NONVANISHING OF SECOND COEFFICIENTS OF HECKE POLYNOMIALS

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Abstract. Let $T_m(N, 2k)$ be the mth Hecke operator on the space $S(N, 2k)$ of cuspforms of weight $2k$ and level N. This paper shows that in all but finitely many cases, which we list, the second coefficient of the characteristic polynomial of $T_2(N, 2k)$ does not vanish when 2 and N are coprime.

1. Introduction

Let $S(N, 2k)$ [15, Section 3.1] denote the space of cuspforms of weight $2k$ and level N with trivial character, and let $s(N, 2k) = \dim S(N, 2k)$. Let $f(z) = \sum_{m=1}^{\infty} a_m q^m, q = e^{2\pi i z}$, be the Fourier expansion of $f \in S(N, 2k)$. For $m \geq 1$, the mth Hecke operator $T_m(N, 2k)$ acts on f by

$$T_m(N, 2k)f(z) = \sum_{n=1}^{\infty} \left(\sum_{\substack{d \mid (m,n) \\ (d,N)=1}} d^{2k-1} a_{mn/d^2} \right) q^n.$$

The study of $T_m(N, 2k)$ is of great interest. For instance, we consider the discriminant function

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n,$$

which is the unique normalized cuspform of weight 12 and level one. Mordell proved Ramunujan's conjectures on the multiplicativity and recurrence of $\tau(n)$ by invoking the fact that $T_m(N, 2k)$ acts on Δ by $T_m(1, 12)(\Delta) = \tau(m)\Delta$ (see [16]). Lehmer [10] conjectured that $\tau(m) \neq 0$ for any $m \geq 1$. Let $\operatorname{Tr} T_m(N, 2k)$ denote the trace of $T_m(N, 2k)$ on the space $S(N, 2k)$. The Lehmer conjecture can then be reinterpreted as follows: $\operatorname{Tr} T(1, 12) \neq 0$. More broadly, the "Generalized Lehmer Conjecture" predicts that $\operatorname{Tr} T_m(N, 2k) \neq 0$ for $2k \geq 16$ or $2k = 12$ and $(m, N) = 1$ (see Rouse [14]). In [14], Rouse proved that the "Generalized Lehmer Conjecture" holds true when $m = 2$. Recently, the nonvanishing of $\operatorname{Tr} T_3(1, 2k)$ was also established in [3].

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A closely related problem is to study the Hecke polynomial $T_m(N,2k)(x)$, the characteristic polynomial of $T_m(N,2k)$. If we write the Hecke polynomial as

$$T_m(N,2k)(x) = \sum_{n=0}^{d_{2k}} (-1)^n a_n x^{d_{2k}-n},$$

where $d_{2k} = \dim S(N,2k)$, then $\operatorname{Tr} T_m(N,2k)$ is the coefficient a_1 of $T_m(N,2k)(x)$. The "Generalized Lehmer Conjecture" then claims that a_1, the first coefficient, does not vanish in general. In this paper, we investigate a_2, the second coefficient of Hecke polynomials, and show its nonvanishing in general. To state our result in direct terms, we first introduce some notation. Let $\alpha_1, \cdots, \alpha_{d_{2k}}$ denote the eigenvalues of $T_m(N,2k)$. Then it is not hard to see that

$$a_2 = \sum_{1 \le i < j \le d_{2k}} \alpha_i \alpha_j.$$

We shall establish the following results on the nonvanishing of a_2 in various scenarios. First, when $N = 1$ and $m = 2$, we have the following complete nonvanishing result on a_2.

Theorem 1.1. *Suppose that $2k \ge 24$ and $2k \ne 26$, that is $\dim S(1,2k) \ge 2$. Let $\alpha_1, ..., \alpha_{d_{2k}}$ be the eigenvalues of the Hecke operator $T_2(1,2k)$ on $S(1,2k)$, where $d_{2k} = s(1,2k)$. Then*

$$\sum_{1 \le i < j \le d_{2k}} \alpha_i \alpha_j \ne 0.$$

Next, when $N = 1$ and $m > 1$ is arbitrary, we obtain an asymptotic result.

Theorem 1.2. *For $m \ge 2$, let $\alpha_1, ..., \alpha_{d_{2k}}$ be the eigenvalues of the Hecke operator $T_m(1,2k)$ on $S(1,2k)$. Then*

$$\sum_{1 \le i < j \le d_{2k}} \alpha_i \alpha_j \ne 0,$$

when $k \ge 72m^4$ for m square and when $k \ge 258m^3$ otherwise.

Lastly, for any odd $N > 1$ and $m = 2$ we get the following complete result.

Theorem 1.3. *Let $\alpha_1, \ldots, \alpha_{d_{2k}}$ be the eigenvalues of the Hecke operator $T_2(N,2k)$ on $S(N,2k)$ with $\gcd(N,2) = 1$ and $N > 1$. Then*

$$\sum_{1 \le i < j \le d_{2k}} \alpha_i \alpha_j \ne 0,$$

except for the values of (N,k) in Table 4.4.

Loosely speaking, Theorem 1.1, Theorem 1.3 and Table 4.4 imply that the a_2 coefficient of $T_2(N,2k)(x)$ does not vanish provided that $\dim S(N,2k) \ge 2$, $2 \nmid N$, and $2k \ge 4$. When $2k = 2$, there are only three exceptions: $(N,2k) = (33,2), (37,2), (57,2)$.

Our method is first to express the coefficient $a_2 = \sum_{1 \le i,j \le d_{2k}} \alpha_i \alpha_j$ in an explicit form (Proposition 2.1) in terms of traces of T_2 and T_4. By resorting to the Eichler-Selberg trace formula of Hecke operators, this reduces the nonvanishing problem to studying the "main terms" (contributed by the trace of T_4) and "error term" according to their asymptotic behaviors as $k \to \infty$ and later as $N \to \infty$.

The paper is organized as follows. In Section 2, we examine the a_2 coefficient of the Hecke polynomial of T_2 for $N = 1$. For k sufficiently large, we demonstrate that nonvanishing of a_2 occurs, and then we demonstrate computationally the nonvanishing for $2k \geq 24$ and $2k \neq 26$, thus completing the proof of Theorem 1.1. The approach here shall serve as a prototype for more general cases. In Section 3, we consider the mth Hecke operator for arbitrary $m \geq 2$ and $N = 1$. We distinguish the behavior of the coefficient a_2 depending on whether m is a perfect square and prove Theorem 1.2 accordingly. In Section 4, we focus on the Hecke polynomial of $T_2(N, 2k)$ for arbitrary odd level $N > 1$. We show that for either N or k sufficiently large, nonvanishing of the a_2 coefficient occurs, and we compute for which remaining cases counterexamples exist. This allows us to establish Theorem 1.3 completely. Finally, in the concluding remarks, we discuss some potential applications of the methodology of this paper to the coefficients of other Hecke polynomials as well as to the new subspace.

2. The case of $T_2(1, 2k)$

In this section, we focus on the Hecke operator $T_2(1, 2k)$, show the nonvanishing of a_2 and prove Theorem 1.1. Before proving this, we prove a useful proposition that applies to any level N and will be used throughout the paper. Then we briefly review some basic facts about modular forms of level one and introduce some preparatory lemmas.

2.1. **Preliminaries**

To simplify notation, we write $T_m = T_m(N, 2k)$ when the level and weight are clear from the context. Then $T_1 = \mathbf{1}$ is the identity, and the Hecke operators on $S(N, 2k)$ satisfy [8, Proposition 3.33]

$$T_m T_n = \sum_{d|(m,n)} d^{2k-1} \cdot T_{mn/d^2}, \tag{2.1}$$

where $\gcd(n, N) = \gcd(m, N) = 1$. We use this relation to construct an explicit formula for a_2.

Proposition 2.1. *Let $\alpha_1, ..., \alpha_{d_{2k}}$ be the eigenvalues of the Hecke operator T_m on $S(N, 2k)$ and let $\gcd(N, m) = 1$. Here, $m = p_1^{r_1} \cdots p_s^{r_s}$ is the prime factorization of m and $d_{2k} = \dim S(N, 2k)$. Then*

$$a_2 = \sum_{1 \leq i < j \leq d_{2k}} \alpha_i \alpha_j = \frac{1}{2} \left[(\operatorname{Tr} T_m)^2 - \sum_{\substack{i_1, \ldots, i_s \\ 0 \leq i_j \leq r_j}} (p_1^{i_1} \cdots p_s^{i_s})^{2k-1} \operatorname{Tr} T_{p_1^{2(r_1 - i_1)} \cdots p_s^{2(r_s - i_s)}} \right].$$

In particular, if $m = 2$, then we have

$$\sum_{1 \le i < j \le d_{2k}} \alpha_i \alpha_j = \frac{1}{2} \left[(\operatorname{Tr} T_2)^2 - \operatorname{Tr} T_4 \right] - 2^{2k-2} d_{2k}. \tag{2.2}$$

Proof. It is straightforward to compute that

$$\sum_{1 \le i < j \le d_{2k}} \alpha_i \alpha_j = \frac{1}{2} \left[\left(\sum_i \alpha_i \right)^2 - \sum_j \alpha_j^2 \right] = \frac{1}{2} \left[(\operatorname{Tr} T_m)^2 - \operatorname{Tr}(T_m \cdot T_m) \right]. \tag{2.3}$$

