Algebra 1: Tutorial 2

When you answer these questions practise your proof writing.
Be clear, concise, and complete.

Question 1: Automorphism Group

Let $\mathrm{Aut}(G) := \{\text{isomorphisms from G to G}\}$. Prove $\mathrm{Aut}(G)$ is a group under homomorphism composition. Isomorphisms from a group to itself are called automorphisms.

Question 2: Isomorphisms as "equality"

Isomorphisms of groups are meant to be the notion of "equality" on the class of all groups. Check that the notion of isomorphism is indeed an equivalence relation on the class of all groups.

Let G be a group. Prove that the relation $a \sim b$ if $b = gag^{-1}$ for some $g \in G$ is an equivalence relation. That is to say, conjugation is an equivalence relation.

Question 3: Partial Converse of Lagrange's Theorem

Find all subgroups of $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \ldots, \mathbb{Z}/p\mathbb{Z}$, for p prime. What can you say about the converse of Lagrange's theorem if the order of G is prime?

Question 4: Further Partial Converse of Lagrange's Theorem

Let G be finite cyclic of order n. If $k|n$, then show G has a subgroup of order k. What can you say about the converse of Lagrange's theorem for finite cyclic groups?

Question 5: Converse of Lagrange's Theorem is False in General

Let $G := A_4$. What is $|A_4|$? Does A_4 have a subgroup of order 6? [Hint: Suppose it does, denote it H. What is the index $[A_4 : H]$? How many left cosets of H are there? How many elements of order 3 are in H?] In general, then, is the converse of Lagrange's theorem true?

Question 6: Decomposing Groups as Products

Consider the subgroups $H := \{0, 4, 8\}$ and $K := \{0, 3, 6, 9\}$ of $\mathbb{Z}/12\mathbb{Z}$ — why are they subgroups? Prove $HK = \mathbb{Z}/12\mathbb{Z}$, and $H \cap K = \{0\}$. What can you conclude about the structure of $\mathbb{Z}/12\mathbb{Z}$?

Question 7: Groups in Number Theory

Find all squares mod 4. Conclude that $x^2 - 4y^2 = 2$ has no solutions in the integers. Group theory can help us understand number theory: it is not all about symmetry and permutations.

Question 8: Automorphism Example

If G is a finite group, under what circumstances is $\varphi : G \to G$ where $x \mapsto x^2$ an automorphism of G?