A Child's Garden of Fractional Derivatives

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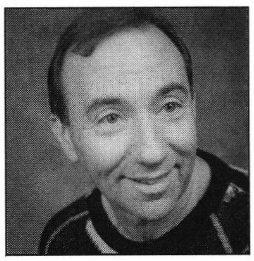

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Introduction

We are all familiar with the idea of derivatives. The usual notation

$$\frac{df(x)}{dx} \quad \text{or} \quad D^1 f(x), \qquad \frac{d^2 f(x)}{dx^2} \quad \text{or} \quad D^2 f(x)$$

is easily understood. We are also familiar with properties like

$$D[f(x) + f(y)] = Df(x) + Df(y).$$

But what would be the meaning of notation like $\frac{d^{1/2}f(x)}{dx^{1/2}}$ or $D^{1/2}f(x)$? Most readers will not have encountered a derivative of "order $1/2$" before, because almost none of the familiar textbooks mention it. Yet the notion was discussed briefly as early as the eighteenth century by Leibnitz. Other giants of the past including L'Hospital, Euler, Lagrange, Laplace, Riemann, Fourier, Liouville, and others at least toyed with the idea. Today a vast literature exists on this subject called the "fractional calculus". Two text books on the subject at the graduate level have appeared recently, **[9]** and **[11]**. Also two collections of papers delivered at conferences are found in **[7]** and **[14]**. A set of very readable seminar notes has been prepared by Wheeler **[15]**, but these have not been published.

It is the purpose of this paper to introduce the fractional calculus in a gentle manner. Rather than the usual definition–lemma–theorem approach, we explore the idea of a fractional derivative by first looking at examples of familiar nth order derivatives like $D^n e^{ax} = a^n e^{ax}$ and then replacing the natural number n by other numbers like $1/2$. In this way, like detectives, we will try to see what mathematical structure might be hidden in the idea. We will avoid a formal definition of the fractional derivative until we have first explored the possibility of various approaches to the notion. (For a quick look at formal definitions see the excellent expository paper by Miller **[8]**.)

As the exploration continues, we will at times ask the reader to ponder certain questions. The answers to these questions are found in the last section of this paper.

So just what is a fractional derivative? Let us see. . . .

Fractional derivatives of exponential functions

We will begin by examining the derivatives of the exponential function e^{ax} because the patterns they develop lend themselves to easy exploration. We are familiar with the expressions for the derivatives of e^{ax}. $D^1 e^{ax} = ae^{ax}$, $D^2 e^{ax} = a^2 e^{ax}$, $D^3 e^{ax} = a^3 e^{ax}$, and in general, $D^n e^{ax} = a^n e^{ax}$ when n is an integer. Could we replace n by $1/2$ and write $D^{1/2} e^{ax} = a^{1/2} e^{ax}$? Why not try? Why not go further and let n be an irrational number like $\sqrt{2}$, or a complex number like $1 + i$?

We will be bold and write

$$D^{\alpha} e^{ax} = a^{\alpha} e^{ax} \tag{1}$$

for any value of α, integer, rational, irrational, or complex. It is interesting to consider the meaning of (1) when α is a negative integer. We naturally want $e^{ax} = D(D^{-1}(e^{ax}))$. Since $e^{ax} = D\left(\frac{1}{a}(e^{ax})\right)$, we have $D^{-1}(e^{ax}) = \int e^{ax}\,dx$. Similarly, $D^{-2}(e^{ax}) = \int\int e^{ax}\,dx\,dx$, so is it reasonable to interpret D^{α} when α is a negative integer $-n$ as the nth iterated integral. D^{α} represents a derivative if α is a positive real number and an integral if α is a negative real number.

Notice that we have not yet given a definition for a fractional derivative of a general function. But if that definition is found, we would expect our relation (1) to follow from it for the exponential function. We note that Liouville used this approach to fractional differentiation in his papers **[5]** and **[6]**.

Questions

Q1 In this case does $D^{\alpha}(c_1 e^{a_1 x} + c_2 e^{a_2 x}) = c_1 D e^{a_1 x} + c_2 D e^{a_2 x}$?

Q2 In this case does $D^{\alpha} D^{\beta} e^{ax} = D^{\alpha+\beta} e^{ax}$?

Q3 Is $D^{-1} e^{ax} = \int e^{ax}\,dx$, and is $D^{-2} e^{ax} = \int\int e^{ax}\,dx\,dx$, (as listed above) really true, or is there something missing?

Q4 What general class of functions could be differentiated fractionally by means of the idea contained in (1)?

