CS-541: Artificial Intelligence
Lecture 8

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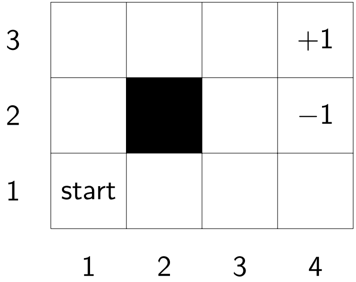
Recap

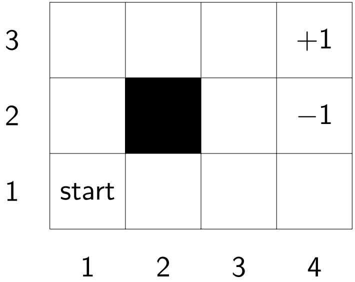
Grid world is 3×4 grid
Start state is at (0,0)
Reward $+1$ at (4,3)
Reward -1 at (4,2)



Recap

For any state, three possible moves

- up: 0.8
- left: 0.1
- right: 0.1



Recap

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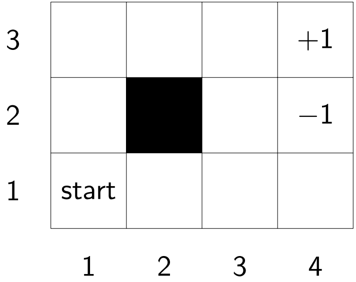


How to find the optimal utility?

Recap

For any state, three possible moves

- up: ~~0.8~~ ?
- left: ~~0.1~~ ?
- right: ~~0.1~~ ?

3				$+1$
2				-1
1	start			
	1	2	3	4

Unknown transition probability. How to find the optimal utility?

Recap

Q-Learning:
Start with some Q-values
Iteratively update the Q-values using the following equation

$$\text{for each } (s, a, r, s')$$
$$\hat{Q}_{opt}(s, a) \leftarrow (1 - \eta)\hat{Q}_{opt}(s, a) + \eta\Big(r + \gamma \max_{a' \in Actions(s')} \hat{Q}_{opt}(s', a')\Big)$$

Joint Probability

2 random variables: sunshine $S \in \{0, 1\}$ and rain $R \in \{0, 1\}$
The table represents the probability when each variable has the corresponding value
The total probabilities sum to 1

s	r	$\mathbb{P}(S, R)$
0	0	0.20
0	1	0.08
1	0	0.70
1	1	0.02

Uppercase letters denote the random variable and lowercase letters denote the values

Marginal Probability

s	$\mathbb{P}(S = s)$
0	0.28
1	0.72

Aggregate all the probabilities for the specific value of the random variable
$\mathbb{P}(S = 0) = \mathbb{P}(S = 0, R = 0) + \mathbb{P}(S = 0, R = 1) = 0.20 + 0.08$
The total marginal probabilities should sum to 1

Conditional Probability

s	$\mathbb{P}(S = s \mid R = 1)$
0	0.8
1	0.2

Select only the probabilities for the specific value of the random variable and normalize them

The total normalized conditional probabilities should sum to 1

Probabilistic Inference

Variables: sunshine (S), rain (R), wind (W), humidity (H)
Joint distribution: $\mathbb{P}(S, R, W, H)$
Probabilistic Inference:
Condition on evidence (wind,humidity): $W = 1,\ H = 1$
Interested in query (rain?): R

$$\mathbb{P}(\underbrace{R}_{query} \mid \underbrace{W = 1, H = 1}_{condition})$$

S is marginalized out

Probabilistic Inference

$$\mathbb{P}(\underbrace{R}_{\textit{query}} \mid \underbrace{W = 1, H = 1}_{\textit{condition}})$$

1) we observe some variables (**evidence**)
2) we are interested another set of variables which we didn't observe (**query**)
3) The process of answering this query is called **probabilistic inference**

Overview

How to specify a joint distribution $\mathbb{P}(X_1, \cdots, X_n)$ **compactly**?
- Bayesian network

How to compute queries $\mathbb{P}(R | T = 1, A = 1)$ **efficiently**?
- Variable elimination, Gibbs sampling etc.

Alarm Network

Earthquakes and **Burglary** are independent random variables
An alarm can go off for either of the two

Suppose you get an alarm
If the news reports there was an **earthquake**,
will the probability of **burglary** increase, decrease or remain same?

