Algebra 1: Tutorial 5

When you answer these questions practise your proof writing.
Be clear, concise, and complete.

Question 1: Cayley's Theorem

Prove Cayley's Theorem:

Every finite group is isomorphic to a subgroup of S_n*, for some* $n \in \mathbb{N}$.

Question 2: Automorphisms Define Groups Actions

Suppose G and H are groups such that H is isomorphic to a subgroup of $\mathrm{Aut}(G)$ i.e. $H \hookrightarrow \mathrm{Aut}(G)$. Define a group action of H on G.

Question 3: Class Equation

Let G be a group. Prove conjugation by $g \in G$ defines an action of G on itself. Use this to prove $|G| = \sum_{\text{orbits}} |\mathrm{cl}(x)|$, where $\mathrm{cl}(x)$ denotes the conjugacy class of x under the action. This is called the *class equation* of G. This is just saying that a group action partitions the set on which it is acting: in this case the set is the group itself.

Suppose G is a group of order 21 and contains an element x such that $|\mathrm{cl}(x)| = 3$. What is the order of x? [Hint: use the "counting formula" and the fact that $x \in \mathrm{Z}(x)$]

Question 4:Symmetries of the Square

Let $G = D_8$ be the dihedral group of symmetries of the square.

1. What is the stabilizer of a vertex? of an edge?
2. G operates on the set of two elements consisting of the diagonal lines. What is the stabilizer of a diagonal?

Question 5: The Operation on Cosets

Let H be the stabilizer of the index $\mathbf{1}$ for the operation of the symmetric group $G = S_n$ on the set of indices $\{\mathbf{1}, \ldots, \mathbf{n}\}$. Describe the left cosets of H in G and the map in the Orbit-Stabilizer Theorem in this case.