Trigonometric functions : sine and cosine.

We are familiar with the derivatives of the sine function: $D^0 \sin x = \sin x$, $D^1 \sin x = \cos x$, $D^2 \sin x = -\sin x$. . . . This presents no obvious pattern from which to find $D^{1/2} \sin x$. However, graphing the functions discloses a pattern. Each time we differentiate, the graph of $\sin x$ is shifted $\pi/2$ to the left. Thus differentiating $\sin x$ n times results in the graph of $\sin x$ being shifted $n\pi/2$ to the left and so $D^n \sin x = \sin\left(x + \frac{n\pi}{2}\right)$. As before, we will replace the positive integer n with an arbitrary α. So, we now have an expression for the general derivative of the sine function, and we can deal similarly with the cosine:

$$D^{\alpha}\sin(x)=\sin\left(x+\frac{\alpha\pi}{2}\right),\qquad D^{\alpha}\cos(x)=\cos\left(x+\frac{\alpha\pi}{2}\right).\tag{2}$$

After finding (2), it is natural to ask if these guesses are consistent with the results of the previous section for the exponential. For this purpose we can use Euler's expression, $e^{ix}=\cos x+i\sin x$. Using (1) we can calculate

$$D^{\alpha}e^{ix}=i^{\alpha}e^{ix}=e^{(i\pi\alpha/2)}e^{ix}=e^{i(x+(\pi/2)\alpha)}=\cos\left(x+\frac{\pi}{2}\alpha\right)+i\sin\left(x+\frac{\pi}{2}\alpha\right),$$

which agrees with (2).

Question

Q5 What is $D^{\alpha}\sin(ax)$?

Derivatives of x^p

We now look at derivatives of powers of x. Starting with x^p we have:

$$\begin{aligned}D^0x^p=x^p,\quad D^1x^p=px^{p-1},\quad D^2x^p=p(p-1)x^{p-2}\cdots\\ D^nx^p=p(p-1)(p-2)\cdots(p-n+1)x^{p-n}.\end{aligned}\tag{3}$$

Multiplying the numerator and denominator of (3) by $(p-n)!$ results in

$$\begin{aligned}D^nx^p&=\frac{p(p-1)(p-2)\cdots(p-n+1)(p-n)(p-n-1)\cdots1}{(p-n)(p-n-1)\cdots1}x^{p-n}\\ &=\frac{p!}{(p-n)!}x^{p-n}.\end{aligned}\tag{4}$$

This is a general expression for D^nx^p. To replace the positive integer n by the arbitrary number α we may use the gamma function. The gamma function gives meaning to $p!$ and $(p-n)!$ in (4) when p and n are not natural numbers. The gamma function was introduced by Euler in the 18th century to generalize the notion of $z!$ to non-integer values of z. Its definition is $\Gamma(z)=\int_0^{\infty}e^{-t}t^{z-1}\,dt$, and it has the property that $\Gamma(z+1)=z!$.

We can rewrite (4) as

$$D^nx^p=\frac{\Gamma(p+1)x^{p-n}}{\Gamma(p-n+1)},$$

which makes sense if n is not an integer, so we put

$$D^{\alpha}x^p=\frac{\Gamma(p+1)x^{p-\alpha}}{\Gamma(p-\alpha+1)}\tag{5}$$

for any α. With (5) we can extend the idea of a fractional derivative to a large number of functions. Given any function that can be expanded in a Taylor series in powers of x,

$$f(x)=\sum_{n=0}^{\infty}a_nx^n,$$

assuming we can differentiate term by term we get

$$D^{\alpha}f(x)=\sum_{n=0}^{\infty}a_nD^{\alpha}x^n=\sum_{n=0}^{\infty}a_n\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha}. \tag{6}$$

This final expression presents itself as a possible candidate for the definition of the fractional derivative for the wide variety of functions that can be expanded in a Taylor's series in powers of x. However, we will soon see that it leads to contradictions.

Question

Q6 Is there a meaning for $D^{\alpha}f(x)$ in geometric terms?

A mysterious contradiction

We wrote the fractional derivative of e^x as

$$D^{\alpha}e^x=e^x. \tag{7}$$

Let us now compare this with (6) to see if they agree. From the Taylor Series, $e^x=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$, (6) gives

$$D^{\alpha}e^x=\sum_{n=0}^{\infty}\frac{x^{n-\alpha}}{\Gamma(n-\alpha+1)}. \tag{8}$$

But (7) and (8) do not match unless α is a whole number! When α is a whole number, the right side of (8) will be the series for e^x, with different indexing. But when α is not a whole number, we have two entirely different functions. We have discovered a contradiction that historically has caused great problems. It appears as though our expression (1) for the fractional derivative of the exponential is inconsistent with our formula (6) for the fractional derivative of a power.