Note also that by (2.1),

$$\begin{aligned} T_m \cdot T_m &= \sum_{d \mid m} d^{2k-1} \cdot T_{m^2/d^2} \\ &= \sum_{\substack{i_1, \ldots, i_s \\ 0 \le i_j \le r_j}} (p_1^{i_1} \cdots p_s^{i_s})^{2k-1} T_{p_1^{2(r_1 - i_1)} \cdots p_s^{2(r_s - i_s)}}. \end{aligned} \tag{2.4}$$

Plugging (2.4) into (2.3) gives the desired result. □

In order to use Proposition 2.1 for level one, we need the level one formula of d_{2k}. The dimension d_{2k} of $S(1, 2k)$, which can be found in various sources such as [6, p. 88], is given by

$$d_{2k} = \begin{cases} \lfloor \frac{k}{6} \rfloor - 1 & k \equiv 1 \pmod 6,\ k > 1, \\ \lfloor \frac{k}{6} \rfloor & \text{otherwise.} \end{cases} \tag{2.5}$$

Following Zagier's work in [9, Theorem 2], we use the following version of the Eichler-Selberg trace formula on $\mathrm{SL}_2(\mathbb{Z})$: for $2k \ge 4$ and $m \ge 1$, the trace of the Hecke operator T_m on $S(1, 2k)$ is given by

$$\operatorname{Tr} T_m = -\frac{1}{2} \sum_{|t| \le 2\sqrt{m}} P_{2k}(t, m) H(4m - t^2) - \frac{1}{2} \sum_{d \mid m} \min(d, m/d)^{2k-1}, \tag{2.6}$$

where $P_{2k}(t, m)$ is the coefficient of x^{2k-2} in the power series expansion of $(1 - tx + mx^2)^{-1}$ and $H(N)$ denotes the Nth Hurwitz class number. We also have

$$P_{2k}(t, m) = \frac{\rho^{2k-1} - \overline{\rho}^{2k-1}}{\rho - \overline{\rho}}, \tag{2.7}$$

where $\rho + \overline{\rho} = t$ and $\rho \cdot \overline{\rho} = m$.

Lemmas 2.2, 2.3, and 2.4 will be used to bound the first sum in the trace formula (2.6).

Lemma 2.2. *For $t^2 \neq 4m$,*

$$|P_{2k}(t, m)| \le \frac{2m^{k-1/2}}{\sqrt{|t^2 - 4m|}}.$$

Proof. From (2.7), note that $|\rho| = \sqrt{m}$ and $|\rho - \overline{\rho}|^2 = |(\rho + \overline{\rho})^2 - 4\rho\overline{\rho})| = |t^2 - 4m|$. From this observation, one obtains

$$|P_{2k}(t, m)| \leq \frac{2|\rho|^{2k-1}}{|\rho - \overline{\rho}|} = \frac{2m^{k-1/2}}{\sqrt{|t^2 - 4m|}},$$

as desired. □

Lemma 2.3. *For $k \geq 2$ and $t \neq 0$,*

$$P_{2k}(2t, t^2) = t^{2k-2}(2k - 1).$$

Proof. As noted above, $P_{2k}(2t, t^2)$ is the coefficient of x^{2k-2} in the power series expansion of $(1 - 2tx + t^2x^2)^{-1} = (1 - tx)^{-2}$. Differentiating $(1 - tx)^{-1} = \sum_{n \geq 0}(tx)^n$ on both sides yields

$$\frac{1}{(1 - tx)^2} = \frac{1}{t} \sum_{n=0}^{\infty} \frac{d}{dx}(tx)^n = \sum_{n=1}^{\infty} nt^{n-1}x^{n-1}.$$

Letting $n = 2k - 1$ gives the desired result. □

For N a nonnegative integer with $N \equiv 0$ or $3 \pmod 4$, the Hurwitz class number $H(N)$ is given by [5, Definition 2.1]:

$$H(N) := \begin{cases} \frac{\sqrt{N}}{\pi} L(1, -N) & N > 0, \\ \zeta(-1) & N = 0, \end{cases} \tag{2.8}$$

where $L(s, -N) = \sum_{n \geq 1} \left(\frac{-N}{n}\right) n^{-s}$ is the L-series for the Kronecker symbol $\chi_{-N}(n) := \left(\frac{-N}{n}\right)$ and $\zeta(-1) = -\frac{1}{12}$.

Lemma 2.4. *For $N > 0$, the following bound on $H(N)$ holds:*

$$|H(N)| \leq \frac{N \log N}{\pi}.$$

Proof. It suffices to give the upper bound on $L(1, -N)$. Using partial summation [11, Theorem 6.8], we get

$$\sum_{n \leq x} \left(\frac{-N}{n}\right) n^{-1} = x^{-1} \cdot \sum_{n \leq x} \left(\frac{-N}{n}\right) + \int_1^x \sum_{n \leq t} \left(\frac{-N}{n}\right) \frac{dt}{t^2}. \tag{2.9}$$

Note that $-N \equiv 0, 1 \pmod 4$, so $\left(\frac{-N}{\cdot}\right)$ is a primitive character with conductor N. By Pólya's inequality [1, Theorem 8.21], for all $x \geq 1$ we have

$$\left| \sum_{n \leq x} \left(\frac{-N}{n}\right) \right| \leq \sqrt{N} \log N. \tag{2.10}$$

Applying (2.10) to (2.9) and letting x go to infinity gives

$$|L(1,-N)| \leq \int_1^\infty \left| \sum_{n\leq t} \left(\frac{-N}{n}\right) \right| \frac{dt}{t^2} \leq \sqrt{N}\log N. \tag{2.11}$$

Plugging (2.11) into (2.8) gives the desired result. □

2.2. Asymptotics in k

Proposition 2.5. *Let $\alpha_1, ..., \alpha_{d_{2k}}$ be the eigenvalues of the Hecke operator T_2 on $S(1,2k)$. Then*

$$\sum_{1\leq i<j\leq d_{2k}} \alpha_i\alpha_j = 4^{k-2}(-k+C),$$

with $|C| < k$ for all $k \geq 66$.

Proof. First, we rewrite (2.2) as

$$\sum_{1\leq i<j\leq d_{2k}} \alpha_i\alpha_j = \frac{1}{2}\left((\mathrm{Tr}\, T_2)^2 - \mathrm{Tr}\, T_4\right) - 2^{2k-2}d_{2k} = 4^{k-2}(C-k).$$

Here C is a constant given by

$$C = 4^{-k+2}\left(\frac{1}{2}\left((\mathrm{Tr}\, T_2)^2 - \left(\mathrm{Tr}\, T_4 - \frac{4^{k-1}}{12}2k\right)\right) - 2^{2k-2}\delta\right),$$

where by (2.5),

$$d_{2k} = \frac{k}{6} + \delta \text{ and } |\delta| < 2. \tag{2.12}$$

Now, it suffices to show that $C < 66$.

We first bound $(\mathrm{Tr}\, T_2)^2$. The trace formula (2.6) for $m = 2$ becomes

$$\mathrm{Tr}\, T_2 = -\frac{1}{2}P_{2k}(0,2) - P_{2k}(1,2) - \frac{1}{2}P_{2k}(2,2) - 1. \tag{2.13}$$

Applying Lemma 2.2 to the non-trivial terms in (2.13), we obtain that

$$\begin{aligned}
\left|-\frac{1}{2}P_{1,2k}(0,2)\right| &\leq \left|-\frac{1}{2}\cdot\frac{2\cdot 2^{k-1/2}}{\sqrt{8}}\right| \leq 2^{k-2},\\
|-P_{2k}(1,2)| &\leq \frac{2\cdot 2^{k-1/2}}{\sqrt{7}} = \frac{1}{\sqrt{7}}\cdot 2^{k+1/2},\\
\left|-\frac{1}{2}P_{2k}(2,2)\right| &\leq \frac{2\cdot 2^{k-1/2}}{2\sqrt{4}} = 2^{k-3/2}.
\end{aligned}$$

Plugging these bounds into (2.13),

$$|\mathrm{Tr}\, T_2| \leq 2^k\left(\frac{1}{4} + \frac{2}{\sqrt{14}} + \frac{1}{2\sqrt{2}}\right) + 1 \leq 2\cdot 2^k\left(\frac{1}{4} + \frac{2}{\sqrt{14}} + \frac{1}{2\sqrt{2}}\right),$$

which implies that

$$|\operatorname{Tr} T_2|^2 \le 4 \cdot 4^k \left(\frac{1}{4} + \frac{2}{\sqrt{14}} + \frac{1}{2\sqrt{2}}\right)^2 \le 4^{k-2}\left(2 + \frac{16}{\sqrt{14}} + \frac{4}{\sqrt{2}}\right)^2. \tag{2.14}$$

