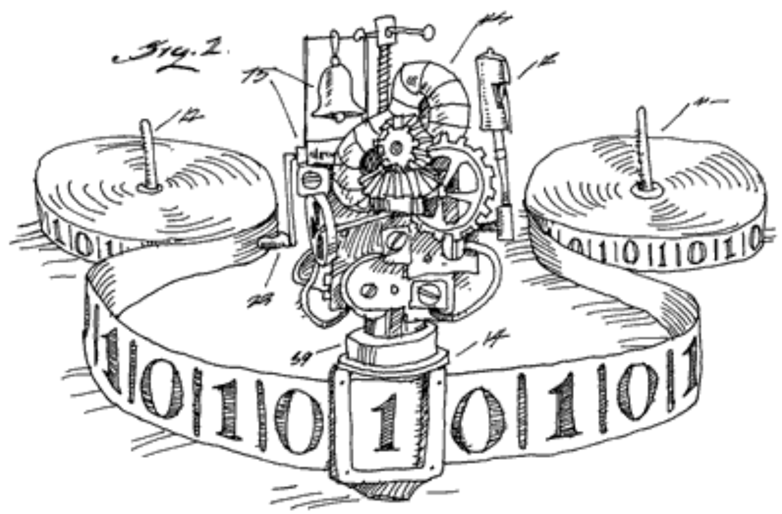
SHANNON MEETS TURING

Pei Wu

April. 2023

Theory of Computation

A. Turing



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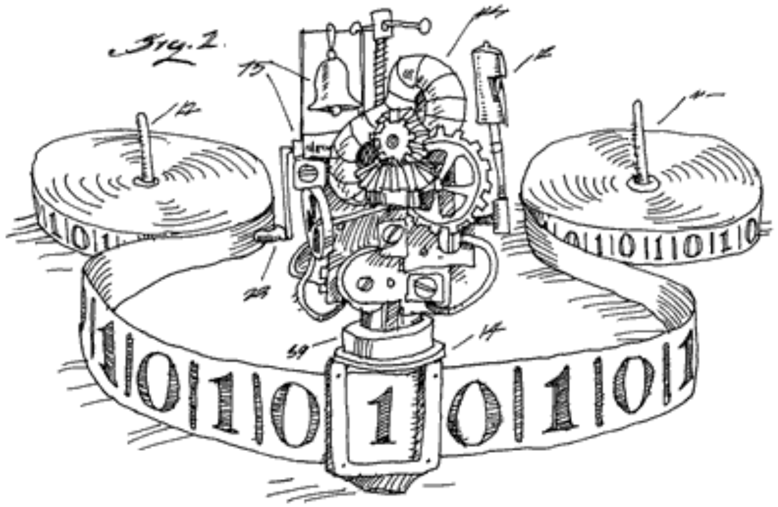

deterministic polynomial-time

non-determinism

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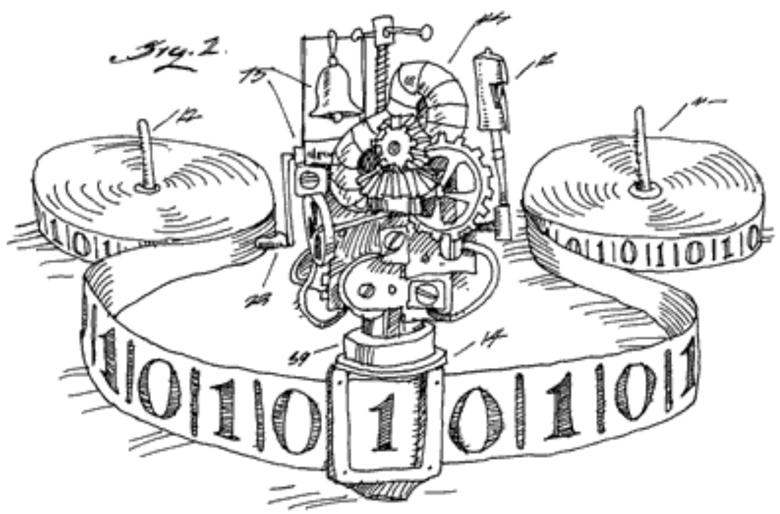

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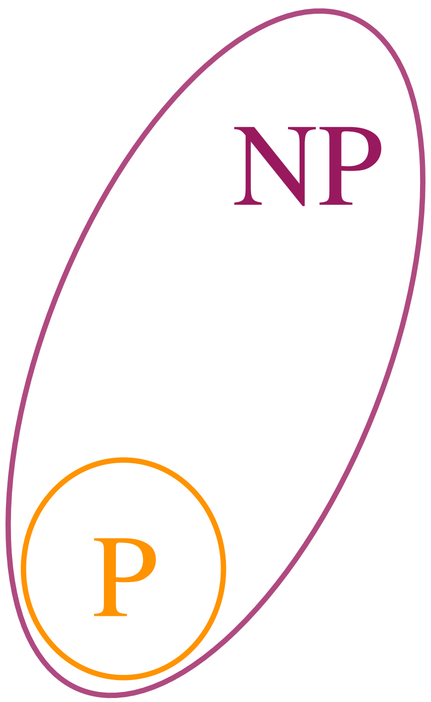

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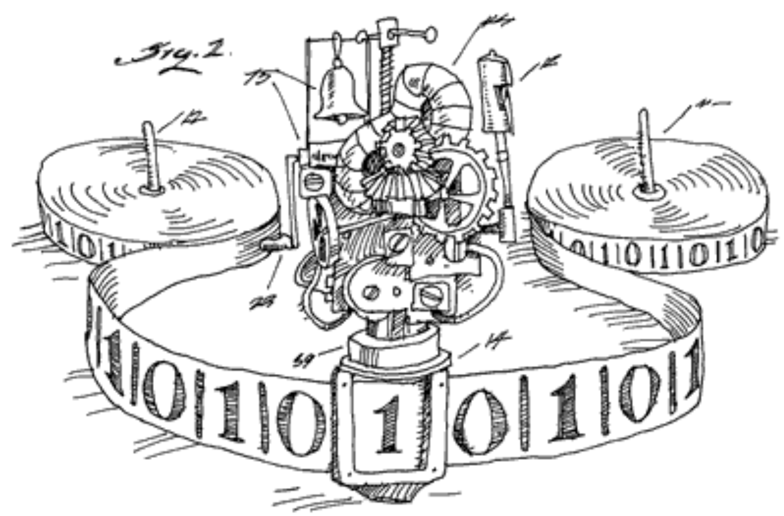
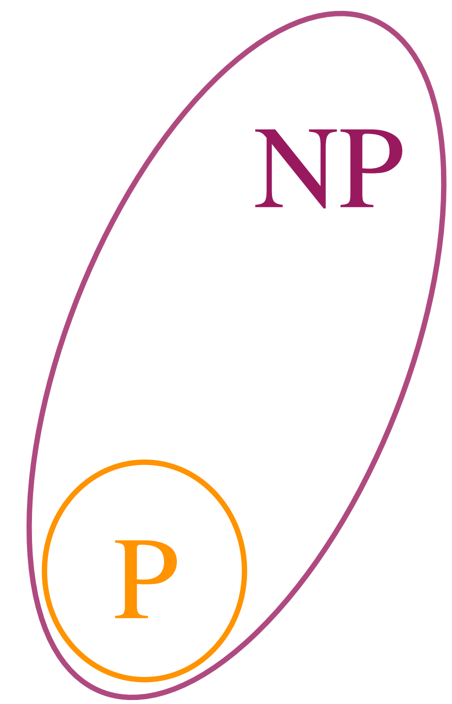


deterministic polynomial-time

non-determinism

randomness



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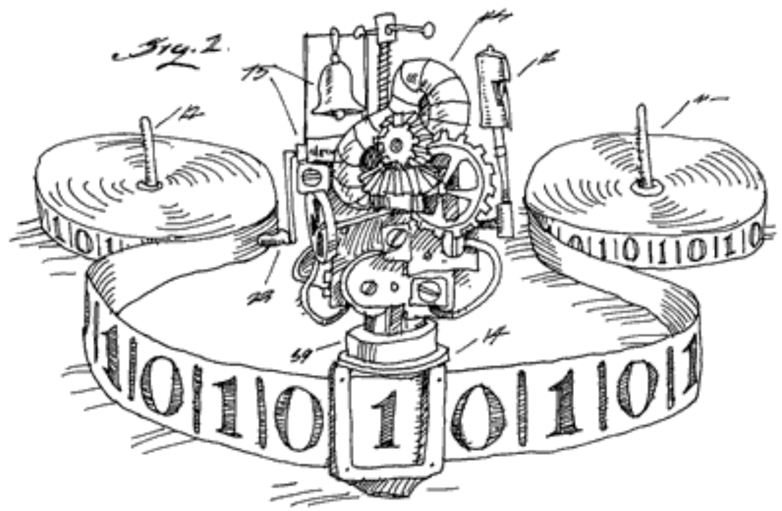

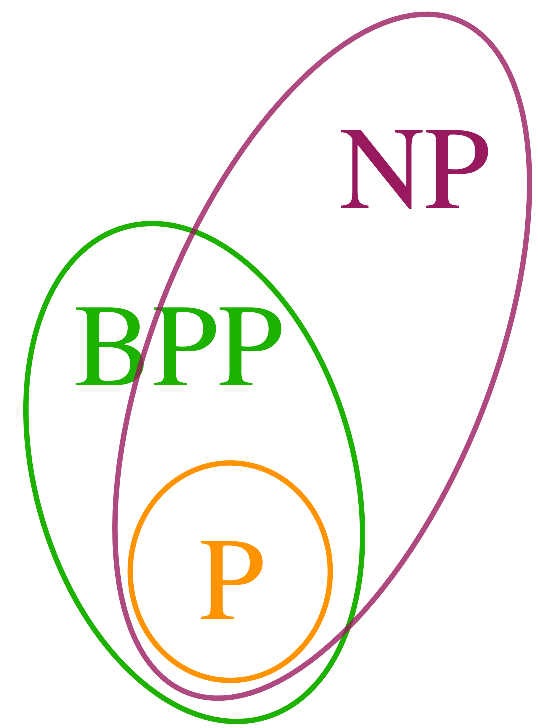

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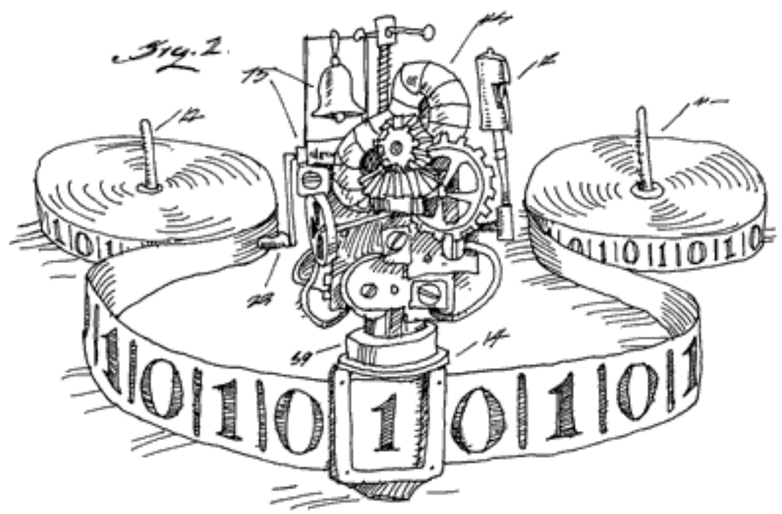

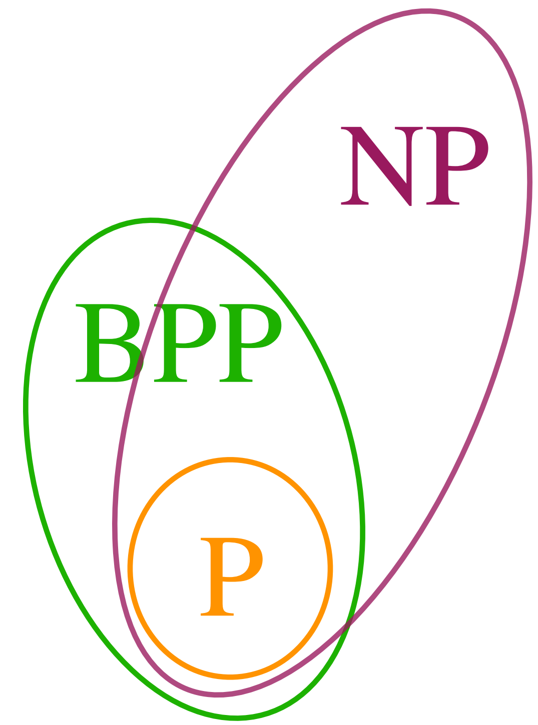

deterministic polynomial-time

non-determinism

randomness

quantum

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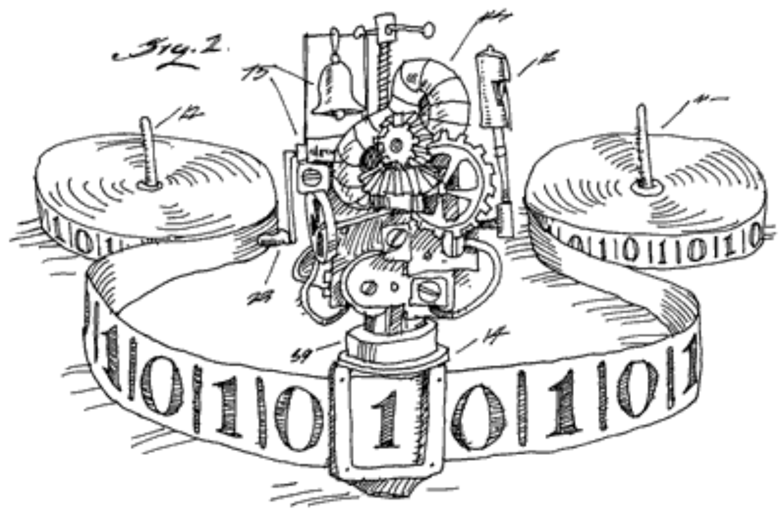

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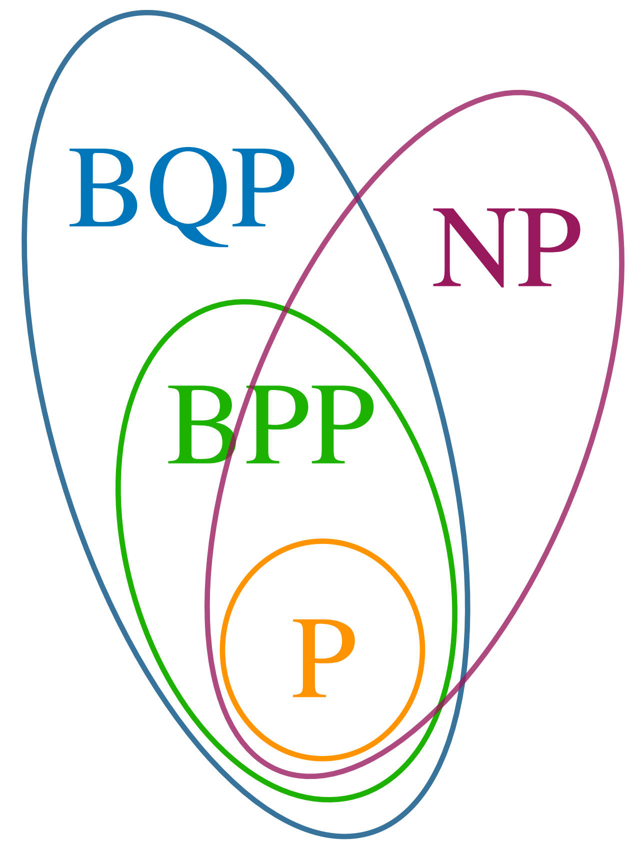
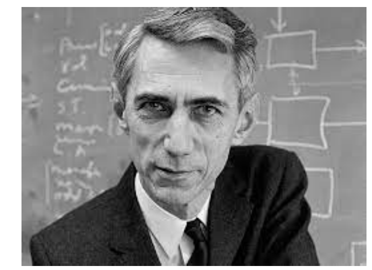


Theory of Communication (one-way)

Reprinted with corrections from *The Bell System Technical Journal,*
Vol. 27, pp. 379–423, 623–656, July, October, 1948.



A Mathematical Theory of Communication

By C. E. SHANNON

INTRODUCTION

THE recent development of various methods of modulation such as PCM and PPM which exchange bandwidth for signal-to-noise ratio has intensified the interest in a general theory of communication. A

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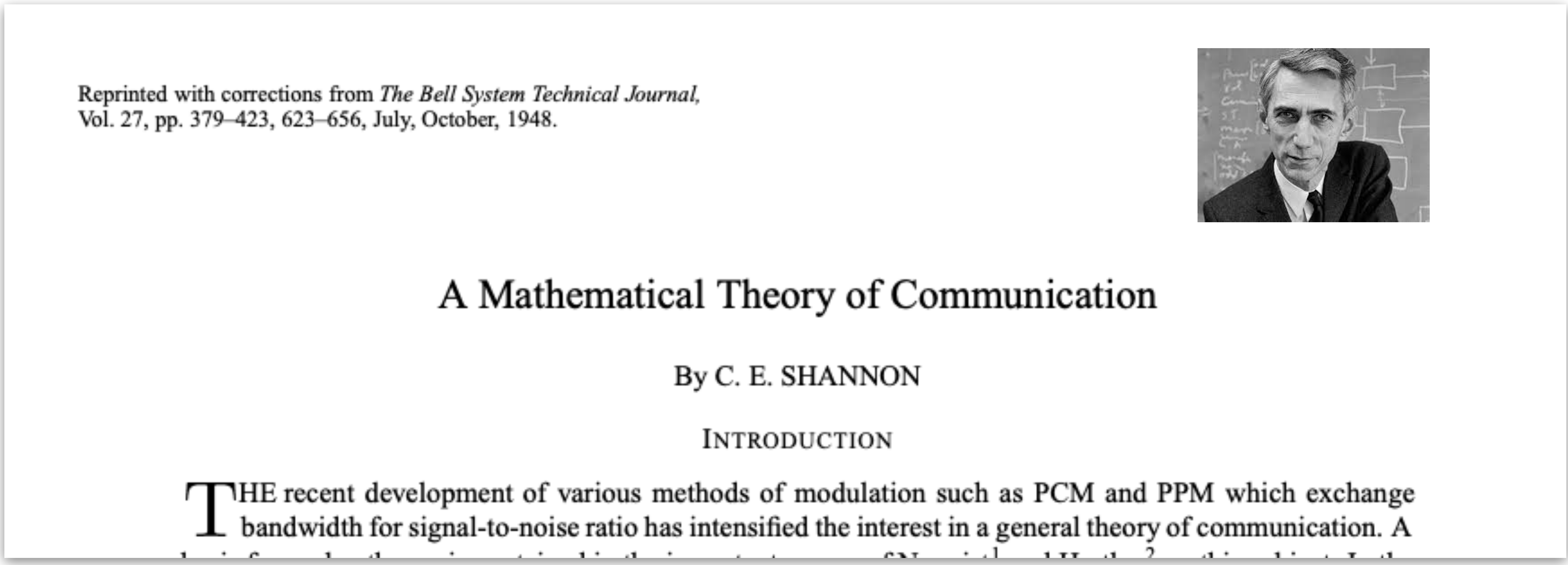

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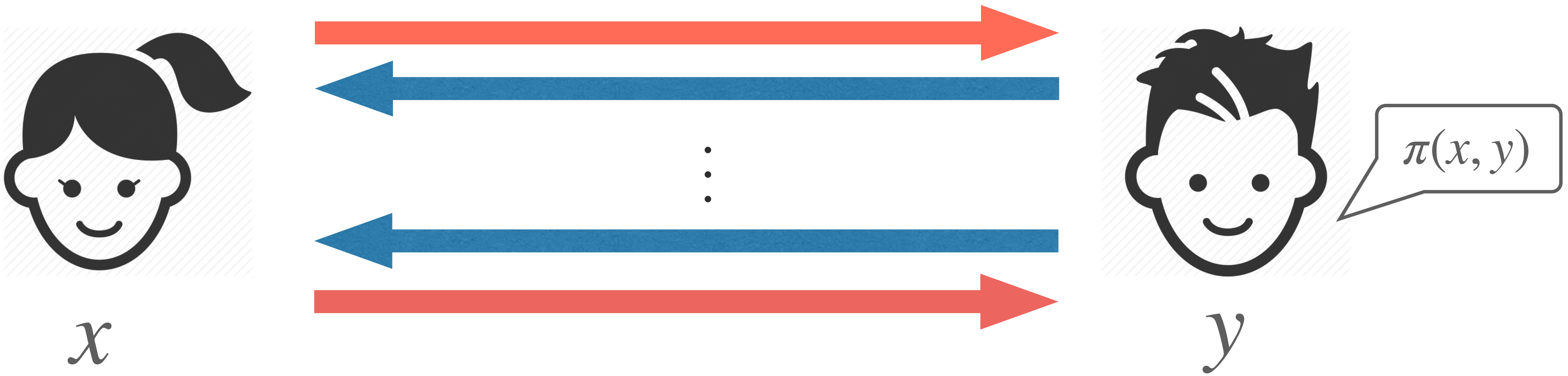
x

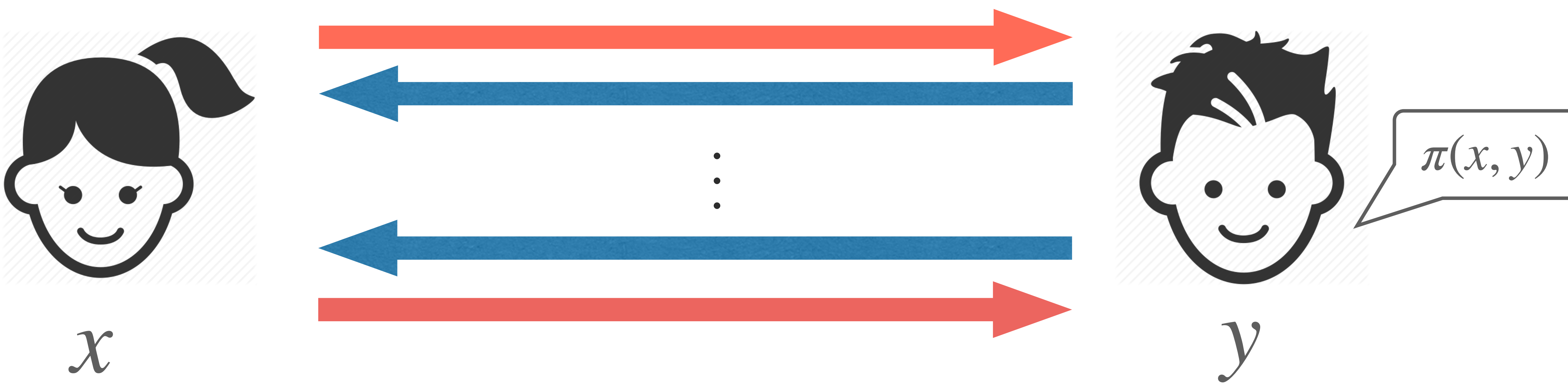
Theory of Communication (interactive)



Theory of Communication (interactive)

A. Yao '79

$f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}$



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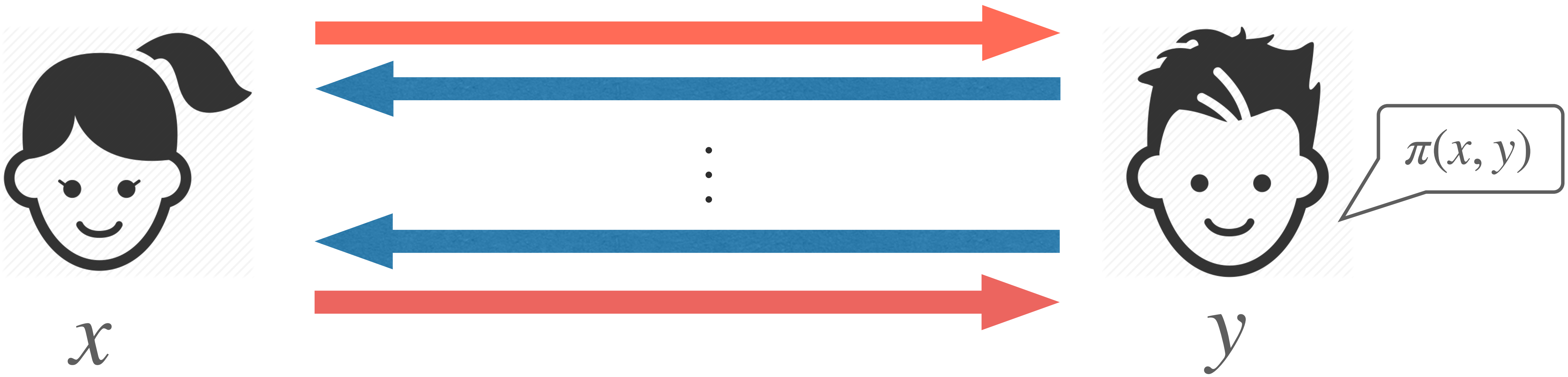


A trivial, $O(n)$-communication solution

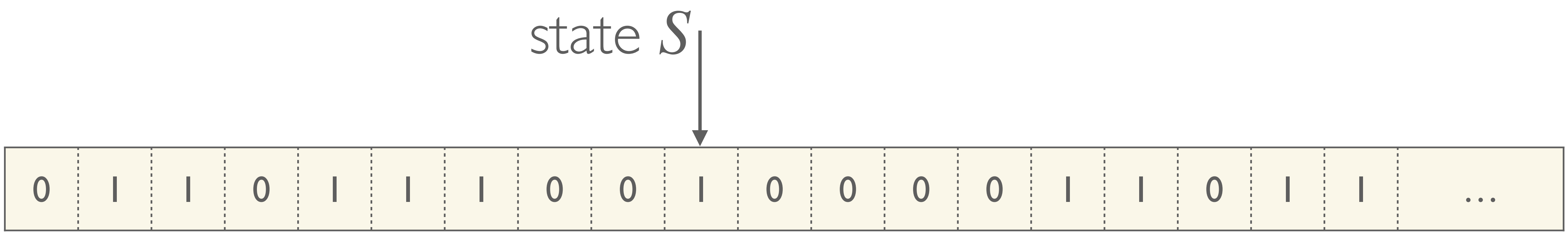
Theory of Communication (interactive)

Theory of Communication (interactive)

Central in cs:

circuits complexity,
streaming algorithm,
learning theory,
differential privacy,
computational economics
...

