$\langle x \rangle$ in $\mathbb{Q}[x]$ is maximal

We show

$$\mathbb{Q}[x]/\langle x \rangle \quad \text{is a field.}$$

By theorem proved in text we have

R/I is a ring iff I is an ideal of R. Also $1 \in \mathbb{Q}[x]$ is a unit in $\mathbb{Q}[x]/\langle x \rangle$.

Now we need to show $U\left(\mathbb{Q}[x]/\langle x \rangle\right) = \mathbb{Q}[x]/\langle x \rangle$

Let $f \in \mathbb{Q}[x]/\langle x \rangle$.

Then f has the form

$$f = \sum_{i \in \mathbb{N}_0} a_i x^i + \langle x \rangle \qquad a_i \in \mathbb{Q}$$

$$= \sum_{i \in \mathbb{N}_0} \left(a_i x_i + \langle x \rangle \right)$$

$$= \overline{\sum_{i \in \mathbb{N}_0}} \; a_i \quad \in \mathbb{Q}. \qquad \text{So}$$

$$f^{-1} = \left(\overline{\sum_{i \in \mathbb{N}_0}} \; a_i \right)^{-1} \qquad \square$$