Journal of Computational Physics 293 (2015) 4–13

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

What is a fractional derivative?

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ARTICLE INFO

Article history:
Received 6 June 2014
Received in revised form 3 July 2014
Accepted 15 July 2014
Available online 22 July 2014

Keywords:
Fractional derivative
Riesz potential
Fractional calculus

ABSTRACT

This paper discusses the concepts underlying the formulation of operators capable of being interpreted as fractional derivatives or fractional integrals. Two criteria for required by a fractional operator are formulated. The Grünwald–Letnikov, Riemann–Liouville and Caputo fractional derivatives and the Riesz potential are accessed in the light of the proposed criteria. A Leibniz rule is also obtained for the Riesz potential.

1. Introduction

The search on the Internet using the keyword "fractional" gives $\sim 7.5 \cdot 10^6$ results. After adding the word "derivative" the number reduces to $\sim 10^5$. However, when our query becomes "What is a fractional derivative?" we obtain merely 13 references. We verify that those papers consist of introductions to several types of definitions, without providing a useful answer to the above question. On the other hand, the name "fractional" is frequently added to several classical procedures and operators, such as the fractional: quantum Hall effect, Josephson effect, Fourier transform, frequency and others [1]. Furthermore, the concepts of "fractional derivative" (FD) and "fractional integral" (FI) appear associated with a considerable number of distinct operators [2], but it is questionable whether they obey the most adequate properties of an FD. In a recent paper [3] a new definition of FD was proposed. This new operator called "conformable" FD [4] has some properties that are distinct from those usual in other formulations. Another example is the so-called "local" FD [5]. This state of affairs leads us to ask: *How can we say that a given operator is an FD? How do we recognise an FD?*

The search for assertive answers is fruitless since no concrete results are found and we do not have a criteria for defining an FD. For example, in [6] the authors attempt to explain what is an FD by means of examples and verifying the emerging difficulties, but no criterion is formulated. The analysis by means of geometric concepts is also of small relevance, since there are several interpretations [7–21], but the discussion seems far from stabilised.

There is an interesting answer formulated by Bertran Ross in 1975 [22]. He suggested the criterion (^{1}**P**) for the FD based on the following properties:

^{1}**P1** The derivative of an analytic function is analytic.
^{1}**P2** When the order is integer FD gives the same result as the ordinary derivative. (Backward compatibility.)
^{1}**P3** The zero order derivative of a function returns the function itself.

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http://dx.doi.org/10.1016/j.jcp.2014.07.019

^{1}P4 The operator must be linear.
^{1}P5 The index law holds, that is, $D^\alpha D^\beta f(t) = D^{\alpha+\beta} f(t)$ for $\alpha < 0$ and $\beta < 0$.

Meanwhile Fractional Calculus (FC) verified a considerable development, with the introduction of novel tools, a significant number of new applications and its generalisation to discrete-time domains [23–25]. Clearly, this state of affairs requires a change in the above criterion and it is the matter to be discussed in this study.

One preliminary remark. In current literature is normal to use the terms "derivative" (for positive orders) and "integral" (for negative orders). We consider that the adoption of the word "integral" may be misleading in the scope of this study, since we are referring to a primitive (a function) and not really to an integral (a number). To avoid confusion we will use for all cases the word "derivative" excepting well established names as Riemann–Liouville integral.

Bearing these ideas in mind this paper is organised as follows. Section 2 formulates the criteria for operators being considered as FD. Section 3 analyses several formulations, namely the Grünwald–Letnikov, Riemann–Liouville and Caputo derivatives, and the Riesz and Feller potentials. Finally, Section 4 outlines the main conclusions.

2. A criterion for fractional derivative

We may discuss about the need for a criterion classifying an operator as an FD or not. This question is of interest when thinking in applied sciences, where it is important to guarantee that the formulae and algorithms are generalisations *in facto* of the corresponding integer order. In this line of thought, we establish in the sequel criteria relevant for such formulation.

2.1. Some considerations

Let us analyse the criterion **^{1}P** formulated by Bertran Ross. Its topics are almost natural and easily acceptable in the perspective of FC before the developments that were verified during the last decade and when we consider shift-invariant derivatives. In what concerns other derivatives, like the scale invariant derivatives (e.g., Hadamard [26], or quantum [27]), we will not consider them here. Meanwhile FC was the object of a considerable evolution particularly in applied sciences. Derivatives with a smaller historical visibility, such as the Grünwald–Letnikov formulation, became popular and part of FC common applications. Furthermore, the extension to discrete domains was accomplished in recent years (see [28–30, 25] and references therein). This means that the analyticity characteristic of a function and its derivatives is of limited importance. Besides, the derivative of an analytic function is not necessarily analytic, since we can compute an infinite number of derivatives thanks to the use of the generalised function (distribution) theory [31,27]. Therefore, it seems that the analyticity should not be considered when thinking FD.