Now we bound $\operatorname{Tr} T_4$. The trace formula (2.6) for $m = 4$ gives

$$\operatorname{Tr} T_4 = -\frac{3}{4}P_{2k}(0,4) - 2P_{2k}(1,4) - \frac{4}{3}P_{2k}(2,4) - P_{2k}(3,4) + \frac{1}{12}P_{2k}(4,4) - 2^{2k-1} - 1, \tag{2.15}$$

where $P_{2k}(4,4) = 4^{k-1}(2k-1)$ by Lemma 2.2. We follow a similar process to bound the remaining terms in (2.15) as follows:

$$\begin{aligned}
\left|-\frac{3}{4}P_{2k}(0,4)\right| &\le \left|\frac{3}{4}\cdot\frac{2\cdot 4^{k-1/2}}{\sqrt{16}}\right| \le 3\cdot 4^{k-2},\\
|2P_{2k}(1,4)| &\le 2\cdot\frac{2\cdot 4^{k-1/2}}{\sqrt{15}} = \frac{32}{\sqrt{15}}4^{k-2},\\
\left|-\frac{4}{3}P_{2k}(2,4)\right| &\le \frac{4}{3}\cdot\frac{2\cdot 4^{k-1/2}}{2\sqrt{3}} = \frac{32}{3\sqrt{3}}4^{k-2},\\
|-P_{2k}(3,4)| &\le \frac{2\cdot 4^{k-1/2}}{\sqrt{7}} = \frac{16}{\sqrt{7}}4^{k-2}.
\end{aligned}$$

From these bounds, we have that

$$\begin{aligned}
|\operatorname{Tr} T_4| &\le 4^{k-2}\left(3 + \frac{32}{\sqrt{15}} + \frac{32}{3\sqrt{3}} + \frac{16}{\sqrt{7}}\right) + \frac{1}{12}P_{2k}(4,4) + 4^{k-1/2} + 1\\
&\le 4^{k-2}\left(3 + \frac{32}{\sqrt{15}} + \frac{32}{3\sqrt{3}} + \frac{16}{\sqrt{7}} + 8\right) + \frac{1}{6}k4^{k-1} + \frac{1}{12}4^{k-1} + 1\\
&\le 4^{k-2}\left(3 + \frac{32}{\sqrt{15}} + \frac{32}{3\sqrt{3}} + \frac{16}{\sqrt{7}} + 8 + \frac{1}{3} + 1\right) + \frac{2}{3}k4^{k-2}\\
&= 4^{k-2}\left(\frac{32}{\sqrt{15}} + \frac{32}{3\sqrt{3}} + \frac{16}{\sqrt{7}} + \frac{37}{3}\right) + \frac{2}{3}k4^{k-2}.
\end{aligned} \tag{2.16}$$

Thus, we obtain the upper bound for $\operatorname{Tr} T_4$.

Combining the bounds given in (2.12), (2.14), and (2.16), one obtains

$$\begin{aligned}
|C| &= \left|4^{-k+2}\left(\frac{1}{2}\left(\operatorname{Tr} T_2^2 - \left(\operatorname{Tr} T_4 - \frac{4^{k-1}}{12}2k\right)\right) - 2^{2k-2}\delta\right)\right|\\
&\le \frac{1}{2}4^{-k+2}\left(4^{k-2}\left(\left(2 + \frac{16}{\sqrt{14}} + \frac{4}{\sqrt{2}}\right)^2 + \frac{32}{\sqrt{15}} + \frac{32}{3\sqrt{3}} + \frac{16}{\sqrt{7}} + \frac{37}{3}\right)\right) + 8\cdot 4^{k-2}\\
&\le \left(\left(\frac{2}{\sqrt{2}} + \frac{16}{\sqrt{28}} + 2\right)^2 + \frac{16}{\sqrt{15}} + \frac{16}{3\sqrt{3}} + \frac{8}{\sqrt{7}} + \frac{37}{6} + 8\right)\\
&\approx 65.85,
\end{aligned}$$

as desired. □

Proposition 2.5 proves that the a_2 coefficient of $T_2(1, 2k)$ does not vanish when $k \geq 66$. There are only finitely many cases in Theorem 1.1 that are not covered by Proposition 2.5, and these we verified by computer [4] (using formulae from [9, Theorem 2] and [2, Lemma 2]). Therefore, the proof of Theorem 1.1 is complete.

3. The case of $T_m(1, 2k)$

In this section, we prove Theorem 1.2, showing the nonvanishing of the a_2 coefficient of the Hecke polynomial $T_m(x) = T_m(1, 2k)(x)$ for an arbitrary m. As a corollary, we will also show that for m a perfect square and k sufficiently large, $\operatorname{Tr} T_m \neq 0$. This is a step in the direction of the Generalized Lehmer Conjecture.

Lemma 3.1. *Let $m \geq 1$ be a perfect square. Then,*

$$\operatorname{Tr} T_m = \frac{km^{k-1}}{6} + B_{m,k} \cdot m^{k-1},$$

with $|B_{m,k}| \leq \frac{35}{4\pi} m^{3/2} \log(4m)$.

Proof. Applying the Eichler-Selberg trace formula given in (2.6),

$$\operatorname{Tr} T_m = -\frac{1}{2} \sum_{|t| \leq 2\sqrt{m}} P_{2k}(t, m) H(4m - t^2) - \frac{1}{2} \sum_{d|m} \min(d, m/d)^{2k-1}.$$

Note that

$$\begin{aligned}
\left| -\frac{1}{2} \sum_{d|m} \min(d, m/d)^{2k-1} \right| &\leq \frac{1}{2} \sum_{d|m} (\sqrt{m})^{2k-1} \\
&\leq \frac{1}{2} m (\sqrt{m})^{2k-1} \\
&= \frac{1}{2} m^{k+1/2}. \qquad (3.1)
\end{aligned}$$

Since m is a perfect square, $\sqrt{m} \in \mathbb{Z}$. It follows that

$$\begin{aligned}
-\frac{1}{2} \sum_{|t| \leq 2\sqrt{m}} P_{2k}(t, m) H(4m - t^2) &= - H(0) P_{2k}(2\sqrt{m}, m) - \frac{1}{2} \sum_{|t| < 2\sqrt{m}} P_{2k}(t, m) H(4m - t^2) \\
&= \frac{(\sqrt{m})^{2k-2}}{12}(2k-1) - \frac{1}{2} \sum_{|t| < 2\sqrt{m}} P_{2k}(t, m) H(4m - t^2) \\
&= \frac{km^{k-1}}{6} - \frac{m^{k-1}}{12} - \frac{1}{2} \sum_{|t| < 2\sqrt{m}} P_{2k}(t, m) H(4m - t^2), \qquad (3.2)
\end{aligned}$$

where the second equality is from Lemma 2.3. We write

$$A := \left| \operatorname{Tr} T_m - \frac{km^{k-1}}{6} \right|.$$

Combining (3.1) and (3.2) and using Lemmas 2.2 and 2.4, one obtains

$$\begin{aligned}
A &\leq \frac{m^{k-1}}{12} + \frac{1}{2}m^{k+1/2} + \left|\frac{1}{2}\sum_{|t|<2\sqrt{m}} P_{2k}(t,m)H(4m-t^2)\right| \\
&\leq \frac{m^{k-1}}{12} + \frac{1}{2}m^{k+1/2} + \frac{1}{2}\sum_{|t|<2\sqrt{m}} \frac{2m^{k-1/2}}{\sqrt{4m-t^2}}\left(\frac{(4m-t^2)\log(4m-t^2)}{\pi}\right) \\
&\leq \frac{m^{k-1}}{12} + \frac{1}{2}m^{k+1/2} + \frac{1}{\pi}\sum_{|t|<2\sqrt{m}} m^{k-1/2}(4m)^{1/2}\log(4m) \\
&\leq \frac{m^{k-1}}{12} + \frac{1}{2}m^{k+1/2} + \frac{8}{\pi}m^{1/2}(m^{k-1/2})m^{1/2}\log(4m) \\
&= \frac{m^{k-1}}{12} + \frac{1}{2}m^{k+1/2} + \frac{8}{\pi}m^{k+1/2}\log(4m) \\
&\leq \frac{35}{4\pi}m^{k+1/2}\log(4m).
\end{aligned}$$

Then if we take $B_{m,k} = Am^{-k+1}$, the result follows. □

Corollary 3.2. *For m a perfect square and $k > 6 \cdot \frac{35}{4\pi}m^{3/2}\log(4m) = \frac{105}{2\pi}m^{3/2}\log(4m)$,*

$$\operatorname{Tr} T_m \neq 0.$$

Proof. From Lemma 3.1,

$$\operatorname{Tr} T_m = \frac{km^{k-1}}{6} + B_{m,k} \cdot m^{k-1},$$

with $|B_{m,k}| \leq \frac{35}{4\pi}m^{3/2}\log(4m)$. From this equation and bound, we can see that when $k > 6 \cdot \frac{35}{4\pi}m^{3/2}\log(4m) = \frac{105}{2\pi}m^{3/2}\log(4m)$, the trace is nonvanishing. □

Lemma 3.3. *Suppose $m \geq 1$ is not a perfect square. Then,*

$$|\operatorname{Tr} T_m| \leq \frac{11}{\pi}m^{k+1/2}\log(4m).$$

Proof. This proof is similar to the proof of Lemma 3.1, so we just outline it. Again,

$$\operatorname{Tr} T_m = -\frac{1}{2}\sum_{|t|<2\sqrt{m}} P_{2k}(t,m)H(4m-t^2) - \frac{1}{2}\sum_{d|m}\min(d,m/d)^{2k-1}.$$

