Algebra 1: Tutorial 9

When you answer these questions practise your proof writing.
Be clear, concise, and complete.

Question 1: Examples of Modules

Give an example of a:

- $\mathbb{Z}$-module
- $\mathbb{Z}/4\mathbb{Z}$-module
- sub-$\mathbb{Z}$-module of the $\mathbb{Z}$-module $\mathbb{Z}$
- $\mathbb{C}$-module
- sub-$\mathbb{R}$-module of the $\mathbb{R}$-module $\mathbb{C}$.
- non-vector space free module.

Question 2: Not All Modules are Free

While modules are the natural generalisation of vector spaces over fields to "vector spaces" over general rings, some times our intuitions do *not* carry over:

- Give a module without a basis i.e. a non-free module.
- Give a finite (as a set) module.

Question 3: Torsion

One of the interesting properties that general modules have which obstructs our intuitions from vector spaces (modules over fields) is *torsion*: a non-trivial element $x \in M$ of an R-module M is said to be a *torsion element* if there exists a non-trivial element $r \in R$ such that $rm = 0 \in M$. Give a module with non-trivial torsion.

Question 4: Modules as Endomorphisms of Groups

[Warning: non-commutative rings ahead] Prove that giving an R-module structure on M is equivalent to defining a ring homomorphism $\varphi : R \to \mathrm{End}(M)$. The ring on the right is the *endomorphism* ring of M, which is the (in general, non-commutative) ring of group homomorphisms from M to itself.

Question 5: Modules in Number Theory

The Abelian group $\mathbb{Z}[\sqrt{2}] := \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ is a $\mathbb{Z}$ module. Prove it is free. What does it look like as a sub-set of the plane? These $\mathbb{Z}$-modules play an important part in algebraic number theory, and solving Diophantine equations. In particular their *geometric* properties tell us a lot!

Question 6: Linear Algebra of Commutative Rings

Diagonalise, into a matrix with integer entries, the following matrices, and determine the matrices which diagonalise them.

$$A := \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$$

$$B := \begin{pmatrix} 3 & 1 & -4 \\ 2 & -3 & 1 \\ -4 & 6 & 2 \end{pmatrix}$$