Bayesian Network (Alarm)

Suppose you get an alarm
If the news reports there was an **earthquake**,
will it change the probability of **burglary**?

Joint probability: $\mathbb{P}(E, B, A)$

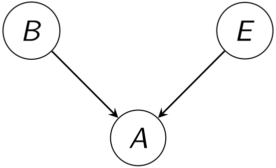
Questions:

$$\mathbb{P}(B = 1 | A = 1) \qquad ? \qquad \mathbb{P}(B = 1 | A = 1, E = 1)$$

Bayesian Network (Alarm)

b	p(B=b)
1	ϵ
0	$1-\epsilon$

e	p(E=e)
1	ϵ
0	$1-\epsilon$



Bayesian Network (Alarm)

b	e	a	$p(a \mid b, e)$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

Bayesian Network (Alarm)

$$
\begin{aligned}
p(b) &= \epsilon \cdot [b = 1] + (1 - \epsilon)[b = 0] \\
p(e) &= \epsilon \cdot [e = 1] + (1 - \epsilon)[e = 0] \\
p(a|b, e) &= [a = (b \vee e)]
\end{aligned}
$$

$$
\mathbb{P}(B = b, E = e, A = a) \stackrel{\text{def}}{=\!=} p(b)p(e)p(a|b, e)
$$

Note:

$\mathbb{P}$ is reserved for the joint distribution over random variables

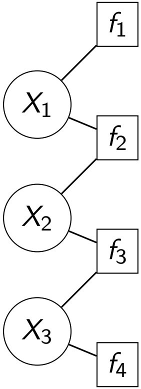
p denotes the local conditional distribution (factor)

Factor Graph

A bipartite graphs which can represent the factorization of a function

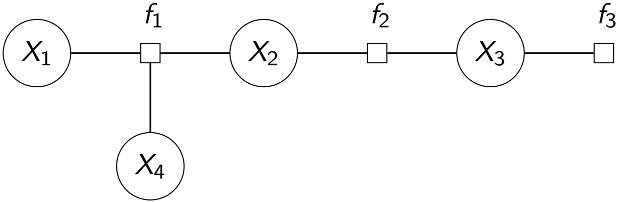
$$\mathbb{P}(X_1, X_2, \cdots, X_n) = \prod_{i=1}^{n} f_i(S_i)$$

$$S \subseteq \{X_1, X_2, \cdots X_4\}$$



• Joint probability written as a product of functions f
• Each function f depends on the subset of the random variables
• Simplify the probability distributions
• Used to visualize and describe independence relations

Factor Graph

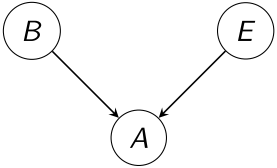


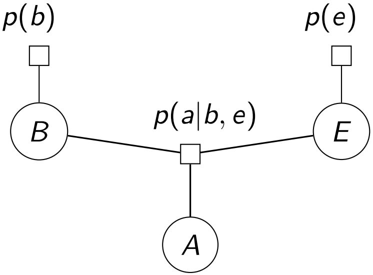
X_1, X_2 and X_4 depend on function f_1, and so on

Joint probability can be written as a product of these functions

$$\mathbb{P}(X_1, X_2, X_3, X_4) = f_1(X_1, X_2, X_4) f_2(X_2, X_3) f_3(X_3)$$

Bayesian Network (Alarm)





$$\mathbb{P}(B = b, E = e, A = a) = p(b)p(e)p(a|b, e)$$

Bayesian networks are special cases of factor graphs

A single factor connects all parents

Probabilistic Inference (Alarm)

Probability of B

b	p(B=b)
1	ϵ
0	$1-\epsilon$

Probability of E

e	p(E=e)
1	ϵ
0	$1-\epsilon$

Joint Probability of B, E, A

b	e	a	$\mathbb{P}(B, E, A)$
0	0	0	$(1-\epsilon)^2$
0	0	1	0
0	1	0	0
0	1	1	$(1-\epsilon)\epsilon$
1	0	0	0
1	0	1	$\epsilon(1-\epsilon)$
1	1	0	0
1	1	1	ϵ^2