Note that here, as m is not a square and in the first summation t is an integer, we have that $|t| \neq 2\sqrt{m}$. Then again by a process similar to (3.1) and (3.2) and using Lemmas 2.2, and 2.4,

$$\begin{aligned}
|\operatorname{Tr} T_m| &\leq \left|\frac{1}{2}\sum_{|t|<2\sqrt{m}} P_{2k}(t,m)H(4m-t^2)\right| + \frac{1}{2}\sum_{d|m}\min(d,m/d)^{2k-1} \\
&\leq \frac{1}{2}m^{k+1/2} + \frac{1}{2}\sum_{|t|<2\sqrt{m}} 2m^{k-1/2}(4m-t^2)^{1/2}\frac{\log(4m)}{\pi}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}m^{k+1/2} + \frac{1}{\pi}(4\sqrt{m}+1)m^{k-1/2}(4m)^{1/2}\log(4m) \\
&= \frac{1}{2}m^{k+1/2} + \frac{2}{\pi}m^{k}\log(4m) + \frac{8}{\pi}m^{k+1/2}\log(4m) \\
&\leq \frac{11}{\pi}m^{k+1/2}\log(4m),
\end{aligned}$$

which completes the proof. □

Lemma 3.4. *Let $m \geq 1$. Then*

$$\mathrm{Tr}(T_m \cdot T_m) = G_{m,k} \cdot km^{2k-1} + C_{m,k} \cdot m^{2k-1},$$

where $\frac{1}{6} \leq |G_{m,k}| \leq 2m$ and $|C_{m,k}| \leq 16m^3$.

Proof. By (2.1),

$$T_m \cdot T_m = \sum_{d|m} d^{2k-1} T_{m^2/d^2}.$$

It follows that

$$\begin{aligned}
\mathrm{Tr}(T_m \cdot T_m) &= \mathrm{Tr}\left(\sum_{d|m} d^{2k-1} T_{m^2/d^2}\right) \\
&= \sum_{d|m} d^{2k-1}\,\mathrm{Tr}\,T_{m^2/d^2} \\
&= m^{2k-1}d_{2k} + \sum_{\substack{d|m \\ d\neq m}} d^{2k-1}\,\mathrm{Tr}\,T_{m^2/d^2}.
\end{aligned}$$

We now apply Lemma 3.1 to write

$$\mathrm{Tr}\,T_{m^2/d^2} = \frac{k(m/d)^{2(k-1)}}{6} + B_{m^2/d^2,k} \cdot (m/d)^{2(k-1)},$$

and we write $d_{2k} = \frac{k}{6} - \delta$. Hence

$$\begin{aligned}
\mathrm{Tr}(T_m \cdot T_m) &= m^{2k-1}d_{2k} + \sum_{\substack{d|m \\ d\neq m}} d^{2k-1} \cdot \mathrm{Tr}\,T_{(m/d)^2} \\
&= \frac{km^{2k-1}}{6} - m^{2k-1}\delta + \sum_{\substack{d|m \\ d\neq m}} d^{2k-1}\left(\frac{k(m/d)^{2k-2}}{6} + B_{m^2/d^2,k} \cdot (m/d)^{2k-2}\right) \\
&= \frac{km^{2k-1}}{6} - m^{2k-1}\delta + \sum_{\substack{d|m \\ d\neq m}} \frac{kdm^{2k-2}}{6} + \sum_{\substack{d|m \\ d\neq m}} B_{m^2/d^2,k} \cdot dm^{2k-2} \\
&= \frac{k}{6}\left(m^{2k-1} + \sum_{\substack{d|m \\ d\neq m}} dm^{2k-2}\right) - m^{2k-1}\delta + \sum_{\substack{d|m \\ d\neq m}} B_{m^2/d^2,k} \cdot dm^{2k-2}.
\end{aligned}$$

Also,

$$\begin{aligned} m^{2k-1}+\sum_{\substack{d\mid m\\ d\neq m}} dm^{2k-2} &\leq m^{2k-1}+\sum_{\substack{d\mid m\\ d\neq m}} m^{2k-1} \\ &\leq m^{2k-1}+m^{2k} \\ &\leq 2m^{2k}. \end{aligned} \tag{3.3}$$

Then by applying the bound for $B_{m,k}$ in Lemma 3.1 and noting that $|\delta|\leq 2$,

$$\begin{aligned} \left| -m^{2k-1}\delta+\sum_{\substack{d\mid m\\ d\neq m}} dm^{2k-2}\cdot B_{m^2/d^2,k}\right| &\leq m^{2k-1}\delta+\sum_{\substack{d\mid m\\ d\neq m}} dm^{2k-2}\cdot \frac{35}{4\pi}((m/d)^2)^{3/2}\log(4(m/d)^2) \\ &= m^{2k-1}\delta+\frac{35}{2\pi}\sum_{\substack{d\mid m\\ d\neq m}} d^{-2}m^{2k+1}\log(2m/d) \\ &\leq 2m^{2k-1}+\frac{35}{\pi}m^{2k+3/2}\log(2m) \\ &\leq \frac{36}{\pi}m^{2k+3/2}\log(2m) \\ &\leq 16m^{2k+2}. \end{aligned} \tag{3.4}$$

By using the bound from (3.4), one obtains

$$|C_{m,k}\cdot m^{2k-1}| = \left| -m^{2k-1}\delta+\sum_{\substack{d\mid m\\ d\neq m}} dm^{2k-2}\cdot B_{m^2/d^2,\ k}\right| \leq 16m^{2k+2},$$

so $|C_{m,k}|\leq 16m^3$. Using the bound from (3.3),

$$\frac{km^{2k-1}}{6}\leq \left|G_{m,k}\cdot km^{2k-1}\right|\leq \frac{km^{2k}}{3},$$

so $\frac{1}{6}\leq |G_{m,k}|\leq 2m$. □

After finding these bounds, we are prepared to use them to find a bound for the main term and a bound for the error term of the a_2 coefficient for $T_m(1,2k)$. The work below follows the same general idea presented before.

Proposition 3.5. *Suppose $m\geq 1$ is not a perfect square. Then*

$$\frac{1}{2}\left((\operatorname{Tr}T_m)^2-\operatorname{Tr}(T_m\cdot T_m)\right)=\frac{m^{2k-1}}{2}(E_{m,k}-k\cdot G_{m,k}),$$

where $|E_{m,k}|\leq 43m^3$ and $|G_{m,k}|\geq \frac{1}{6}$ as before.

Proof. By using Lemmas 3.3 and 3.4, we can write

$$\frac{1}{2}\left((\operatorname{Tr} T_m)^2 - \operatorname{Tr}(T_m \cdot T_m)\right) = \frac{1}{2}\left((\operatorname{Tr} T_m)^2 - C_{m,k}\cdot m^{2k-1} - G_{m,k}\cdot km^{2k-1}\right).$$

Again by these lemmas, we have

$$\begin{aligned}\left|\frac{1}{2}(\operatorname{Tr} T_m)^2 - \frac{1}{2}(C_{m,k})m^{2k-1}\right| &\le \frac{1}{2}\left(\frac{11}{\pi}m^{k+1/2}\log(4m)\right)^2 + \frac{1}{2}\cdot 16m^{2k+2}\\ &= \frac{m^{2k-1}}{2}\left(\frac{121}{\pi^2}\log(4m)^2m^2 + 16m^3\right).\end{aligned}$$

Notice that

$$\frac{121}{\pi^2}\log(4m)^2 < \frac{11^3}{5\pi^2}m < 27m,$$

since $m \ge 2$. Hence,

$$\left|\frac{1}{2}(\operatorname{Tr} T_m)^2 - \frac{1}{2}C_{m,k}\cdot m^{2k-1}\right| \le \frac{m^{2k-1}}{2}(27m^3+16m^3) = \frac{m^{2k-1}}{2}(43m^3).$$

Thus, $|E_{m,k}| \le 43m^3$. Also, from Lemma 3.4, $|G_{m,k}| \ge \frac{1}{6}$. $\square$

Using the results from Proposition 3.5, it is straightforward to see that for $k \ge 258m^3 \ge |E_{m,k}/G_{m,k}|$, the second coefficient of the Hecke polynomial of T_m is nonvanishing in the case that m is not a perfect square.

Proposition 3.6. *Suppose $m \ge 4$ is a perfect square. Then*

$$\frac{1}{2}\left((\operatorname{Tr} T_m)^2 - \operatorname{Tr}(T_m \cdot T_m)\right) = \frac{km^{2k-2}}{2}\left(F_{m,k} + \frac{k}{36}\right),$$

where $|F_{m,k}| \le 2m^4$ for $k \ge 256$.

