Algebra 1: Tutorial 6

When you answer these questions practise your proof writing.
Be clear, concise, and complete.

Question 1: Underlying Group Structures in Rings

Pick your favourite group. Can you find a "natural" ring structure on it? [Recall that a ring is a commutative group under addition with another operation, multiplication. So can you see a multiplication on your favourite group which induces a ring structure? Perhaps the group already has multiplication, in which case you need the additive structure.]

Question 2: Examples

Give an example of

- a ring which is not a field, but has a non-trivial (not ± 1) unit.
- a ring in which all non-zero elements are units.
- an ideal of $\mathbb{Z}$. Is it principal?
- a ring with no zero divisors and only the trivial (not ± 1) units.

Question 3: Non-Commutative Example

Some mathematicians prefer to relax the definition of a ring a little bit. Many do not require the *multiplication* to be commutative. Can you give an example of such a *non-commutative ring*?

Question 4: Kernels are Ideals

If $\phi : R \to R'$ is a ring homomorphism, prove $\ker(\phi)$ is an ideal of R.

Question 5: Homomorphisms Determined by Generators

Give a homomorphism $\phi : \mathbb{R}[x] \to \mathbb{R}$ which fixes elements of $\mathbb{R}$. [This can be done by applying the *substitution principle* with the identity map $i : \mathbb{R} \to \mathbb{R}$]

Question 6: Units and Trivial Ideals

If $u \in I$ is a unit in an ideal $I \subseteq R$, what can you say about I?

Question 7: Primes Factor in Extensions

If $i = \sqrt{-1}$, show $\mathbb{Z}[i] := \{a + bi \mid a, b \in \mathbb{Z}\}$ is a ring with the standard multiplication of complex numbers. Can you factor 2 in $\mathbb{Z}[i]$?

Question 8: Zero Divisors

For which n does $\mathbb{Z}/n\mathbb{Z}$ contain zero-divisors? When is $\mathbb{Z}/n\mathbb{Z}$ a field?

Question 9: Ideal of Vanishing

Let $a \in \mathbb{C}$ and define the set $I_a := \{f \in \mathbb{C}[x] \mid f(a) = 0\} \subseteq \mathbb{C}[x]$. Prove I_a is an ideal in $\mathbb{C}[x]$. Moreover, prove it is principal.