Probabilistic Inference (Alarm)

b	e	a	$\mathbb{P}(B, E, A)$
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1	1	0	0
1	1	1	ϵ^2

Questions:

$$\mathbb{P}(B=1) = \epsilon(1-\epsilon) + \epsilon^2 = \epsilon$$

$$\mathbb{P}(B=1|A=1) = \frac{\epsilon(1-\epsilon)+\epsilon^2}{\epsilon(1-\epsilon)+\epsilon^2+(1-\epsilon)\epsilon} = \frac{1}{2-\epsilon}$$

$$\mathbb{P}(B=1|A=1,E=1) = \frac{\epsilon^2}{\epsilon^2+(1-\epsilon)\epsilon} = \epsilon$$

Probabilistic Inference (Alarm)

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Earthquake decrease the probability of burglary!*

* This is not a causal statement!

Probabilistic Inference (Alarm)

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Questions:

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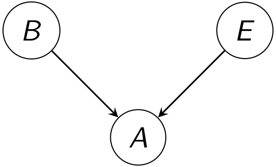
Bayesian Network

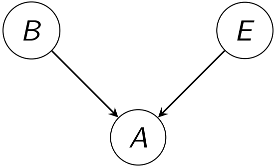


If two causes positively influence an effect.
If we condition on an effect and further condition on one causes
This reduces the probability of the second cause

$$\mathbb{P}(B = 1 | A = 1, E = 1) < \mathbb{P}(B = 1 | A = 1)$$

Note: Only if causes are independent

Conditional Independence



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Bayesian Network (definition)

Let $X = (X_1, \cdots, X_n)$ are random variables
Bayesian Network is a directed acyclic graph which specifies joint distribution over X as a product of local conditional distributions

$$\mathbb{P}(X_1 = x_1, \cdots, X_n = x_n) \stackrel{\text{def}}{=\!=} \prod_{i=1}^{n} p(x_i | x_{parents(i)})$$

Special Properties

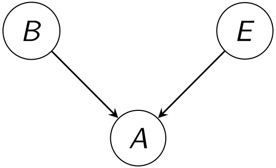
All local distributions satisfy:

$$\mathbb{P}(B = b, E = e) \stackrel{\text{def}}{=\!=} \sum_{x_i} p(x_i | x_{parents(i)}) = 1 \text{ if for each } x_{parents(i)}$$

Implications:

- Consistency of sub-Bayesian networks
- Consistency of conditional distributions

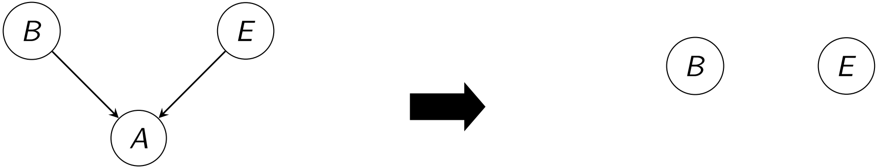
Consistency of sub-Bayesian Networks



$$
\begin{aligned}
\mathbb{P}(B=b, E=e) &\stackrel{\text{def}}{=\!=} \sum_a \mathbb{P}(B=b, E=e, A=a) \\
&\stackrel{\text{def}}{=\!=} \sum_a p(b)p(e)p(a|b,e) \\
&= p(b)p(e)\sum_a p(a|b,e) \\
&= p(b)p(e)
\end{aligned}
$$

Consistency of sub-Bayesian Networks

Marginalization of a leaf node yields a Bayesian network without the node



Consistency of sub-Bayesian Networks

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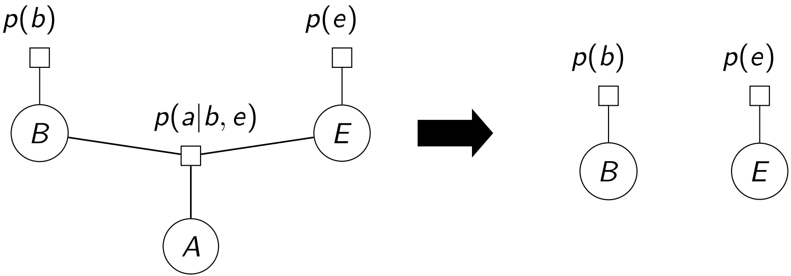
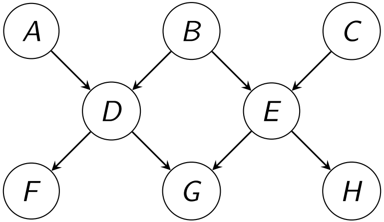