Proof. By using Lemmas 3.1 and 3.4, we can write

$$\begin{aligned}\frac{1}{2}((\operatorname{Tr} T_m)^2 - \operatorname{Tr}(T_m\cdot T_m)) =& \frac{1}{2}\left(\left(\frac{km^{k-1}}{6} + B_{m,k}\cdot m^{k-1}\right)^2 - \left(G_{m,k}\cdot km^{2k-1} + C_{m,k}\cdot m^{2k-1}\right)\right)\\ =&\frac{1}{2}\left(\frac{k^2m^{2k-2}}{36} + \frac{B_{m,k}\cdot km^{2k-2}}{3} + (B_{m,k})^2m^{2k-2}\right)\\ &-\frac{1}{2}\left(G_{m,k}\cdot km^{2k-1} + C_{m,k}\cdot m^{2k-1}\right)\\ =&\frac{km^{2k-2}}{2}\left(\frac{k}{36} + \frac{B_{m,k}}{3} - G_{m,k}\cdot m + \frac{(B_{m,k})^2}{k} - \frac{C_{m,k}\cdot m}{k}\right).\end{aligned}$$

Here, by Lemmas 3.1 and 3.4, $|G_{m,k}| \leq 2m$, $|B_{m,k}| \leq \frac{35}{4\pi} m^{3/2} \log(4m)$, and $|C_{m,k}| \leq 16m^3$. We now find a bound for $F_{m,k} = \frac{B_{m,k}}{3} - (G_{m,k})m + \frac{(B_{m,k})^2}{k} - \frac{(C_{m,k})m}{k}$. Also, since we are searching for a bound when k is sufficiently large, we assume that $k \geq 256 = 4^4$. Then,

$$\begin{aligned} |F_{m,k}| &= \left| \frac{B_{m,k}}{3} - G_{m,k} \cdot m + \frac{B^2}{k} - \frac{C_{m,k} \cdot m}{k} \right| \\ &\leq \left| \frac{B_{m,k}}{3} \right| + |G_{m,k} \cdot m| + \frac{(B_{m,k})^2}{256} + \left| \frac{C_{m,k} \cdot m}{256} \right| \\ &\leq \frac{35}{12\pi} m^{3/2} \log(4m) + 2m^2 + \frac{35^2}{256 \cdot 16\pi^2} m^3 \log(4m)^2 + \frac{16}{256} m^4 \\ &\leq \frac{35}{16\pi} m^{5/2} + 2m^2 + m^4 + \frac{m^4}{16} \\ &\leq 2m^4. \end{aligned}$$

This finishes the proof. □

Using the results from Proposition 3.6, it is straightforward to see that for $k \geq 72m^4 \geq |36F_{m,k}|$, the second coefficient of the Hecke polynomial of T_m is nonvanishing when m is a square. Hence, combined with Proposition 3.5, we have proven Theorem 1.2.

4. The case of $T_2(N, 2k)$

In this section, we prove Theorem 1.3 by showing the nonvanishing of the a_2 coefficient of $\operatorname{Tr} T_2(N, 2k)$ when either N or k is sufficiently large. In contrast to Sections 2 and 3, we must develop asymptotic bounds on the a_2 coefficient of $T_2(N, 2k)$ in both N and k, thereby leaving finitely many cases below these thresholds to check via computer. Throughout this section, we shall assume that $N > 1$ (the case of $N = 1$ has been dealt with in Section 2) and that N is odd.

We introduce the trace formula for Hecke operators $T_m(N, 2k)$ using Serre's notation from [15, eq. (34)]:

$$\operatorname{Tr}(T_m(N, 2k)) = A_{1,m} + A_{2,m} + A_{3,m} + A_{4,m}, \tag{4.1}$$

where we adapt from Knightly and Li [8, p. 370-371] and Serre [15, p. 82-84] the formulae for the terms:

$$\begin{aligned} A_{1,m} &= \begin{cases} \frac{2k-1}{12} \psi(N) m^{k-1} & \text{if } m \text{ is a perfect square,} \\ 0 & \text{otherwise,} \end{cases} \\ A_{2,m} &= -\frac{1}{2} \sum_{t^2 < 4m} P_{2k}(t, m) \sum_n h_w \left(\frac{t^2 - 4m}{n^2} \right) \mu(t, n, m), \\ A_{3,m} &= -\frac{1}{2} \sum_{d \mid m} \min(d, m/d)^{2k-1} \sum_\tau{}' \varphi(\gcd(\tau, N/\tau)), \end{aligned}$$

$$A_{4,m} = \begin{cases} \sum_{\substack{c|m \\ \gcd(N,m/c)=1}} c & \text{if } k=1, \\ 0 & \text{otherwise.} \end{cases}$$

Here, we have

- $\psi(N) = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N\prod_{p|N}\left(1+\frac{1}{p}\right)$ [6, p. 21],
- n in $A_{2,m}$ runs through all positive integers such that $n^2 \mid (t^2-4m)$,
- $h_w\left(\frac{t^2-4m}{n^2}\right)$ the weighted class number of discriminant $\frac{t^2-4m}{n^2}$; for our purposes, these are given explicitly in [8, p. 345] (a table of relevant values is included below),
- $\mu(t,n,m)$ is defined to be the number of solutions $c \in (\mathbb{Z}/N\mathbb{Z})^\times$ to the equation $c^2 - tc + m \equiv 0 \pmod N$,
- φ is the Euler totient function.
- $\sum_\tau' \varphi(\gcd(\tau, N/\tau))$ means the summation is over all positive divisors τ of N such that $\gcd(\tau, N/\tau) \mid (d - \frac{m}{d})$.

Note that for $s \equiv 2, 3 \pmod 4$, $h_w(s) = 0$. The following table from [8, p. 345] shows the first few nonzero values of $h_w(s)$:

TABLE 4.1. Weighted Class Numbers

s	$h_w(s)$	s	$h_w(s)$
-3	1/3	-11	1
-4	1/2	-12	1
-7	1	-15	2
-8	1	-16	1

Let $\left(\frac{a}{p}\right)$ denote the Legendre symbol. The following formula for $s(N, 2k) = \dim S(N, 2k)$ comes from [6, p. 88]:

$$s(N,2k) = \begin{cases} (2k-1)(g-1) + \lfloor \frac{k}{2} \rfloor \varepsilon_2 + \lfloor \frac{2k}{3} \rfloor \varepsilon_3 + (k-1)\varepsilon_\infty & 2k \geq 4, \\ g & 2k = 2, \end{cases} \tag{4.2}$$

$$\varepsilon_2 = \prod_{p|N}\left(1 + \left(\frac{-1}{p}\right)\right),$$

$$\varepsilon_3 = \prod_{p|N}\left(1 + \left(\frac{-3}{p}\right)\right),$$

$$\varepsilon_\infty = \sum_{\tau|N} \varphi\left(\gcd\left(\tau, N/\tau\right)\right),$$

$$g = 1 + \frac{\psi(N)}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_\infty}{2},$$

where ε_2 and ε_3 respectively denote the number of elliptic points of order 2 and 3 for $\Gamma_0(N)$, ε_∞ is the number of cusps for $\Gamma_0(N)$, and g is the genus of the compactified modular curve $X_0(N)$.

4.1. **Lemmas and Calculations**

We introduce here a handful of lemmas and calculations common to both of the bounds for a_2 that we will develop in terms of N and k. Firstly, the formula for $\sum_{i<j} \alpha_i\alpha_j$ is as in (2.2):

$$\sum_{1\le i<j\le s(N,2k)} \alpha_i\alpha_j = \frac{1}{2}\left[(\operatorname{Tr} T_2(N,2k))^2 - \operatorname{Tr} T_4(N,2k)\right] - 2^{2k-2}s(N,2k).$$

Using (4.1), the trace formulae for $T_2(N,2k)$ and $T_4(N,2k)$ are

$$\begin{aligned}\operatorname{Tr} T_2(N,2k) = &-\frac{1}{2}\sum_{t^2<8} P_{2k}(t,2)\sum_n h_w\left(\frac{t^2-8}{n^2}\right)\mu(t,n,2)\\ &-\frac{1}{2}\sum_{d\mid 2}\min(d,2/d)^{2k-1}{\sum_\tau}' \varphi(\gcd(\tau,N/\tau)) + A_{4,2},\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\operatorname{Tr} T_4(N,2k) =&\frac{2k-1}{12}\psi(N)4^{k-1} - \frac{1}{2}\sum_{t^2<16} P_{2k}(t,4)\sum_n h_w\left(\frac{t^2-16}{n^2}\right)\mu(t,n,4)\\ &-\frac{1}{2}\sum_{d\mid 4}\min(d,4/d)^{2k-1}{\sum_\tau}' \varphi(\gcd(\tau,N/\tau)) + A_{4,4}.\end{aligned} \tag{4.4}$$

For $T_2(N,2k)$,

$$A_{4,2} \le \sum_{\substack{c\mid 2,\\ \gcd(N,2/c)=1}} c \le 1+2 = 3, \tag{4.5}$$

and for $T_4(N,2k)$,

$$A_{4,4} \le \sum_{\substack{c\mid 4\\ \gcd(N,4/c)=1}} c \le 1+2+4 = 7. \tag{4.6}$$

Lemma 4.1. *Let $\omega(N)$ denote the number of distinct prime divisors of N. Then*

$$|\mu(t,1,2)| \le 2^{\omega(N)}\sqrt{8-t^2}, \quad \textit{and} \quad |\mu(t,1,4)|,\ |\mu(t,2,4)| \le 2^{\omega(N)}\sqrt{16-t^2}.$$

Proof. Recall that $\mu(t,n,m)$ is the number of solutions of c in $(\mathbb{Z}/N\mathbb{Z})^\times$ to the equation $c^2 - tc + m \equiv 0 \pmod N$. We then use the bound from Serre [15, Lemma 2], which he attributes to Huxley [7]:

$$\mu(t,n,m) \le 2^{\omega(N)}\sqrt{4m-t^2}.$$

Note from this that $\mu(t,n,m)$ and $\mu(-t,n,m)$ have the same bound. $\square$

The following bounds for $2^{\omega(N)}$ and $\sigma_0(N)$ are useful for many of the terms appearing in the trace expansions. Although these bounds are well-known, we include them here with proofs for lack of a good reference in the literature.

