Algebra 1: Tutorial 3

When you answer these questions practise your proof writing.
Be clear, concise, and complete.

Question 1: Abelian Groups of Order 12

The cyclic group $(\mathbb{Z}/12\mathbb{Z})$ and the product groups $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$ and $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/6\mathbb{Z})$ all have order 12. Which of these groups are isomorphic?

Question 2: Products of General Cyclic Groups

Try to generalise the previous question. Let m and n be positive integers, and consider the product $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$. Under what circumstances is this cyclic?

Question 3: Correspondence Theorem – Cyclic Groups

Let G be the cyclic group of order 12 generated by g. Define a group homomorphism $\varphi : \mathbb{Z} \to G$ which maps 1 to g. Verify that φ is surjective. What is the exact correspondence given by the correspondence theorem?

Question 4: Subgroups of Product Groups

Let G, G' be groups with subgroups H, H'. Show that $H \times H'$ is a subgroup of $G \times G'$, and furthermore, that if H and H' are both normal, then $H \times H'$ is normal. Is it always the case that subgroups of $G \times G'$ are of the form $H \times H'$ for some $H \subset G$, $H' \subset G'$?

Question 5: Correspondence Preserves Indexes

Let $H \subset G$ correspond to $H' \subset G'$ via the correspondence theorem. Prove that the indexes $[G : H]$ and $[G' : H']$ are equal.

Question 6: Classifying Groups of Low Order

Can you classify all groups of order ≤ 6 (up to isomorphism)? [Hint: for order 6, there are three possibilities for the maximum order of an element: 6, 3 or 2.]