Consistency of local conditionals

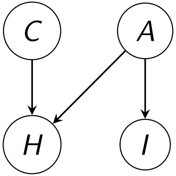
Local conditional distributions are the true conditional distributions.



$$\underbrace{\mathbb{P}(D = d | A = a, B = b)}_{\text{from probabilistic inference}} = \underbrace{p(d|a, b)}_{\text{by definition}}$$

Medical Diagnosis

Cold or Allergies?
If you have cough and itchy eyes, do you have a cold?



Random variables:
Cold (C), allergy (A), cough (H), and itchy eyes (I)

Medical Diagnosis

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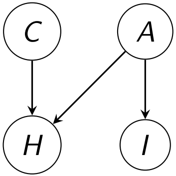

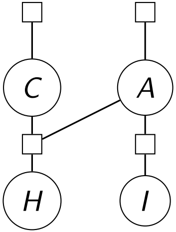

Medical Diagnosis

Cold or Allergies?

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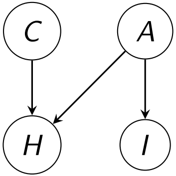


Random variables:

Cold (C), allergy (A), cough (H), and itchy eyes (I)

$$\mathbb{P}(C = c, A = a, H = h, I = i) = p(c)p(a)p(h|c,a)p(i|a)$$

$$\mathbb{P}(C = 1|H = 1, I = 1) = ?$$

$$\mathbb{P}(C = 1|H = 1) = ?$$

Medical Diagnosis

Cold or Allergies?

If you have cough and itchy eyes, do you have a cold?

c	$p(c)$
1	0.1
0	0.9

Table: Cold

a	$p(a)$
1	0.2
0	0.8

Table: Allergy

a	i	$p(i\|a)$
1	1	0.9
1	0	0.1
0	0	0.9
0	1	0.1

Table: Itchy eyes

c	a	h	$p(h\|c,a)$
1	0	1	0.9
1	0	0	0.1
0	1	1	0.9
0	1	0	0.1
1	1	1	0.9
1	1	0	0.1
0	0	1	0.1
0	0	0	0.9

Table: Cough

Medical Diagnosis

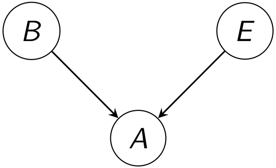
$$\mathbb{P}(C = c, A = a, H = h, I = i) = p(c)p(a)p(h|c, a)p(i|a)$$

$$\begin{aligned}
\mathbb{P}(C = 1|H = 1, I = 1) &= \sum_{a \in A} \mathbb{P}(C = c, A = a, H = h, I = i) \\
&= \sum_{a \in A} p(c)p(a)p(h|c, a)p(i|a)
\end{aligned}$$

Whiteboard

Probabilistic Programming

Probabilistic program: A randomized program that sets the random variable



$$B \sim \mathit{Bernoulli}(\epsilon)$$
$$E \sim \mathit{Bernoulli}(\epsilon)$$
$$A = B \vee E$$

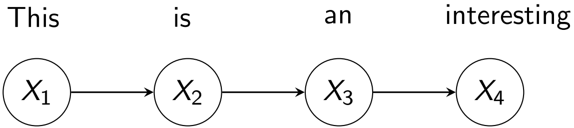
```
def bernoulli(epsilon):
    return random.random()<epsilon

def alarm(epsilon):
    B = bernoulli(epsilon)
    E = bernoulli(epsilon)
    A = (B or E)
    return A
```

Probabilistic Programming: Language Modeling

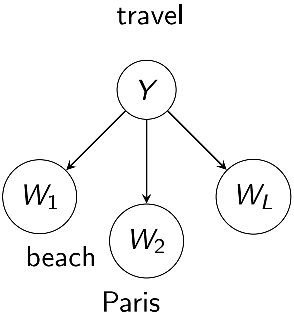
Probabilistic program: A randomized program that sets the random variable

for each position $i = 1, 2, \cdots, n$
    generate words $X_i \sim p(X_i | X_{i-1})$



- Given the previous word, what is the next word?

