be considered as a special case of transition probability

$$T(s, a, s') = \begin{cases} 1 \text{ if } s' = Succ(s, a) \\ 0 \text{ otherwise} \end{cases}$$

Markov Decision Process

$\text{Cost}(s, a) \rightarrow \text{Reward}(s, a, s')$
Instead of minimizing the cost, we maximize the reward
Negating one is equivalent to the other

Markov Decision Process

$\mathbf{T(}s, a, s'\mathbf{)}$: probability of s' if action a is taken from state s

s	a	s'	$T(s, a, s')$
s_{start}	$Quit$	s_{end}	1
s_{start}	$Question$	s_{end}	$2/3$
s_{start}	$Question$	s_{start}	$1/3$

Markov Decision Process

$\mathbf{T(}s, a, s'\mathbf{)}$: probability of s' if action a is taken from state s

s	a	s'	$\mathrm{T}(s, a, s')$
s_{start}	$Quit$	s_{end}	1
s_{start}	$Question$	s_{end}	$^2/_3$
s_{start}	$Question$	s_{start}	$^1/_3$

To re-iterate:
Sum of probabilities from a given state s by making an action a is 1

$$\sum_{s' \in states} T(s, a, s') = 1$$

Successors: states s' where $T(s, a, s') > 0$

Markov Decision Process

$\mathbf{T}(s, a, s')$: probability of s' if action a is taken from state s

s	a	s'	$\mathrm{T}(s, a, s')$
s_{start}	$Quit$	s_{end}	1
s_{start}	$Question$	s_{end}	$^2/_3$
s_{start}	$Question$	s_{start}	$^1/_3$

Sum of probabilities from a given state s by making an action a is 1

Policy

Policy: gives an action a for a given $\pi : s \to s$

For deterministic search problems, we wanted the optimal sequence of actions from start to goal
For MDP, we want the optimal policy $\pi^* : s \to a$ which maximizes the reward Reward(s, a, s')

Grid World!

Our world is 3×4 grid

Start state is at (0,0)

Reward $+1$ at (4,3)

Reward -1 at (4,2)

3				$+1$
2				-1
1	start			
	1	2	3	4

Grid World!

For any state, three possible moves

- up: 0.8
- left: 0.1
- right: 0.1

3				$+1$
2		■		-1
1	start			
	1	2	3	4

Grid World!

3	$\rightarrow$	$\rightarrow$	$\rightarrow$	$+1$
2	$\uparrow$		$\uparrow$	-1
1	$\uparrow$	$\leftarrow$	$\leftarrow\uparrow$	$\leftarrow$
	1	2	3	4

Optimal policy for $\gamma < -0.04$
There are two optimal policies for state (3,1)

Discount

Additive discount utility

Let say the path is $s_0, a_1 r_1 s_1, a_2 r_2 s_2, \cdot$ (sequence of state, action, and reward)

The utility with discount γ is:

$\mathrm{R}(s, a, s') + \gamma \mathrm{R}(s, a, s') + \gamma^2 \mathrm{R}(s, a, s') + \cdots$ where $\gamma \in [0, 1]$

γ is based on how important current reward is compared to the future reward

Discount

Solving the problem of infinite stream of rewards
Geometric series: $1 + \gamma + \gamma^2 + \ldots = 1/(1 - \gamma)$
Assume rewards bounded by $\pm R_{max}$
Then $r_0 + \gamma_1 r_1 + \gamma_2 r_2 + \ldots$ is bounded by $\pm R_{max}/(1 - \gamma)$

Policy Evaluation

The **utility** is the discounted sum of rewards on the path.
Optimal policy: $\pi^*(s) =$ optimal actions from state s
It gives highest $U_\pi(s)$ for any π

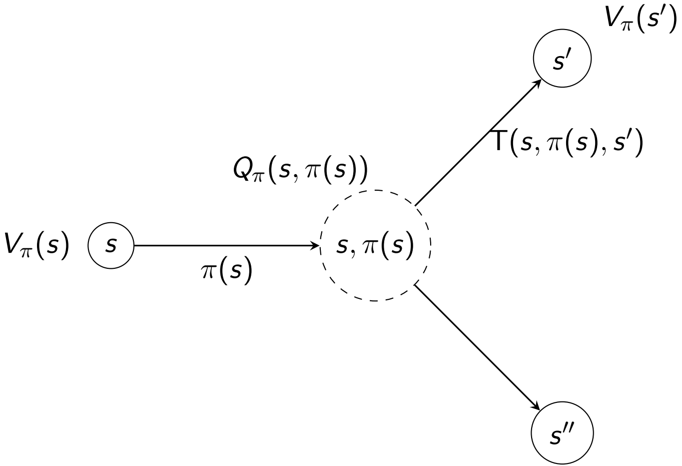
$$U_\pi(s) = R(s, a, s') + (s, a, s') + \gamma^2 R(s, a, s') + \cdots$$

Policy Evaluation

For a given policy π, we have two variable associated with it:

- Value of the policy $V_\pi(s)$
- Q-value of the policy $Q_\pi(s, \pi(s))$

Markov Decision Process



Policy Evaluation

For a given policy π, we have two variable associated with it:

- Value of the policy $V_\pi(s)$
- Q-value of the policy $Q_\pi(s, \pi(s))$

The value can be thought of as the label for the nodes representing the states and the Q-value as the label for the chance nodes

Policy Evaluation

Value is the expected utility from following policy π from state s
Q-value is the expected utility of taking action a from state s, and then following policy π.