An example



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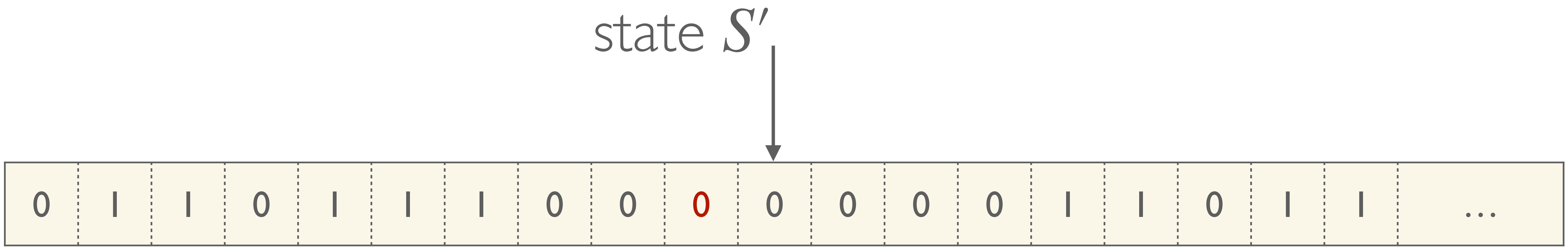

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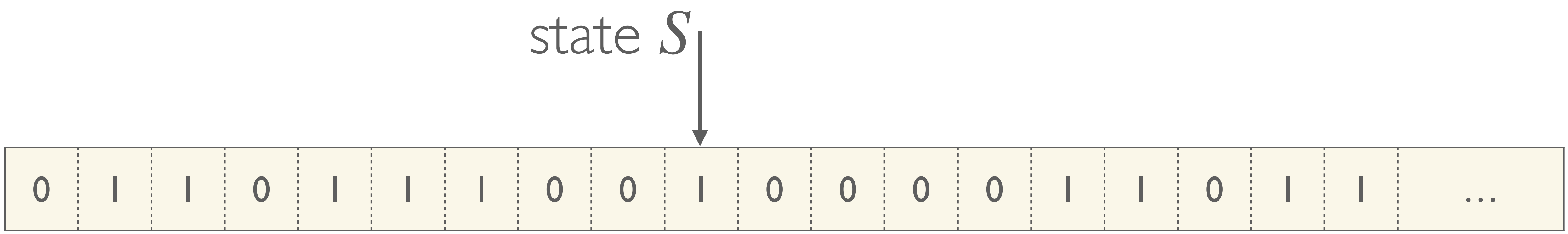

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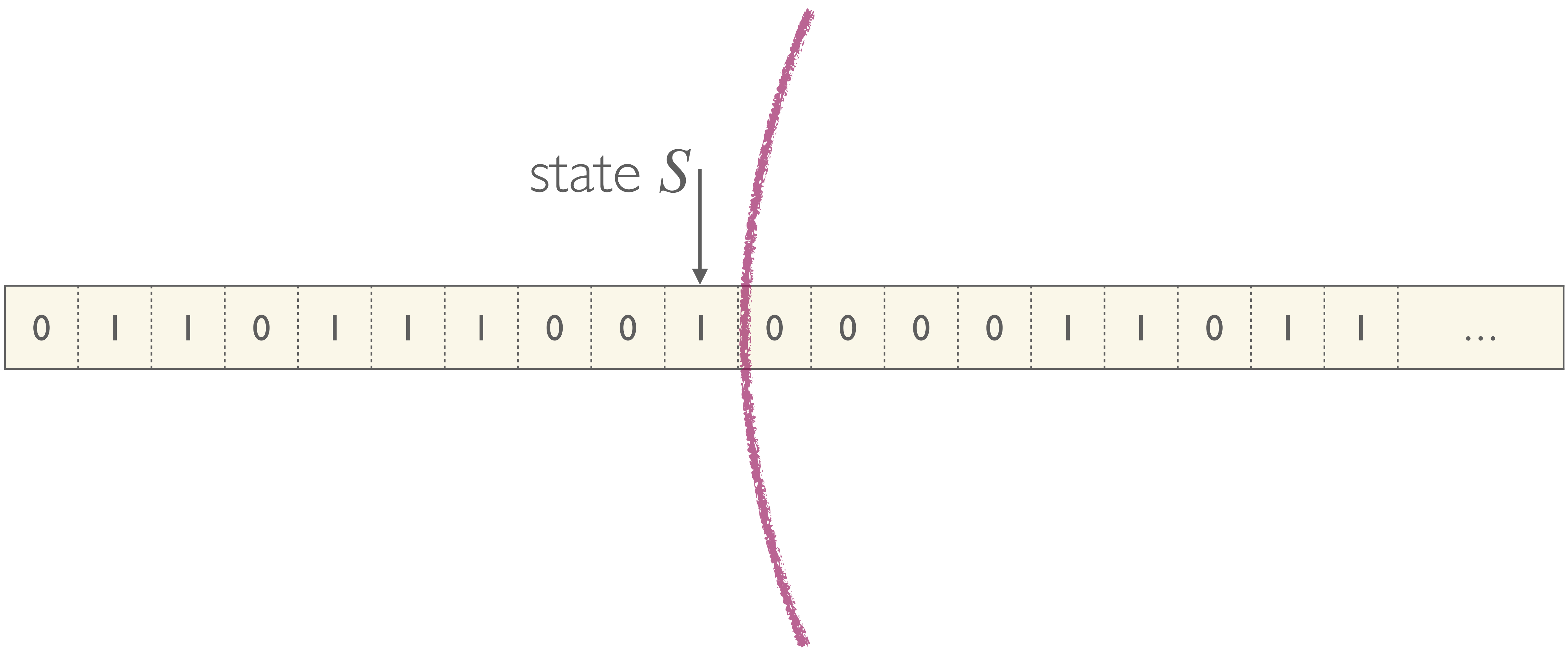

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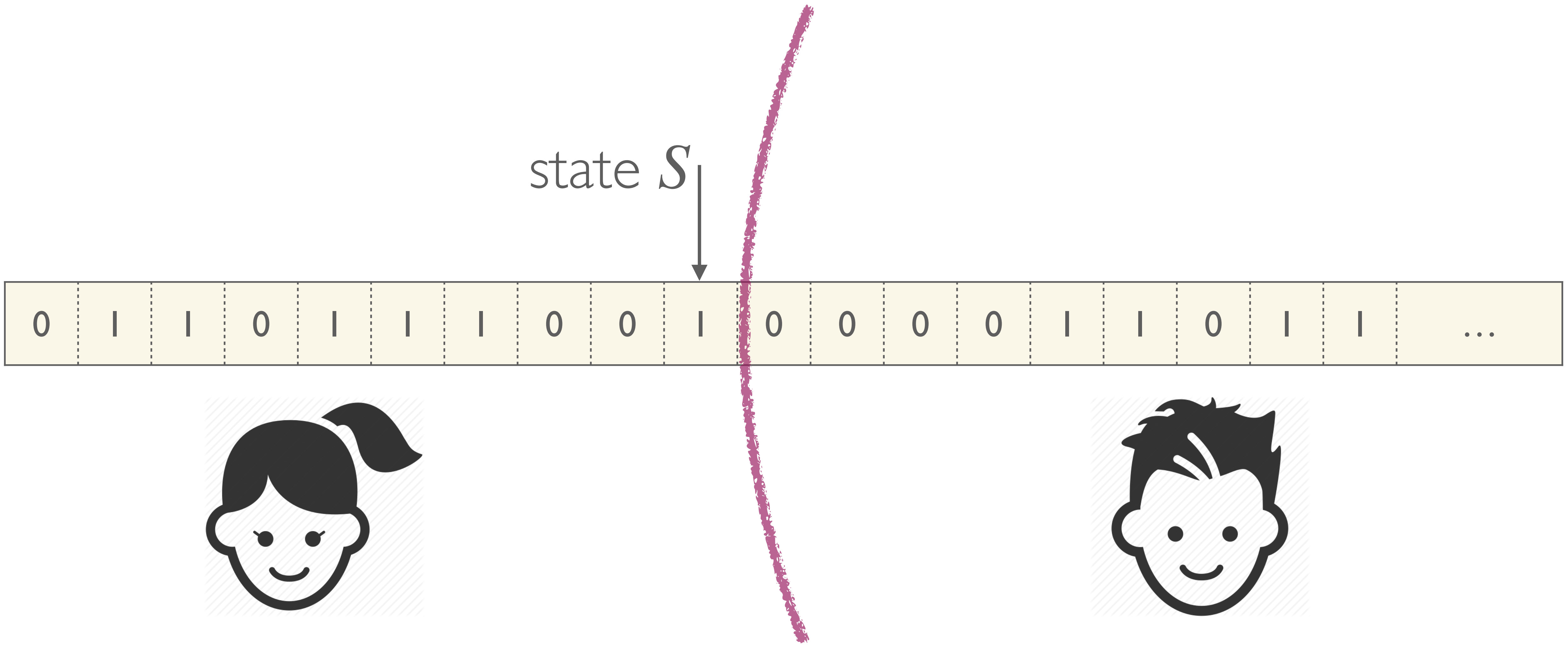

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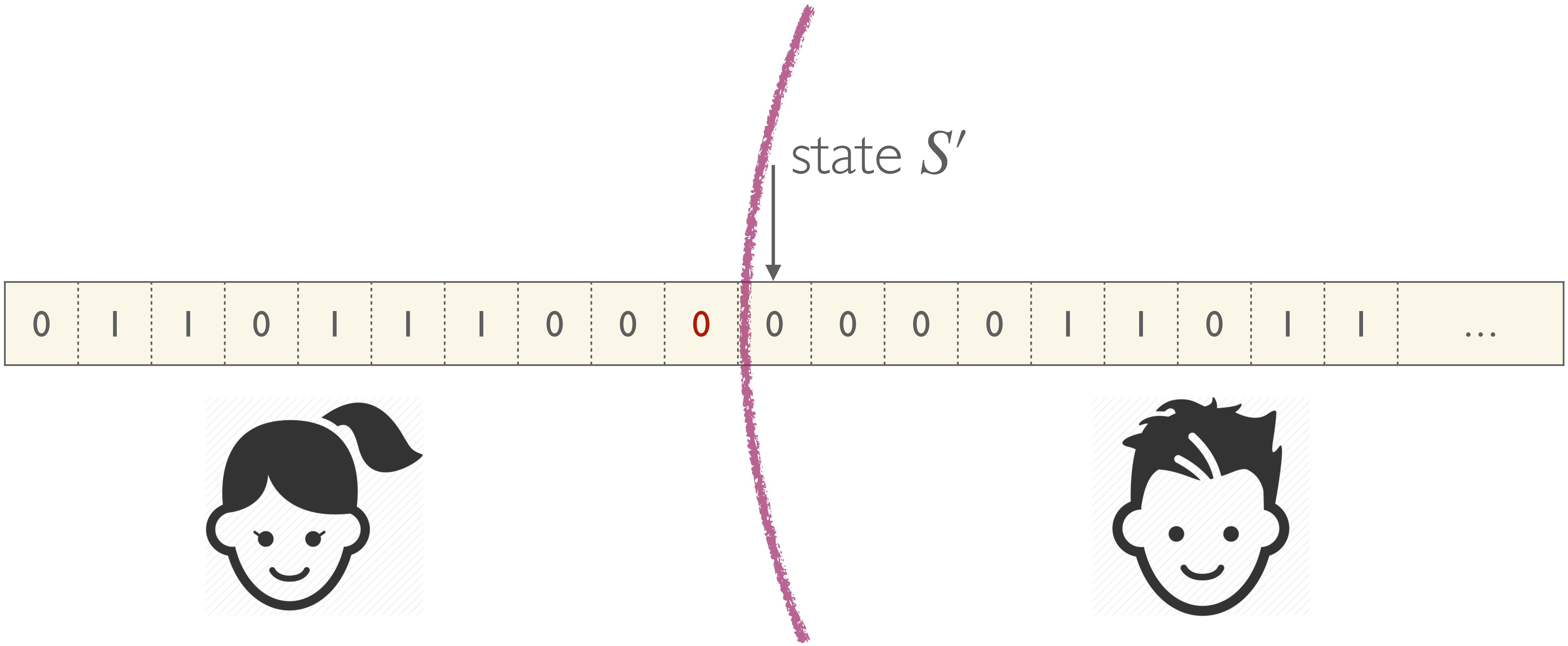

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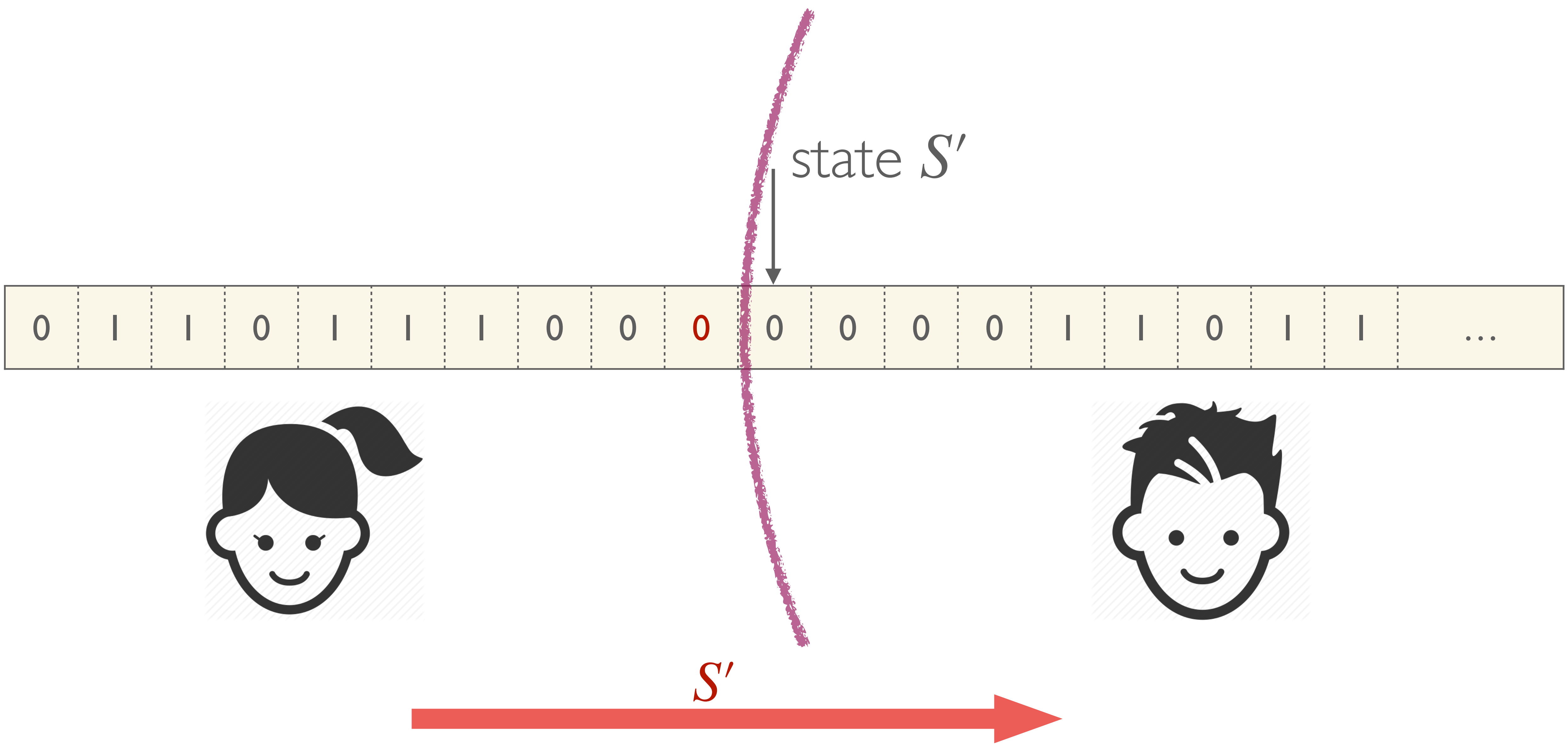

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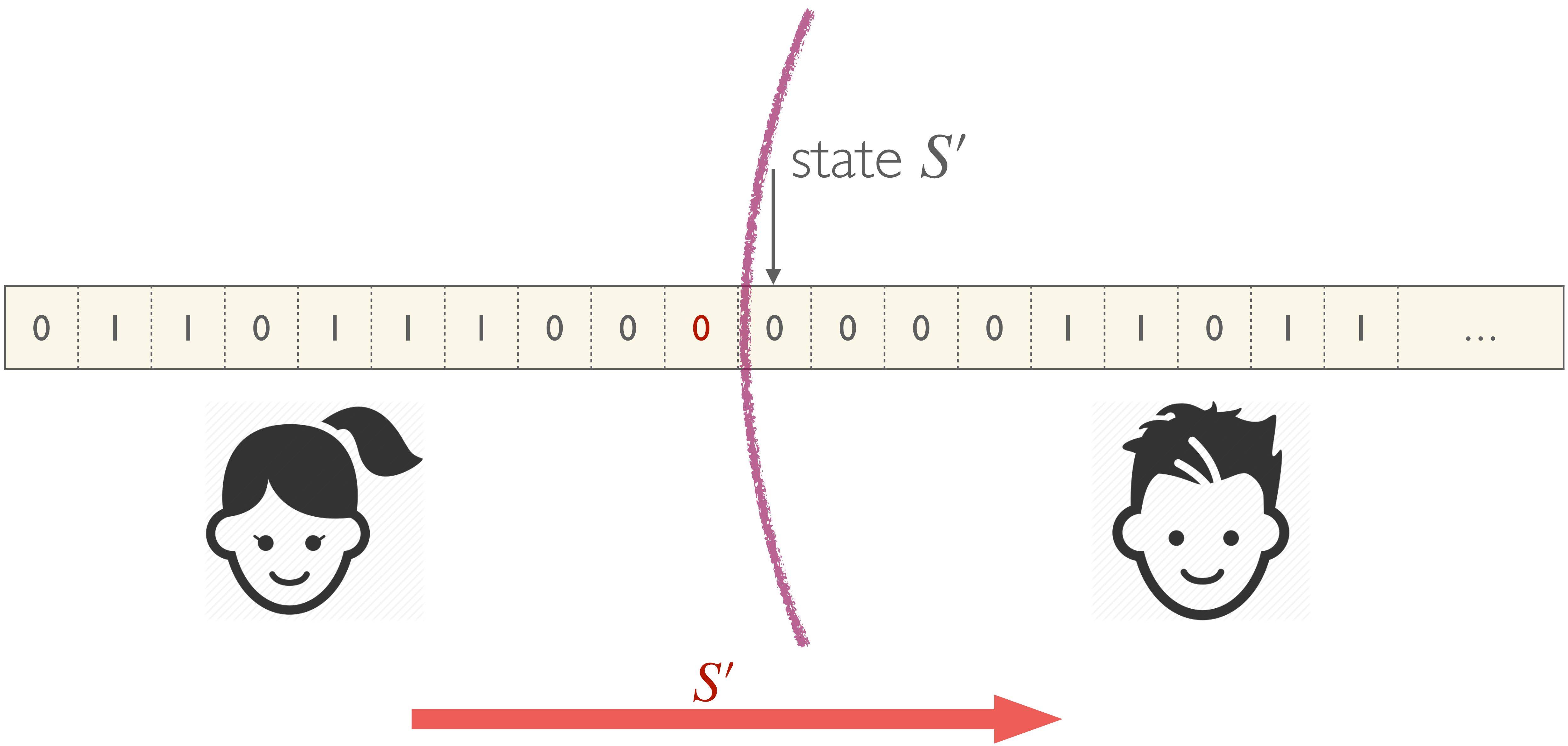


communication ≈ running time

Communication Complexity

[Babai-Frankl-Simon ’86]

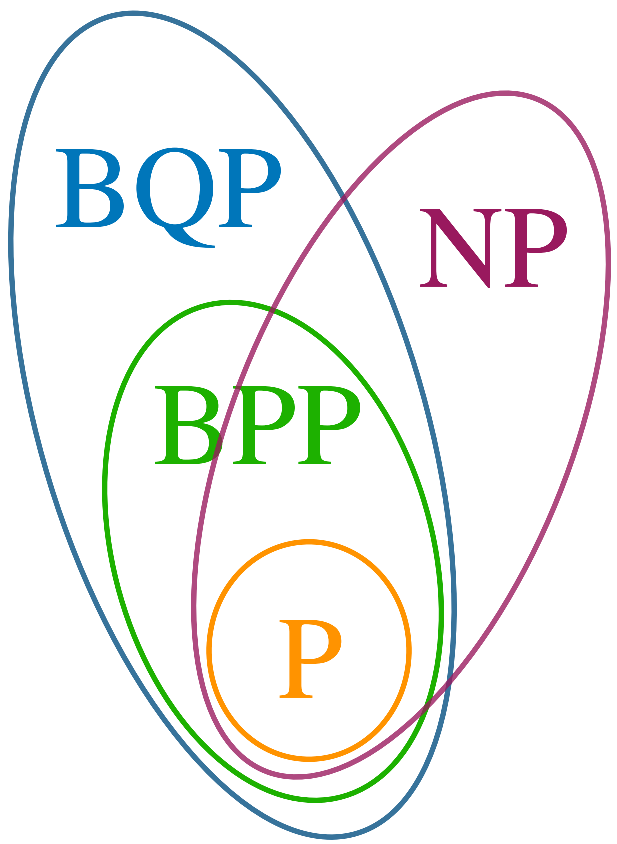
P: deterministic communication

NP: non-deterministic communication

BPP: randomized communication
(bounded-error)

BQP: quantum communication

PP: randomized communication
(unbounded-error)



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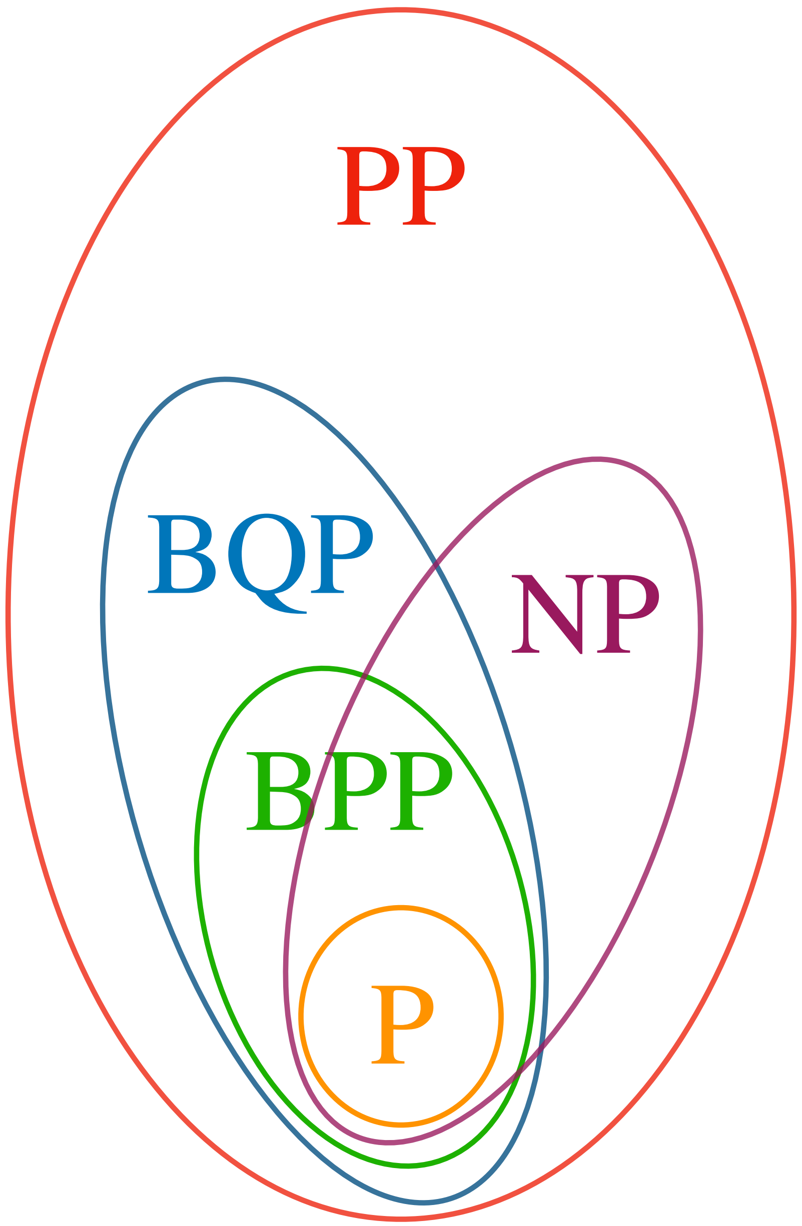

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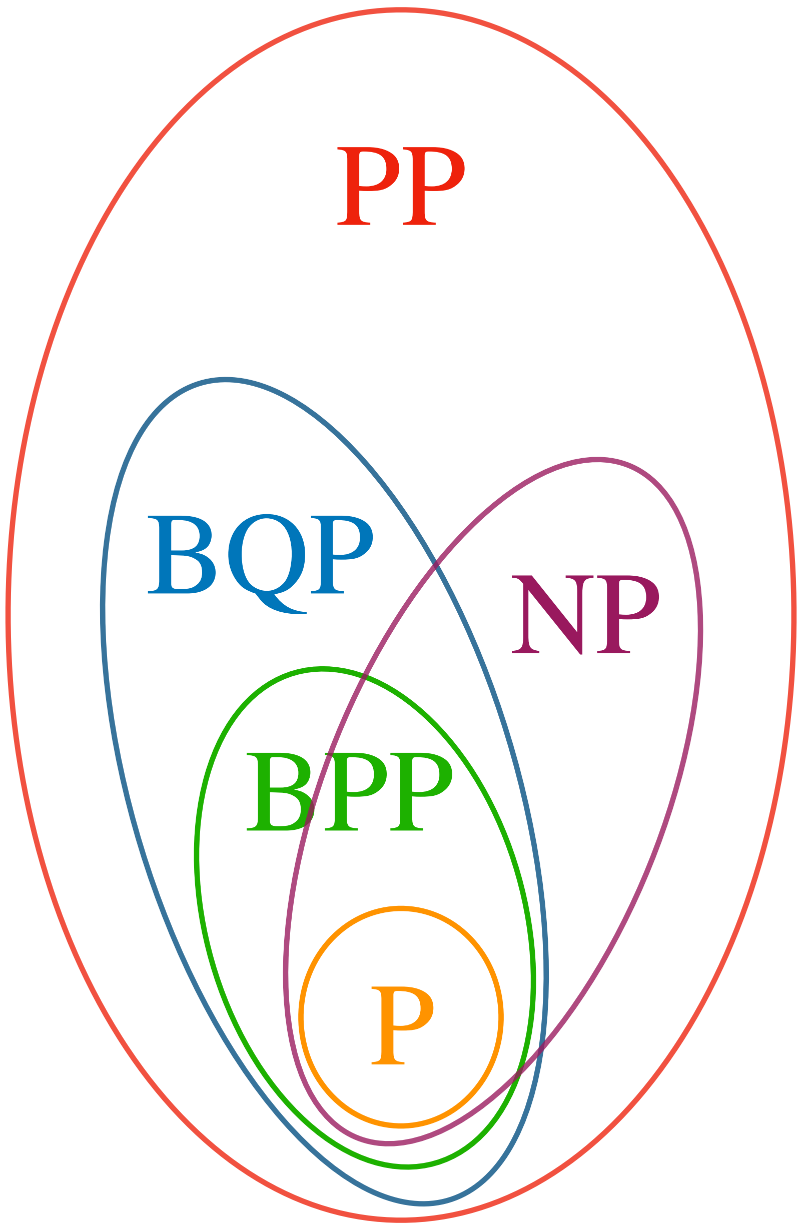
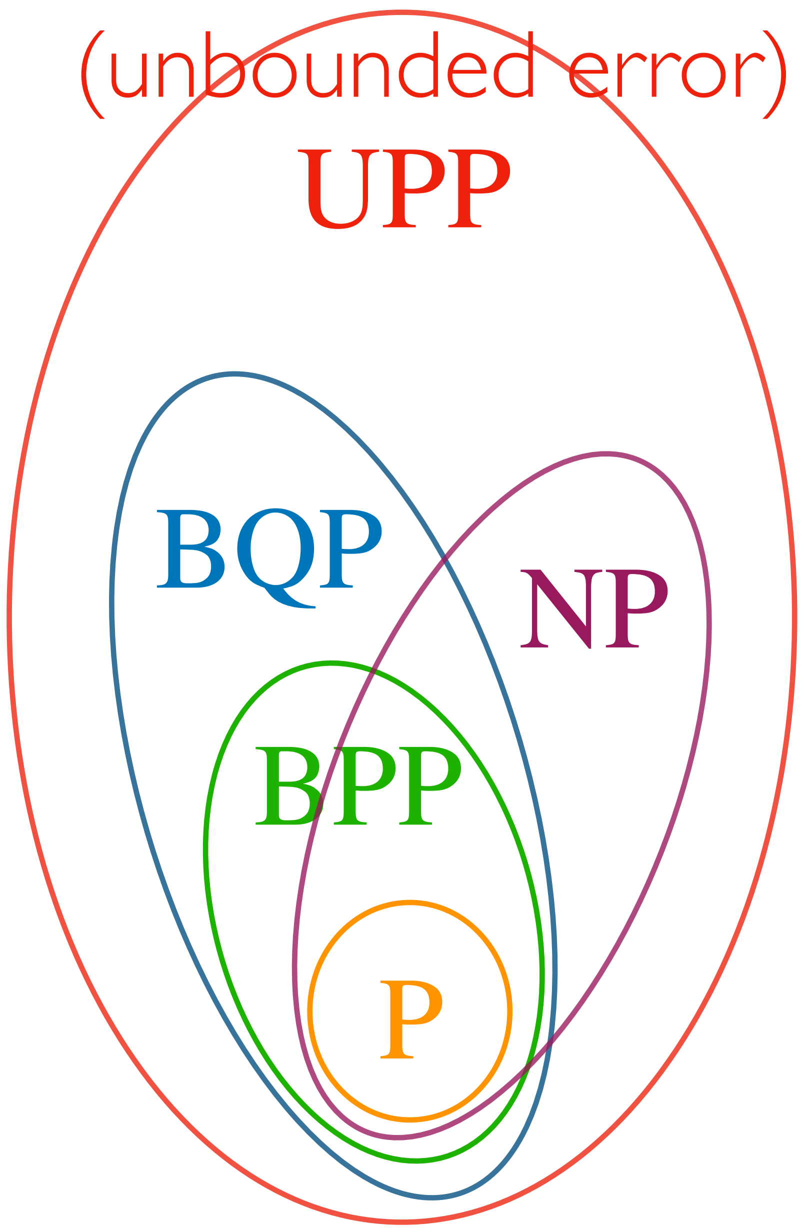


