Math 415B Midterm 2 Practice Problems (Posted)

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April 15, 2019

Question 1

Let R, S be commutative unital rings and $\phi : R \to S$ a ring homomorphism. Let $I \triangleleft S$ be an ideal.

(a) NTS I prime in $S \implies \phi^{-1}(I)$ prime in R.

Proof. Assume I is a prime ideal in S. If $\phi^{-1}(I) = R$ then as R is unital $1_R \in R = \phi^{-1}(R)$ so as ϕ is a homomorphism $\phi(1_R) = 1_S \in I$ but then $I = S$ contradicting out assumption of primality. So, $\phi^{-1}(I) \subsetneq R$.

Let $a, b \in \phi^{-1}(I)$ arbitrarily; then as ϕ is a homomorphism $\phi(a - b) = \phi(a) - \phi(b) \in I$ since I is an ideal containing $\phi(a), \phi(b)$. Let $r \in R$ arbitrarily; then as ϕ is a homomorphism, $\phi(ra) = \phi(r)\phi(a) \in I$ since $\phi(r) \in S$ and $\phi(a) \in I$ and I is an ideal in S. By commutativity and the preceeding logic we conclude $\phi^{-1}(I)$ is a proper ideal of R.

Let $h, j \in R$ be such that $hj \in \phi^{-1}(I)$. Then as ϕ is a homomorphism, $\phi(hj) = \phi(h)\phi(j) \in I$; but then as I is prime $\phi(h) \in I$ or $\phi(j) \in I$, so $\phi^{-1}(\phi(h)) \subseteq \phi^{-1}(I)$ or $\phi^{-1}(\phi(j)) \subseteq \phi^{-1}(I)$, so $h \in \phi^{-1}(I)$ or $j \in \phi^{-1}(I)$. But h, j were arbitrary so, combined with the fact that $\phi^{-1}(I)$ is a proper ideal of R, we conclude that $\phi^{-1}(I)$ is a prime ideal of R and we are done. $\square$

(b) NTS I maximal in $S \implies \phi^{-1}(I)$ maximal in R.

Proof. Assume I is a maximal ideal in S. Consider the natural homomorphism $\rho : R \to S/I$ defined by $r \mapsto \phi(r) + I$. By Theorem 15.3,

$$R/\text{Ker}(\sigma) = R/\phi^{-1}(I) \approx \sigma(R) = \phi(R)/\phi(\phi^{-1}(I))$$

Since I is a maximal ideal of S and $\phi(R)$ is a subring of S, $\phi(R) \cap I$ is a maximal ideal of $\phi(R)$. So by Theorem 14.4, $\phi(R)/(\phi(R) \cap I) = \phi(R)/\phi(\phi^{-1}(I))$ is a field. Then $R/\phi^{-1}(I)$ is isomorphic to a field, therefore $R/\phi^{-1}(I)$ is a field. So again by Theorem 14.4, as R is a ring and $\phi^{-1}(I)$ an ideal in R it follows that $\phi^{-1}(I)$ must be a maximal ideal of R. $\square$

Question 2

Show that the homomorphic image of a PID is a PID

Proof. Assume that S is a PID, R is a ring, and $\phi : S \to R$ is a homomorphism. Let $I \triangleleft \phi(S)$ be an arbitrary ideal of $\phi(S)$. Then $\phi^{-1}(I)$ is an ideal of S. But S is a PID, so we can express $\phi^{-1}(I) = \langle i \rangle$ for some $i \in \phi^{-1}(I)$. So, for any $j \in \phi^{-1}(I)$, we have that $j = ri$ for some $r \in R$. Then $\phi(j) = \phi(r)\phi(i)$ where $\phi(i) \in \phi(\phi^{-1}(I)) \subseteq I$; so $\phi(\phi^{-1}(I)) = \langle \phi(i) \rangle$. But ϕ is onto, so $\phi(\phi^{-1}(I)) = I$, therefore $I = \langle \phi(i) \rangle$. So I is principal. But I was arbitrary so every ideal of $\phi(R)$ is principal; hence as ϕ, R, S were arbitrary (with only the restrictions that ϕ be a homomorphism and S a PID) we conclude in general that the homomorphic image of a PID is a PID, and we are done. $\square$