Math 415B Midterm 2 Practice Problems (Posted)

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Question 1

Let R, S be commutative unital rings and $\phi : R \to S$ a ring homomorphism. Let $I \triangleleft S$ be an ideal.

(a) NTS I prime in $S \implies \phi^{-1}(I)$ prime in R.

Proof. Assume I is a prime ideal in S. If $\phi^{-1}(I) = R$ then as R is unital $1_R \in R = \phi^{-1}(R)$ so as ϕ is a homomorphism $\phi(1_R) = 1_S \in I$ but then $I = S$ contradicting out assumption of primality. So, $\phi^{-1}(I) \subsetneq R$.

Let $a, b \in \phi^{-1}(I)$ arbitrarily; then as ϕ is a homomorphism $\phi(a - b) = \phi(a) - \phi(b) \in I$ since I is an ideal containing $\phi(a), \phi(b)$. Let $r \in R$ arbitrarily; then as ϕ is a homomorphism, $\phi(ra) = \phi(r)\phi(a) \in I$ since $\phi(r) \in S$ and $\phi(a) \in I$ and I is an ideal in S. By commutativity and the preceeding logic we conclude $\phi^{-1}(I)$ is a proper ideal of R.

Let $h, j \in R$ be such that $hj \in \phi^{-1}(I)$. Then as ϕ is a homomorphism, $\phi(hj) = \phi(h)\phi(j) \in I$; but then as I is prime $\phi(h) \in I$ or $\phi(j) \in I$, so $\phi^{-1}(\phi(h)) \subseteq \phi^{-1}(I)$ or $\phi^{-1}(\phi(j)) \subseteq \phi^{-1}(I)$, so $h \in \phi^{-1}(I)$ or $j \in \phi^{-1}(I)$. But h, j were arbitrary so, combined with the fact that $\phi^{-1}(I)$ is a proper ideal of R, we conclude that $\phi^{-1}(I)$ is a prime ideal of R and we are done. $\square$

(b) NTS I maximal in $S \implies \phi^{-1}(I)$ maximal in R.

Proof. Assume I is a maximal ideal in S. Consider the natural homomorphism $\rho : R \to S/I$ defined by $r \mapsto \phi(r) + I$. By Theorem 15.3,

$$R/\text{Ker}(\sigma) = R/\phi^{-1}(I) \approx \sigma(R) = \phi(R)/\phi(\phi^{-1}(I))$$

Since I is a maximal ideal of S and $\phi(R)$ is a subring of S, $\phi(R) \cap I$ is a maximal ideal of $\phi(R)$. So by Theorem 14.4, $\phi(R)/(\phi(R) \cap I) = \phi(R)/\phi(\phi^{-1}(I))$ is a field. Then $R/\phi^{-1}(I)$ is isomorphic to a field, therefore $R/\phi^{-1}(I)$ is a field. So again by Theorem 14.4, as R is a ring and $\phi^{-1}(I)$ an ideal in R it follows that $\phi^{-1}(I)$ must be a maximal ideal of R. $\square$

Question 2

Show that the homomorphic image of a PID is a PID

Proof. Assume that S is a PID, R is a ring, and $\phi : S \to R$ is a homomorphism. Let $I \triangleleft \phi(S)$ be an arbitrary ideal of $\phi(S)$. Then $\phi^{-1}(I)$ is an ideal of S. But S is a PID, so we can express $\phi^{-1}(I) = \langle i \rangle$ for some $i \in \phi^{-1}(I)$. So, for any $j \in \phi^{-1}(I)$, we have that $j = ri$ for some $r \in R$. Then $\phi(j) = \phi(r)\phi(i)$ where $\phi(i) \in \phi(\phi^{-1}(I)) \subseteq I$; so $\phi(\phi^{-1}(I)) = \langle \phi(i) \rangle$. But ϕ is onto, so $\phi(\phi^{-1}(I)) = I$, therefore $I = \langle \phi(i) \rangle$. So I is principal. But I was arbitrary so every ideal of $\phi(R)$ is principal; hence as ϕ, R, S were arbitrary (with only the restrictions that ϕ be a homomorphism and S a PID) we conclude in general that the homomorphic image of a PID is a PID, and we are done. $\square$

Question 3

Show that the polynomial $2X + 1 \in \mathbb{Z}_4[X]$ has a multiplicative inverse in $\mathbb{Z}_4[X]$.