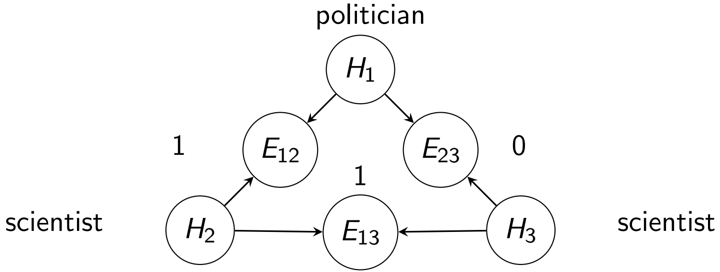
Probabilistic Programming: Topic Modeling




```
generate label Y ~ p(Y)
    for each position i = 1, ..., L:
        generate word W_i ~ p(W_i|Y)
```

- Given a text document, what is the topic?

Probabilistic Programming: Social Network Analysis



for each person $i = 1, \cdots, n$:

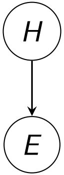
  generate person type $H_i \sim p(H_i)$

for each pair of people $i \neq j$:

  generate connectedness $E_{ij} \sim p(E_{ij} | H_i, H_j)$

- Given a social network (graph over n people), what types of people are there?

Summary so far



- Many different types of models
- Come up with scenarios of how the data (input) was generated through quantities of interest (output)
- Opposite of how we normally do classification!

Probabilistic Inference (definition)

Input

Bayesian network: $\mathbb{P}(X_1, \cdots, X_n)$

Evidence: $E = e$ where $E \subseteq X$ is subset of variables

Query: $Q \subseteq X$ is subset of variables

$$\Downarrow$$

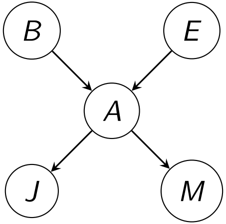
Output

$$\mathbb{P}(Q = q | E = e) \text{ for all values of } q$$

For example: If coughing but no itchy eyes, do you have a cold?

$$\mathbb{P}(C | H = 1, I = 0)$$

Enumeration



$$\mathbb{P}(B|J=j, M=m) = \sum_{a,e} \mathbb{P}(B, J=j, M=m, A, E)$$

$$\sum_{a,e} p(B)p(e)p(a|B,e)p(j|a)p(m|a)$$

So inference in Bayes nets means computing sums of products of numbers

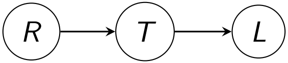
Enumeration

Random Variables: Rain (R), Traffic (T) and Late (L)

$$R \longrightarrow T \longrightarrow L$$

Enumeration

Random Variables: Rain (R), Traffic (T) and Late (L)



$$\mathbb{P}(R=r, T=t, L=l) = \mathbb{P}(R=r)\mathbb{P}(T=t|R=r)\mathbb{P}(L=l|T=t)$$
$$\stackrel{\text{def}}{=\!=} p(r)p(t|r)p(l|t)$$

$P(R)$	
+r	0.1
-r	0.9

$P(T\|R)$		
+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L\|T)$		
+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Enumeration

Random Variables: Rain (R), Traffic (T) and Late (L)

$$R \rightarrow T \rightarrow L$$

$$\begin{aligned}\mathbb{P}(R = r, T = t, L = l) &= \mathbb{P}(R = r)\mathbb{P}(T = t|R = r)\mathbb{P}(L = l|T = t) \\ &\stackrel{\text{def}}{=\!=} p(r)p(t|r)p(l|t)\end{aligned}$$

$$\mathbb{P}(L = l) = ?$$

Enumeration

Random Variables: Rain (R), Traffic (T) and Late (L)

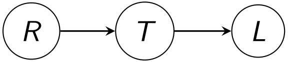
$$R \longrightarrow T \longrightarrow L$$

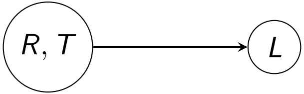
$$\begin{aligned}\mathbb{P}(R = r, T = t, L = l) &= \mathbb{P}(R = r)\mathbb{P}(T = t|R = r)\mathbb{P}(L = l|T = t) \\ &\stackrel{\text{def}}{=\!=} p(r)p(t|r)p(l|t)\end{aligned}$$

$$\mathbb{P}(L = l) = ?$$

Enumeration

Step1: Join factors (similar to database join)