Recently another important condition was formulated by Vasily Tarasov [32–34]: *A fractional operator must verify the generalised Leibniz rule*. He proposed this rule for making a distinction between those operators that we currently classify as FD and others recently published, that use such designation, but not verifying the Leibniz rule, as it is the case of the Jumarie FD [35] and the local FD [5]. Due to the importance and consequences of this rule we follow it in the sequel.

2.2. The criteria

Based on the criterion described previously we propose the following *wide sense criterion* (WSC):
An operator is considered as an FD in WSC if it enjoys the properties **^{2}P** defined as:

^{2}P1 Linearity
The operator is linear.
^{2}P2 Identity
The zero order derivative of a function returns the function itself.
^{2}P3 Backward compatibility
When the order is integer, FD gives the same result as the ordinary derivative.
^{2}P4 The index law holds

$$D^\alpha D^\beta f(t) = D^{\alpha+\beta} f(t) \tag{1}$$

for $\alpha < 0$ and $\beta < 0$.
^{2}P5 Generalised Leibniz rule

$$D^\alpha\big[f(t)g(t)\big] = \sum_{i=0}^{\infty} \binom{\alpha}{i} D^i f(t) D^{\alpha-i} g(t). \tag{2}$$

As it is clear when $\alpha = N \in \mathbb{Z}^+$ we obtain the classical Leibniz rule. Another generalisation of this rule [36], such as $D^\alpha[f(t)g(t)] = \sum_{i=-\infty}^{+\infty} \binom{\alpha}{i+\beta} D^{\beta+i} f(t) D^{\alpha-\beta-i} g(t)$, is possible. However, expression (2) is adopted due to its usefulness in the sequel.

The index law property can be modified to include positive orders. This leads to the *strict sense criterion* (SSC). Therefore, criterion ^{3}**P** has five conditions, where ^{2}**P4** is modified to:

^{3}**P4** The index law

$$D^\alpha D^\beta f(t) = D^{\alpha+\beta} f(t) \tag{3}$$

for any α and β.

Note: The SSC in the form ^{3}**P4**, presented by (3), may be not considered as a direct property of FD. In this perspective, expression (3) is not a requirement (property) for the operator itself (fractional derivative). It is a requirement (property) for the domain of the operator, that, the space of functions. The "philosophical controversy" if the properties are from the operators, or of the functions, is not addressed in this paper.

2.3. An example

The conformable FD is defined in [3] as:

$$T_\alpha f(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon} \tag{4}$$

for $t > 0$.

Based on the SSC and WSC criteria we verify that:

- The zero order derivative of a function does not return the function. In fact
$$T_0(f)(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon t) - f(t)}{\epsilon} = \lim_{\epsilon\to 0} \frac{f[t(1+\epsilon)] - f(t)}{1+\epsilon} \cdot \lim_{\epsilon\to 0} \frac{1+\epsilon}{\epsilon}. \tag{5}$$
The first factor in (5) is the order 1 quantum derivative [27] that coincides with $Df(t)$. The second factor goes to ∞.
- The index law does not hold, that is, $T_\alpha T_\beta f(t) \neq T_{\alpha+\beta} f(t)$ for any α and β. To see it we only have to apply (4) twice
$$T_\alpha T_\beta f(t) = \lim_{\epsilon\to 0} \frac{f[t+\epsilon t^{1-\alpha} + (\epsilon t^{1-\beta})^{1-\beta}] - f(t+\epsilon t^{1-\beta}) - f(t+\epsilon t^{1-\alpha}) + f(t)}{\epsilon^2} \neq T_{\alpha+\beta}.$$
- The operator does not verify the generalised Leibniz rule. In fact, it verifies $D^\alpha[fg] = [D^\alpha f]\cdot g + f\cdot[D^\alpha g]$ (see [3]).

In the perspective of the criteria the conformable fractional derivative is not an FD.

3. Analysis of several important definitions

In this section we analyse several important definitions, namely the Grünwald–Letnikov, Riemann–Liouville, and Caputo derivatives (GL-FD, RL-FD and C-FD) and the Riesz potential.

3.1. The Grünwald–Letnikov derivative

We define FD by the limit of the fractional incremental ratio [27]

$$ {}^{GL}D^\alpha_\theta f(z) = e^{-i\alpha\theta} \lim_{|h|\to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z-kh)}{|h|^\alpha}, \tag{6}$$

where $h = |h|e^{j\theta}$ is a complex number, such that $\theta \in (-\pi, \pi]$. This expression is a generalisation of the classical GL-FD [27]. If $t \in \mathbb{R}$, then we define the forward and backward derivatives given by:
Forward derivative ($h = |h|$)

$$ {}^{GL}D^\alpha_f f(t) = \lim_{|h|\to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t-kh)}{|h|^\alpha}. \tag{7}$$

Backward derivative ($h = -|h|$)

$$ {}^{GL}D^\alpha_b f(t) = e^{-i\alpha\pi} \lim_{|h|\to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t+k|h|)}{|h|^\alpha}. \tag{8}$$

Due to the similarity of both expressions we will consider the forward derivative only in the description of the above properties.