Unbounded-error communication

[Babai-Frankl-Simon '86]



In communication world,

$$\mathrm{P} \subsetneq \mathrm{BPP} \subseteq \mathrm{BQP} \subsetneq \mathrm{UPP},$$
$$\mathrm{P} \subsetneq \mathrm{NP} \subsetneq \mathrm{UPP}\,.$$

Unbounded-error communication

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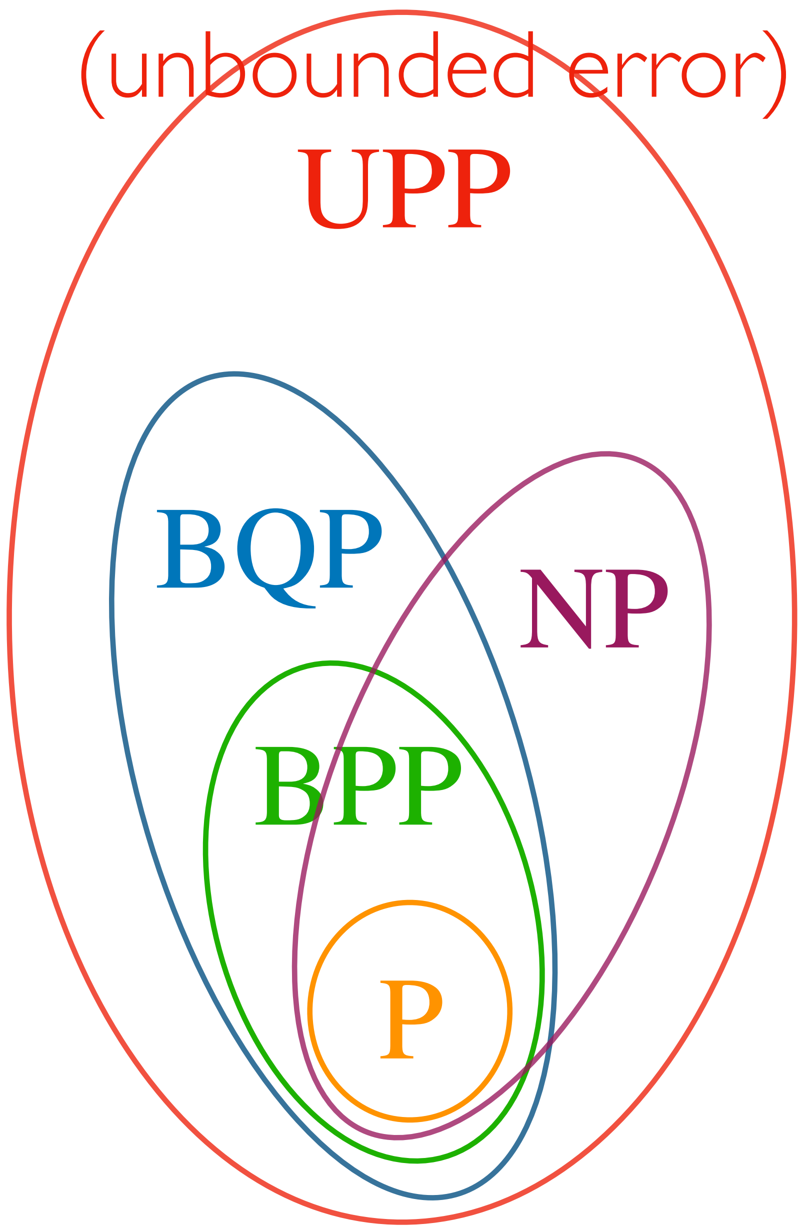

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Roadmap

- Unbounded-error communication
- $\mathbf{BQP}$ vs. $\mathbf{BPP}$ communication

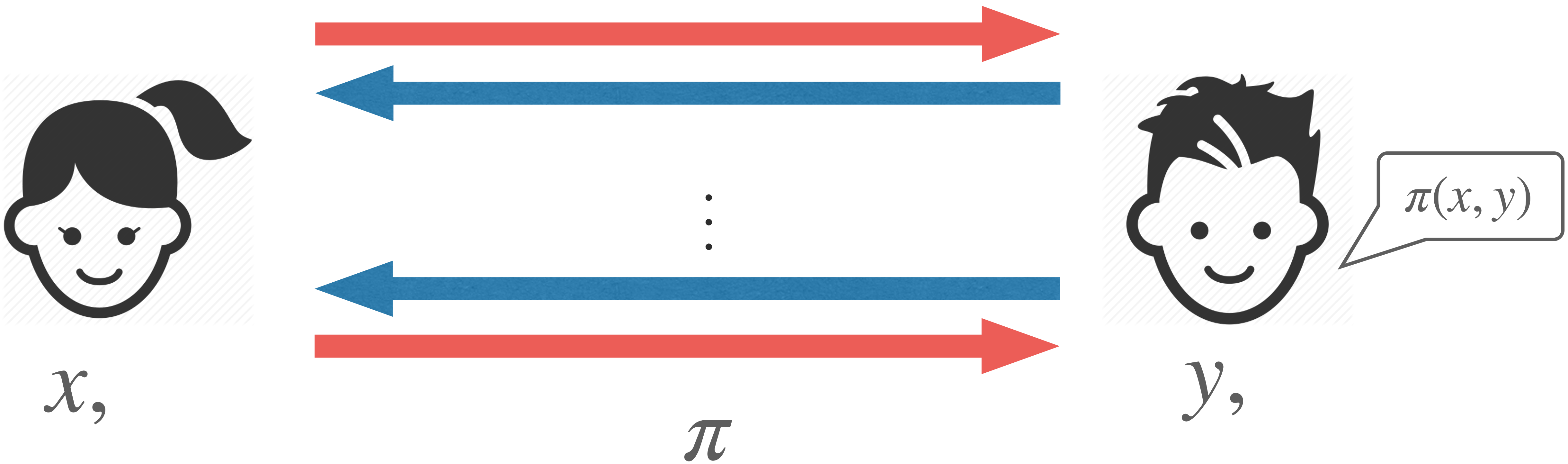
Roadmap

- Unbounded-error communication
- BQP vs. BPP communication

Unbounded-error communication

[Babai-Frankl-Simon '86]

$f : X \times Y \to \{0,1\}$



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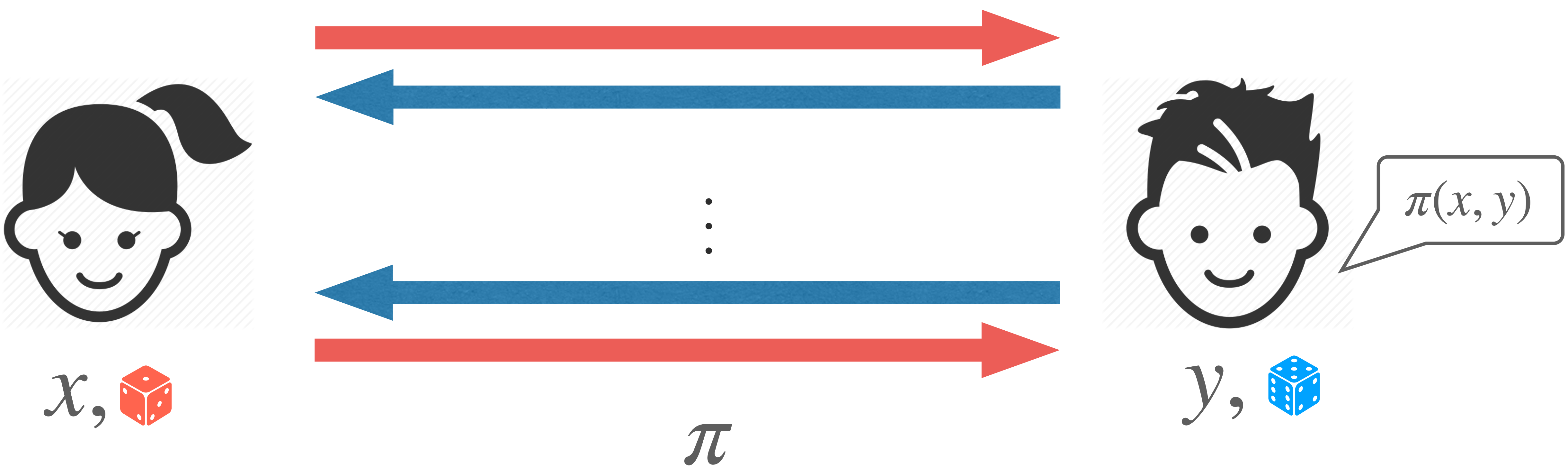

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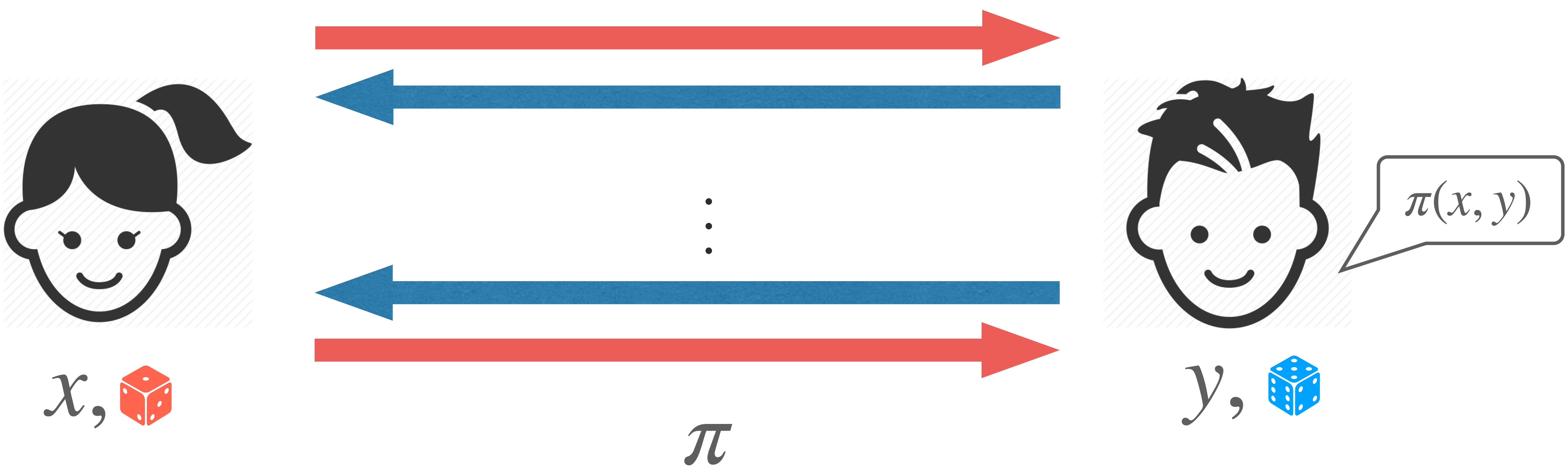

Correctness: $\Pr[\pi(x, y) = f(x, y)] > \frac{1}{2}, \ \forall x, y\,.$

Unbounded-error communication

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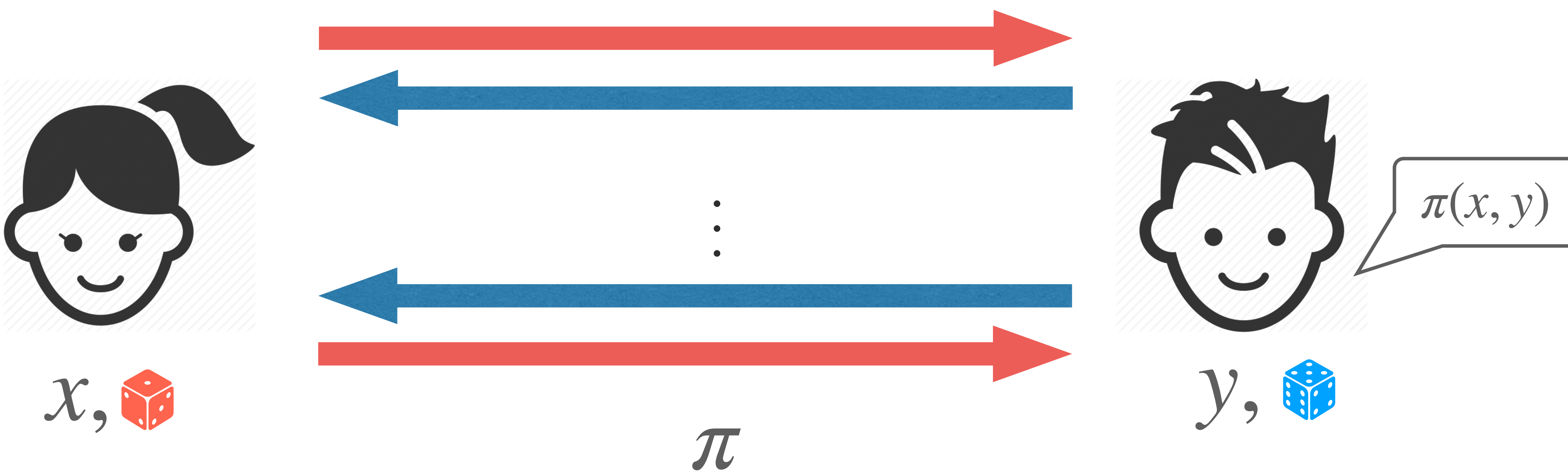

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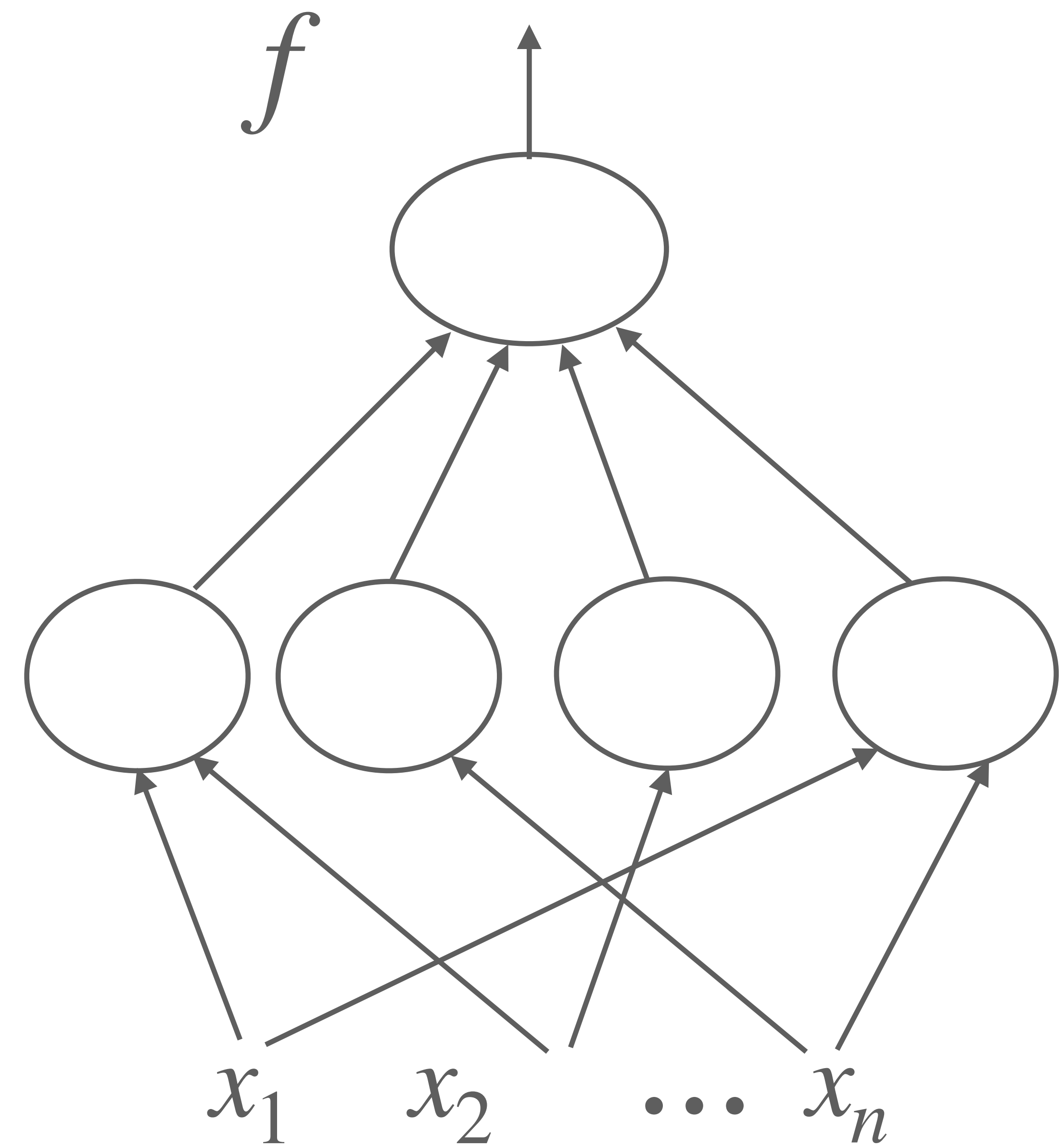
Barely larger than guess

Unbounded-error communication

$f : \{0,1\}^n \to \{0,1\}$

A simple neural network



Unbounded-error communication

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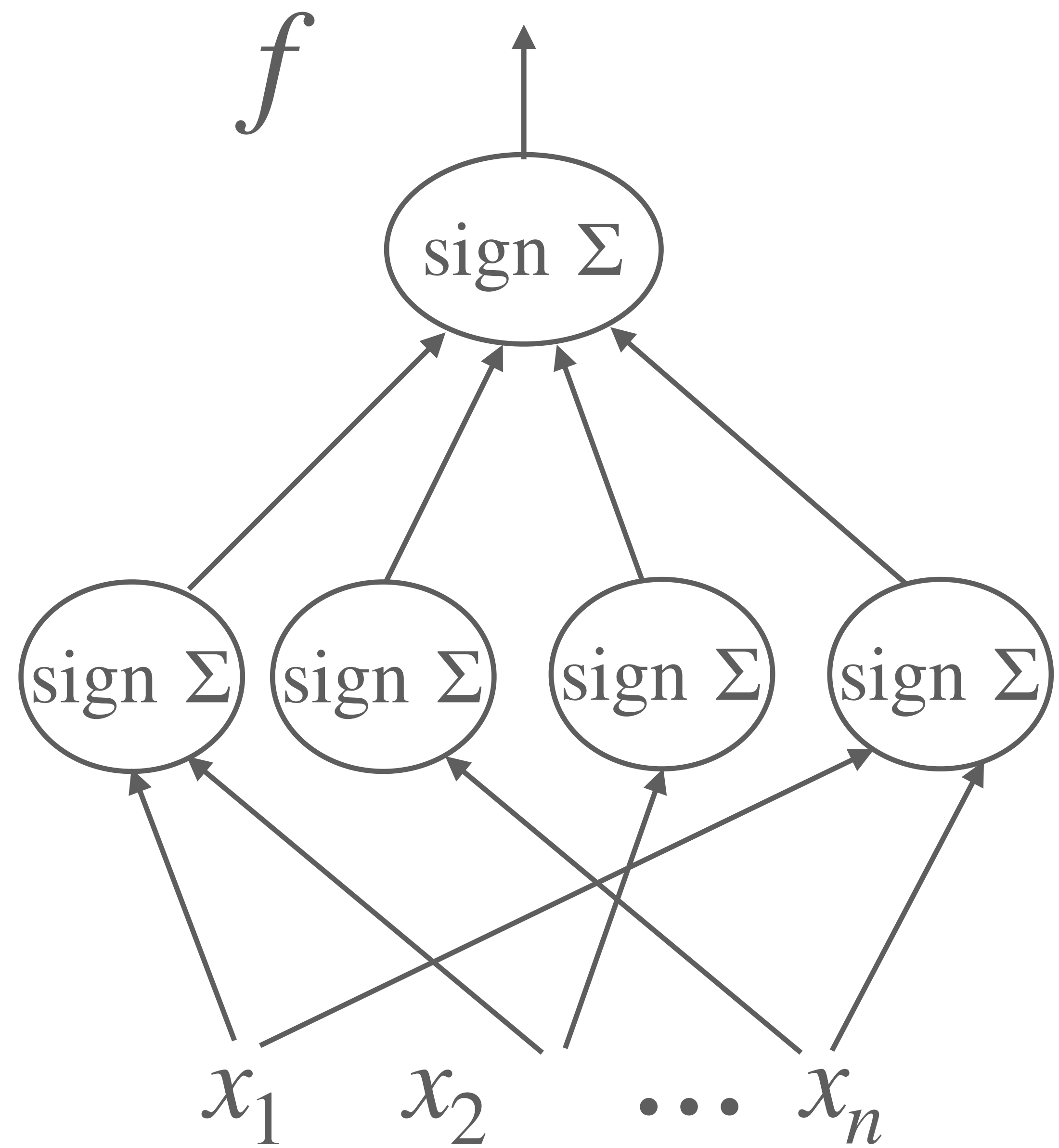

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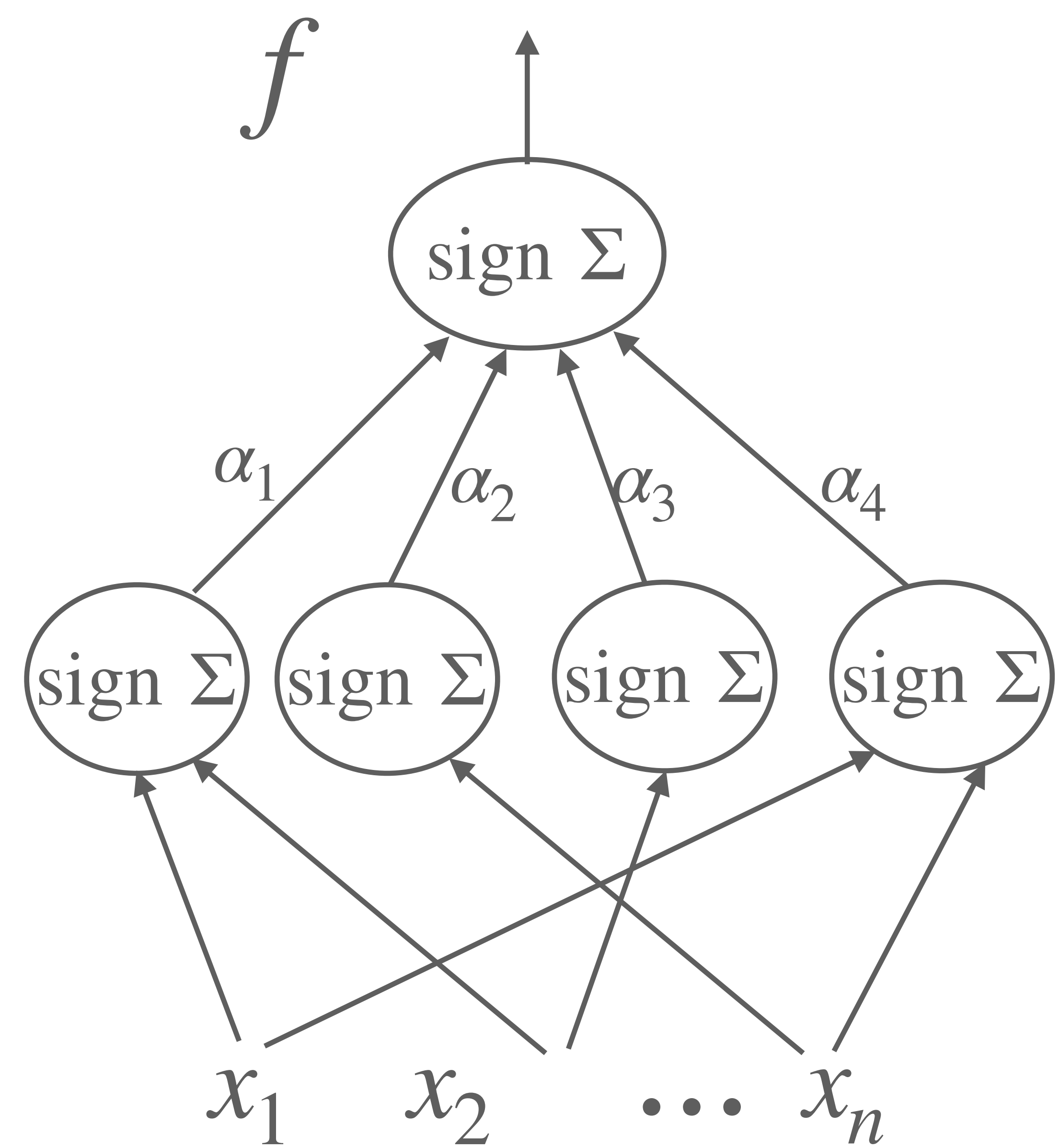