$P(R)$	
+r	0.1
-r	0.9

$\times$

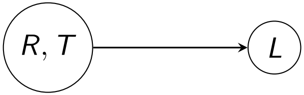
$P(T \mid R)$		
+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

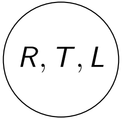
$\rightarrow$

$P(R, T)$		
+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81

Enumeration

Step1: Join factors





$P(T \mid R)$		
+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$\times$

$P(L \mid T)$		
+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

$\rightarrow$

$P(R, T, L)$			
+r	+t	+l	0.024
+r	+t	-l	0.056
+r	-t	+l	0.002
+r	-t	-l	0.018
-r	+t	+l	0.027
-r	+t	-l	0.063
-r	-t	+l	0.081
-r	-t	-l	0.729

Enumeration

Step2: Eliminate factors (remember marginalization)

R, T, L $\qquad\qquad$ T, L

$P(R, T, L)$			
+r	+t	+l	0.024
+r	+t	-l	0.056
+r	-t	+l	0.002
+r	-t	-l	0.018
-r	+t	+l	0.027
-r	+t	-l	0.063
-r	-t	+l	0.081
-r	-t	-l	0.729

$\xrightarrow{\text{Sum out } R}$

$P(L\|T)$		
+t	+l	0.051
+t	-l	0.119
-t	+l	0.083
-t	-l	0.747

Enumeration

Step2: Eliminate factors (remember marginalization)

T, L $\qquad\qquad$ L

$P(L \vert T)$		
+t	+l	0.051
+t	-l	0.119
-t	+l	0.083
-t	-l	0.747

$\xrightarrow{\text{Sum out } T}$

$P(L)$	
+l	0.134
-l	0.866

Variable Elimination

$$X_1 \to X_2 \to X_3 \to X_4$$

Query: $\mathbb{P}(X_3 = x_3 | X_2 = 5)$ for all x_3

$$
\begin{aligned}
\textbf{Tedious way!} &= \frac{\mathbb{P}(X_3 = x_3, X_2 = 5)}{\mathbb{P}(X_2 = 5)} \\
&\propto \mathbb{P}(X_3 = x_3, X_2 = 5) \\
&\propto \sum_{x_1, x_4} \mathbb{P}(X_1 = x_1, X_2 = 5, X_3 = x_3, X_4 = x_4) \\
&\propto \sum_{x_1, x_4} p(x_1) p(x_2 = 5 | x_1) p(x_3 | x_2 = 5) p(x_4 | x_3) \\
&\propto \left(\sum_{x_1} p(x_1) p(x_2 = 5 | x_1) \right) p(x_3 | x_2 = 5) \sum_{x_4} p(x_4 | x_3) \\
&\propto p(x_3 | x_2 = 5)
\end{aligned}
$$

Variable Elimination

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4$$

Query: $\mathbb{P}(X_3 = x_3 | X_2 = 5)$ for all x_3

Faster way!

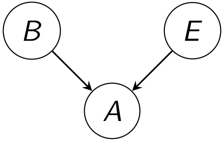
Variable Elimination

Query: $\mathbb{P}(Q|E = e)$ for all x_3

General Probabilistic Inference Strategy

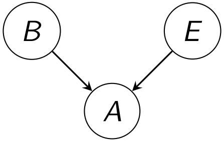
1. Marginalize (remove) non-ancestors of Q or E
2. Convert Bayesian network to factor graph
3. Condition on evidence ($E = e$)
4. Marginalize disconnected
5. Run probabilistic inference algorithm (manual, variable elimination etc.)