This inconsistency is one reason the fractional calculus is not found in elementary texts. In the traditional calculus, where α is a whole number, the derivative of an elementary function is an elementary function. Unfortunately, in the fractional calculus this is not true. The fractional derivative of an elementary function is usually a higher transcendental function. For a table of fractional derivatives see [3].

At this point you may be asking what is going on? The mystery will be solved in later sections. Stay tuned. . . .

Iterated integrals

We have been talking about repeated derivatives. Integrals can also be repeated. We could write $D^{-1}f(x)=\int f(x)\,dx$, but the right-hand side is indefinite. We will instead write $D^{-1}f(x)=\int_0^x f(t)\,dt$. The second integral will then be $D^{-2}f(x)=\int_0^x\int_0^{t_2}f(t_1)\,dt_1\,dt_2$.

The region of integration is the triangle in Figure 1. If we interchange the order of integration, the right-hand diagram in Figure 1 shows that

$$D^{-2}f(x)=\int_0^x\int_{t_1}^x f(t_1)\,dt_2\,dt_1.$$

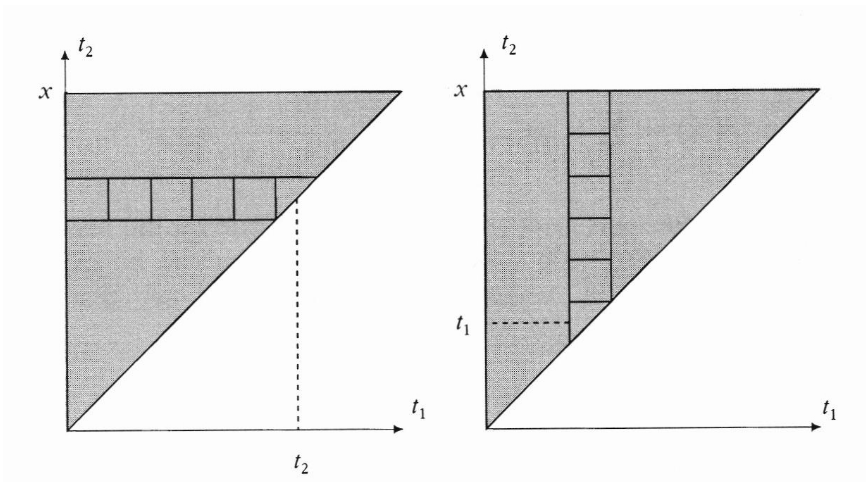

Figure 1

Since $f(t_1)$ is not a function of t_2, it can be moved outside the inner integral, so

$$D^{-2}f(x) = \int_0^x f(t_1)\int_{t_1}^x dt_2\,dt_1 = \int_0^x f(t_1)(x-t_1)\,dt_1$$

or

$$D^{-2}f(x) = \int_0^x f(t)(x-t)\,dt.$$

Using the same procedure we can show that

$$D^{-3}f(x) = \frac{1}{2}\int_0^x f(t)(x-t)^2\,dt, \quad D^{-4}f(x) = \frac{1}{2\cdot 3}\int_0^x f(t)(x-t)^3\,dt,$$

and, in general,

$$D^{-n}f(x) = \frac{1}{(n-1)!}\int_0^x f(t)(x-t)^{n-1}\,dt.$$

Now, as we have previously done, let us replace the $-n$ with arbitrary α and the factorial with the gamma function to get

$$D^{\alpha}f(x) = \frac{1}{\Gamma(-\alpha)}\int_0^x \frac{f(t)\,dt}{(x-t)^{\alpha+1}}. \tag{9}$$

This is a general expression (using an integral) for fractional derivatives that has the potential of being used as a definition. But there is a problem. If $\alpha > -1$, the integral is improper. This occurs because as $t \to x$, $x-t \to 0$. The integral diverges for every $\alpha \geq 0$. When $-1 < \alpha < 0$, the improper integral converges, so if α is negative there is no problem. Since (9) converges only for negative α, it is truly a fractional integral.

