Math 415B Midterm 2 Practice Problems (Posted)

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Question 1

Let R, S be commutative unital rings and $\phi : R \to S$ a ring homomorphism. Let $I \triangleleft S$ be an ideal.

(a) NTS I prime in $S \implies \phi^{-1}(I)$ prime in R.

Proof. Assume I is a prime ideal in S. If $\phi^{-1}(I) = R$ then as R is unital $1_R \in R = \phi^{-1}(R)$ so as ϕ is a homomorphism $\phi(1_R) = 1_S \in I$ but then $I = S$ contradicting out assumption of primality. So, $\phi^{-1}(I) \subsetneq R$. Let $a, b \in \phi^{-1}(I)$ arbitrarily; then as ϕ is a homomorphism $\phi(a - b) = \phi(a) - \phi(b) \in I$ since I is an ideal containing $\phi(a), \phi(b)$. Let $r \in R$ arbitrarily; then as ϕ is a homomorphism, $\phi(ra) = \phi(r)\phi(a) \in I$ since $\phi(r) \in S$ and $\phi(a) \in I$ and I is an ideal in S. By commutativity and the preceeding logic we conclude $\phi^{-1}(I)$ is a proper ideal of R. Let $h, j \in R$ be such that $hj \in \phi^{-1}(I)$. Then as ϕ is a homomorphism, $\phi(hj) = \phi(h)\phi(j) \in I$; but then as I is prime $\phi(h) \in I$ or $\phi(j) \in I$, so $\phi^{-1}(\phi(h)) \subseteq \phi^{-1}(I)$ or $\phi^{-1}(\phi(j)) \subseteq \phi^{-1}(I)$, so $h \in \phi^{-1}(I)$ or $j \in \phi^{-1}(I)$. But h, j were arbitrary so, combined with the fact that $\phi^{-1}(I)$ is a proper ideal of R, we conclude that $\phi^{-1}(I)$ is a prime ideal of R and we are done. $\square$

(b) NTS I maximal in $S \implies \phi^{-1}(I)$ maximal in R.