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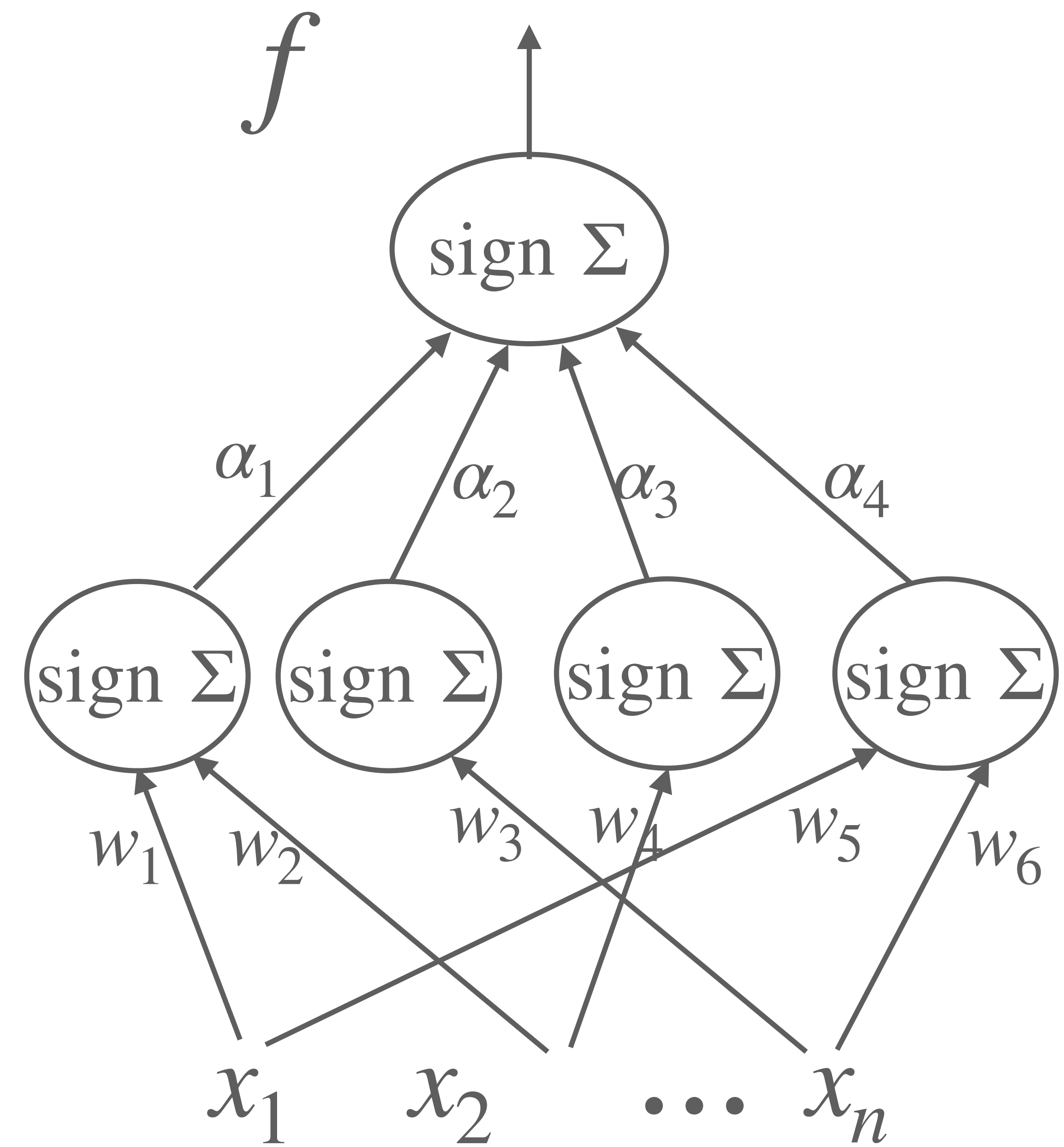

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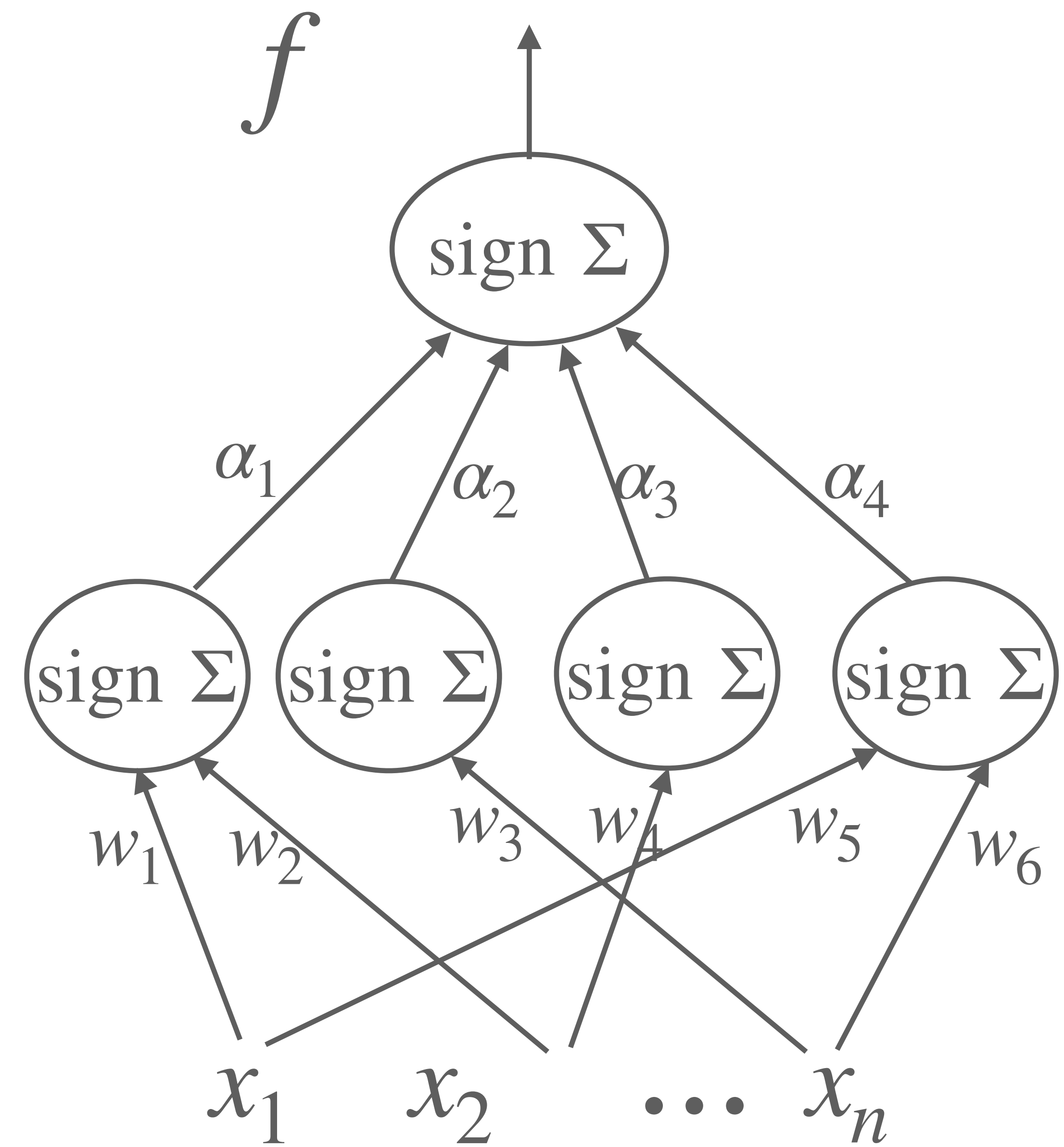


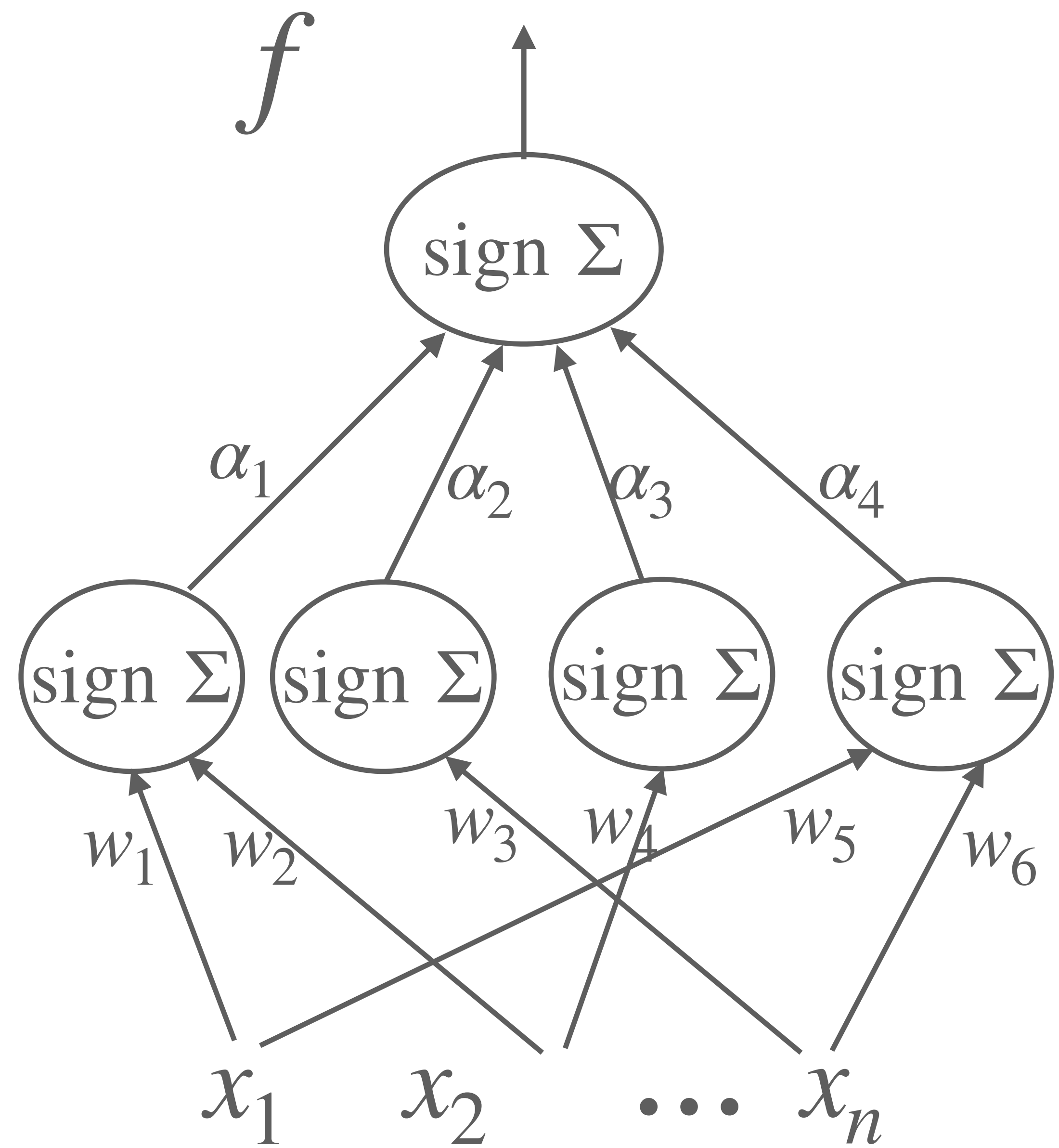
Unbounded-error communication

$f : \{0,1\}^n \to \{0,1\}$

$x_1 \ldots x_{n/2}$

$x_{n/2+1} \cdots x_n$

A simple neural network



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$x_1 \ldots x_{n/2}$

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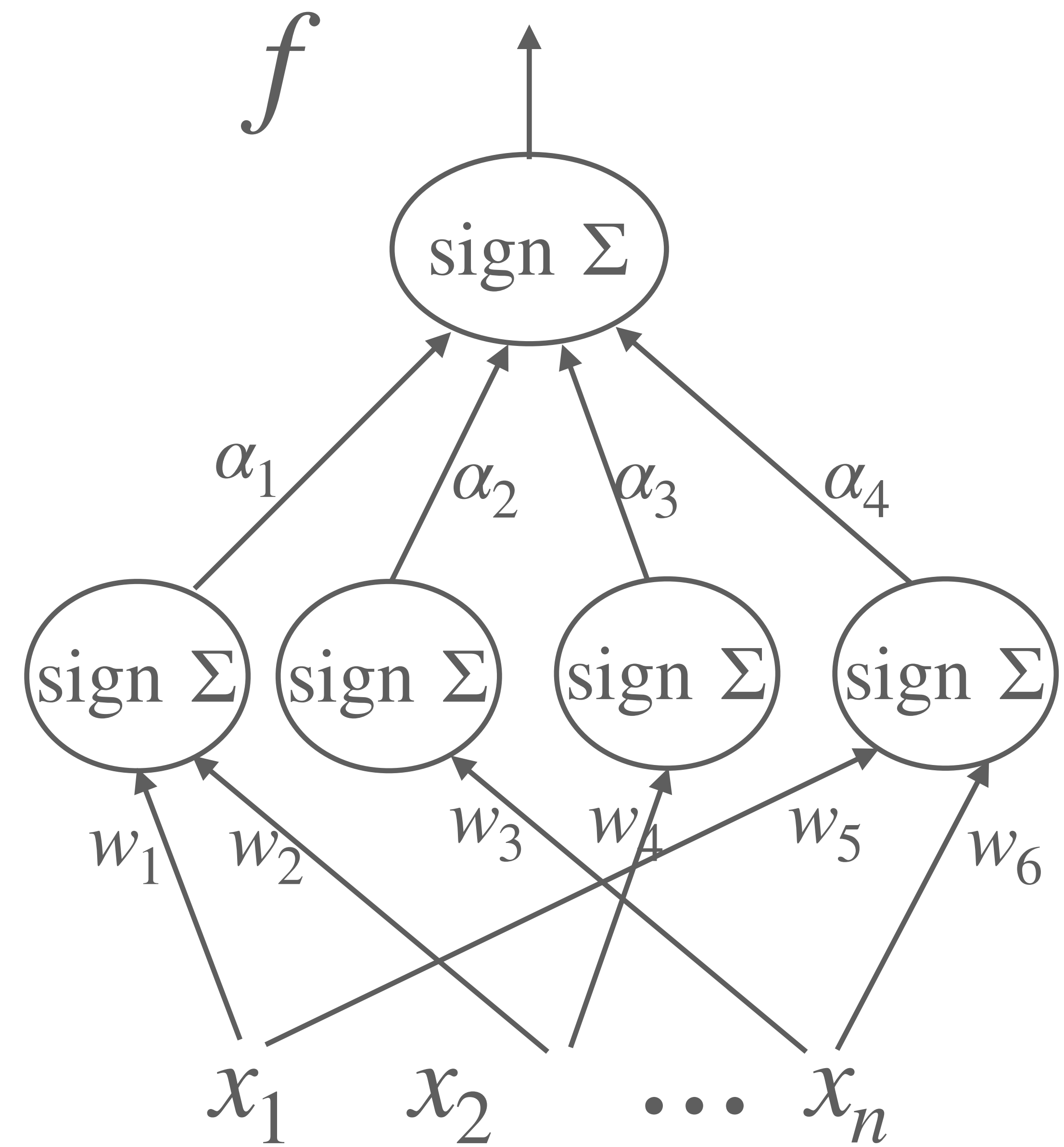
Theorem * (Forster et al. '01).

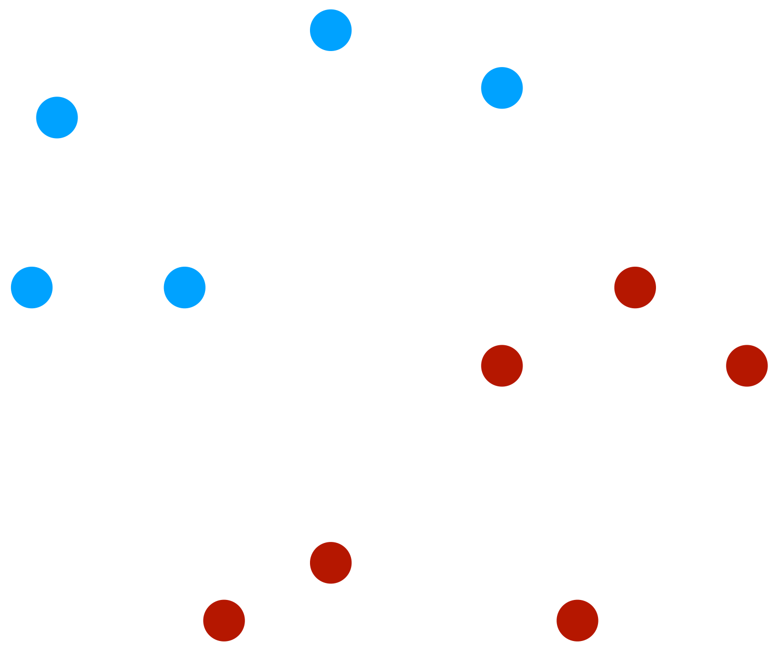
size$(f) \gtrsim 2^{\Omega(U(f))}$.

A simple neural network



Unbounded-error communication

Learn halfspaces

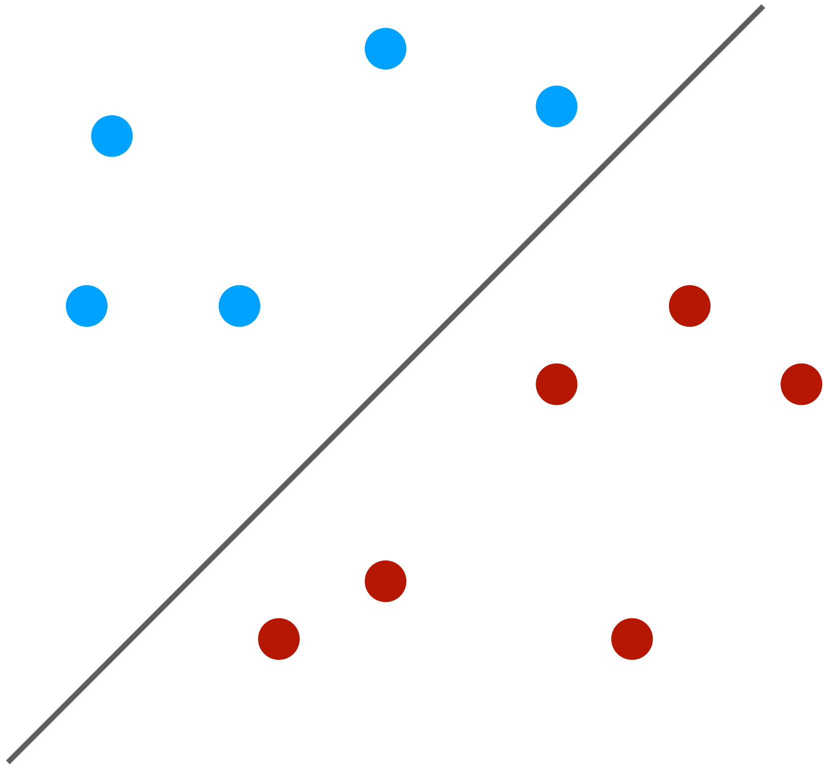


$$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_0 \geq 0$$

learn the coefficients a

Unbounded-error communication

Learn halfspaces



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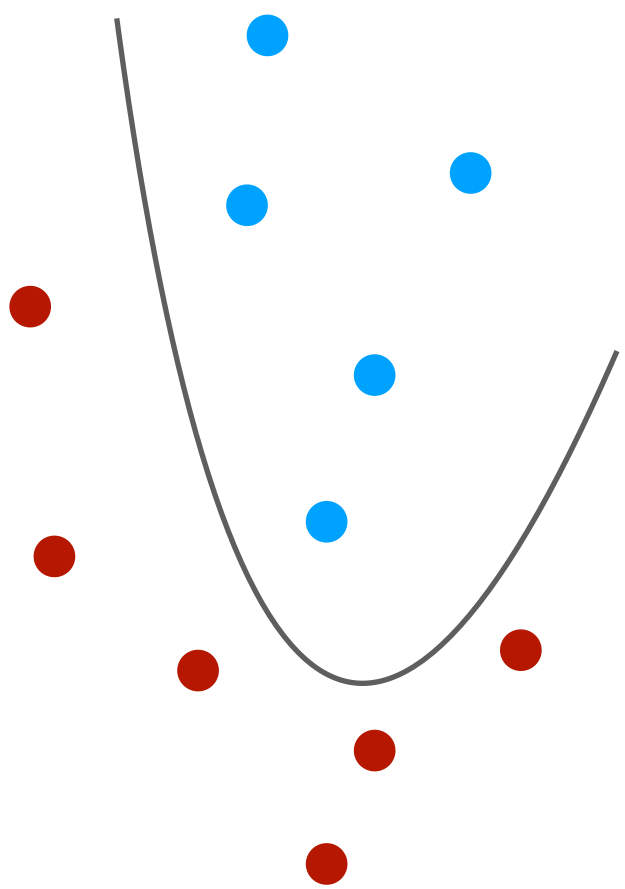
Unbounded-error communication

Learn low degree polynomials



$$a_1x_1 + a_2x_2 + a_3x_3 + a_{12} \cdot x_1x_2 + \\ a_{13} \cdot x_1x_3 + a_{23} \cdot x_2x_3 \geq 0$$

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Unbounded-error communication

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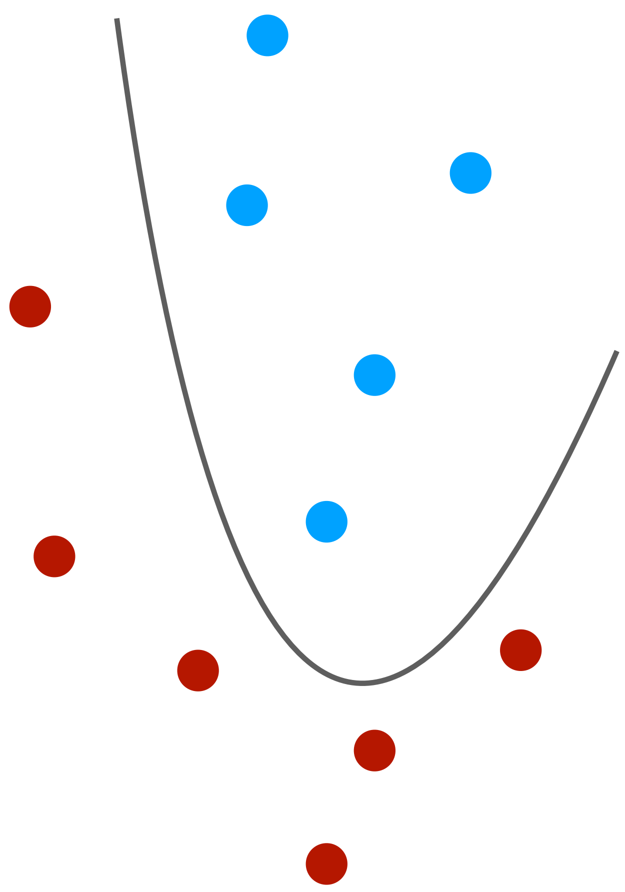

$$a_1x_1 + a_2x_2 + a_3x_3 + a_{12}\cdot \overset{y_{12}}{\cancel{x_1x_2}} +$$
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Unbounded-error communication

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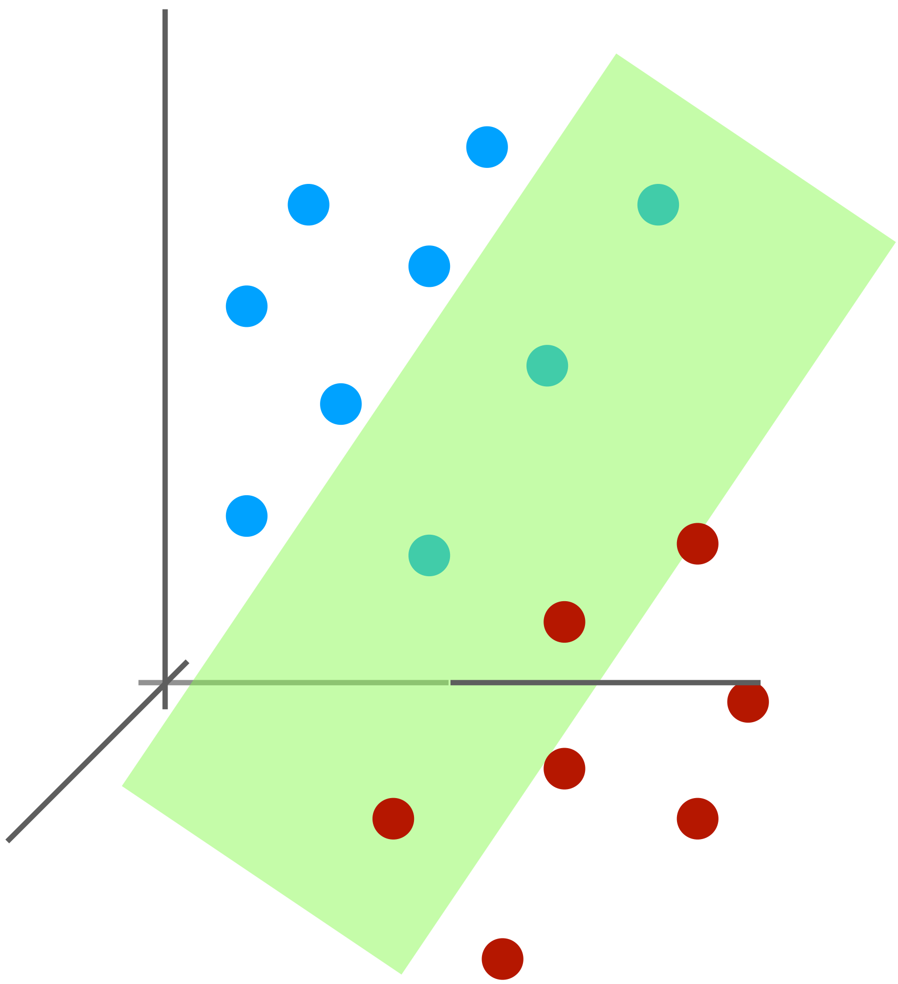

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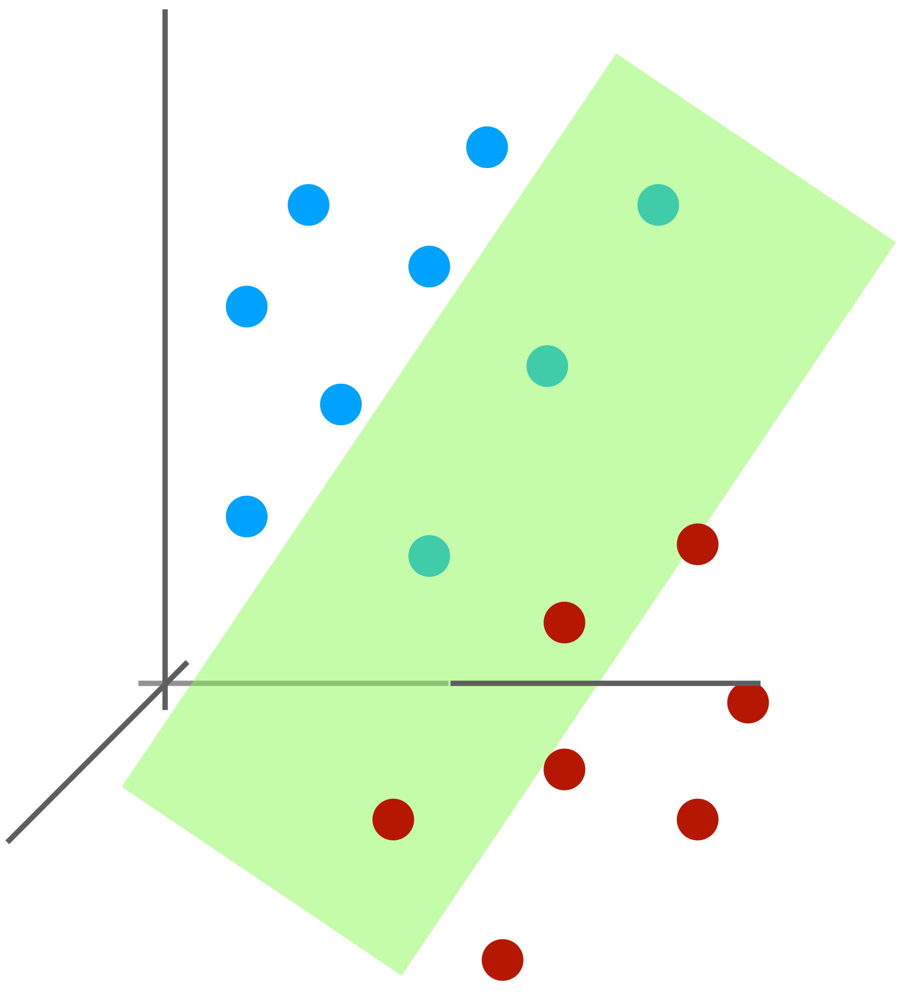