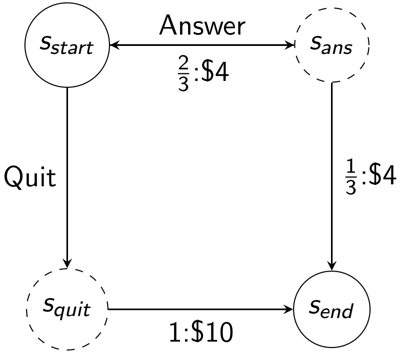
$$V_\pi(s) = E[V_\pi(s)] = \begin{cases} 0 \text{ if } \mathit{isEnd}(s) \\ Q_\pi(s) \text{ otherwise} \end{cases}$$

$$Q_\pi(s) = \sum_{s'} T(s'|s, a)[R(s, a, s') + \gamma V(s')]$$

Policy Evaluation

Let the policy π be '*Answer*':

$$
\begin{aligned}
V_\pi(s_{end}) &= 0 \\
V_\pi(s_{start}) &= Q_\pi(s_{start}, Answer) \\
&= \frac{1}{3}(4 + V_\pi(s_{end})) + \frac{2}{3}(4 + V_\pi(s_{start})) \\
\implies \quad V_\pi(s_{start}) &= \frac{1}{3}(4) + \frac{2}{3}(4 + V_\pi(s_{start}))
\end{aligned}
$$



Closed form solution:

$$
\begin{aligned}
3V_\pi(s_{start}) &= 4 + 2 \cdot 4 + 2V_\pi(s_{start}) \\
V_\pi(s_{start}) &= 12
\end{aligned}
$$

Policy Evaluation

Given the recursion $V^*(s) = \max_a Q^*(s, a)$

Value:

$$V^*(s) = \max_{a \in Actions(s)} \sum_{s'} \{P(s'|s, a)[R(s, a, s') + \gamma V(s')]\}$$

Q-value:

$$\begin{aligned} Q^*(s, a) &= \sum_{s'} \{P(s'|s, a)[R(s, a, s') + \gamma V(s')]\} \\ &= \sum_{s'} \{P(s'|s, a)[R(s, a, s') + \gamma \max_{a'} Q(s', a')]\} \end{aligned}$$

Markov Decision Process

Solving MDPs:

- Value Iteration
- Policy Iteration

Policy Iteration

$V_\pi^{(0)}(s) \leftarrow 0$
```
for  i = 1 ··· t_max
     for  each  state  s
```
$$V_\pi^{(t)}(s) \leftarrow \sum_{s'} T(s'|s,a)[R(s,\pi(s),s') + \gamma V_\pi^{(t-1)}(s')]$$

Policy Iteration

$V_\pi^{(0)}(s) \leftarrow 0$

`for` $i = 1 \cdots t_{max}$

`for each state` s

$$V_\pi^{(t)}(s) \leftarrow \underbrace{\sum_{s'} T(s'|s,a)[R(s,\pi(s),s') + \gamma V_\pi^{(t-1)}(s')]}_{Q_\pi^{(t-1)}(s)}$$

Policy Evaluation

How many iterations (t_{max})?
Repeat until there is no/very little change

$$\max_{s \in states} |V_\pi^{(t)}(s) - V_\pi^{(t-1)}(s)| \leq \epsilon$$

Only save the last two iterations, $V_\pi^{(t)}$ & $V_\pi^{(t-1)}$

Policy Iteration

$V_\pi^{(0)}(s) \leftarrow 0$
`for` $i = 1 \cdots t_{max}$
    `for each state` s
        $V_\pi^{(t)}(s) \leftarrow \sum_{s'} T(s'|s,a)[R(s,\pi(s),s') + \gamma V_\pi^{(t-1)}(s')]$

Total states: S
Actions per state: A
Total successor (with $\mathrm{T}(s'|s,a) > 0$): S'

Complexity: $O(SS' t_{max})$

Policy Iteration

Let the policy π be '*Answer*':

$$V_\pi^{(t)}(s_{end}) = 0$$
$$V_\pi^{(t)}(s_{start}) = \frac{1}{3}(4 + V_\pi^{(t-1)}(s_{end})) + \frac{2}{3}(4 + V_\pi^{(t-1)}(s_{start}))$$