Variable Elimination: Alarm



Query: $\mathbb{P}(B)$

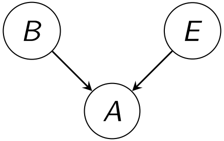
Variable Elimination: Alarm



Query: $\mathbb{P}(B)$

- Marginalize out A and E

Variable Elimination: Alarm

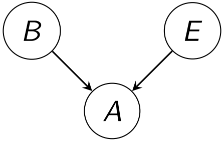


Query: $\mathbb{P}(B)$

- Marginalize out A and E

Query: $\mathbb{P}(B|A=1)$

Variable Elimination: Alarm



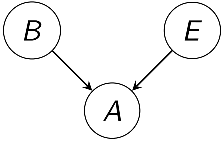
Query: $\mathbb{P}(B)$

- Marginalize out A and E

Query: $\mathbb{P}(B|A=1)$

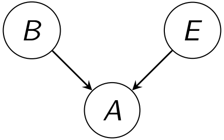
- Condition on $A = 1$

Variable Elimination: Alarm



Query: $\mathbb{P}(B|A=1) = ?$

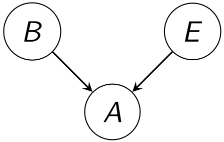
Variable Elimination: Alarm



Query: $\mathbb{P}(B|A = 1) =?$

$$\mathbb{P}(B = b|A = 1) \propto p(b)f(b)$$
$$f(b) = \sum_{e} p(e)p(a = 1|b, e)$$

Variable Elimination: Alarm



Query: $\mathbb{P}(B|A=1) = ?$

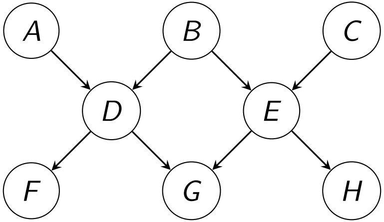
$$\mathbb{P}(B=b|A=1) \propto p(b)f(b)$$

$$\mathbb{P}(B=1|A=1) = \epsilon(\epsilon + (1-\epsilon)) = \epsilon$$

$$\mathbb{P}(B=0|A=1) = (1-\epsilon)(\epsilon + 0) = \epsilon(1-\epsilon)$$

$$\implies \mathbb{P}(B=b|A=1) = \frac{\epsilon}{\epsilon + \epsilon(1-\epsilon)} = \frac{1}{2-\epsilon}$$

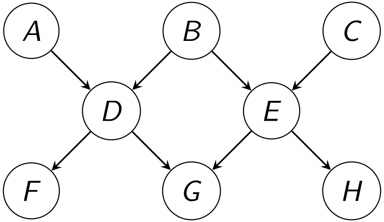
Variable Elimination



Query: $\mathbb{P}(C|B = b)$

Query: $\mathbb{P}(C, H|E = e)$

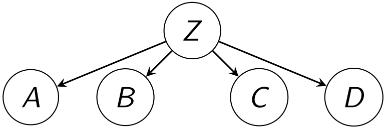
Variable Elimination



Query: $\mathbb{P}(C|B=b)$
- Marginalize out everything else, note $C \perp\!\!\!\perp B$

Query: $\mathbb{P}(C, H|E=e)$
- Marginalize out A, D, F, G, note $C \perp\!\!\!\perp H|E$

Order Matters

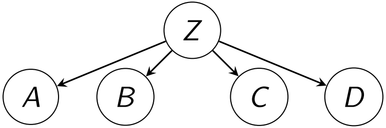


Order the terms Z, A, B, C, D

$$
\begin{aligned}
P(D) &= \sum_{z,a,b,c} p(z)p(a|z)p(b|z)p(c|z)p(D|z) \\
&= \sum_{z} p(z) \sum_{a} p(a|z) \sum_{b} p(b|z) \sum_{c} p(c|z)p(D|z)
\end{aligned}
$$

- Largest factor has 2 variables (D, Z)

Order Matters



Order the terms A, B, C, D, Z

$$\begin{aligned} P(D) &= \sum_{z,a,b,c} p(a|z)p(b|z)p(c|z)p(D|z)p(z) \\ &= \sum_a \sum_b \sum_c \sum_z p(a|z)p(b|z)p(c|z)p(D|z)p(z) \end{aligned}$$

- Largest factor has 4 variables (A, B, C, D)
- In general, with n leaves, factor of size 2^n

Recap

Bayesian Networks
Probabilistic programming
Probabilistic Inference
Enumeration Method
Variable Elimination Method
Variable Elimination Method with factor graph

References

Stuart Russell and Xiaodong Song (2021)
CS 188 — Introduction to Artificial Intelligence
University of California, Berkeley

Chelsea Finn and Nima Anari (2021)
CS221 — Artificial Intelligence: Principles and Techniques
Stanford University

The End