Lemma 4.2. *Let $\sigma_0(N)$ denote the number of divisors of N. For $N \geq 1$,*

$$2^{\omega(N)} \leq \sigma_0(N), \quad 2^{\omega(N)} \leq \frac{7}{4}\sqrt{N},$$
$$\sigma_0(N) \leq 9N^{1/4}, \quad \textit{and} \quad 2^{\omega(N)} \leq \frac{11}{2}N^{1/4}.$$

Proof. Let $N = \prod_{i=1}^{\omega(N)} p_i^{e_i}$ be the prime factorization of N. Then we can write

$$\sigma_0(N) = \prod_{i=1}^{\omega(N)} (e_i + 1)$$

Hence the first inequality $\sigma_0(N) \geq 2^{\omega(N)}$ follows from the fact that $e_i \geq 1$.

We now show that $2^{\omega(N)} \leq \frac{7}{4}\sqrt{N}$. Again letting $N = \prod_{i=1}^{\omega(N)} p_i^{e_i}$,

$$\frac{2^{\omega(N)}}{\frac{7}{4}\sqrt{N}} = \frac{4}{7}\prod_{i=1}^{\omega(N)} \frac{2}{p_i^{e_i/2}} \leq \frac{4}{7}\prod_{i=1}^{\omega(N)} \frac{2}{p_i^{1/2}}.$$

Notice that if $p_i \geq 5$, clearly $\frac{2}{p_i^{1/2}} < 1$. Hence, it suffices to check that

$$\frac{4}{7}\cdot\frac{2}{2^{1/2}} \leq 1, \quad \frac{4}{7}\cdot\frac{2}{3^{1/2}} \leq 1, \quad \text{and} \quad \frac{4}{7}\cdot\frac{4}{2^{1/2}\cdot 3^{1/2}} \leq 1.$$

All three of the above inequalities are true. Thus for all N,

$$\frac{2^{\omega(N)}}{\frac{7}{4}\sqrt{N}} = \frac{4}{7}\prod_{i=1}^{\omega(N)} \frac{2}{p_i^{e_i/2}} \leq 1.$$

We now show that $\sigma_0(N) \leq 9N^{1/4}$ for $N \geq 17$. From [12, Theorem 1], we have that

$$\log(\sigma_0(N)) \leq \frac{1.5379\log(N)\log(2)}{\log\log(N)},$$

for $N \geq 3$. We use this to demonstrate that for $N \geq 17$,

$$\frac{1.5379\log(2)}{\log\log(N)} \leq \frac{\log(9)}{\log(N)} + \frac{1}{4}.$$

For ease of notation, let $f(N) = \frac{1.5379\log(2)}{\log\log(N)}$ and $g(N) = \frac{\log(9)}{\log(N)}$. Note that the functions f and g are both strictly decreasing for $N \geq e^e$ and $N \geq e$ respectively. We will use this fact to prove that the above inequality is always true for $N \geq 17$.

To do so we first calculate when $f(N) = \frac{1}{4}$. This occurs when $N = e^{64\cdot 2^{379/2500}}$; call this value n_0. Then $g(n_0)$ is about equal to .030907, and $g(n_0) \geq .0309$; let $c_0 = .0309$. Since g is strictly decreasing, for $N \leq n_0$, $g(N) \geq c_0$. Thus, we can now check when $f(N) = \frac{1}{4} + .0309$ and repeat this process. We find that for $N = n_1 = e^{32\cdot 2^{1334/2809}}$, we have $f(N) = \frac{1}{4} + .0309$ and $g(n_1)$ is about equal to .049404 and $g(n_1) \geq .0494$. We continue to repeat this process. A table of relevant values is included below, where c_i is a value such that $g(n_i) \geq c_i$.

Table 4.2. $\sigma_0(N)$ Bound

i	n_i	$f(n_i)$	c_i	i	n_i	$f(n_i)$	c_i
0	$e^{64\cdot 2^{379/2500}}$	$1/4$	.0309	-	-	-	-
1	$e^{32\cdot 2^{1334/2809}}$	$1/4 + .0309$	.0494	11	$e^{16\cdot 2^{943/3609}}$	$1/4 + .1109$	.1145
2	$e^{32\cdot 2^{409/2994}}$	$1/4 + .0494$	.0624	12	$e^{16\cdot 2^{799/3645}}$	$1/4 + .1145$	.1179
3	$e^{16\cdot 2^{2883/3124}}$	$1/4 + .0624$	.0724	13	$e^{16\cdot 2^{51/283}}$	$1/4 + .1179$	.1212
4	$e^{16\cdot 2^{191/248}}$	$1/4 + .0724$	.0805	14	$e^{16\cdot 2^{531/3712}}$	$1/4 + .1212$	.1243
5	$e^{16\cdot 2^{2159/3305}}$	$1/4 + .0805$	.0873	15	$e^{16\cdot 2^{407/3743}}$	$1/4 + .1243$	.1273
6	$e^{16\cdot 2^{1887/3373}}$	$1/4 + .0873$	.0931	16	$e^{16\cdot 2^{41/539}}$	$1/4 + .1273$	.1302
7	$e^{16\cdot 2^{1655/3431}}$	$1/4 + .0931$	.0982	17	$e^{16\cdot 2^{171/3802}}$	$1/4 + .1302$	.1331
8	$e^{16\cdot 2^{1451/3482}}$	$1/4 + .0982$	.1028	18	$e^{16\cdot 2^{55/3831}}$	$1/4 + .1331$	.1359
9	$e^{16\cdot 2^{181/504}}$	$1/4 + .1028$	.1070	19	$e^{8\cdot 2^{3802/3859}}$	$1/4 + .1359$	.1387
10	$e^{16\cdot 2^{157/510}}$	$1/4 + .1070$	.1109	20	$e^{8\cdot 2^{22/23}}$	$1/4 + .1387$	$-$

Then, $e^{8\cdot 2^{22/23}} < 5526162$, so it suffices to check the the inequality above for values of N between 17 and 5526162. This is easily verified by Mathematica, hence the inequality holds. Then

$$\log(\sigma_0(N)) \leq \frac{1.5379\log(2)\log(N)}{\log\log(N)} \leq \log(9) + \frac{\log(N)}{4},$$
$$\sigma_0(N) \leq N^{\frac{1.5379\log(2)}{\log\log(N)}} \leq 9N^{1/4}.$$

This inequality holds for all $N \geq 17$, but by a simple check comparing the first 16 values of $\sigma_0(N)$ and $9N^{1/4}$, it is straightforward to see that $\sigma_0(N) \leq 9N^{1/4}$ for all $N \geq 1$.

For the final bound, we use [13, Theorem 11] which states

$$\omega(N) \leq \frac{1.3841\log(N)}{\log\log(N)}, \text{ for } N \geq 3.$$

We then follow the same process as the previous bound by showing that for $N \geq 783$, the following is true

$$\frac{1.3841\log(2)}{\log\log(N)} \leq \frac{\log(11/2)}{\log(N)} + \frac{1}{4}.$$

By a similar verification as the previous bound, this implies the desired result for $N \geq 783$; for code that does this verification see [4]. Then as before, it suffices to run a check for each value between 1 and 783 to see that $2^{\omega(N)} \leq \frac{11}{2}N^{1/4}$ for all positive N. We verified using Sage that this is true, so the result is complete. □

Proposition 4.3. *Recall that in the trace formula (4.1), when d runs through all divisors of m, τ in $\sum_\tau' \varphi(\gcd(\tau, N/\tau))$ runs through all positive divisors of N such that $\gcd(\tau, N/\tau) \mid (d - \frac{m}{d})$. When $m = 2$ and $d = 1$ or 2,*

$$\sum_\tau{}' \varphi(\gcd(\tau, N/\tau)) \leq \sigma_0(N).$$

When $m = 4$ and $d = 1$ or 4,

$$\sum_\tau{}' \varphi(\gcd(\tau, N/\tau)) \leq 2\sigma_0(N).$$

Proof. First, consider the summations in which $d = 1, 2$ and $m = 2$. Then $d - \frac{2}{d} = \pm 1$. We sum over all positive $\tau \mid N$ such that $\gcd(\tau, N/\tau) \mid \pm 1$. Therefore, $\gcd(\tau, N/\tau) = 1$. Also, in the extreme case, each possible τ will exhibit the property $\gcd(\tau, N/\tau) = 1$, which means there are at most $\sigma_0(N)$ of the τ over which we are summing. Hence,

$$\sum_\tau{}' \varphi(\gcd(\tau, N/\tau) \leq \varphi(1)\sigma_0(N) = \sigma_0(N).$$

