Algebra 1: Tutorial 7

For this tutorial, the first three questions are more important than the last two questions. When you answer these questions practise your proof writing. **Be clear, concise, and complete.**

Question 1: The Eisenstein Integers

Let $\omega = e^{2\pi i/3}$, and let us try to adjoin this element to the ring $\mathbb{Z}$. Verify that ω is a root to the monic polynomial $x^2 + x + 1$. What would you quotient $\mathbb{Z}[x]$ by to produce $\mathbb{Z}[\omega]$? Can you give a $\mathbb{Z}$-basis for $\mathbb{Z}[\omega]$?

Question 2: Simplification of Presentations

Identify (simplify the presentation of) each of the following rings:

- $\mathbb{Q}[x, y]/\langle y - x^2 \rangle$;
- $\mathbb{Q}[x, y]/\langle y - x^3, x - 3 \rangle$;
- $\mathbb{Q}[x, y]/\langle y - x^2, x^2 + 1 \rangle$.

Be specific about the isomorphism used at each stage of your simplification.

Question 3: Generators of Rings

Find a $\mathbb{Z}$-basis for the ring $\mathbb{Z}[x]/\langle f \rangle$, when f is: $f(x) = x^2 + 1$, $f(x) = x^2 - 2$, $f(x) = x - 4$, and $f(x) = x^4 - 1$.

Question 4: Adjoining Inverses of Zero Divisors

[Artin Exercise 11.5.3] Describe the ring obtained by adjoining an inverse of 2 to the ring $\mathbb{Z}/12\mathbb{Z}$ [Hint: $\langle 12, 2x - 1 \rangle = \langle 3, x + 1 \rangle$ in $\mathbb{Z}[x]$].

Question 5: Function Algebras

Let $I_a := \{f \in \mathbb{C}[x] \mid f(a) = 0\} = \langle x - a \rangle \subseteq \mathbb{C}[x]$. Determine the quotient ring $\mathbb{C}[x]/I_a$.