Algebra 1: Tutorial 2

When you answer these questions practise your proof writing.
Be clear, concise, and complete.

Question 1: Automorphism Group

Let $\text{Aut}(G) := \{\text{isomorphisms from G to G}\}$. Prove $\text{Aut}(G)$ is a group under homomorphism composition. Isomorphisms from a group to itself are called automorphisms.

Question 2: Partial Converse of Lagrange's Theorem

Find all subgroups of $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}, \ldots, \mathbb{Z}/p\mathbb{Z}$, for p prime. What can you say about the converse of Lagrange's theorem if the order of G is prime?

Question 3: Further Partial Converse of Lagrange's Theorem

Let G be finite cyclic of order n. If $k|n$, then show G has a subgroup of order k. What can you say about the converse of Lagrange's theorem for finite cyclic groups?

Question 4: Converse of Lagrange's Theorem is False in General

Let $G := A_4$. What is $|A_4|$? Does A_4 have a subgroup of order 6? [Hint: Suppose it does, denote it H. What is the index $[A_4 : H]$? How many left cosets of H are there? How many elements of order 3 are in H?] In general, then, is the converse of Lagrange's theorem true?

Question 5: Groups in Number Theory

Find all squares mod 4. Conclude that $x^2 - 4y^2 = 2$ has no solutions in the integers. Group theory can help us understand number theory: it is not all about symmetry and permutations.

Question 6: Automorphism Example

If G is a finite group, under what circumstances is $\varphi : G \to G$ where $x \mapsto x^2$ an automorphism of G?

(Optional) Challenge Question: Conjugation is Injective

Let G be a group and pick some fixed $g \in G$. Prove that the map $\varphi : G \to G$ which sends x to $g^{-1}xg$ is an isomorphism. We call such maps, i.e. maps that act by conjugation, inner automorphisms.

Suppose $G = (\mathbb{Z}, +)$, find an automorphism of $\mathbb{Z}$ which is not an inner automorphism. Try this with other groups you know.