Linearity. It is a direct characteristic of this operator.

Additivity and commutativity. We are going to apply (7) twice for any two orders. We have

$$ {}^{GL}D_f^{\alpha}\left[{}^{GL}D_f^{\beta} f(t)\right] = {}^{GL}D_f^{\beta}\left[{}^{GL}D_f^{\alpha} f(t)\right] = {}^{GL}D_\theta^{\alpha+\beta} f(t). \tag{9}$$

This statement is proven in [27].

Neutral element. If we consider $\beta = -\alpha$ in (9), then we obtain:

$$ {}^{GL}D_f^{\alpha}\left[{}^{GL}D_f^{-\alpha} f(t)\right] = {}^{GL}D_f^{0} f(t) = f(t) \tag{10}$$

or again by (9)

$$ {}^{GL}D_f^{-\alpha}\left[{}^{GL}D_f^{\alpha} f(t)\right] = {}^{GL}D_f^{0} f(t) = f(t). \tag{11}$$

When $\alpha \to 0$ the binomial coefficients $\binom{\alpha}{k}$ are all null excepting the first, for $k = 0$, which is one. Therefore, we recover $f(t)$.

Backward compatibility. When $\alpha = n \in \mathbb{N}$ we obtain

$$ {}^{GL}D_f^{n} f(t) = \lim_{h\to 0} \frac{\sum_{k=0}^{n}(-1)^k\binom{n}{k} f(t-kh)}{h^n}.$$

This is the expression we obtain by the repeated application of the first order derivative. If $\alpha = -n \in \mathbb{N}$ we must note that the binomial coefficients may become infinite. To avoid this problem we express them in terms of the Pochhamer symbol $\binom{\alpha}{k} = (-1)^k \frac{(-\alpha)_k}{k!}$. If $\alpha = -n$, then we have $\binom{-n}{k} = (-1)^k \frac{(n)_k}{k!}$ and we can write

$$ {}^{GL}D_f^{-n} f(t) = \lim_{h\to 0} \sum_{k=0}^{n} \frac{(n)_k}{k!} f(t-kh)h^n.$$

This is the expression we obtain when adopting an n-th repeated summation [27]. With the limit computation is essentially a Riemann integral definition. Let us now assume that the summation on the right hand side is uniformly convergent. We can move the computation of the limit towards inside the sum

$$ {}^{GL}D_f^{-n} f(t) = \sum_{k=0}^{n} \lim_{h\to 0} \frac{(n)_k}{k!} f(t-kh)h^n.$$

With $\tau = kh$ and $(n)_k = \frac{\Gamma(n+k)}{\Gamma(n)}$ we obtain $\frac{(n)_k}{k!} = \frac{\Gamma[n+\tau/h]}{(n-1)!\Gamma(\tau/h)}$. The ratio of two gamma functions has a well known asymptotic expression [37] that gives

$$ \frac{\Gamma[n+\tau/h]}{\Gamma(\tau/h)} \approx (\tau/h)^{n-1}$$

as $h \to 0$. In this conditions the summation becomes an integral with $d\tau = h$ and $t = nh$

$$ {}^{GL}D_f^{-n} f(t) = \int_0^\infty f(t-\tau)\tau^{n-1} d\tau,$$

that is the result we obtain by repeated integration.

Derivative of a product. Consider the product of two functions: $f(t) = \varphi(t)\cdot\psi(t)$ defined for $t \in \mathbb{R}$. Assume that one function is analytic in a given region. We have:

$$ {}^{GL}D_f^{\alpha}\left[\varphi(t)\psi(t)\right] = \sum_{n=0}^{\infty}\binom{\alpha}{n}\varphi^{(n)}(t)\,{}^{GL}D_f^{\alpha-n}\psi(t), \tag{12}$$

that is, we obtain the generalised Leibniz rule.

Consequently, the GL-FD verifies both the WSC and the SSC criteria.

3.2. Riemann–Liouville fractional derivative

In this subsection we give the definitions of the RL-FD in the real line and we present some of their properties.

Let $\Omega = [a, b]$ $(-\infty < a < b < \infty)$ be an interval on the real axis $\mathbb{R}$. The *Riemann–Liouville fractional integral* (RL-FI) $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha \in \mathbb{R}^+$ are defined by

$$ {}^{RL}I_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad (x > a; \alpha > 0) \tag{13} $$

and

$$ {}^{RL}I_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)dt}{(t-x)^{1-\alpha}} \quad (x < b; \alpha > 0), \tag{14} $$

respectively. These integrals are called the *left-sided and the right-sided RL-FI*.