Iteration (t)	$V_\pi^{(t)}(s_{end})$	$V_\pi^{(t)}(s_{start})$
0	0.00	0.00
1	0.00	4.00
2	0.00	6.67
3	0.00	8.44
100	0.00	12.00

$$V_\pi^{(t)}(s_{start}) = 12$$

Optimal Value

Goal: try to get directly at maximum expected utility

$V_{opt}(s) =$ is the maximum value obtained by any policy

Optimal Value

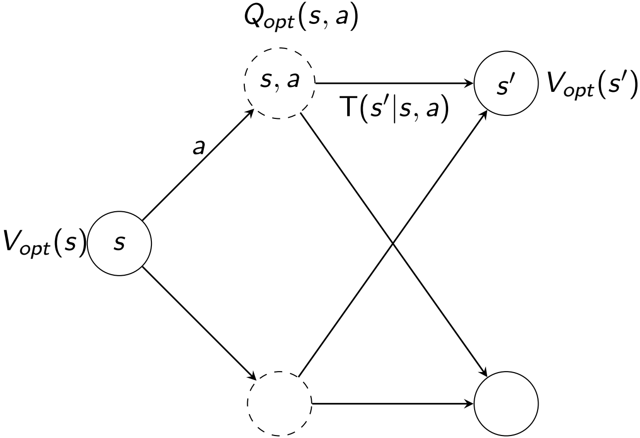
Given the recursion $V_{opt}(s) = \max_a Q_{opt}(s, a)$
Value:

$$V_{opt}(s) = \max_{a \in Actions(s)} \sum_{s'} \{T(s'|s, a)[R(s, a, s') + \gamma V_{opt}(s')]\}$$

Q-value:

$$\begin{aligned} Q_{opt}(s, a) &= \sum_{s'} \{T(s'|s, a)[R(s, a, s') + \gamma V_{opt}(s')]\} \\ &= \sum_{s'} \{T(s'|s, a)[R(s, a, s') + \gamma \max_{a'} Q_{opt}(s', a')]\} \end{aligned}$$

Optimal Value



Optimal Value

Policy evaluation used the action from a fixed policy π
Now we pick the action which maximizes the Q-value $Q_{opt}(s)$

$$V_{opt}(s) = \begin{cases} 0 \text{ if } \mathit{isEnd}(s) \\ \max_{a \in Actions(s)} Q_{opt}(s) \text{ otherwise} \end{cases}$$

$$Q_{opt}(s) = \sum_{s'} T(s, a, s')[R(s, a, s') + \gamma V_{opt}(s')]$$

Optimal Policy

As for any state s, $Q_\pi(s)$ gives you the value of taking the policy $\pi(s)$
Therefore, **Optimal policy** π_{opt} in state s is the one which gives the largest value for $Q_{opt}(s)$

$$\pi_{opt}(s) = \underset{a \in Actions(s)}{\arg\max}\ Q_{opt}(s)$$

Value Iteration

$V_{opt}^{(0)}(s) \leftarrow 0$

for $i = 1 \cdots t_{max}$

for each state s

$V_{opt}^{(t)}(s) \leftarrow \max_{a \in Actions(s)} \sum_{s'} T(s,a,s')[R(s,\pi(s),s') + \gamma V_{opt}^{(t-1)}(s')]$

Value Iteration

$V_{opt}^{(0)}(s) \leftarrow 0$
for $i = 1 \cdots t_{max}$
    for each state s
        $V_{opt}^{(t)}(s) \leftarrow \max_{a \in Actions(s)} \underbrace{\sum_{s'} T(s, a, s')[R(s, \pi(s), s') + \gamma V_{opt}^{(t-1)}(s')]}_{Q_{opt}^{(t-1)}(s)}$

Value Iteration

$V_{opt}^{(0)}(s) \leftarrow 0$
for $i = 1 \cdots t_{max}$
    for each state s
        $V_{opt}^{(t)}(s) \leftarrow \max_{a \in Actions(s)} \sum_{s'} T(s, a, s')[R(s, \pi(s), s') + \gamma V_{opt}^{(t-1)}(s')]$

Total states: S
Actions per state: A
Total successor: S'

Complexity: $O(SAS' t_{max})$

Value Iteration

$V_{opt}^{(0)}(s) \leftarrow 0$

`for` $i = 1 \cdots t_{max}$

`for each state` s

$V_{opt}^{(t)}(s) \leftarrow \max_{a \in Actions(s)} \sum_{s'} T(s, a, s')[R(s, \pi(s), s') + \gamma V_{opt}^{(t-1)}(s')]$

argmax instead of **max** will give the optimal policy π_{opt}

Value Iteration

Iteration (t)	$V_{opt}^{(t)}(s_{end})$	$V_{opt}^{(t)}(s_{start})$	$\pi_{opt}(s_{end})$	$\pi_{opt}(s_{start})$
0	0.00	0.00	-	-
1	0.00	10.00	-	Quit
2	0.00	10.67	-	Answer
3	0.00	11.11	-	Answer
100	0.00	12.00	-	Answer

$$V_{\pi}^{(t)}(s_{start}) = 12$$

Recap

Deterministic vs Stochastic

Markov Decision Process

- Transition
- Reward
- Policy
- Discount

Policy value & Q-value

Solving MDPs

- Policy Iteration
- Value Iteration

References

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The End