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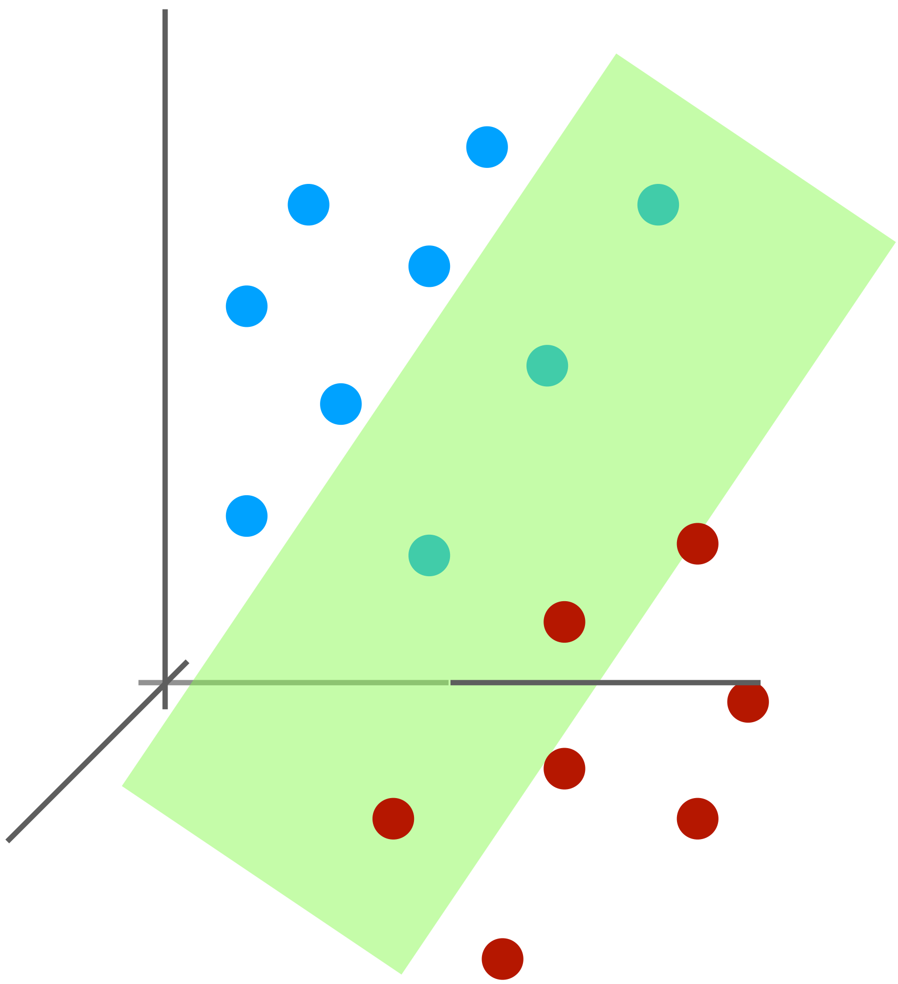
Def. $f:\{0,1\}^n \to \{0,1\}$,
$\deg_{\pm}(f)$: min degree of a separating curve

learn the coefficients a

Unbounded-error communication

Embedding into spaces with larger dimension



Dimension complexity

$\mathscr{C}$ concept class,

$\mathbf{dc}(\mathscr{C})$ minimum dimension for such embedding

Unbounded-error communication

Embedding into spaces with
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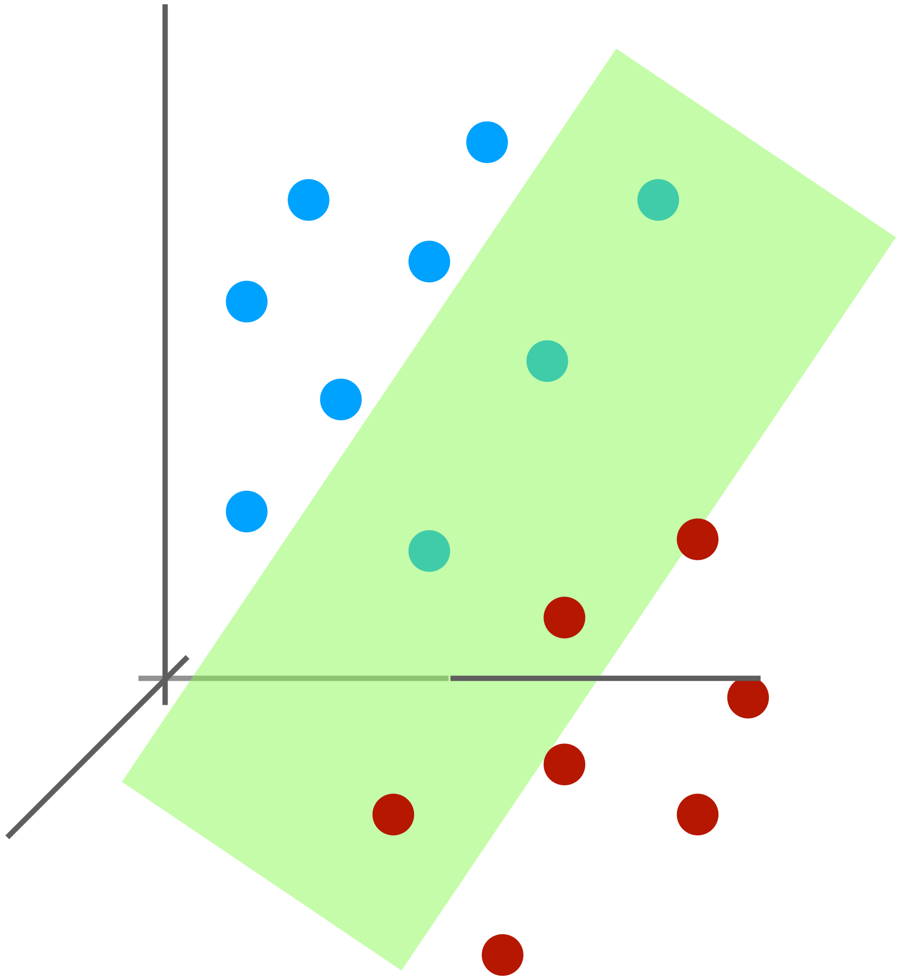

Dimension complexity

$\mathscr{C}$ concept class,

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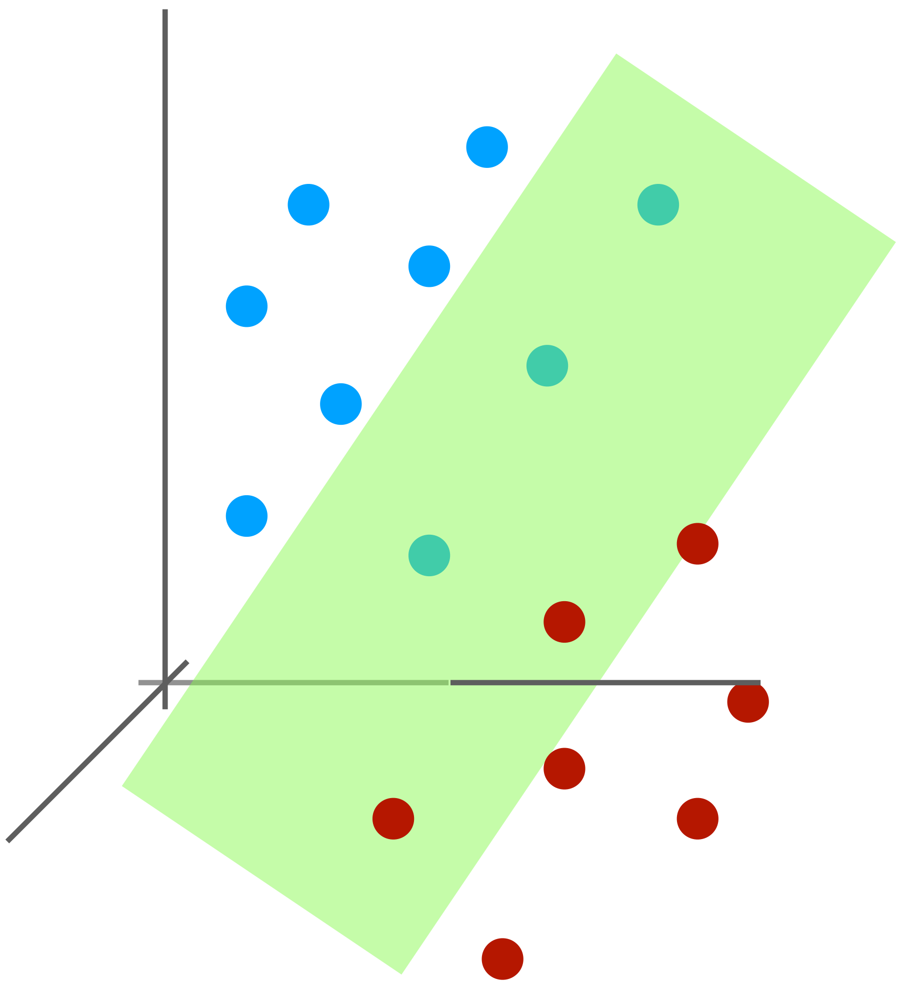
for such embedding

Surprisingly powerful!
Captures many results in PAC
learning model.

Unbounded-error communication

Embedding into spaces with larger dimension



Dimension complexity

$\mathscr{C}$ concept class,

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Fact (folklore).

$\mathrm{dc}(\mathscr{C}) = 2^{\Theta(U(M_{\mathscr{C}}))}$,

where $M_{\mathscr{C}}(f, x) = f(x)$.

$f \in \mathscr{C}$ $\quad\quad$ x

goal: output f(x)

Unbounded-error communication

Theorem (Sherstov-W. 19)

$\mathrm{U}(\mathrm{AC}^0) \geq \Omega(n^{1-\epsilon})$.

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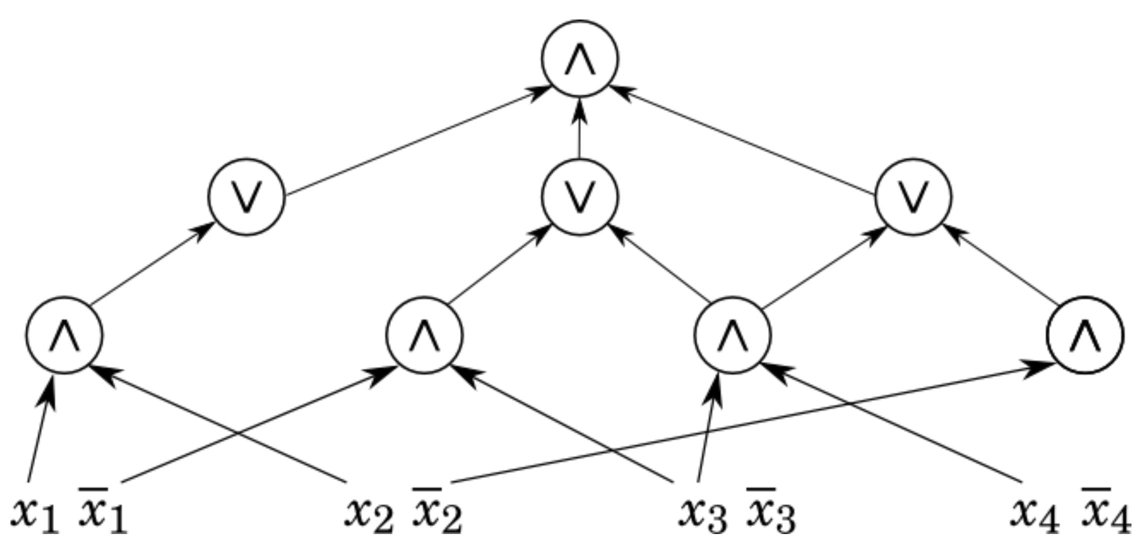

AC^0: constant depth, polynomial #gates ($\wedge$, $\vee$, $\neg$)

Unbounded-error communication

Theorem (Sherstov-W. 19)

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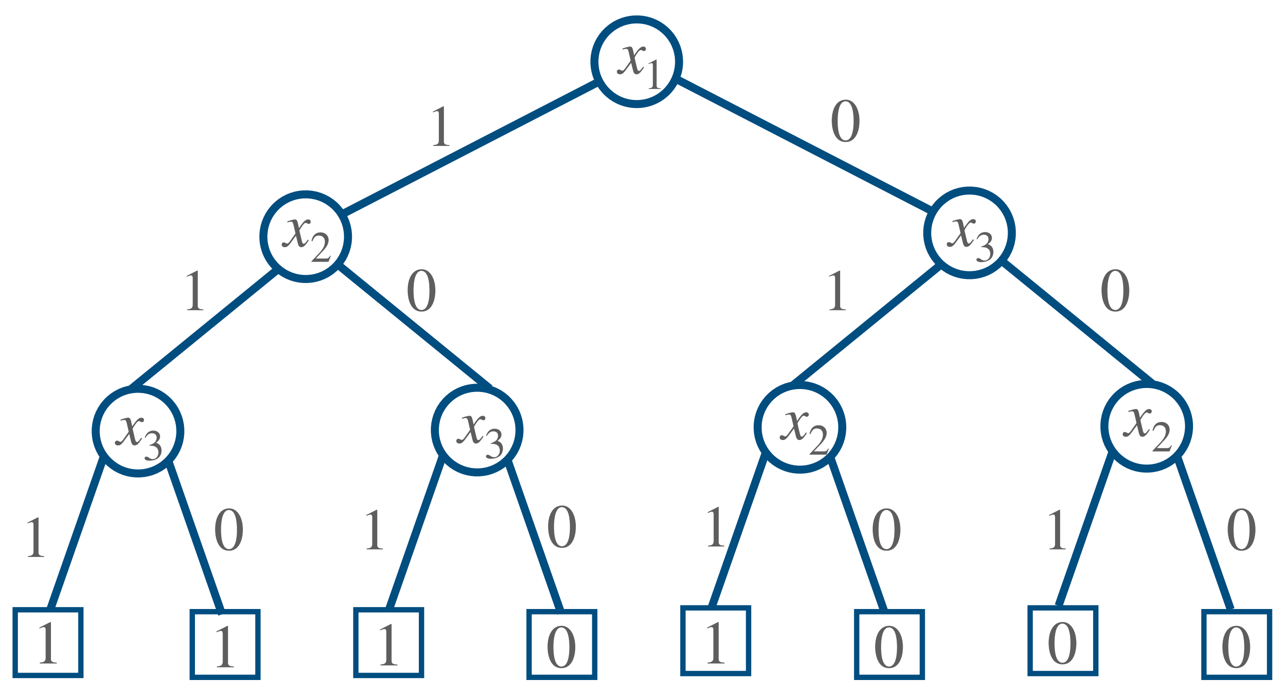

a decision tree

Unbounded-error communication

Theorem (Sherstov-W. 19)
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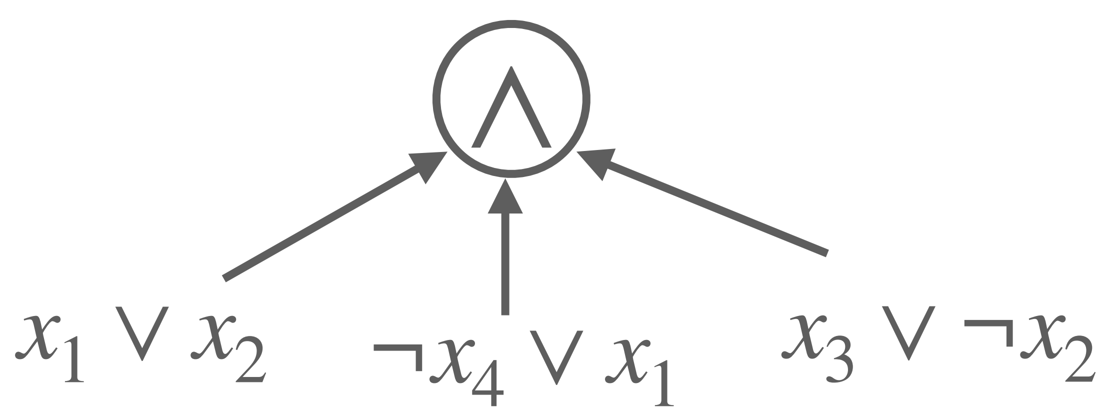
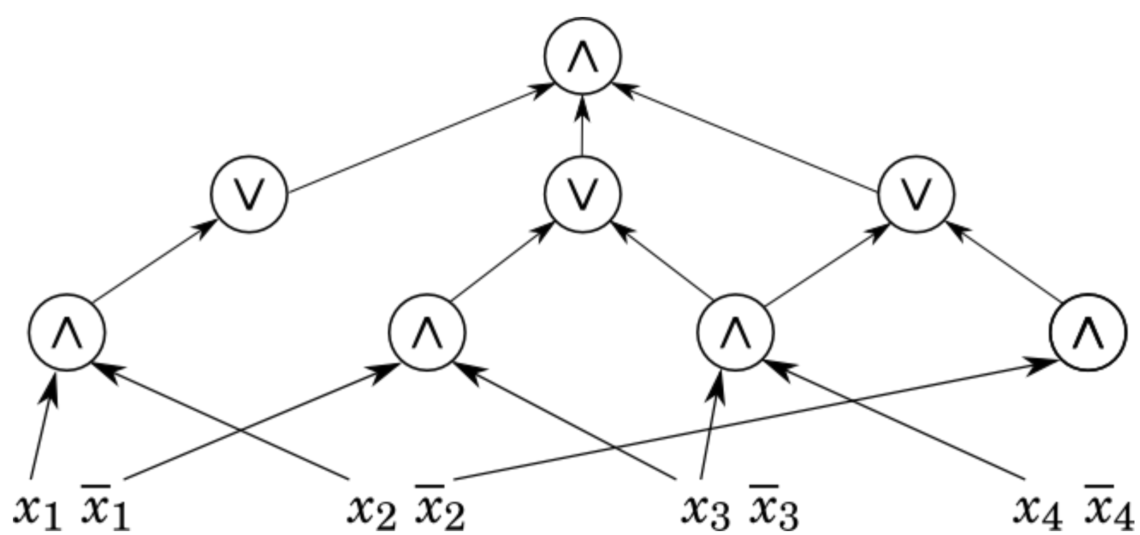


a CNF

Unbounded-error communication

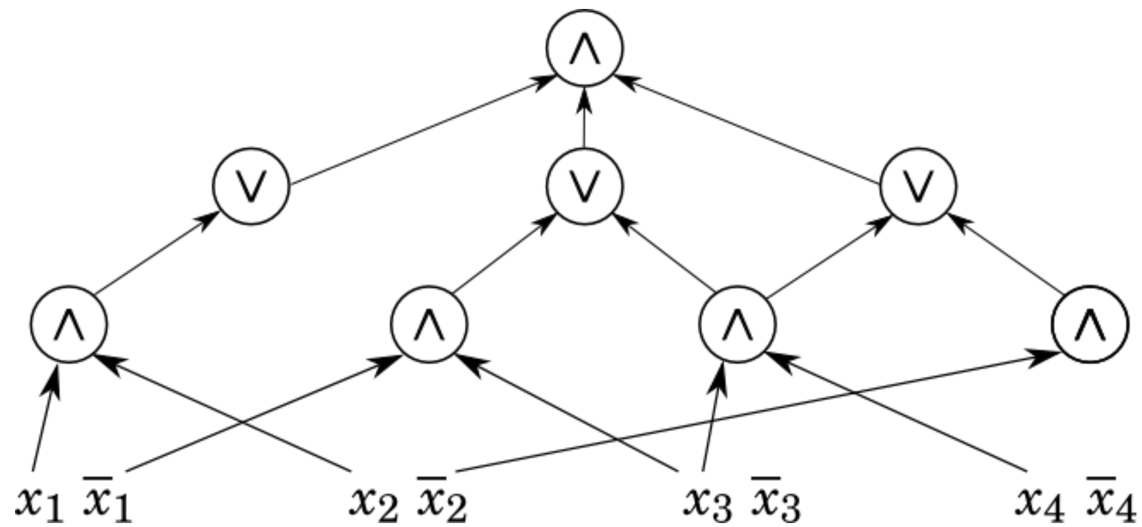
Theorem (Sherstov-W. 19)

$\mathrm{U}(\mathrm{AC}^0) \geq \Omega(n^{1-\epsilon})$.



AC^0: constant depth, polynomial #gates ($\wedge$, $\vee$, $\neg$)

Constant depth circuits (AC^0)



Circuits lower bound "P vs NP"

[FSS84, Ajt83, Yao85, Has86, Aar10, RS10, LV11, BIL12, IMP12, Has14, AA15, LRR17, Ros18, Vio18]

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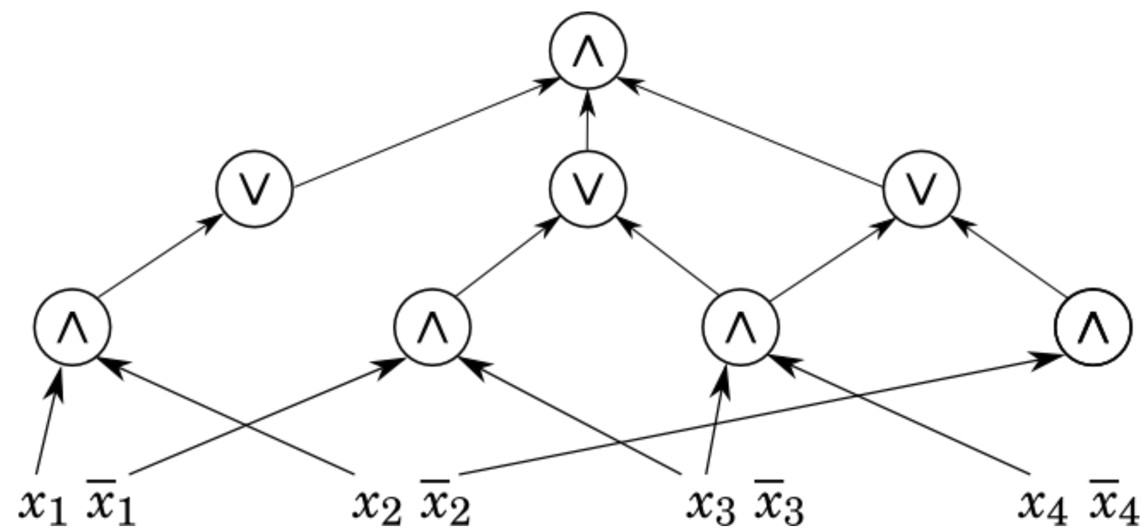

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"P vs BPP"

[LN90, Nis91, Baz07, Raz08, Bra09, ETT10, GMR13, TX13, Tal14, CSV15, HS16, Tal17, ST18, DHH18, Lyu22]

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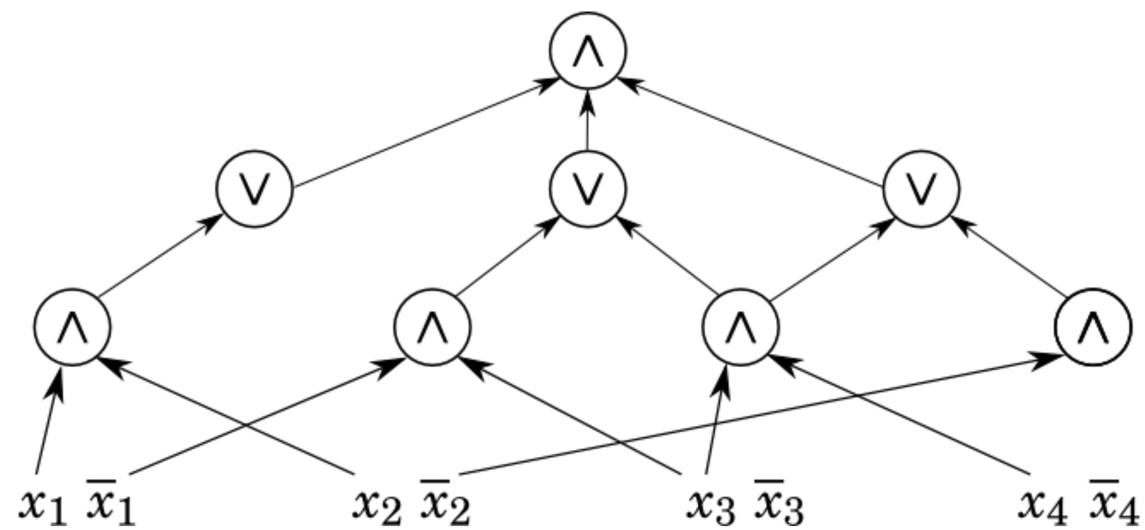

Quantum supremacy?

[AS04, Amb07, ACR+10, BM10, Rei10, Bel12, BS13, RT19]

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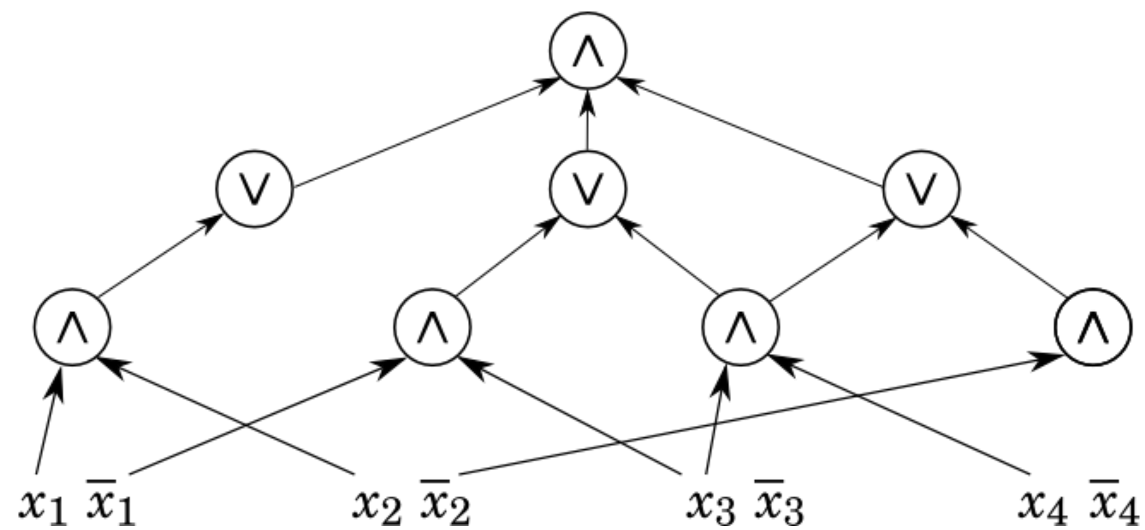

Quantum supremacy? [AS04, Amb07, ACR+10, BM10, Rei10, Bel12, BS13, RT19]

Learning [LMN93, Jac02, BES03, OS03, KOS04, KS04, LMSS07, AMY16, DRG17, AGS20]

……

Threshold degree of AC^0

Theorem (Sherstov-W. 19).
$\deg_{\pm}(\mathrm{AC}^0) = \Omega(n^{1-\epsilon})$.

