Algebra 1: Tutorial 10

When you answer these questions practise your proof writing.
Be a group, ring, or module.

Question 1: Presentation Matrices

What is the presentation matrix of the following abelian groups:

- The abelian group generated by x, y, with the single relation $19x + 13y = 0$;
- The abelian group generated by x, y, z, with the single relation $19x + 13y = 0$.

Question 2: Presentation Matrices

Identify the $\mathbb{Z}$-module presented by the following presentation matrices:

$$\begin{bmatrix} 2 & 5 \\ 4 & 10 \end{bmatrix}; \qquad \begin{bmatrix} 2 & -6 & 0 \\ -6 & 12 & 0 \end{bmatrix}.$$

Question 3: Cyclic Modules

A module is said to be *cyclic* if it can be generated by exactly one element. Prove that a cyclic module is isomorphic to $R/(a)$ for some principal ideal (a).

Question 4: Torsion submodules

Let M be a $\mathbb{Z}$-module. Let T consist of all $x \in M$ such that there exists a $r \neq 0$ in $\mathbb{Z}$ with $rx = 0$. Prove that T is a submodule of M. The set T is often called the *torsion submodule* of M. What is the torsion submodule of $\mathbb{Z}^2 \oplus \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2017\mathbb{Z}$?

Question 5: A Non-Noetherian Ring

Consider the ring $C(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$, where addition and multiplication of functions is performed pointwise, i.e. $(f+g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$. Show that it is not a Noetherian ring.

Question 6: Modules over a Quotient Ring

Let R be a ring, I be an ideal, and M be a R-module. Under what conditions can we make M into an R/I-module via $(r + I)x = rx$, $r \in I$, $x \in M$?