Practice Problems II Zero Level Mathematical Foundation M Tech AIML NSP1 S1 25

AIML_MFML Mid semester Assignment (Birla Institute of Technology and Science, Pilani)

Practice Problems-II

1 Finding Min/Max, Nature of the critical points

1. To optimize highway traffic, engineers model the traffic flow as

$$F(v) = v(100 - v)$$

where:

- $F(v)$: number of vehicles per hour,
- v: vehicle speed in km/h, with $0 < v < 100$.

Then,

(a) Find the **critical point(s)** of $F(v)$.

(b) Use the **second derivative test** to determine its nature (maximum or minimum).

(c) Compute the **maximum or minimum traffic flow** and interpret the result.

2. Suppose the cabin temperature of an aircraft is initially maintained at 22°C. Due to severe turbulence at cruising altitude, a sudden structural failure occurs, causing rapid **depressurization** of the cabin. As a result, the cabin temperature begins to fall sharply toward the outside atmospheric temperature of -50°C. This temperature drop is modeled by the function:

$$T(t) = -50 + 72e^{-kt}$$

where:

- $T(t)$: Cabin temperature (in °C) at time t minutes after the incident,
- $k > 0$: A cooling constant (depends on factors such as insulation and cabin breach severity),
- $t \geq 0$: Time in minutes after depressurization.

(a) Find the first derivative $T'(t)$, representing the rate at which the temperature changes.

(b) Determine the time at which the temperature is dropping the fastest.

(c) Use the second derivative test to classify the nature of this critical point.

(d) Explain the physical significance of this analysis in the context of emergency response.

2 Limits

1. Let $f(x) = x \quad$ for $x = 1, 2, 3, \ldots$. Then, what is the value of $\lim_{x\to 1} f(x)$? Justify your answer.

2. Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x < 1 \\ 3, & x = 1 \\ x+1, & x > 1. \end{cases}$$

   Then, evaluate $\lim_{x\to 1} f(x)$.

3 Continuity

1. Consider the above problem (problem no.2 in Limits section). Is $f(x)$ continuous at $x = 1$? Justify your answer.

2. Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} kx^2+1, & x < 1 \\ 2x+3, & 1 \le x < 3 \\ 7, & x \ge 3 \end{cases}$$

   (a) Find the value of k for which $f(x)$ is continuous at $x = 1$.

   (b) Determine whether $f(x)$ is continuous at $x = 3$. Justify your answer.

4 Derivatives

1. Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} x^2, & x \le 2 \\ mx+c, & x > 2. \end{cases}$$

   Then, find the values of m and c such that $f(x)$ is continuous and differentiable at $x = 2$

2. Let the functions be defined as:

$$f(x) = \frac{x^2+3x-1}{x+2}, \quad g(x) = \sin(x^2+1).$$

   (a) Find $f'(1)$, the derivative of $f(x)$ at $x = 1$, using the **quotient rule**.

   (b) Find $g'(1)$, the derivative of $g(x)$ at $x = 1$, using the **chain rule**.