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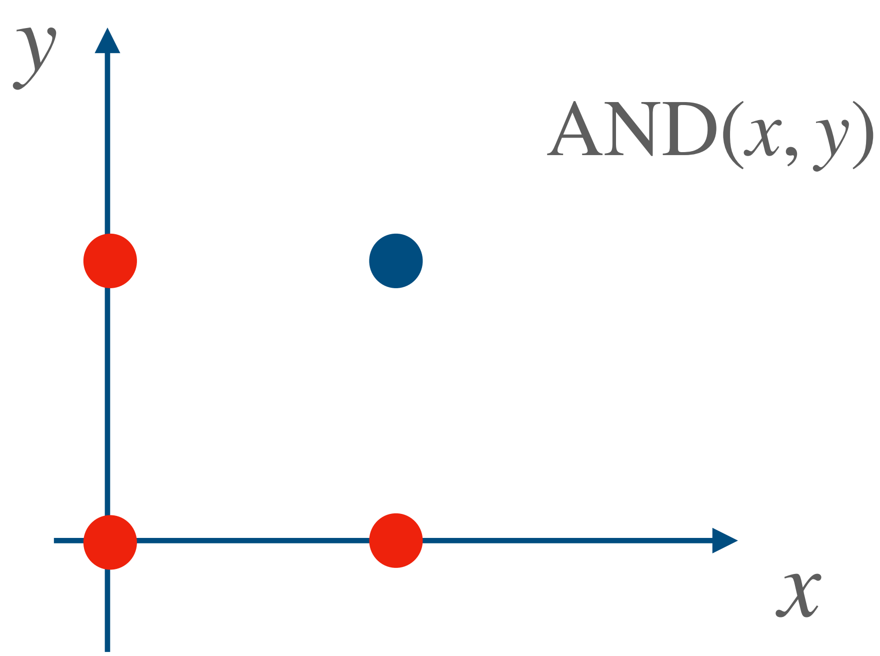

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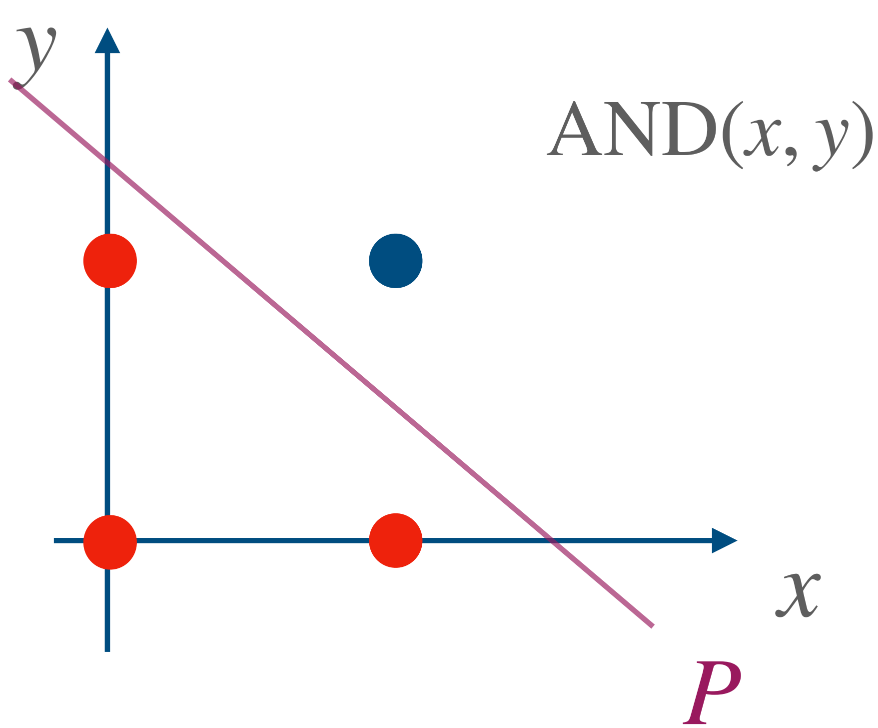

$(n - 1/2 - x_1 - x_2 - \cdots - x_n)$

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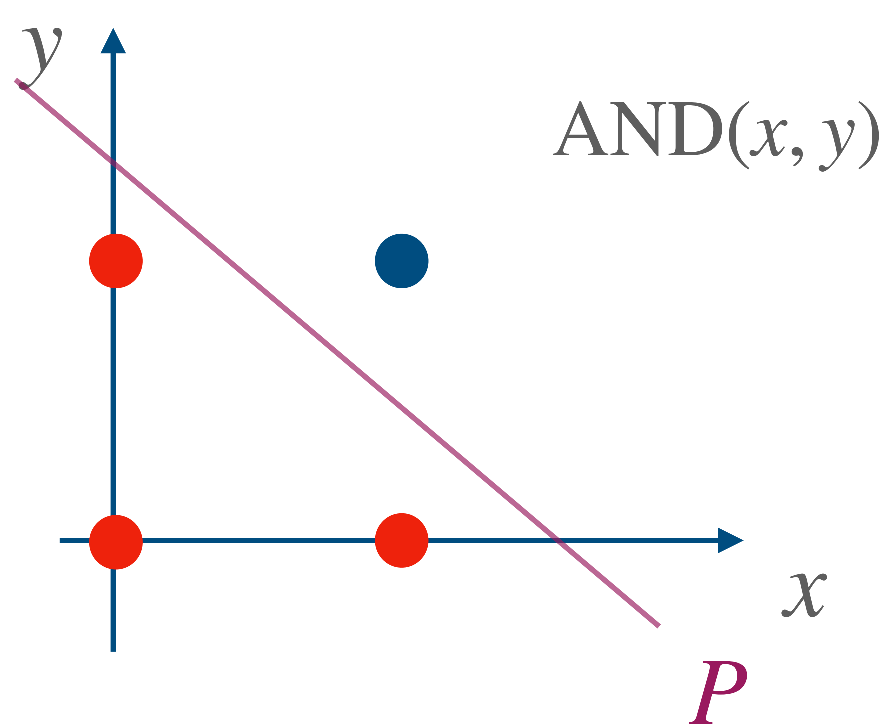

$$(-1)^{\mathrm{AND}(x)} \cdot (n - 1/2 - x_1 - x_2 - \cdots - x_n) > 0$$

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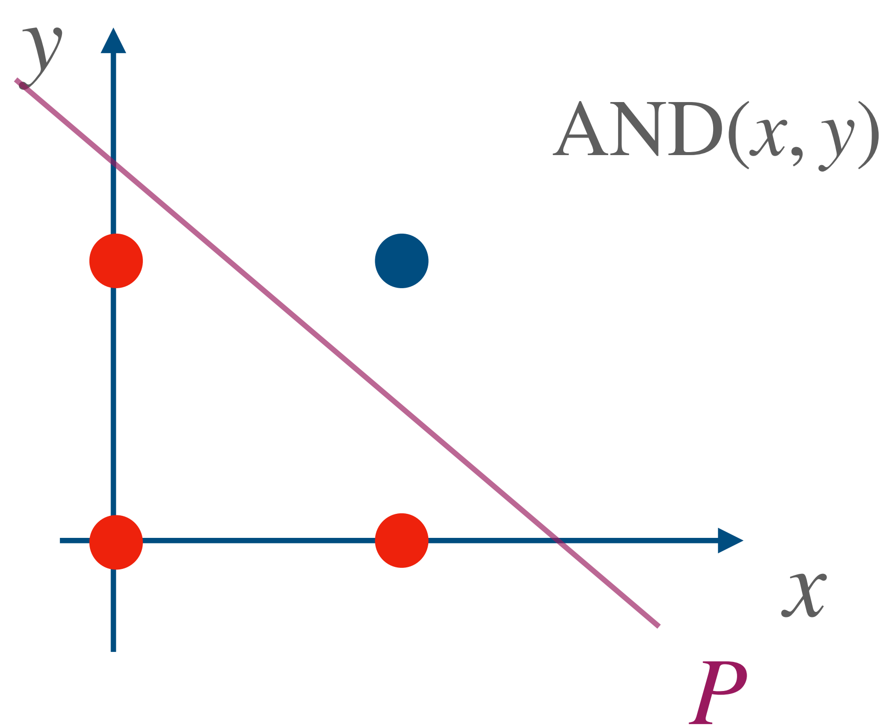

$(-1)^{\mathrm{AND}(x)} \cdot (n - 1/2 - x_1 - x_2 - \cdots - x_n) > 0$

$\deg_{\pm}(\mathrm{AND}(x)) = 1.$

Threshold degree of AC^0

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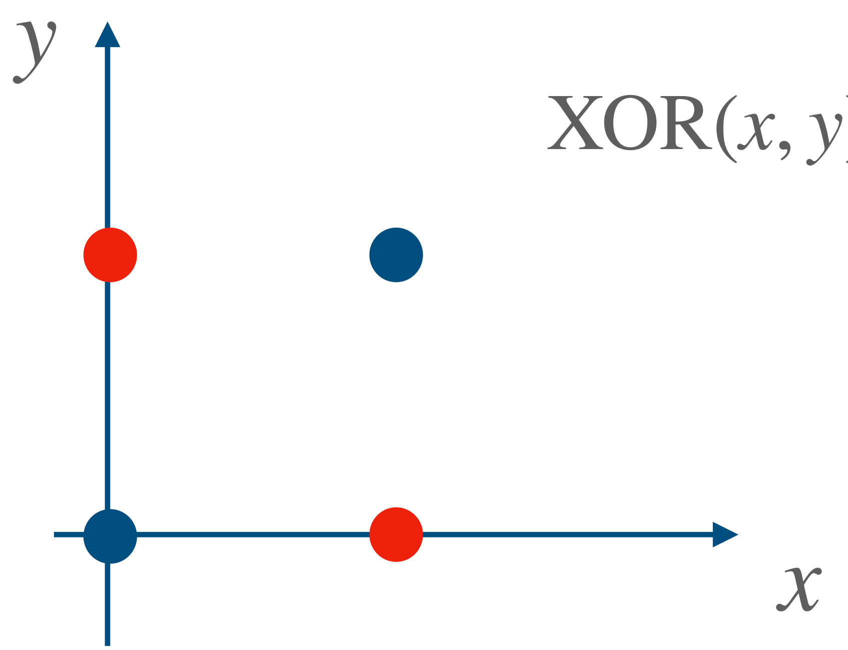

$\mathrm{XOR}(x, y)$

Threshold degree of AC^0

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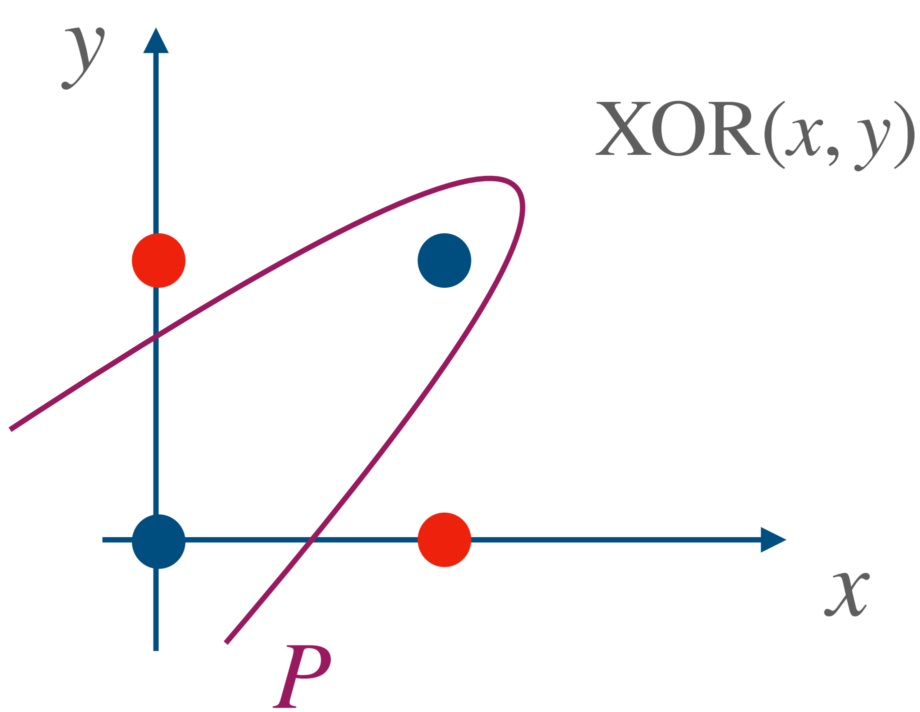

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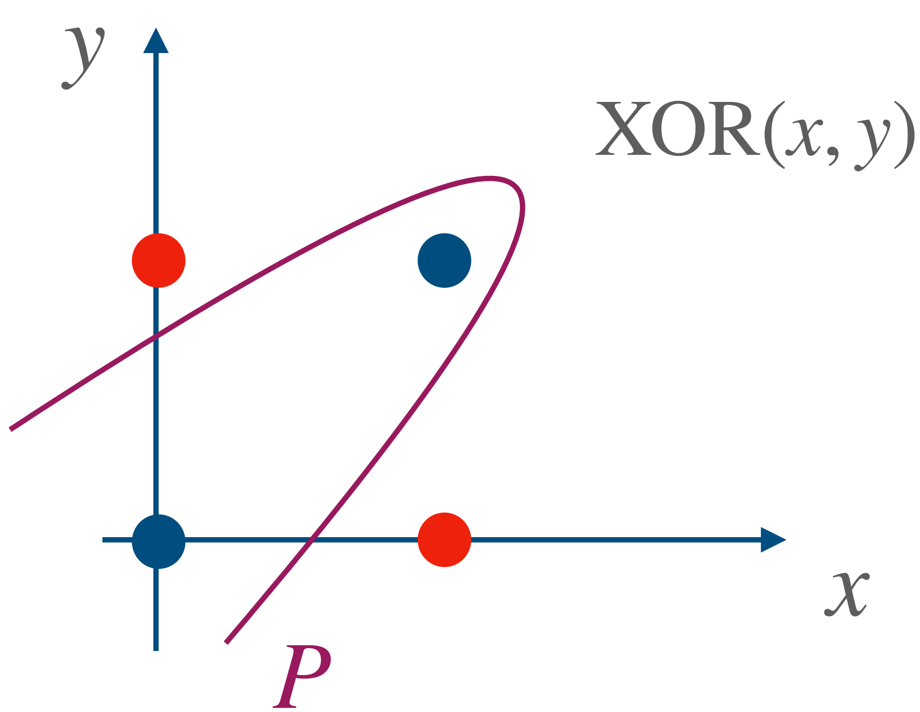

$\deg_{\pm}(\mathrm{XOR}(x)) = n$.

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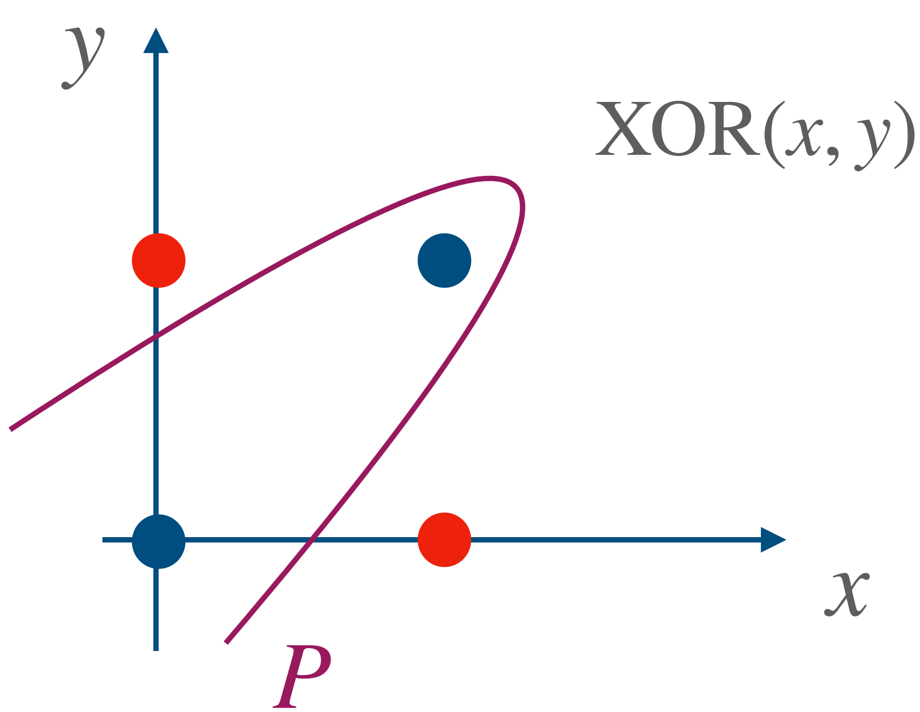

Prob. Minsky-Papert 69
Max threshold degree of AC^0?

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reference	threshold degree	depth
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O'Donnell-Servedio 03	$\Omega(n^{1/3} \log^{\frac{2(k-2)}{3}} n)$	k
Sherstov 14	$\Omega(n^{\frac{k-1}{2k-1}})$	k
Sherstov 15	$\Omega(\sqrt{n})$	4
Bun-Thaler 18	$\tilde{\Omega}(\sqrt{n})$	3
Sherstov-W. 19	$\tilde{\Omega}(n^{1-\frac{2}{k+1}})$	k

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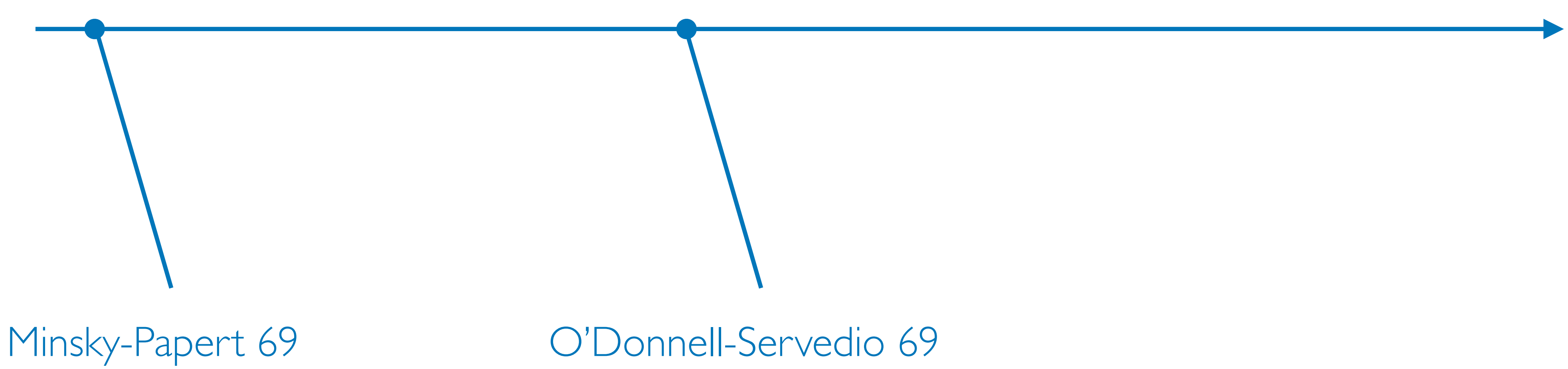

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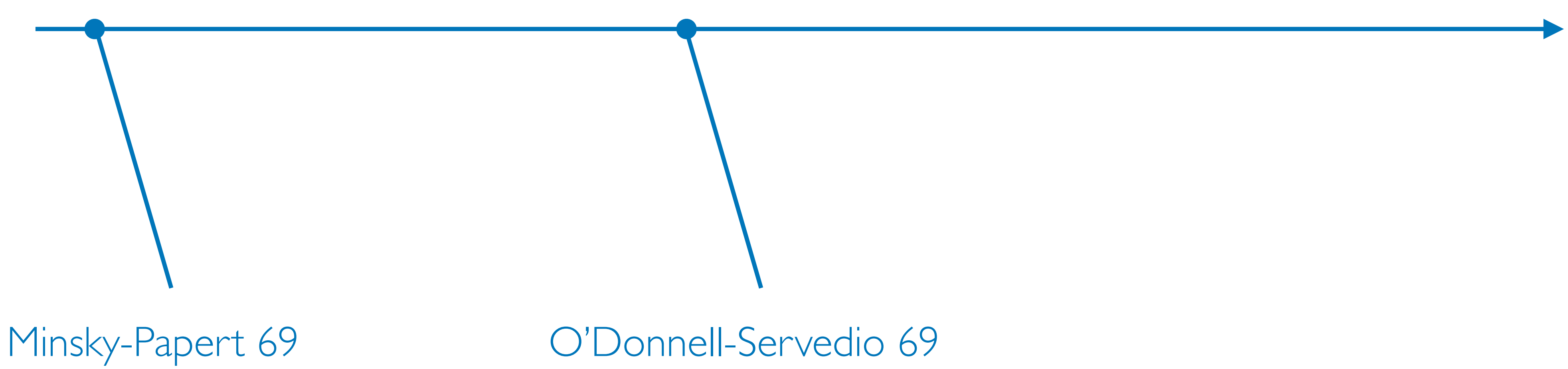

Proof Sketch: Hardness amplification

Given $f : \{0,1\}^n \to \{0,1\}, \qquad \deg_{\pm}(f) = n^{1-\epsilon}$

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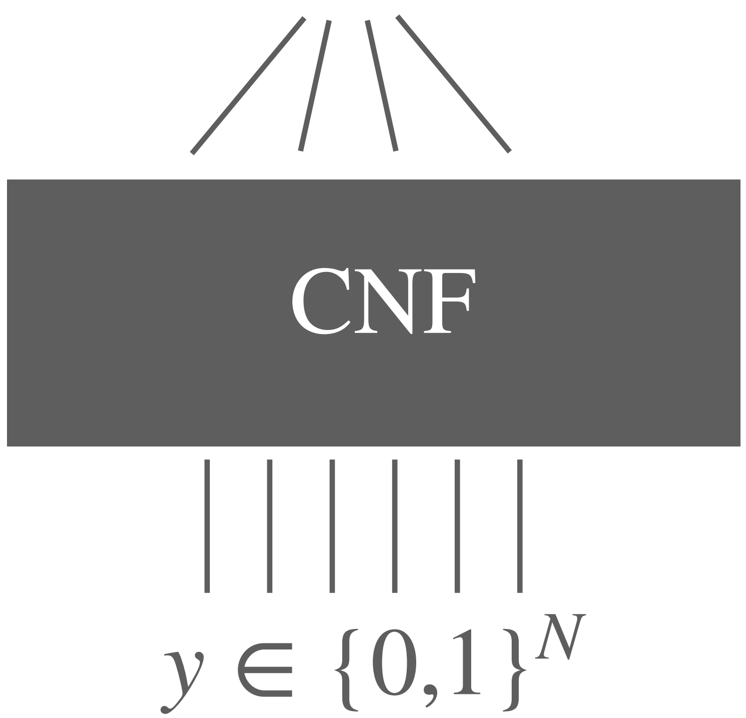
Then $F = f$



Proof Sketch: Hardness amplification

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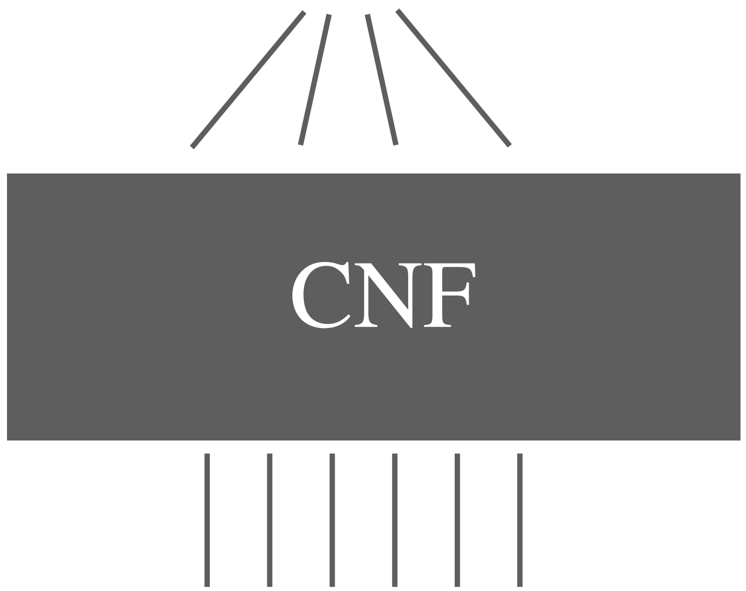


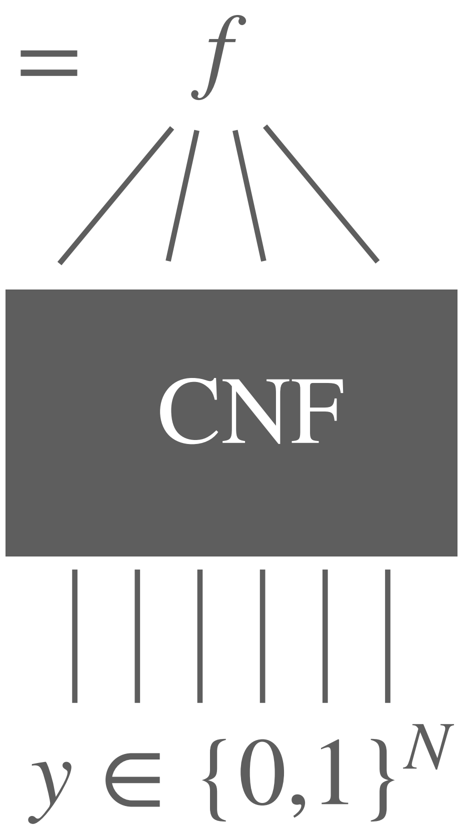
$y \in \{0,1\}^N$

$$\deg_\pm(f \circ \mathrm{CNF}_m) \geq n^{1-\epsilon} \cdot m$$

Proof Sketch: Compression

Given $f : \{0,1\}^n \to \{0,1\}$, $\qquad \deg_{\pm}(f) = n^{1-\epsilon}$

Then $F =$



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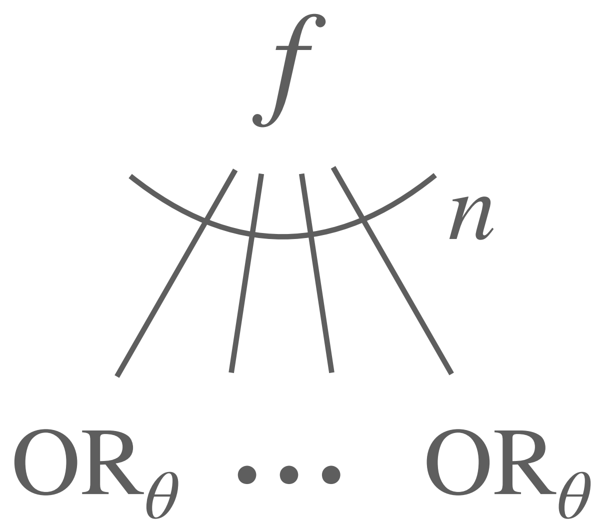
Compression: input transformation

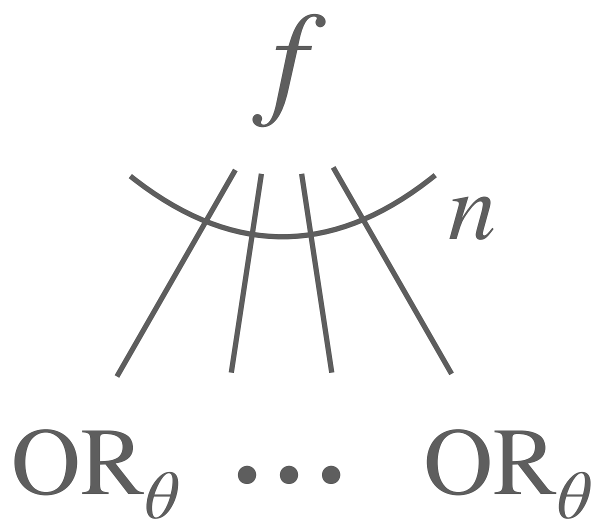


$\boldsymbol{\theta}$

0	1	1	0	0	1	1
0	0	1	1	0	1	0
1	0	1	0	1	0	1
0	1	0	1	1	0	0
0	0	1	0	0	1	1
1	1	0	0	0	1	0
1	1	0	1	1	0	0
0	0	1	1	0	0	1