Proof.

$$(2X + 1)(1 - 2X) = (1 + 2X)(1 - 2X) = 1 - 4X^2$$

$$4 + 0 = 4 = 0 \text{ mod } 4 \implies -4 = 0 \in \mathbb{Z}_4[X]$$

$$\implies 1 - 4X^2 = 1 + 0X^2 = 1 \in \mathbb{Z}_4[X] \implies (2X + 1)^{-1} = (1 - 2X) \in \mathbb{Z}_4[X]$$

□

Question 4

Prove that the ideal $\langle X \rangle \subseteq \mathbb{Q}[X]$ is maximal.

Proof. Let $a(X) + \langle X \rangle \in \mathbb{Q}[X] + \langle X \rangle$ arbitrarily. If $\deg(a(X)) > 0$, then we can set $a(X) = a_m X^m + a_{m-1} X^{m-1} + ... + a_1 X + a_0 = X(a_m X^{m-1} + a_{m-1} X^{m-2} + ... + a_1) + a_0$ so $a(X) + \langle X \rangle = a_0 + \langle X \rangle$. So, $\deg(a(X)) = 0$. Then $a(X) = a_0 \in \mathbb{Q}$ is just a rational constant. If it is non-zero, then it must be of the form $a_0 = t_0/b_0$ for some $t_0, b_0 \in \mathbb{Z} - \{0\}$. Then it has the multiplicative inverse b_0/t_0 in $\mathbb{Q}$, so:

$$(a_0 + \langle X \rangle)(\frac{b_0}{t_0} + \langle X \rangle) = a_0 \frac{b_0}{t_0} + \langle X \rangle = \frac{b_0}{a_0} \frac{a_0}{b_0} + \langle X \rangle = 1 + \langle X \rangle$$

Since $1 \notin \langle X \rangle$ given that $\deg(1) = 0 < \deg(X) = 1$ it follows that this must be the identity in $\mathbb{Q}[X]/\langle X \rangle$. So then $a(X)^{-1} = (b_0/t_0)X^0$, so $a(X) \in U(\mathbb{Q}[X]/\langle X \rangle)$, so $U(\mathbb{Q}[X]/\langle X \rangle) = \mathbb{Q}[X]/\langle X \rangle$. Moreover, X is irreducible and therefore $\mathbb{Q}[X]/\langle X \rangle$ has no zero divisors. So then $\mathbb{Q}[X]/\langle X \rangle$ is a ring in which every non-zero element is a unit and there are no zero divisors, so it's a field, so $\langle X \rangle \subseteq \mathbb{Q}[X]$ is a maximal ideal of $\mathbb{Q}[X]$ by Theorem 14.4 and we are done. □

Question 5

Find a polynomial with integer coefficients that has $1/2$ and $-1/3$ as zeros.

Proof.

$$(X - \frac{1}{2})2(X + \frac{1}{3})3 = (2X - 1)(3X + 1) = 6X^2 - 3X + 2X - 1 = 6X^2 - X - 1$$

$$6(\frac{1}{2})^2 - \frac{1}{2} - 1 = \frac{6}{4} - \frac{2}{4} - 1 = \frac{4}{4} - 1 = 0$$

$$6(\frac{-1}{3})^2 - \frac{-1}{3} - 1 = 6(\frac{1}{9}) + \frac{3}{9} - \frac{9}{9} = \frac{6 + 3 - 9}{9} = \frac{9 - 9}{9} = 0$$

So $f(X) = 6X^2 - X - 1$ solves the problem. □

Question 6

Suppose that $f(X) = X^n + a_{n-1}X^{n-1} + ... + a_0 \in \mathbb{Z}[X]$. If $r \in \mathbb{Q}$ is a rational number such that $X - r$ divides $f(X)$ then show that r is an integer.

Proof. Assume that $r \in \mathbb{Q}$ is a rational number such that $X - r \mid f(X) = X^n + a_{n-1}X^{n-1} + ... + a_0 \in \mathbb{Z}[X]$.

TODO □