The RL-FD $D_{a+}^{\alpha} f$ and $D_{b-}^{\alpha} f$ of order $\alpha \in \mathbb{R}_0^+$ are defined by

$$ {}^{RL}D_{a+}^{\alpha} f(x) := \left(\frac{d}{dt}\right)^n I_{a+}^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}} \quad \big(n = [\alpha]+1; x > a\big) \tag{15} $$

and

$$ {}^{RL}D_{b-}^{\alpha} f(x) := \left(-\frac{d}{dt}\right)^n I_{b-}^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_x^b \frac{f(t)dt}{(t-x)^{\alpha-n+1}} \quad \big(n = [\alpha]+1; x < b\big), \tag{16} $$

respectively, where $[\alpha]$ means the integral part of α.

In the following we will continue with the left RL operator only.

Linearity. It is also a linear operator as it is immediate to observe.

Additivity and commutativity. In the following we will assume that $t \in [a, b]$ and $f(t) \in L_p(a, b)$ for $(1 \leqq p \leqq \infty)$. With this hypothesis the below described relations hold almost everywhere on $[a, b]$ [26,36].

- If $\alpha > 0$ and $\beta > 0$, then

$$ {}^{RL}I_{a+}^{\alpha} I_{a+}^{\beta} f(t) = I_{a+}^{\alpha+\beta} f(t). \tag{17} $$

- The FD is the inverse operation of the left RL-FI.

$$ {}^{RL}D_{a+}^{\alpha}\, {}^{RL}I_{a+}^{\alpha} f(t) = f(t). \tag{18} $$

- If $\alpha > \beta > 0$, then

$$ {}^{RL}D_{a+}^{\beta}\, {}^{RL}I_{a+}^{\alpha} f(t) = {}^{RL}I_{a+}^{\alpha-\beta} f(t). \tag{19} $$

- Let $\alpha \geq 0$, and $m \in \mathbb{N}$. If the FD $D_{a+}^{\alpha} f(t)$ and $D_{a+}^{\alpha+m} f(t)$ exist, then

$$ D^{m\,RL} D_{a+}^{\alpha} y(t) = {}^{RL}D_{a+}^{\alpha+m} y(t). \tag{20} $$

- Let $\alpha > 0$, $n = [\alpha] + 1$, and ${}^{RL}I_{a+}^{n-\alpha} f(t)$ be the RL-FI of order $n - \alpha$.
  (a) If $1 \leq p \leq \infty$ and $f(t) \in I_{a+}^{\alpha}(L_p)$, then

$$ {}^{RL}I_{a+}^{\alpha}\, {}^{RL}D_{a+}^{\alpha} f(t) = f(t). \tag{21} $$

  (b) If $f(t) \in L_1(a, b)$ and $I_{a+}^{n-\alpha} f(t) \in AC^n[a, b]$, then

$$ {}^{RL}I_{a+}^{\alpha}\, {}^{RL}D_{a+}^{\alpha} f(t) = f(t) - \sum_{j=1}^{n} \frac{f_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha - j + 1)} (x-a)^{\alpha-j}. \tag{22} $$

  In particular, if $\alpha = n \in \mathbb{N}$, then [26,36]:

$$ {}^{RL}I_{a+}^{n} D_{a+}^{n} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k. \tag{23} $$

- Let $\alpha > 0$ and $\beta > 0$ be such that $n - 1 < \alpha \leq n$ and $m - 1 < \beta \leq m$ with $(n, m \in \mathbb{N})$ and $\alpha + \beta < n$. For $f \in L_1(a, b)$ and ${}^{RL}I_{a+}^{n-\alpha} f(t) \in AC^m([a, b])$ we have

$$ {}^{RL}D_{a+}^{\alpha}\, {}^{RL}D_{a+}^{\beta} f(t) = {}^{RL}D_{a+}^{\alpha+\beta} f(t) - \sum_{j=1}^{m} {}^{RL}D_{a+}^{\beta-j} f(a+) \frac{(x-a)^{-j-\alpha}}{\Gamma(1-j-\alpha)} \tag{24} $$

and

$$ {}^{RL}I_{a+}^{\beta}\,{}^{RL}D_{a+}^{\beta} f(t) = f(t) - \sum_{j=1}^{m} \frac{{}^{RL}I_{a+}^{m-\beta} f^{(m-j)}(a+)}{\Gamma(\beta - j + 1)} (x-a)^{\beta - j}. \tag{25} $$

Neutral element.

