Algebra 1: Tutorial 8

When you answer these questions practise your proof writing.
Be clear, concise, and complete.

Question 1: Idempotents are Zero Divisors

Recall that an idempotent of a ring R is an element $e \in R$ such that $e^2 = e$. The identities 0 and 1 of a ring are always idempotents, we call these *trivial idempotents*. Prove all non-trivial idempotents in a ring R are zero divisors. Prove that an integral domain can't be decomposed as a product of smaller rings.

Question 2: Idempotents Break Rings Up

If A and B are rings, then show $(a, b) \in A \times B$ is an idempotent if and only if a and b are idempotents in A and B respectively. Find all idempotents in (i) $\mathbb{R}^2$ (ii) $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ (iii) $\mathbb{Z} \times \mathbb{Q}$

Question 3: Product Rings Example I

Decompose $\mathbb{Z}/6\mathbb{Z}$ as a product of rings. Decompose $\mathbb{Z}/12\mathbb{Z}$ as a product of rings.

Question 4: Ideal of Vanishing are Maximal

Recall the ideal $I_a := \{f \in \mathbb{C}[x] \mid f(a) = 0\} = \langle x - a\rangle \subseteq \mathbb{C}[x]$ from the last few tutorials. Prove this ideal is a maximal ideal.

This proves there is a bijection between points of $\mathbb{C}$ and maximal ideals of $\mathbb{C}[x]$: this is the beginning point of algebraic geometry.

Question 5: Quotient Ring Examples I

Prove $f(x) = x^2 - 2$ is irreducible over $\mathbb{Q}$. As a consequence prove $\mathbb{Q}[x]/\langle x^2 - 2\rangle$ is a field. Identify this field with a subfield of $\mathbb{R}$. Do the same with (i) $g(x) = x^2 - 3$ (ii) $h(x) = x^3 - 2$. One can't always realise such quotient fields as subfields of $\mathbb{R}$: for example the quotient $\mathbb{Q}[x]/\langle x^2 + 1\rangle$ can't be imbedded into $\mathbb{R}$ — why? — instead, realise this as a subfield of $\mathbb{C}$.

Prove $\mathbb{Z}/3\mathbb{Z}[x]/\langle x^2 + 1\rangle$ is a field. How many elements does it have?