## Algebra 1: Tutorial 6

When you answer these questions practise your proof writing.
**Be clear, concise, and complete.**

### Question 1: Examples

Give an example of

- a ring which is not a field, but has a non-trivial (not $\pm 1$) unit.
- a ring in which all non-zero elements are units.
- an ideal of $\mathbb{Z}$. Is it principal?
- a ring with no zero divisors and only the trivial units.

### Question 2: Non-Commutative Example

Some mathematicians prefer to relax the definition of a ring a little bit. Many do not require the *multiplication* to be commutative. Can you give an example of such a *non-commutative ring*?

### Question 3: Units and Trivial Ideals

If $u \in I$ is a unit in an ideal $I \subseteq R$, what can you say about $I$?

### Question 4: Division in Polynomial Rings

For which positive integers $n$ does $x^2 + x + 1$ divide $x^4 + 3x^3 + x^2 + 7x + 5$ in $[\mathbb{Z}/(n)][x]$?

### Question 5: Ideal of Vanishing

Let $a \in \mathbb{C}$ and define the set $I_a := \{f \in \mathbb{C}[x] \mid f(a) = 0\} \subseteq \mathbb{C}[x]$. Prove $I_a$ is an ideal in $\mathbb{C}[x]$. Moreover, prove it is principal.

### Question 6: Frobenius Map

Let $R$ be a ring of characteristic $p$. Prove that the map $R \to R$ defined by $x \mapsto x^p$ is a ring homomorphism. (It is called the *Frobenius map*.)