$$ D_{a+}^{0} f(t) = f(t). $$

This property can be deduced from the above relations, but we will use a distinct approach. It can be shown that the power function has the following Laurent series expansion [31]

$$ \tau^{z} = \frac{(-1)^{n-1}\delta^{(n-1)}(\tau)}{(n-1)!(z+n)} + \tau^{-n} u(\tau) + (z+n)\tau^{-n} \ln\big[\tau u(\tau)\big] + \ldots \tag{26} $$

where z is a complex number in the neighbourhood of n. We divide it by a gamma function

$$ \frac{\tau^{z}}{\Gamma(z+1)} = \frac{(-1)^{n-1}\delta^{(n-1)}(\tau)}{(n-1)!(z+n)\Gamma(z+1)} + \frac{\tau^{-n}}{\Gamma(z+1)} u(\tau) + \frac{(z+n)\tau^{-n}\ln[\tau u(\tau)]}{\Gamma(z+1)} + \ldots. \tag{27} $$

When $z \to -n$ the gamma function has a pole at $-(n-1)$. In such case we can write

$$ \lim_{z\to -n} \frac{\tau^{z}}{\Gamma(z+1)} = (-1)^{n-1}\delta^{(n-1)}(\tau). \tag{28} $$

In particular, when $z = -1$, $n = 0$, we have

$$ \lim_{z\to -1} \frac{\tau^{z}}{\Gamma(z+1)} = \delta(\tau) \tag{29} $$

that confirms the existence of the neural element.

Backward compatibility. When $\alpha = n \in \mathbb{N}$, the definition (13) coincides with the n-th integrals of the form [26,36]

$$ \begin{aligned} {}^{RL}I_{a+}^{n} f(t) &= \int_a^x dt_1 \int_a^{t_1} dt_2 \cdots \int_a^{t_{n-1}} f(t_n)dt_n \\ &= \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)dt \quad (n \in \mathbb{N}) \end{aligned} $$

Concerning the integer order derivative we use relation (29) to obtain

$$ {}^{RL}D_{a+}^{n} f(t) = f^{(n)}(t) \quad (n \in \mathbb{N}), \tag{30} $$

where $f^{(n)}(t)$ is the usual derivative of $y(t)$ of order n.

Derivative of a product. The generalised Leibniz rule coincides with the one presented above (12), but it has a somehow involved proof [36].

3.3. Caputo fractional derivatives

In this section we present the definitions and study the properties of the C-FD. Let $[a, b]$ be a finite interval of the real line, and let ${}^{RL}D_{a+}^{\alpha}[f(t)](t) \equiv {}^{RL}D_{a+}^{\alpha} f(t)$ and ${}^{RL}D_{b-}^{\alpha}[f(t)](t) \equiv {}^{RL}D_{b-}^{\alpha} f(t)$ be the RL-FD $\alpha \in \mathbb{R}^{+}$ defined by (15) and (16), respectively. The C-FD ${}^{C}D_{a+}^{\alpha} f(t)$ and ${}^{C}D_{b-}^{\alpha} f(t)$ of order $\alpha \in \mathbb{R}^{+}$ on $[a, b]$ are defined via the above RL-FD by [36]

$$ {}^{C}D_{a+}^{\alpha} f(t) := {}^{RL}D_{a+}^{\alpha} \left[y(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^{k} \right](t) \tag{31} $$

and

$$ {}^{C}D_{b-}^{\alpha} f(t) := D_{b-}^{\alpha} \left[y(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!}(b-t)^{k} \right](t), \tag{32} $$

respectively, where $n = [\alpha] + 1$. These derivatives are called *left-sided and right-sided C-FD of order* α. If $\alpha \notin \mathbb{N}_0$ and $f(t)$ is a function for which the C-FD ${}^{C}D_{a+}^{\alpha} f(t)$ and ${}^{C}D_{b-}^{\alpha} f(t)$ exist together with the RL-FD ${}^{RL}D_{a+}^{\alpha} f(t)$ and ${}^{RL}D_{b-}^{\alpha} f(t)$, then, in accordance with (31) and (32), they are connected with each other by the following relations [36]:

$$ {}^{C}D^{\alpha}_{a+}f(t) = {}^{RL}D^{\alpha}_{a+}f(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha} \quad \big(n = [\alpha]+1\big) \tag{33} $$

and

$$ {}^{C}D^{\alpha}_{b-}f(t) = {}^{RL}D^{\alpha}_{b-}f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-x)^{k-\alpha} \quad \big(n = [\alpha]+1\big). \tag{34} $$

If $\alpha \notin \mathbb{N}_0$, then the C-FD (33) and (34) coincide with the RL-FD (13) and (14) in the following cases $n = [\alpha]+1$:

$$ {}^{C}D^{\alpha}_{a+}f(t) = {}^{RL}D^{\alpha}_{a+}f(t), \tag{35} $$

if $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$ and

$$ {}^{C}D^{\alpha}_{b-}f(t) = {}^{RL}D^{\alpha}_{b-}f(t), \tag{36} $$

if $f(b) = f'(b) = \cdots = f^{(n-1)}(b) = 0$.

