Algebra 1: Tutorial 4

When you answer these questions practise your proof writing.
Be clear, concise, and complete.

Question 1: Quaternions

Let $G := \{\pm 1, \pm i, \pm j, \pm k\}$ denote the *quaternionic group*. The operations of G are defined by the following relations: $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. Note: the group is *not* commutative.

a) Prove G is in fact a group! Write out the Cayley table.

b) Express G in generator and relation form.

c) Prove $H := \{\pm 1\}$ is a normal subgroup of G.

d) Use the Cayley table, or other means, to determine the group structure of G/H.

Question 2: Circle Group

Let $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$, and consider the group structure on S^1 induced by multiplication of complex numbers. Prove $\mathbb{R}/\mathbb{Z} \cong S^1$. [Hint: use the first isomorphism theorem of groups]

Question 3: Roots of Unity

Consider the polynomial $f(x) = x^n - 1$. Prove that the collection of roots of $f(x)$ in $\mathbb{C}$ forms a group under multiplication of complex numbers, denote it μ_n. Consider the special case $n = 4$. Prove μ_4 is cyclic: can you prove μ_n is cyclic for all n? Prove $H := \{\pm 1\}$ is a normal subgroup of μ_4. Prove $\mu_4/H \cong \mathbb{Z}/2\mathbb{Z}$. Note: μ_n is a finite subgroup of the infinite group S^1 for each n.

Question 4: Generators and Relations I

Let $G := \langle a, b \mid a^3 = e, b^2 = e, aba^{-1}b^{-1} = e\rangle$. To what familiar group is G isomorphic to? What does the final relation tell us about G?

Question 5: Generators and Relations II

Let $G := \langle a, b \mid a^2 = e = b^2, aba = bab\rangle$. Show $G \cong S_3$.