$\boldsymbol{n}$

Compression: input transformation



θ (rows) × n (columns):

0	1	1	0	0	1	1
0	0	1	1	0	1	0
1	0	1	0	1	0	1
0	1	0	1	1	0	0
0	0	1	0	0	1	1
1	1	0	0	0	1	0
1	1	0	1	1	0	0
0	0	1	1	0	0	1

$$\xrightarrow[\{0,1\}^{\theta\times n}\,|_{\leq\theta}]{\textit{Restrict}}$$

0	0	0	0	0	0	0
0	0	0	1	0	0	0
0	0	1	0	0	0	0
0	0	0	0	1	0	0
0	0	0	0	0	1	0
0	1	0	0	0	0	0
1	0	0	0	0	0	0
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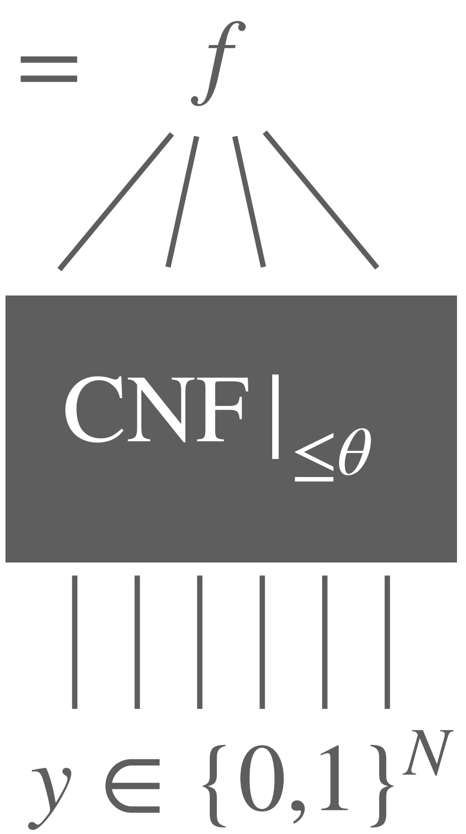


$(f \circ \mathrm{CNF_m})|_{\leq\theta}$

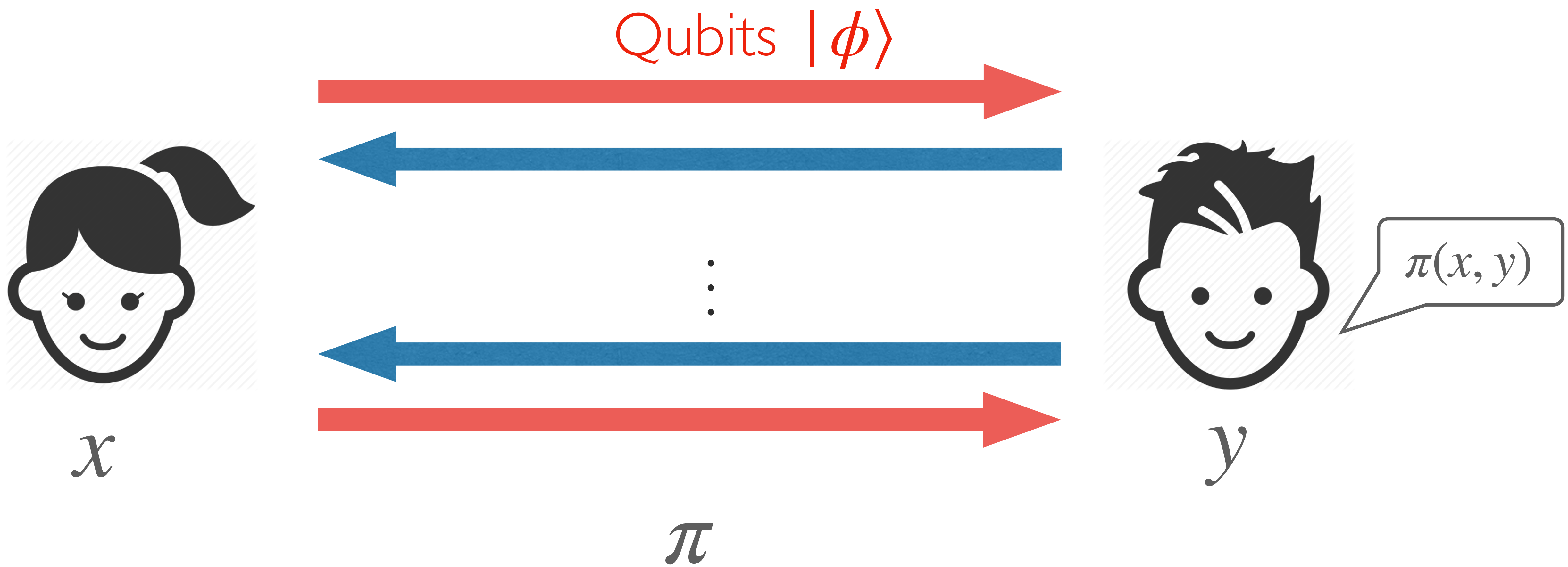
$\deg_{\pm}(\cancel{f \circ \mathrm{CNF}_m}) \geq n^{1-\epsilon} \cdot m$

More tools from duality.

Roadmap

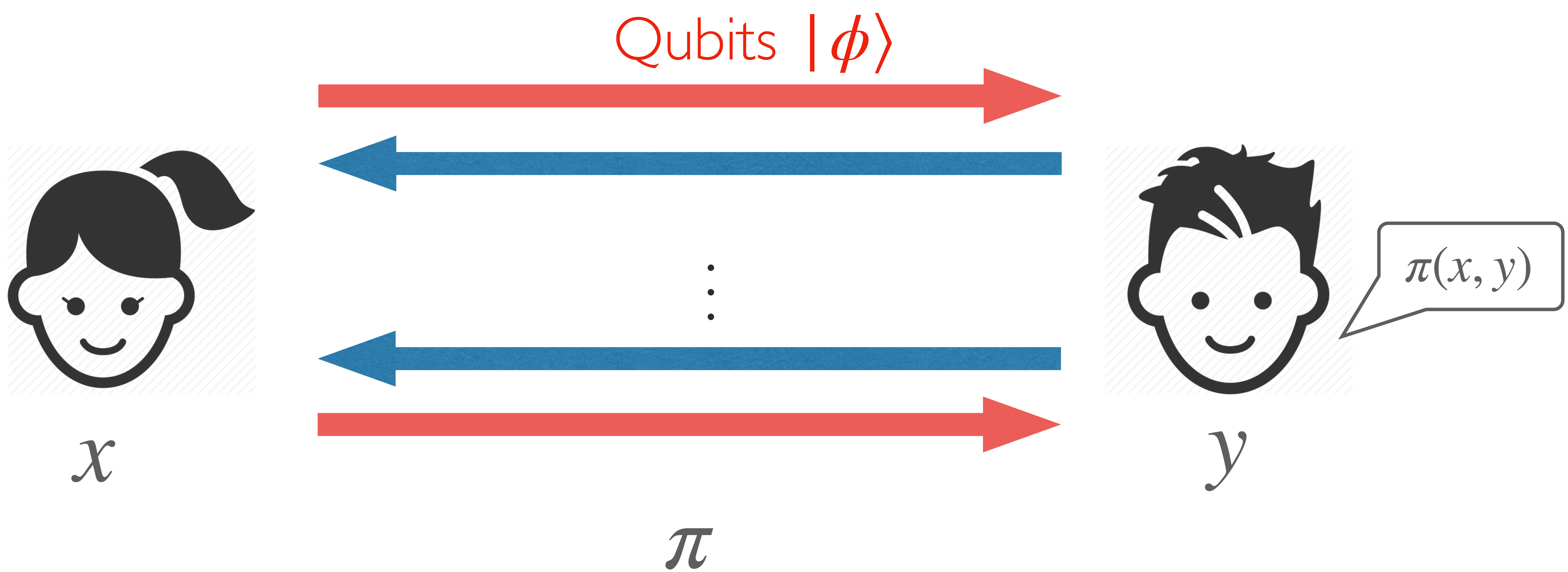
- Unbounded-error communication
- BQP vs. BPP communication

Communication complexity (Quantum)



Communication complexity (Quantum)

Advantage of quantum computation?



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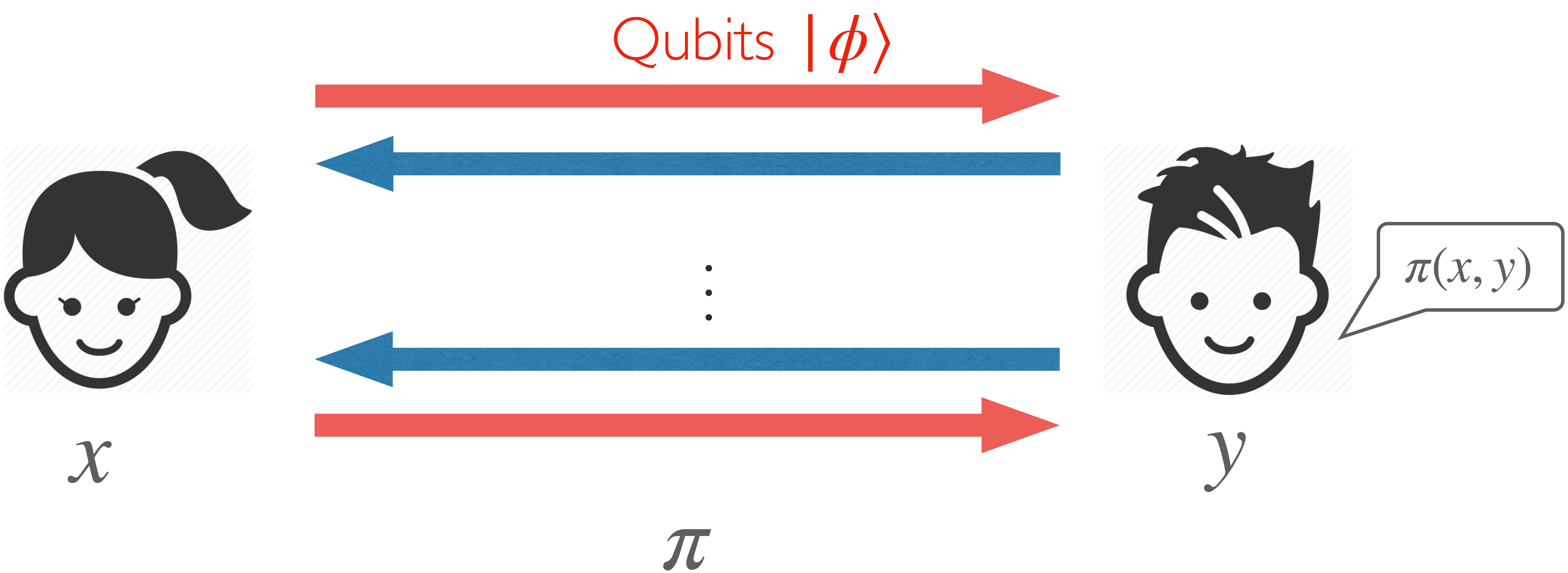

Correctness: $\Pr[\pi(x, y) = f(x, y)] \geq \frac{2}{3}, \forall x, y\,.$

What's the largest separation?

Partial functions $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1,\text{undef}\}$,

	Classical	Quantum
Buhrman et al. '98	$\mathrm{D}(f) = \Omega(n)$	$O(\log n)$

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near-optimal

What's the largest separation?

Total functions $f: \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$,

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Buhrman et al., '98, Razborov, '02	$R(f) \geq \Omega(Q(f)^2)$
Aaronson et al., '15	$R(f) \geq \tilde{\Omega}(Q(f)^{5/2})$
Tal, '19	$R(f) \geq \Omega(Q(f)^{8/3-o(1)})$
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Lifting

In short,

f, hard for query model

$$\Big\downarrow \text{lift}$$

F, hard for communication model

[Raz-McKenzie., '99]

[Goos et al., '15]

[Chattopattyay et al., '19]

Query complexity

a huge unstructured database

0	1	0	0	1	1	1	0	1	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	1	1	1	1	1	0	0	1	1	0	0	0	1	0	1	1

Query complexity

a huge unstructured database

$f($

0	1	0	0	1	1	1	0	1	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	1	1	1	1	1	0	0	1	1	0	0	0	1	0	1	1

$)$

Query complexity

a huge unstructured database

$f($ 0 1 0 0 1 1 1 0 1 1 0 1 0 0 0 1 0 1 0 0 0 1 0 1 1 1 1 1 1 0 0 1 1 0 0 0 1 0 1 1 $)$

query a few locations

Query complexity

a huge unstructured database

$$f(\,0\;1\;0\;0\;1\;1\;1\;0\;1\;1\;0\;1\;0\;0\;0\;1\;0\;1\;0\;0\;0\;1\;0\;1\;1\;1\;1\;1\;1\;0\;0\;1\;1\;0\;0\;0\;1\;0\;1\;1\,)$$

query a few locations

query complexity = min queries

Quantum query complexity

State any unit vector in a fixed Euclidean space

Query $$|\phi\rangle = \sum_{i,w} a_{i,w} |i\rangle |w\rangle$$

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query index

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workspace

query index

Quantum query complexity

State any unit vector in a fixed Euclidean space

workspace

Query $$|\phi\rangle = \sum_{i,w} a_{i,w} |i\rangle |w\rangle$$

query index

$$\downarrow$$

$$|\phi'\rangle = \sum_{i,w} a_{i,w} (-1)^{x_i} |i\rangle |w\rangle$$

Quantum query complexity

State any unit vector in a fixed Euclidean space

workspace

Query $$|\phi\rangle = \sum_{i,w} a_{i,w} |i\rangle |w\rangle$$

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can access all x_i in a single query!

Quantum speedups

Query model captures nearly all quantum breakthroughs:

Deutsch-Jozsa's algorithm

Simon's algorithm

Grover's search

Bernstein-Vazirani's algorithm

Shor's factoring algorithm

……

Largest possible separation?

Partial functions

	Randomized	Quantum
Simon '97	$\Omega(\sqrt{n})$	$O(\log n)$
Aaronson-Ambainis '15	$\tilde{\Omega}(\sqrt{n})$	1
AA '15, BGGS '21	$O_k(n^{1-\frac{1}{k}})$	$k/2$
Tal '19	$\tilde{\Omega}(n^{\frac{2k-2}{3k-1}})$	$k/2$
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Optimal

Largest possible separation?

Total functions

	Randomized vs. Quantum
Grover '69, BBBV '97	$R(f) = \Omega(Q(f)^2)$
Beals et al. '01	$R(f) = O(Q(f)^6)$
Aaronson et al. '16	$R(f) \geq \tilde{\Omega}(Q(f)^{2.5})$
Tal '19	$R(f) \geq Q(f)^{\frac{8}{3}-o(1)}$
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Fourier weight of decision trees

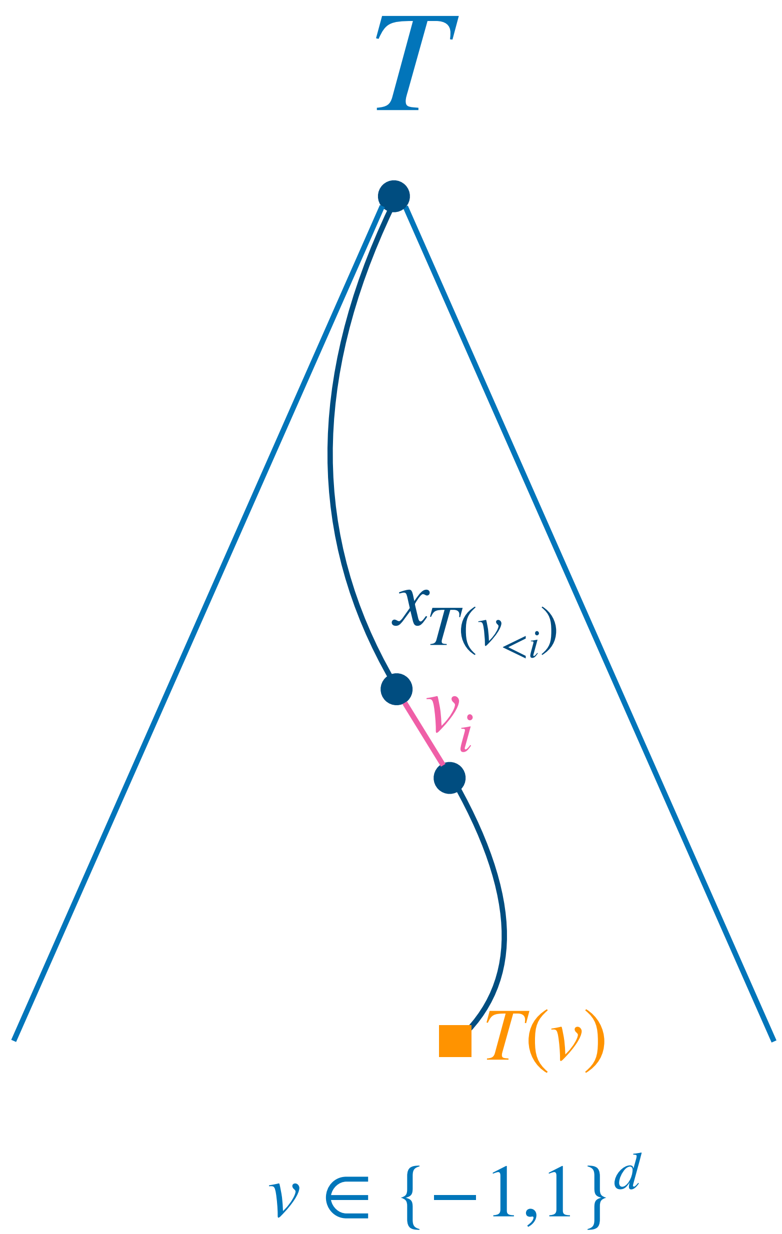
Theorem.

For any decision tree $T : \{-1,1\}^n \to \{0,1\}$ of depth d,

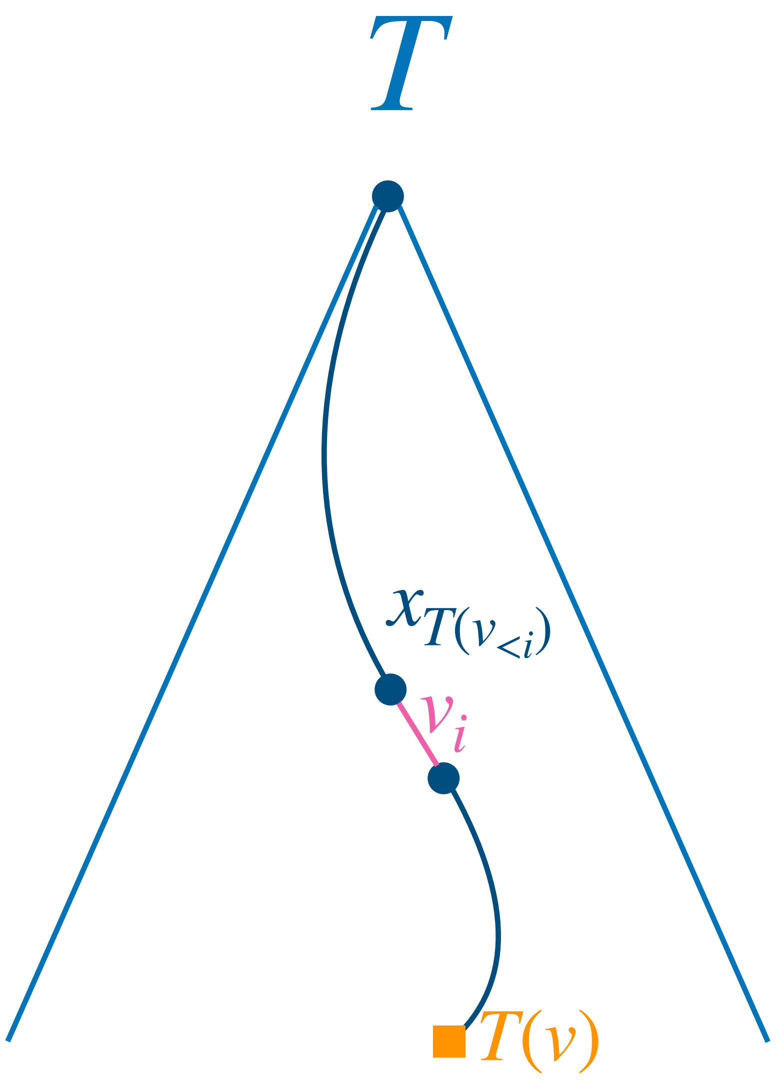
$$\sum_{\substack{S \subseteq \{1,2,\ldots,n\}: \\ |S| = \ell}} |\hat{T}(S)| \leq c^\ell \sqrt{\binom{d}{\ell}(1+\log n)^{\ell-1}}\,.$$

Our approach

Our approach



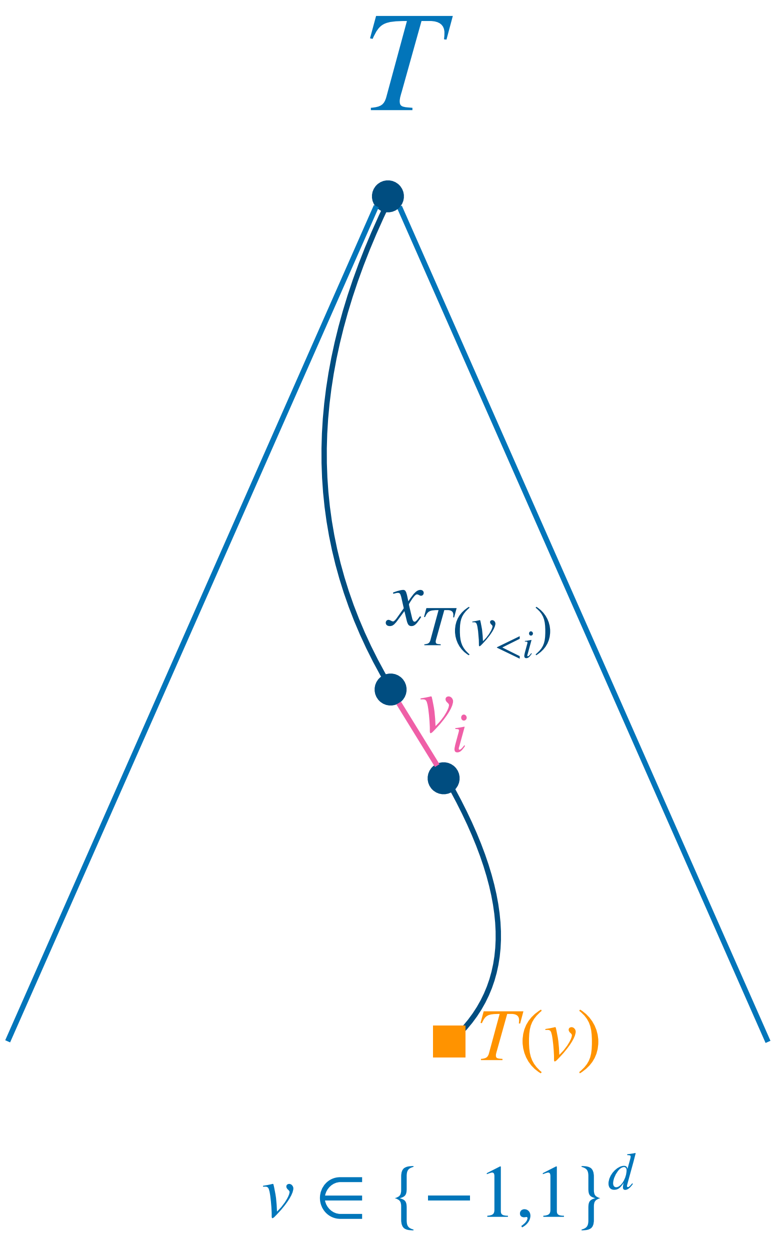
Our approach



$v \in \{-1,1\}^d$

$$T(v)\prod_{i=1}^{d}\frac{1+v_i x_{T(v_{<i})}}{2}$$

Our approach



$$T = \sum_{v \in \{-1,1\}^d} T(v) \prod_{i=1}^{d} \frac{1 + v_i x_{T(v_{<i})}}{2}$$