If $\alpha = n \in \mathbb{N}_0$ and the usual derivative $f^{(n)}(t)$ of order n exists, then ${}^{C}D^{n}_{a+}f(t)$ coincides with $f^{(n)}(t)$, while ${}^{C}D^{n}_{b-}f(t)$ coincides with $f^{(n)}(t)$ aside the constant multiplier $(-1)^n$, $(n \in \mathbb{N})$:

$$ {}^{C}D^{n}_{a+}f(t) = f^{(n)}(t) \tag{37} $$

and

$$ {}^{C}D^{n}_{b-}f(t) = (-1)^n f^{(n)}(t). \tag{38} $$

The C-FD are defined for functions for which the RL-FD of the right-hand sides of (33) and (34) exist. In particular, they are defined for $f(t)$ belonging to the space $AC^n[a,b]$ of absolutely continuous functions. Let $\alpha \geq 0$ and let n as above. If $f(t) \in AC^n[a,b]$, then the C-FD ${}^{C}D^{\alpha}_{a+}f(t)$ and ${}^{C}D^{\alpha}_{b-}f(t)$ exist almost everywhere on $[a,b]$.

(a) If $\alpha \notin \mathbb{N}_0$, then ${}^{C}D^{\alpha}_{a+}f(t)$ and ${}^{C}D^{\alpha}_{b-}f(t)$ are represented by

$$ {}^{C}D^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)dt}{(x-t)^{\alpha-n+1}} =: {}^{RL}I^{n-\alpha}_{a+}D^n f(t) \tag{39} $$

and

$$ {}^{C}D^{\alpha}_{b-}f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)dt}{(t-x)^{\alpha-n+1}} := (-1)^{n RL}I^{n-\alpha}_{b-}D^n f(t) \tag{40} $$

respectively, where $D = d/dt$ and $n = [\alpha]+1$.

(b) If $\alpha = n \in \mathbb{N}_0$, then from (37) and (38) ${}^{C}D^{n}_{a+}f(t) = f^{(n)}(t)$ and ${}^{C}D^{n}_{b-}f(t) = (-1)^n f^{(n)}(t)$.

In the following we shall be working with the left derivative only since its properties are similar.

Thus the following statements hold.

Linearity. It is an obvious characteristic of this derivative.

Additivity and commutativity. We are going to apply (6) twice for two orders. We have

Let $\alpha > 0$, $\beta > 0$, and $n = [\alpha]+1$ with $f(t) \in L_\infty(a,b)$ or $f(t) \in C[a,b]$,

$$ {}^{C}D^{\alpha}_{a+}\,{}^{RL}I^{\alpha}_{a+}f(t) = f(t) - \frac{{}^{RL}I^{\alpha+1-n}_{a+}y(a+)}{\Gamma(n-\alpha)}(x-a)^{n-\alpha}. \tag{41} $$

We have also

$$ {}^{RL}I^{\alpha}_{a+}f^{(k)}(t) = {}^{RL}I^{\alpha-k}_{a+}f(t) \quad (k = 0, 1, \cdots n-1) \tag{42} $$

$$ {}^{C}D^{\alpha}_{a+}\,{}^{RL}I^{\alpha}_{a+}f(t) = {}^{RL}D^{\alpha}_{a+}\,{}^{RL}I^{\alpha}_{a+}f(t) = f(t). \tag{43} $$

Let $\alpha \notin \mathbb{N}$. If $f(t) \in AC^n[a,b]$ $(f(t) \in C^n[a,b])$, then

$$ {}^{RL}I^{\alpha}_{a+}\,{}^{C}D^{\alpha}_{a+}f(t) = {}^{RL}I^{\alpha}_{a+}\,{}^{RL}I^{n-\alpha}_{a+}D^n f(t) = {}^{RL}I^{n}_{a+}D^{n}_{a+}f(t). \tag{44} $$

Let $\alpha \notin \mathbb{N}$. If $f(t) \in AC^n[a,b]$ $(f(t) \in C^n[a,b])$, then

$$ {}^{RL}I^{\alpha}_{a+}\,{}^{C}D^{\alpha}_{a+}f(t) = {}^{RL}I^{\alpha}_{a+}\,{}^{RL}I^{n-\alpha}_{a+}D^n f(t) = {}^{RL}I^{n}_{a+}D^{n}_{a+}f(t). \tag{45} $$

Neutral element. This property is equal to the corresponding RL-FD.

Backward compatibility. When the order is negative integer we are in the situation of the RL-FI.
The derivative case is simple. In fact it is equivalent to the neutral element ($\alpha - n - 1 = 0$).

