

# Math 415B Midterm 2 Practice Problems (Posted)

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## Question 1

Let  $R, S$  be commutative unital rings and  $\phi : R \rightarrow S$  a ring homomorphism. Let  $I \triangleleft S$  be an ideal.

(a) NTS  $I$  prime in  $S \implies \phi^{-1}(I)$  prime in  $R$ .

*Proof.* Assume  $I$  is a prime ideal in  $S$ . If  $\phi^{-1}(I) = R$  then as  $R$  is unital  $1_R \in R = \phi^{-1}(I)$  so as  $\phi$  is a homomorphism  $\phi(1_R) = 1_S \in I$  but then  $I = S$  contradicting our assumption of primality. So,  $\phi^{-1}(I) \subsetneq R$ .

Let  $a, b \in \phi^{-1}(I)$  arbitrarily; then as  $\phi$  is a homomorphism  $\phi(a - b) = \phi(a) - \phi(b) \in I$  since  $I$  is an ideal containing  $\phi(a), \phi(b)$ . Let  $r \in R$  arbitrarily; then as  $\phi$  is a homomorphism,  $\phi(ra) = \phi(r)\phi(a) \in I$  since  $\phi(r) \in S$  and  $\phi(a) \in I$  and  $I$  is an ideal in  $S$ . By commutativity and the preceding logic we conclude  $\phi^{-1}(I)$  is a proper ideal of  $R$ .

Let  $h, j \in R$  be such that  $hj \in \phi^{-1}(I)$ . Then as  $\phi$  is a homomorphism,  $\phi(hj) = \phi(h)\phi(j) \in I$ ; but then as  $I$  is prime  $\phi(h) \in I$  or  $\phi(j) \in I$ , so  $\phi^{-1}(\phi(h)) \subseteq \phi^{-1}(I)$  or  $\phi^{-1}(\phi(j)) \subseteq \phi^{-1}(I)$ , so  $h \in \phi^{-1}(I)$  or  $j \in \phi^{-1}(I)$ . But  $h, j$  were arbitrary so, combined with the fact that  $\phi^{-1}(I)$  is a proper ideal of  $R$ , we conclude that  $\phi^{-1}(I)$  is a prime ideal of  $R$  and we are done.  $\square$

(b) NTS  $I$  maximal in  $S \implies \phi^{-1}(I)$  maximal in  $R$ .

*Proof.* Assume  $I$  is a maximal ideal in  $S$ . Consider the natural homomorphism  $\rho : R \rightarrow S/I$  defined by  $r \mapsto \phi(r) + I$ . By Theorem 15.3,

$$R/\text{Ker}(\rho) = R/\phi^{-1}(I) \approx \rho(R) = \phi(R)/\phi(\phi^{-1}(I))$$

Since  $I$  is a maximal ideal of  $S$  and  $\phi(R)$  is a subring of  $S$ ,  $\phi(R) \cap I$  is a maximal ideal of  $\phi(R)$ . So by Theorem 14.4,  $\phi(R)/(\phi(R) \cap I) = \phi(R)/\phi(\phi^{-1}(I))$  is a field. Then  $R/\phi^{-1}(I)$  is isomorphic to a field, therefore  $R/\phi^{-1}(I)$  is a field. So again by Theorem 14.4, as  $R$  is a ring and  $\phi^{-1}(I)$  an ideal in  $R$  it follows that  $\phi^{-1}(I)$  must be a maximal ideal of  $R$ .  $\square$

## Question 2

Show that the homomorphic image of a PID is a PID

*Proof.* Assume that  $S$  is a PID,  $R$  is a ring, and  $\phi : S \rightarrow R$  is a homomorphism. Let  $I \triangleleft \phi(S)$  be an arbitrary ideal of  $\phi(S)$ . Then  $\phi^{-1}(I)$  is an ideal of  $S$ . But  $S$  is a PID, so we can express  $\phi^{-1}(I) = \langle i \rangle$  for some  $i \in \phi^{-1}(I)$ . So, for any  $j \in \phi^{-1}(I)$ , we have that  $j = ri$  for some  $r \in R$ . Then  $\phi(j) = \phi(r)\phi(i)$  where  $\phi(i) \in \phi(\phi^{-1}(I)) \subseteq I$ ; so  $\phi(\phi^{-1}(I)) = \langle \phi(i) \rangle$ . But  $\phi$  is onto, so  $\phi(\phi^{-1}(I)) = I$ , therefore  $I = \langle \phi(i) \rangle$ . So  $I$  is principal. But  $I$  was arbitrary so every ideal of  $\phi(R)$  is principal; hence as  $\phi, R, S$  were arbitrary (with only the restrictions that  $\phi$  be a homomorphism and  $S$  a PID) we conclude in general that the homomorphic image of a PID is a PID, and we are done.  $\square$

### Question 3

Find a polynomial with integer coefficients that has  $1/2$  and  $-1/3$  as zeros.

*Proof.*

$$(X - \frac{1}{2})2(X + \frac{1}{3})3 = (2X - 1)(3X + 1) = 6X^2 - 3X + 2X - 1 = 6X^2 - X - 1$$

$$6(\frac{1}{2})^2 - \frac{1}{2} - 1 = \frac{6}{4} - \frac{2}{4} - 1 = \frac{4}{4} - 1 = 0$$

$$6(\frac{-1}{3})^2 - \frac{-1}{3} - 1 = 6(\frac{1}{9}) + \frac{3}{9} - \frac{9}{9} = \frac{6+3-9}{9} = \frac{9-9}{9} = 0$$

So  $f(X) = 6X^2 - X - 1$  solves the problem.

□