Algebra 1: Tutorial 8

When you answer these questions practise your proof writing.
Be clear, concise, and complete.

Question 1: Find the kernel and image

Let $\phi : \mathbb{R}[x] \to \mathbb{C} \times \mathbb{C}$ be the homomorphism defined by $\phi(x) = (1, i)$ and $\phi(r) = (r, r)$ for $r \in \mathbb{R}$. Determine the kernel and the image of ϕ.

Question 2: Idempotents Break Rings Up

If A and B are rings, then show $(a, b) \in A \times B$ is an idempotent if and only if a and b are idempotents in A and B respectively. Find all idempotents in (i) $\mathbb{R}^2$ (ii) $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ (iii) $\mathbb{Z} \times \mathbb{Q}$

Question 3: Product Rings Example I

Decompose $\mathbb{Z}/6\mathbb{Z}$ as a product of rings. Decompose $\mathbb{Z}/12\mathbb{Z}$ as a product of rings.

Question 3: Idempotents are Zero Divisors

The identities 0 and 1 of a ring are always idempotents, we call these *trivial idempotents*. Prove all non-trivial idempotents in a ring R are zero divisors. Prove that an integral domain can't be decomposed as a product of smaller rings.

Question 4: Ideal of Vanishing are Maximal

Recall the ideal $I_a := \{f \in \mathbb{C}[x] \mid f(a) = 0\} = \langle x - a \rangle \subseteq \mathbb{C}[x]$ from the last few tutorials. Prove this ideal is a maximal ideal.

This proves there is a bijection between points of $\mathbb{C}$ and maximal ideals of $\mathbb{C}[x]$: this is the beginning point of algebraic geometry.