Derivative of a product. This derivative verifies a generalised Leibniz rule that is slightly different from the GL and the RL. The correct result can be found in [38] and it reads

$$ {}^{C}D_f^{\alpha}\left[\varphi(t)\psi(t)\right] = \sum_{n=0}^{\infty} \binom{\alpha}{n} \varphi^{(n)}(t)\,{}^{C}D_f^{\alpha-n}\psi(t) + \frac{(t-a)^{-n}}{\Gamma(1-n)}\varphi(a)\big(\psi(t)-\psi(a)\big) + \big(D^n\varphi(t)\big)\psi(t). \tag{46}$$

3.4. The Riesz potential

Many authors adopted the so called one-dimensional Riesz operator on the spacial variable with good results. In fact, such expressions are variations of the corresponding operators introduced by Riesz [39] and Feller [40].

Riesz introduced three n-dimensional integral potential operators to obtain explicitly the potential for hyperbolic, elliptic and parabolic Cauchy problems [39]. The most interesting is the following weak singular integral operator $\mathbb{I}^{\alpha}$ defined by Riesz [39] as

$$ {}^{R}\mathbb{I}^{\alpha} f(t) := \frac{1}{\gamma_n(\alpha)} \int\limits_{\mathbb{R}^n} f(y)|x-y|^{\alpha-n} dy, \tag{47}$$

where $0 < \alpha < n$, with $\alpha - n \neq 2k$, $k \in \mathbb{N}_0$, and f is a suitable function. The normalised constant $\gamma_n(\alpha)$ is given by

$$ \gamma_n(\alpha) = \frac{\pi^{\frac{n}{2}} 2^{\alpha}\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)} = 2\frac{\pi^{\frac{n+1}{2}}\Gamma(\alpha)}{\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)}. \tag{48}$$

Expression (47) originates a hypersingular integral when we extend it to negative orders. Due to this fact, the inverse Riesz potential has been done by means of the following integral [36]:

$$ {}^{R}\mathbb{I}^{-\alpha} f(\bar{x}) := \frac{1}{\gamma_n(\alpha)} \int\limits_{\mathbb{R}^n} \frac{(\Delta_y^l f)(\bar{x})}{|\bar{x}-\bar{y}|^{2\alpha+n}} d\bar{y}, \tag{49}$$

where $(\Delta_y^l f)(t)$ is the lth difference of $f(t)$. For $l > 2\alpha$ the above integral is absolutely convergent. Frequently it is adopted $\alpha < 1$ and $l = 1$ yielding [41,42]

$$ {}^{R}\mathbb{I}^{-\alpha} f(\bar{x}) := \frac{1}{\gamma_n(\alpha)} \int\limits_{\mathbb{R}^n} \frac{f(\bar{y})-f(\bar{x})}{|\bar{x}-\bar{y}|^{2\alpha+n}} d\bar{y}. \tag{50}$$

This operator has been used to implement the fractional Laplacian [41,42]. It should be noted that expression (50) does not implement it exactly [36], since its Fourier transform is not precisely $|\bar{k}|^{\alpha}\hat{f}(\bar{k})$. However, we are dealing with FD and, therefore, we consider the Riesz potential defined on $\mathbb{R}$, since in an n-D domain it is no longer a derivative. In this case it is equivalent to the centred FD [43,44].

We start by rewriting the above integral in a slightly different format

$$ {}^{R}\mathbb{I}^{-\alpha} f(t) := \frac{1}{2\cos\left(\alpha\frac{\pi}{2}\right)\Gamma(-\alpha)} \int\limits_{\mathbb{R}} f(x-y)|y|^{-\alpha-1} dy. \tag{51}$$

We introduce a similar integral, that is, the Feller potential [40], defined as:

$$ {}^{F}\mathbb{I}^{-\alpha} f(t) := \frac{1}{2\sin\left(\alpha\frac{\pi}{2}\right)\Gamma(-\alpha)} \int\limits_{\mathbb{R}} f(x-y)|y|^{-\alpha-1} sgn(y) dy. \tag{52}$$

The Riesz potential (51) verifies properties ^{2}**P1**, ^{2}**P2** and ^{2}**P4** [27,45]. To prove properties ^{2}**P2** and ^{2}**P3** we need a result from the distribution theory [31]

$$ |x|^{\sigma} = 2\frac{\delta^{(2m)}(t)}{\sigma+2m+1} + |x|^{-2m-1} + (\sigma+2m+1)|x|^{-2m-1}\ln|x| + \dots $$

where m is a non-negative integer. In our case ($m = 0$) and we can write:

$$ \frac{|x|^{-\alpha-1}}{2\cos(\alpha\frac{\pi}{2})\Gamma(-\alpha)} = \frac{\delta(t)}{(-\alpha)\cos(\alpha\frac{\pi}{2})\Gamma(-\alpha)} + \frac{|x|^{-1}}{\cos(\alpha\frac{\pi}{2})\Gamma(-\alpha)} + \frac{(-\alpha)|x|^{-1}\ln|x|}{\cos(\alpha\frac{\pi}{2})\Gamma(-\alpha)} + \dots. $$