Next, consider the summations in which $d = 1, 4$ and $m = 4$. Then $d - \frac{4}{d} = \pm 3$. We sum over all positive $\tau | N$ such that $\gcd(\tau, N/\tau) \mid \pm 3$. In the extreme case, $\gcd(\tau, N/\tau) = 3$, so

$$\sum_\tau{}' \varphi(\gcd(\tau, N/\tau) \leq \varphi(3)\sigma_0(N) = 2\sigma_0(N),$$

as desired. ∎

Thus far, the only $\sum_\tau' \varphi(\gcd(\tau, N/\tau))$ term that appears in the trace formula which has not been computed or bounded is that with $m = 4$ and $d = 2$. Observe that

$$\sum_\tau{}' \varphi(\gcd(\tau, N/\tau)) = \sum_{\tau | N} \varphi(\gcd(\tau, N/\tau)).$$

This is because $d - \frac{m}{d} = 2 - \frac{4}{2} = 0$, so τ on both sides actually runs through all divisors of N. The next lemma helps us compute this case. Note that by (4.2), this sum is exactly the number of cusps ε_∞, so we can also use this bound when considering ε_∞.

Lemma 4.4. *For $N > 1$,*

$$\varepsilon_\infty = \sum_{\tau | N} \varphi(\gcd(\tau, N/\tau)) \leq 2^{\omega(N)}\sqrt{N} \leq \psi(N)$$

Proof. Let $N = p_1^{e_1} \cdots p_s^{e_s}$. Recall that φ is multiplicative, and that $\varphi(p^e) = p^e - p^{e-1}$ for $e \geq 0$, where we interpret $p^{-1} = 0$. From these facts,

$$\sum_{\tau | N} \varphi(\gcd(\tau, N/\tau)) = \sum_{b_1, \ldots, b_s} \varphi\left(\gcd\left(\prod_{i=1}^s p_i^{b_i}, \prod_{i=1}^s p_i^{e_i - b_i}\right)\right)$$

$$= \sum_{b_1,\dots,b_s} \varphi\left(\prod_{i=1}^{s} p_i^{\min(b_i,e_i-b_i)}\right)$$
$$= \sum_{b_1,\dots,b_s} \prod_{i=1}^{s} \varphi\left(p_i^{\min(b_i,e_i-b_i)}\right),$$

where the summation ranges over all s-tuples $(b_1,\dots,b_s)$ such that $0 \le b_i \le e_i$. Now fix some i with $1 \le i \le s$ and consider the sum as b_i ranges from 0 to e_i. Define

$$S_i := \sum_{0 \le b_i \le e_i} \varphi\left(p_i^{\min(b_i,e_i-b_i)}\right).$$

Then we can write

$$\sum_{b_1,\dots,b_s} \prod_{i=1}^{s} \varphi\left(p_i^{\min(b_i,e_i-b_i)}\right) = \prod_{i=1}^{s} S_i.$$

We proceed with two cases for calculating S_i.

(a) When e_i is odd:

$$\begin{aligned} S_i = \sum_{0 \le b_i \le e_i} \varphi\left(p_i^{\min(b_i,e_i-b_i)}\right) &= 2 \sum_{0 \le b_i \le \frac{e_i-1}{2}} \varphi(p_i^{b_i}) \\ &= 2 \sum_{0 \le b_i \le \frac{e_i-1}{2}} \left(p_i^{b_i} - p_i^{b_i-1}\right) \\ &= 2p^{(e_i-1)/2}. \end{aligned}$$

(b) When e_i is even:

$$\begin{aligned} S_i = \sum_{0 \le b_i \le e_i} \varphi\left(p_i^{\min(b_i,e_i-b_i)}\right) &= \varphi\left(p_i^{e_i/2}\right) + 2 \sum_{0 \le b_i \le \frac{e_i}{2}-1} \varphi(p_i^{b_i}) \\ &= p_i^{e_i/2} - p_i^{e_i/2-1} + 2 \sum_{0 \le b_i \le \frac{e_i}{2}-1} \left(p_i^{b_i} - p_i^{b_i-1}\right) \\ &= p_i^{e_i/2} - p_i^{e_i/2-1} + 2p^{e_i/2-1} \\ &= p_i^{e_i/2} + p_i^{e_i/2-1}. \end{aligned}$$

In both cases, it is easy to see that $|S_i| \le 2p^{e_i/2}$. Finally,

$$\left|\sum_{\tau \mid N} \varphi(\gcd(\tau, N/\tau))\right| = \prod_{i=1}^{s} |S_i| \le 2^s \prod_{i=1}^{s} p_i^{e_i/2} = 2^{\omega(N)}\sqrt{N},$$

where $s = \omega(N)$. This proves the first inequality.

To show the second inequality, observe that the functions $2^{\omega(N)}\sqrt{N}$ and $\psi(N)$ on both sides of the inequality are multiplicative, and it suffices to show the inequality at prime powers, where:

$$2^{\omega(p^\alpha)}\sqrt{p^\alpha} \leq \psi(p^\alpha) = p^\alpha\left(1+\frac{1}{p}\right).$$

If $\alpha \geq 2$, then $2^{\omega(p^\alpha)}\sqrt{p^\alpha} \leq p^{1+\alpha/2} \leq \psi(p^\alpha)$. So it suffices to consider the case where $\alpha = 1$ and $p = 2$; this is easily verified, as

$$2^{\omega(2)} \cdot \sqrt{2} = 2 \cdot \sqrt{2} \leq 3 = 2\left(1+\frac{1}{2}\right).$$

This proves the string of inequalities. □

Proposition 4.5. *If $2k \geq 4$, then*

$$s(N,2k) = \frac{\psi(N)}{6}(k + F_{N,k}),$$

where $|F_{N,k}| \leq 12$.

Proof. We have from (4.2) that

$$\begin{aligned}
s(N,2k) =&(2k-1)(g-1) + \left\lfloor\frac{k}{2}\right\rfloor\varepsilon_2 + \left\lfloor\frac{2k}{3}\right\rfloor\varepsilon_3 + (k-1)\varepsilon_\infty \\
=&2k + \frac{2k\psi(N)}{12} - \frac{k}{2}\varepsilon_2 - \frac{2k}{3}\varepsilon_3 - k\varepsilon_\infty - 2k - 1 - \frac{\psi(N)}{12} + \frac{\varepsilon_2}{4} + \frac{\varepsilon_3}{3} + \frac{\varepsilon_\infty}{2} \\
&+ 1 + \left\lfloor\frac{k}{2}\right\rfloor\varepsilon_2 + \left\lfloor\frac{2k}{3}\right\rfloor\varepsilon_3 + (k-1)\varepsilon_\infty \\
=&\frac{2k\psi(N)}{12} + \left(\left\lfloor\frac{k}{2}\right\rfloor - \frac{k}{2} + \frac{1}{4}\right)\varepsilon_2 + \left(\left\lfloor\frac{2k}{3}\right\rfloor - \frac{2k}{3} + \frac{1}{3}\right)\varepsilon_3 - \frac{\varepsilon_\infty}{2} - \frac{\psi(N)}{12}. \qquad (4.7)
\end{aligned}$$

Recall the following formulae for ε_2 and ε_3 from [6, p. 96]:

$$\varepsilon_2 = \prod_{p|N}\left(1 + \left(\frac{-1}{p}\right)\right), \quad \text{and} \quad \varepsilon_3 = \prod_{p|N}\left(1 + \left(\frac{-3}{p}\right)\right).$$

Hence, $0 \leq \varepsilon_2, \varepsilon_3 \leq 2^{\omega(N)}$ since $0 \leq 1 + \left(\frac{-1}{p}\right) \leq 2$, and $0 \leq 1 + \left(\frac{-3}{p}\right) \leq 2$. Then $\varepsilon_2, \varepsilon_3 \leq 2^{\omega(N)} \leq 2^{\omega(N)}\sqrt{N} \leq \psi(N)$ by Lemma 4.4. Moreover, by the same lemma, $0 \leq \varepsilon_\infty \leq \psi(N)$. Also, one can easily verify through casework that

$$\left|\left\lfloor\frac{k}{2}\right\rfloor - \frac{k}{2} + \frac{1}{4}\right| \leq \frac{1}{4}, \quad \text{and} \quad \left|\left\lfloor\frac{2k}{3}\right\rfloor - \frac{2k}{3} + \frac{1}{3}\right| \leq \frac{1}{3}. \qquad (4.8)$$