Before we leave this section we want to mention that the choice of zero for the lower limit was arbitrary. The lower limit could just as easily have been b. However, the resulting expression will be different. Because of this, many people who work in this field use the notation ${}_bD_x^{\alpha}f(x)$ indicating limits of integration going from b to x. Thus we have from (9)

$${}_bD_x^{\alpha}f(x) = \frac{1}{\Gamma(-\alpha)}\int_b^x \frac{f(t)\,dt}{(x-t)^{\alpha+1}}. \tag{10}$$

Question

Q7 What lower limit of fractional differentiation b will give us the result

$$_bD_x^\alpha (x-c)^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}(x-c)^{p-\alpha}?$$

The mystery solved

Now you may begin to see what went wrong before. We are not surprised that fractional integrals involve limits, because integrals involve limits. Since ordinary derivatives do not involve limits of integration, no one expects fractional derivatives to involve such limits. We think of derivatives as local properties of functions. The fractional derivative symbol D^α incorporates both derivatives (positive α) and integrals (negative α). Integrals are between limits. It turns out that fractional derivatives are between limits also. The reason for the contradiction is that two different limits of integration were being used. Now we can resolve the mystery.

What is the secret? Let's stop and think. What are the limits that will work for the exponential from (1)? Remember we want to write

$$_bD_x^{-1} e^{ax} = \int_b^x e^{ax}\,dx = \frac{1}{a}e^{ax}. \tag{11}$$

What value of b will give this answer? Since the integral in (11) is really

$$\int_b^x e^{ax}\,dx = \frac{1}{a}e^{ax} - \frac{1}{a}e^{ab},$$

we will get the form we want when $\frac{1}{a}e^{ab} = 0$. It will be zero when $ab = -\infty$. So, if a is positive, then $b = -\infty$. This type of integral with a lower limit of $-\infty$ is sometimes called the Weyl fractional derivative. In the notation from (10) we can write (1) as

$$_{-\infty}D_x^\alpha e^{ax} = a^\alpha e^{ax}.$$

Now, what limits will work for the derivative of x^p in (5)? We have

$$_bD_x^{-1} x^p = \int_b^x x^p\,dx = \frac{x^{p+1}}{p+1} - \frac{b^{p+1}}{p+1}.$$

Again we want $\frac{b^{p+1}}{p+1} = 0$. This will be the case when $b = 0$. We conclude that (5) should be written in the more revealing notation

$$_0D_x^\alpha x^p = \frac{\Gamma(p+1)x^{p-\alpha}}{\Gamma(p-\alpha+1)}.$$

So, the expression (5) for $D^\alpha x^p$ has a built-in lower limit of 0. However, expression (1) for $D^\alpha e^{ax}$ has $-\infty$ as a lower limit. This discrepancy is why (7) and (8) do not match. In (7) we calculated $_{-\infty}D_x^\alpha e^{ax}$ and in (8) we calculated $_0D_x^\alpha e^{ax}$.

If the reader wishes to continue this study, we recommend the very fine paper by Miller **[8]** as well as the excellent books by Oldham and Spanier **[11]** and by Miller and Ross **[9]**. Both books contain a short, but very good, history of the fractional calculus with many references. The book by Miller and Ross **[9]** has an excellent discussion of fractional differential equations. Wheelers notes **[14]** are another first rate introduction, which should be made more widely available. Wheeler gives several easily accessible applications, and is particularly interesting to read. Other references of historical interest are **[1, 2, 4, 5, 6, 10, 13]**.

Answers to questions

The following are short answers to the questions throughout the paper.

Q1 Yes, this property does hold.

Q2 Yes, and this is easy to show from relation (2.2)

Q3 Something is missing. That something is the constant of integration. We should have

$$D^{-1}e^{ax} = \int e^{ax}\,dx = a^{-1}e^{ax} + c_1 \quad D^{-2}e^{ax}$$

$$= \iint e^{ax}\,dx = a^{-2}e^{ax} + c_1 x + c_2 \dots .$$

Q4 Let $f(x)$ be expandable in an exponential Fourier series, $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$. Assuming we can differentiate fractionally term by term we get $D^{\alpha}f(x) = \sum_{n=-\infty}^{\infty} c_n (in)^{\alpha} e^{inx}$.

Q5 $D^{\alpha}\sin(ax) = a^{\alpha}\sin(ax + \alpha\pi/2)$.

Q6 We know that $D^1 f(x)$ is geometrically interpreted as the slope of the curve $y = f(x)$ and $D^2 f(x)$ gives us the concavity of the curve. But the third and higher derivatives give us little or no geometric information. Since these are special cases of $D^{\alpha}f(x)$, we are not surprised that there is no easy geometric meaning for the fractional derivative.

Q7 The lower limit of differentiation should be "c".

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