When $\alpha \to 0$ all the terms on the right go also to zero, with exception of the first term, giving:

$$\lim_{\alpha\to 0}\frac{|x|^{-\alpha-1}}{2\cos(\alpha\frac{\pi}{2})\Gamma(-\alpha)} = \delta(t).$$

Similarly, when the order is a negative odd integer we obtain a null result. When looked from the Fourier transform point of view this case corresponds to a logarithmic potential, which is different from (51). Consequently, we discard this case.

Let us now prove the generalised Leibniz rule. We start by assuming that $f(t) = \psi(t)\phi(t)$ and that one of the functions, say ψ, is analytic. We can have $\psi(x-y) = \sum_0^\infty (-1)^k \frac{\psi^{(k)}(t)}{k!} y^k$. Since $y = |y|sgn(y)$, where $sgn(\cdot)$ is the signum function, we can write

$$\begin{aligned}
{}^{R}\mathbb{I}^{-\alpha}\psi\phi(t) &= \frac{1}{\gamma_1(-\alpha)}\int_{\mathbb{R}} \psi(x-y)\phi(x-y)|y|^{-\alpha-1}dy \\
&= \sum_{k=0}^{\infty}(-1)^k\frac{\psi^{(k)}(t)}{k!}\frac{1}{\gamma_1(-\alpha)}\int_{\mathbb{R}}\phi(x-y)|y|^{-\alpha+k-1}sgn^k(y)dy \\
&= \sum_{m=0}^{\infty}\frac{\psi^{(2m)}(t)}{(2m)!}\left(\mathbb{I}_R^{-\alpha+2m}\phi\right)(t) - \sum_{m=0}^{\infty}\frac{\psi^{(2m+1)}(t)}{(2m+1)!}{}^{F}\mathbb{I}^{-\alpha+2m+1}\phi(t).
\end{aligned}$$

This relation can be considered as another generalisation of the Leibniz rule. Therefore, if we discard the negative odd integer orders, then the Riesz potential can be considered as an FD.

In the n-D case, the Riesz potential is not a derivative, but it makes sense to see if it verifies the Leibniz rule. We write

$$\psi(\bar{x}-\bar{y}) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\dots\sum_{k_n=0}^{\infty}\frac{\frac{\partial^{k_1+k_2+\ldots+k_n}(\bar{x})}{\partial^{k_1}\partial^{k_2}\ldots\partial^{k_n}}}{k_1!k_2!\ldots k_n!}y_1^{k_1}y_2^{k_2}\ldots y_n^{k_n}.$$

Proceeding as previously, we obtain a multiple summation involving integrals with the general format

$$\int_{\mathbb{R}}\phi(\bar{x}-\bar{y})y_1^{k_1}y_2^{k_2}\ldots y_n^{k_n}|\bar{y}|^{-\alpha-n}d\bar{y}.$$

It is straightforward to verify that the terms $y_1^{k_1}y_2^{k_2}\ldots y_n^{k_n}|\bar{y}|^{-\alpha-n}$ do not lead to $|\bar{y}|^{-\alpha-n+k}$. Therefore, the above terms cannot be related to Riesz and Feller potentials. In conclusion, the Leibniz rule is not valid in the n-D case. An interesting particular case occurs if $\psi(\bar{x})$ is spherically invariant [46], because ψ has a Taylor series depending only on $|\bar{x}|$ and we obtain a result that is similar to the 1-D case.

4. Conclusions

This paper discussed the concept of fractional derivative and the properties that such operator should obey. Two sets of conditions for classifying a given operator as an FD were proposed, namely the wide sense criterion and the strict sense criterion. Based on those criteria several current formulations of fractional derivatives were analysed. The conformable fractional derivative was verified to fail some topics. On the other hand, the Grünwald–Letnikov, Riemann–Liouville and Caputo fractional derivatives were revisited and analysed under the light of the proposed criteria. Operators proposed by Riesz and Feller were also considered. In the case of the Riesz potential was deduced a new Leibniz rule. Presently there is a considerable number of proposals for the definitions of FD [2]. This paper did not analyse all formulations, but may represent a useful step towards establishing a clear picture in the topic.

Acknowledgements

This work was partially funded by National Funds through the Foundation for Science and Technology of Portugal, under the project PEst-OE/EEI/UI0066/2011. We thank to Carl F. Lorenzo for the suggesting the fractional derivative concept.

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