Algebra 1: Tutorial 4

When you answer these questions practise your proof writing.
Be clear, concise, and complete.

Question 1: Complex Number

Let $H = \{\pm 1,\ \pm i\}$ be the subgroup of $G = \mathbb{C}^\times$ of fourth roots of unity. Describe the costes of H in G explicitly. Is G/H isomorphic to G?

Question 2: Circle Group

Let $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$, and consider the group structure on S^1 induced by multiplication of complex numbers. Prove $\mathbb{R}/\mathbb{Z} \cong S^1$. [Hint: use the first isomorphism theorem of groups]

Question 3: Roots of Unity

Consider the polynomial $f(x) = x^n - 1$. Prove that the collection of roots of $f(x)$ in $\mathbb{C}$ forms a group under multiplication of complex numbers, denote it μ_n. Consider the special case $n = 4$. Prove μ_4 is cyclic: can you prove μ_n is cyclic for all n? Prove $H := \{\pm 1\}$ is a normal subgroup of μ_4. Prove $\mu_4/H \cong \mathbb{Z}/2\mathbb{Z}$. Note: μ_n is a finite subgroup of the infinite group S^1 for each n.

Question 4: Product of infinite cyclic groups

Show, by counterexample, that the product of two infinite cyclic groups is not infinitely cyclic.

Revision Exercises

Exercise 1: Suppose $\phi; G \to G'$ is a group homomorphism. Show that, for all $g \in G$, the order of $\phi(g)$ divides the order of g. Furthermore, prove that they are equal if ϕ is injective.

Exercise 2: Show that if A is a subgroup of G and B is a normal subgroup, then $AB = \{x \mid x = ab \text{ for } a \in A, b \in B\}$ is a subgroup of G.

Exercise 3: An index 2 subgroup of a group is normal. (Third way to prove A_n is normal!)