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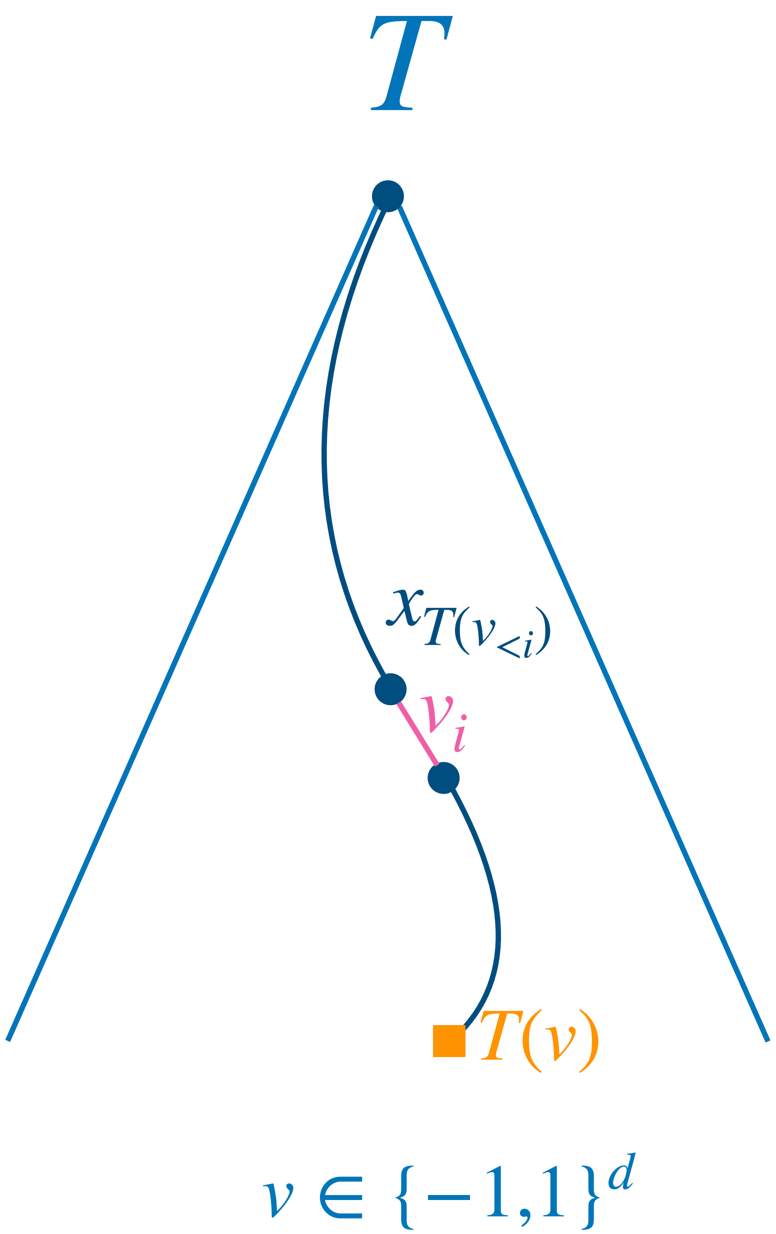
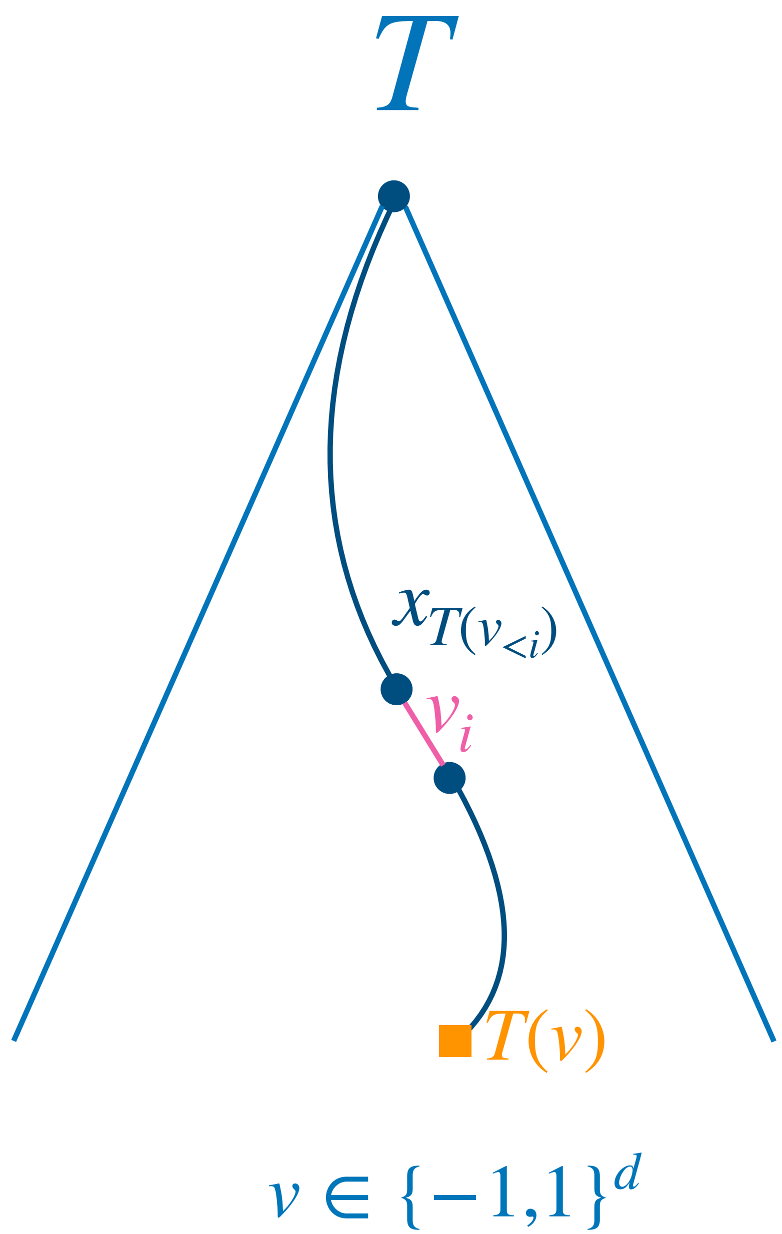


$$T = \sum_{v \in \{-1,1\}^d} T(v) \prod_{i=1}^{d} \frac{1 + v_i x_{T(v_{<i})}}{2}$$

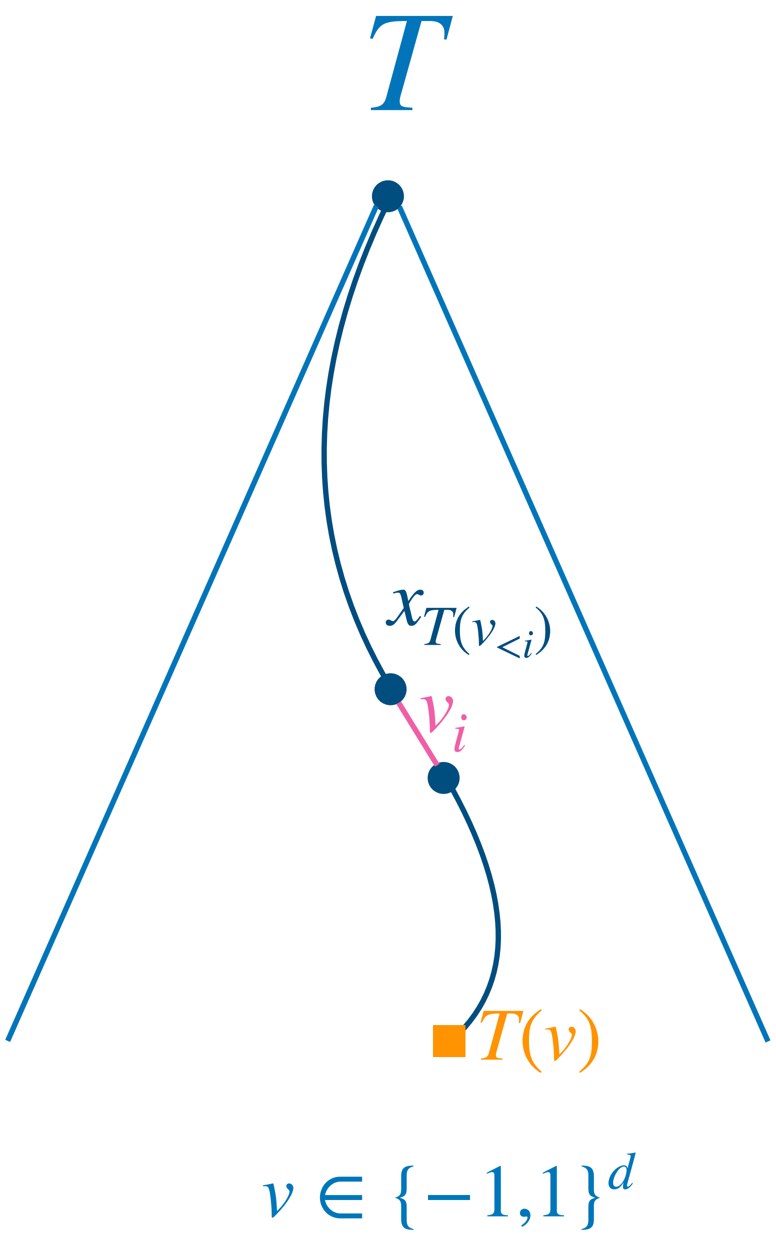
$$= \sum_{v \in \{-1,1\}^d} T(v) 2^{-d} \sum_{S \subseteq \{1,\ldots,d\}} \prod_{i \in S} v_i x_{T(v_{<i})}$$

Our approach



$$
\begin{aligned}
T &= \sum_{v\in\{-1,1\}^d} T(v) \prod_{i=1}^{d} \frac{1 + v_i x_{T(v_{<i})}}{2} \\
&= \sum_{v\in\{-1,1\}^d} T(v) 2^{-d} \sum_{S\subseteq\{1,\ldots,d\}} \prod_{i\in S} v_i x_{T(v_{<i})} \\
&= \sum_{S\subseteq\{1,\ldots,d\}} \sum_{v\in\{-1,1\}^d} T(v) 2^{-d} \prod_{i\in S} v_i x_{T(v_{<i})} \,.
\end{aligned}
$$

Our approach



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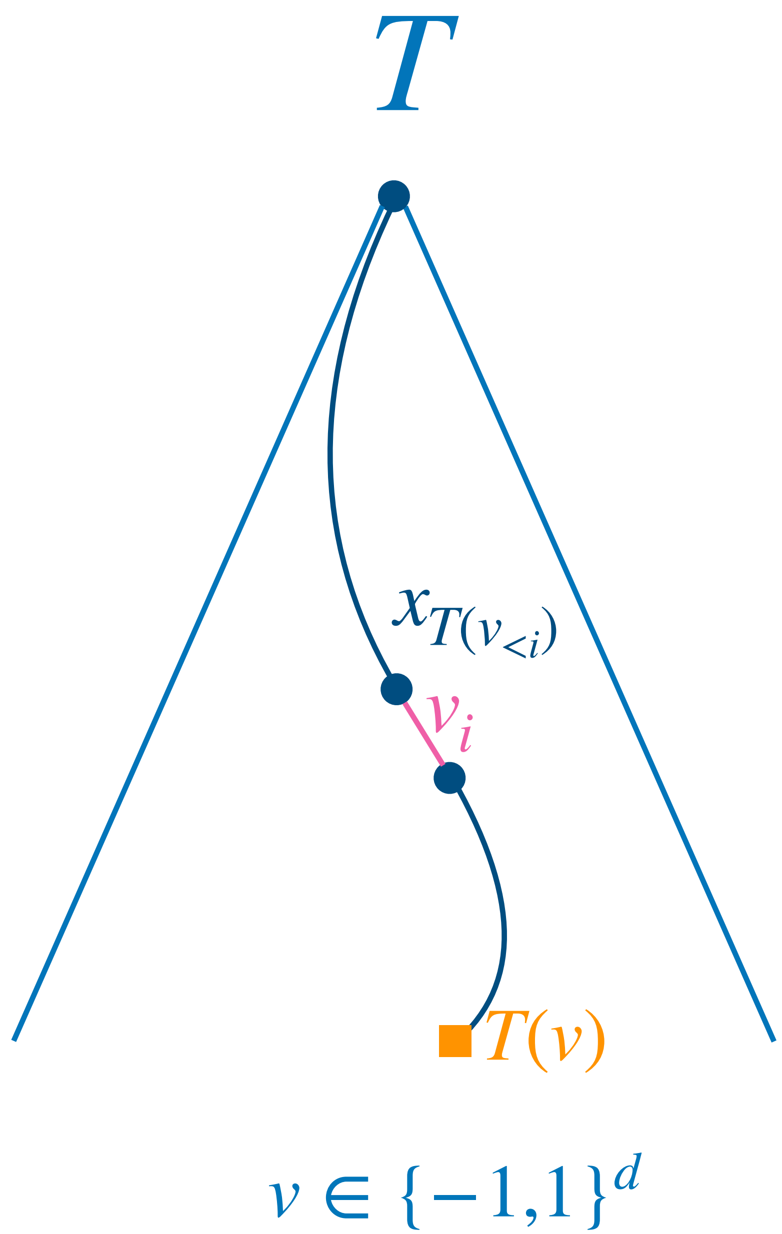
$$L_\ell T = \sum_{S\in\mathscr{P}_{d,\ell}} \sum_{v\in\{-1,1\}^d} T(v) 2^{-d} \prod_{i\in S} v_i x_{T(v_{<i})} \, .$$

Our approach



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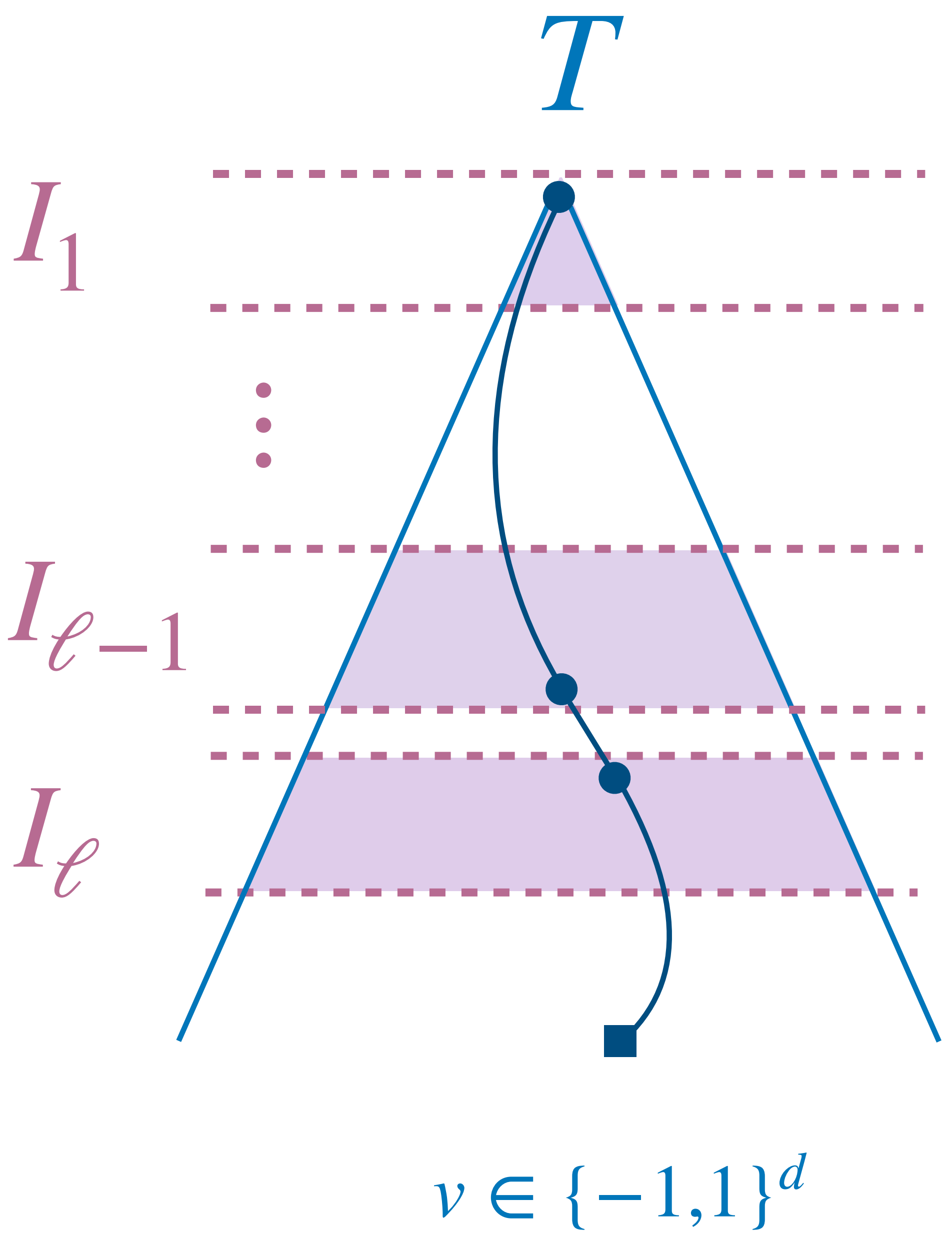

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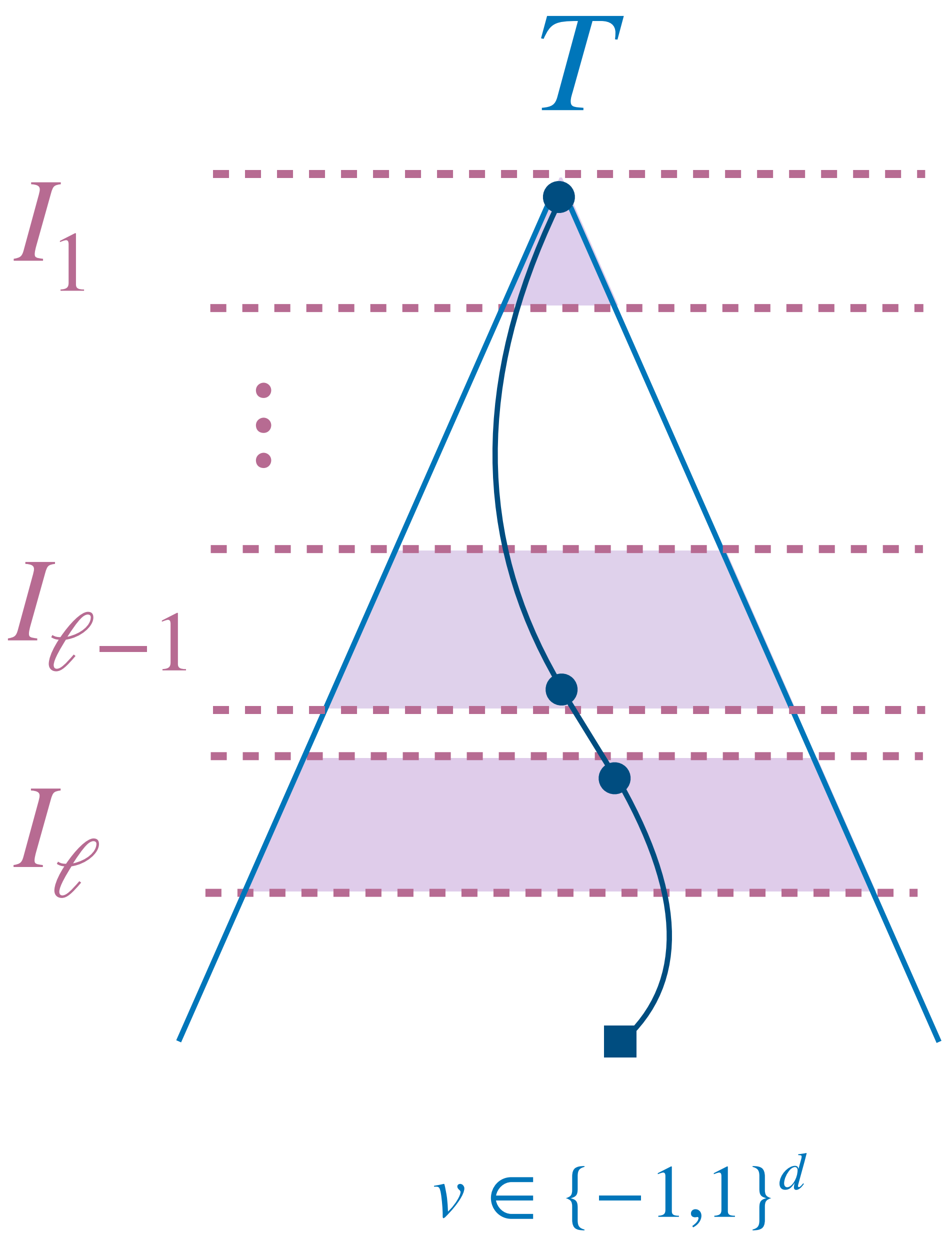
Our approach



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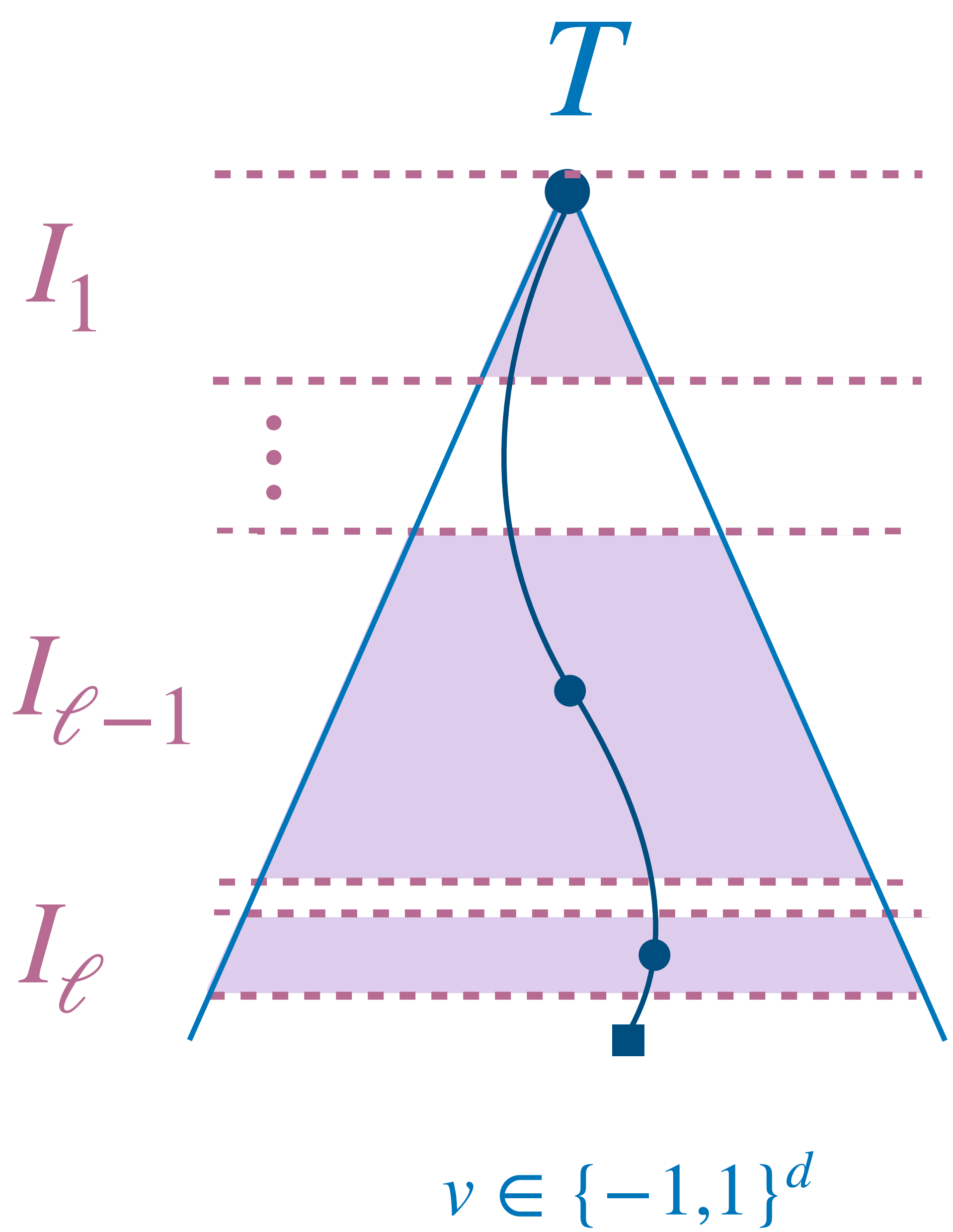
$$T|_{I_1 * I_2 * \ldots * I_\ell} = \sum_{\substack{S \subseteq \{1,\ldots,d\}: \\ |S \cap I_i| = 1}} \sum_{v \in \{-1,1\}^d} T(v) 2^{-d} \prod_{i \in S} v_i x_{T(v_{<i})} .$$

Our approach



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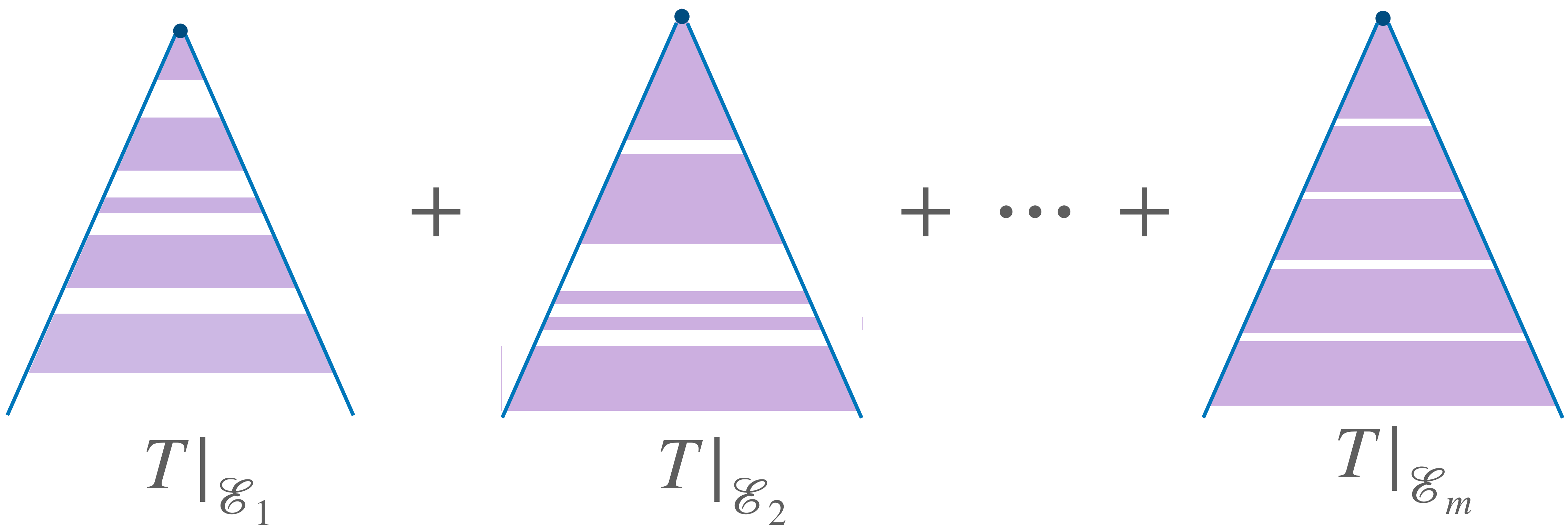
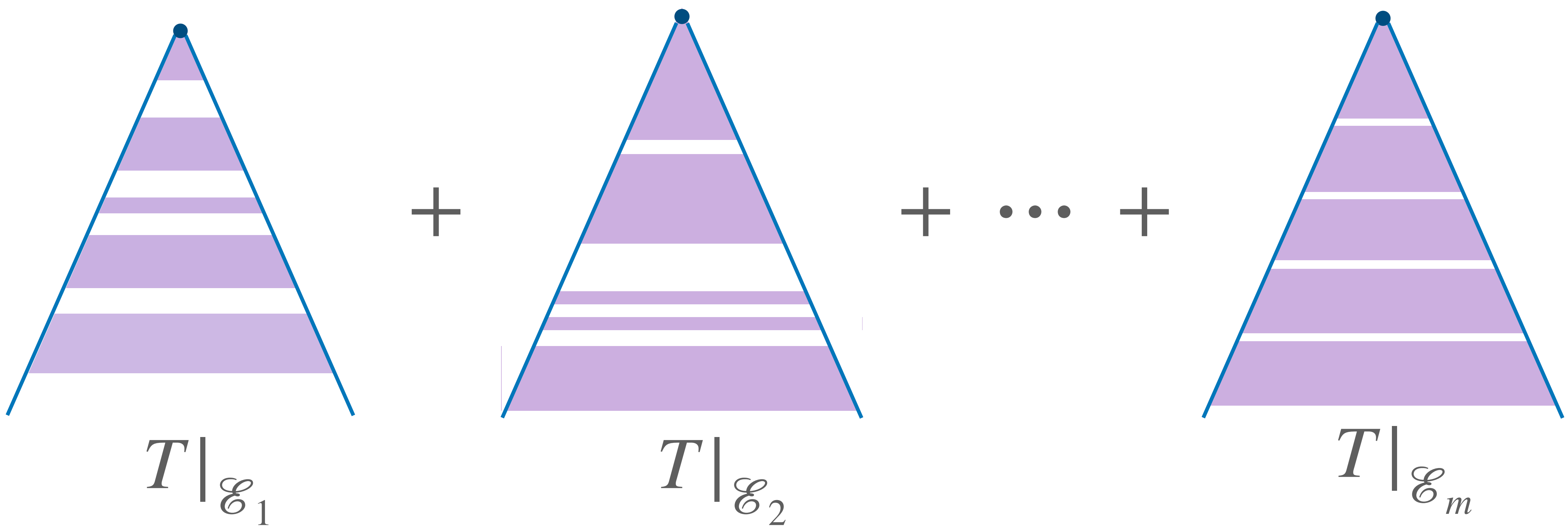
Fourier weight of decision trees

$L_\ell T =$



Fourier weight of decision trees

$L_\ell T =$



$\|\!| L_\ell T |\!\|| \leq \sum |\!\|| T|_{\mathscr{E}_i} |\!\|| .$ (Triangle-inequality)

Grand Challenges

Grand Challenges

1. Quantum v.s. Classical communication
2. Quantum Proof System

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Thank you!