Algebra 1: Tutorial 7

When you answer these questions practise your proof writing.
Be clear, concise, and complete.

Question 0: The Frobenius

Prove $x \mapsto x^5$ is a homomorphism from $\mathbb{Z}/5\mathbb{Z}$ to itself. In fact, this is true for *any* prime. It is called the *Frobenius map*.

Question 1: Simplification of Presentations

Identify (simplify the presentation of) each of the following rings: $\mathbb{Q}[x,y]/\langle y-x^2\rangle$; $\mathbb{Q}[x,y]/\langle y-x^3, x-3\rangle$; $\mathbb{Q}[x,y]/\langle y-x^2, x^2+1\rangle$. Be specific about the isomorphism used at each stage of your simplification.

Question 2: Matrix Ring Example

Denote $M_n(\mathbb{Z})$ to be the ring of matrices with integer entries. Prove the matrices with *even* entries form an ideal. Identify the quotient of $M_n(\mathbb{Z})$ by this ideal.

Question 3: Generators of Rings

Find a $\mathbb{Z}$-basis for the ring $\mathbb{Z}[x]/\langle f\rangle$, when f is: $f(x) = x^2+1$, $f(x) = x^2-2$, $f(x) = x-4$, and $f(x) = x^4-1$

Question 4: Adjoining Inverses of Zero Divisors

[Artin Exercise 11.5.3] Describe the ring obtained by adjoining an inverse of 2 to the ring $\mathbb{Z}/12\mathbb{Z}$ [Hint: $\langle 12, 2x-1\rangle = \langle 3, x+1\rangle$ in $\mathbb{Z}[x]$]

Question 5: Function Algebras

$I_a := \{f \in \mathbb{C}[x] \mid f(a) = 0\} = \langle x-a\rangle \subseteq \mathbb{C}[x]$. Determine the quotient $\mathbb{C}[x]/I_a$.

Question 6: Automorphisms of Polynomial Ring

Determine the automorphisms of $\mathbb{Z}[x]$ [Hint: Recall that the integers are fixed by any such map (substitution principal) so such a morphism is determined by 1 piece of information]