Using the fact that $\varepsilon_2, \varepsilon_3$ and ε_∞ are bounded above by $\psi(N)$ and using (4.8), we get that

$$\begin{aligned}
\left|s(N,2k) - \frac{k\psi(N)}{6}\right| &\leq \left|\left\lfloor\frac{k}{2}\right\rfloor - \frac{k}{2} + \frac{1}{4}\right|\varepsilon_2 + \left|\left\lfloor\frac{2k}{3}\right\rfloor - \frac{2k}{3} + \frac{1}{3}\right|\varepsilon_3 + \frac{\varepsilon_\infty}{2} + \frac{\psi(N)}{12} \\
&\leq \frac{3}{4}\varepsilon_2 + \frac{2}{3}\varepsilon_3 + \frac{\psi(N)}{2} + \frac{\psi(N)}{12}
\end{aligned}$$

$$\leq \frac{\psi(N)}{6}\left(\frac{9}{2}+4+3+\frac{1}{2}\right)$$
$$= \frac{\psi(N)}{6}(12).$$

Hence, $|F_{N,k}| \leq 12$ as desired. □

4.2. **Asymptotics in** k

Proposition 4.6. *For odd* $N > 1$,

$$|\operatorname{Tr} T_2(N,2k)| \leq 3 + 2^{k+3/2}2^{\omega(N)} + \sigma_0(N).$$

Proof. According to (4.3),

$$|\operatorname{Tr} T_2(N,2k)| \leq \frac{1}{2}\left|\sum_{t^2<8} P_{2k}(t,2)\sum_n h_w\left(\frac{t^2-8}{n^2}\right)\mu(t,n,2)\right| + \frac{1}{2}\left|\sum_{d|2}\min(d,2/d)^{2k-1}\sum_{\tau}\varphi(\gcd(\tau,N/\tau))\right| + A_{4,2}.$$

Recall from (4.5) that $0 \leq A_{4,2} \leq 3$. Moreover, using the values in Table 4.1,

$$\begin{aligned}|A_{2,2}| =& \frac{1}{2}\left|\sum_{t^2<8} P_{2k}(t,2)\sum_n h_w\left(\frac{t^2-8}{n^2}\right)\mu(t,n,2)\right| \\ \leq& \frac{1}{2}\left|P_{2k}(2,2)\cdot\frac{1}{2}\left(\mu(2,1,2)+\mu(-2,1,2)\right) + P_{2k}(1,2)(\mu(1,1,2)+\mu(-1,1,2))\right| \\ &+ \frac{1}{2}\left|P_{2k}(0,2)\mu(0,1,2)\right|.\end{aligned}$$

Then combining the bound for $\mu(t,1,2)$ given in Proposition 4.1 with the bound for $P_{2k}(t,2)$ in Lemma 2.2, we obtain

$$\begin{aligned}|A_{2,2}| &\leq \frac{1}{2}|P_{2k}(2,2)|\mu(2,1,2) + |P_{2k}(1,2)|\mu(1,1,2) + \frac{1}{2}|P_{2k}(0,2)|\mu(0,1,2) \\ &\leq \frac{2^{\omega(N)}\sqrt{8-4}}{2}\left(\frac{2\cdot 2^{k-1/2}}{\sqrt{8-4}}\right) + 2^{\omega(N)}\sqrt{8-1}\left(\frac{2\cdot 2^{k-1/2}}{\sqrt{8-1}}\right) + \frac{2^{\omega(N)}\sqrt{8}}{2}\left(\frac{2\cdot 2^{k-1/2}}{\sqrt{8}}\right) \\ &= 2^{k-1/2}2^{\omega(N)}\left(1+2+1\right) \\ &= 2^{k+3/2}2^{\omega(N)}.\end{aligned}$$

Now, consider the third term:

$$A_{3,2} = \frac{1}{2}\left|\sum_{d|2}\min(d,2/d)^{2k-1}{\sum_{\tau}}'\varphi(\gcd(\tau,N/\tau))\right| \leq \frac{1}{2}\sigma_0(N)\sum_{d|2}\min(d,2/d)^{2k-1} = \sigma_0(N),$$

where, with $m = 2$, we have from Proposition 4.3,

$$\left|\sum_{\tau}{}' \varphi(\gcd(\tau, N/\tau))\right| \leq \sigma_0(N).$$

Combining everything above yields

$$|\operatorname{Tr} T_2(N, 2k)| \leq 3 + 2^{k+3/2} 2^{\omega(N)} + \sigma_0(N),$$

as desired. □

Proposition 4.7. *For odd $N > 1$,*

$$\operatorname{Tr} T_4(N, 2k) = \frac{\psi(N)}{6} 4^{k-1} \left(k + E_{N,k}\right),$$

where $|E_{N,k}| \leq 93.5$ if $1 \leq k \leq 3$ and $|E_{N,k}| \leq 68$ if $k \geq 4$.

Proof. Recall from (4.4) that

$$\begin{aligned}\operatorname{Tr} T_4(N, 2k) =& \frac{2k-1}{12}\psi(N)4^{k-1} \\ & - \frac{1}{2}\sum_{t^2<16} P_{2k}(t,4)\sum_{n} h_w\left(\frac{t^2-16}{n^2}\right)\mu(t,n,4) \\ & - \frac{1}{2}\sum_{d|4}\min(d,4/d)^{2k-1}\sum_{\tau}{}'\varphi(\gcd(\tau,N/\tau)) + A_{4,4}.\end{aligned}$$

We have the bound $0 \leq A_{4,4} \leq 7$ as given in (4.6). We write the first term as

$$\frac{2k-1}{12}\psi(N)4^{k-1} = \frac{k}{6}\psi(N)4^{k-1} - \frac{1}{12}\psi(N)4^{k-1}. \tag{4.9}$$

Note that $\frac{k}{6}\psi(N)4^{k-1}$ is the main term of $\operatorname{Tr} T_4(N, 2k)$. Using the values in Table 4.1, the $A_{2,4}$ term is as before:

$$\begin{aligned}A_{2,4} =& -\frac{1}{2}\left(P_{2k}(3,4)(\mu(-3,1,4)+\mu(3,1,4)) + 2P_{2k}(1,4)(\mu(-1,1,4)+\mu(1,1,4))\right) \\ & - \frac{1}{2}\left(P_{2k}(2,4)\left(\mu(-2,1,4)+\mu(2,1,4)+\frac{1}{3}\left(\mu(-2,2,4)+\mu(2,2,4)\right)\right)\right) \\ & - \frac{1}{2}\left(P_{2k}(0,4)\left(\mu(0,1,4)+\frac{1}{2}\mu(0,2,4)\right)\right).\end{aligned} \tag{4.10}$$

Combining the bounds for μ from Proposition 4.1 with those given for $P_{2k}(t, 4)$ in Lemma 2.2,

$$\begin{aligned}|A_{2,4}| \leq& \frac{1}{2}\left(\frac{2\cdot 4^{k-1/2}}{\sqrt{16-9}}\right) 2\cdot 2^{\omega(N)}\sqrt{16-9} + \frac{1}{2}\left(\frac{4\cdot 4^{k-1/2}}{\sqrt{16-1}}\right) 2\cdot 2^{\omega(N)}\sqrt{16-1} \\ & + \frac{1}{2}\left(\frac{2\cdot 4^{k-1/2}}{\sqrt{16-4}}\right)\left(2\cdot 2^{\omega(N)}\sqrt{16-4} + \frac{2}{3}\cdot 2^{\omega(N)}\sqrt{16-4}\right) + \frac{1}{2}\left(\frac{2\cdot 4^{k-1/2}}{\sqrt{16}}\right)\frac{3}{2}\cdot 2^{\omega(N)}\sqrt{16}\end{aligned}$$

$$
\begin{aligned}
&= 4^{k-1/2} \cdot 2^{\omega(N)} \left(2 + 4 + \frac{8}{3} + \frac{3}{2} \right) \\
&= \frac{61}{6} \cdot 4^{k-1/2} \cdot 2^{\omega(N)}. \qquad (4.11)
\end{aligned}
$$

Now, we bound the third term. Recall that

$$
A_{3,4} = -\frac{1}{2} \sum_{d|4} \min(d, 4/d)^{2k-1} \sum_{\tau}{}' \varphi(\gcd(\tau, N/\tau)).
$$

Then,

$$
A_{3,4} = -\frac{1}{2} \sum_{\tau}{}' \varphi(\gcd(\tau, N/\tau)) - 2^{2k-2} \sum_{\tau}{}' \varphi(\gcd(\tau, N/\tau)) - \frac{1}{2} \sum_{\tau}{}' \varphi(\gcd(\tau, N/\tau)). \quad (4.12)
$$

Here, the left and right summations over τ in (4.12) have $d = 1$ and $d = 4$, respectively. By Proposition 4.3, both of these sums are bounded by $2\sigma_0(N)$. Now consider the middle summation in (4.12) in which $d = 2$. Since $\sum' \varphi(\gcd(\tau, N/\tau)) = \sum_{\tau|N} \varphi(\gcd(\tau, N/\tau))$ in this case, by Lemma 4.4 it follows that

$$
\left| -2^{2k-2} \sum_{\tau}{}' \varphi(\gcd(\tau, N/\tau) \right| \leq 2^{2k-2} \cdot 2^{\omega(N)} \sqrt{N} \leq 2^{2k-2} \psi(N).
$$

Overall, we get

$$
|A_{3,4}| \leq 2\sigma_0(N) + 2^{2k-2} \psi(N). \qquad (4.13)
$$

We then combine the inequalities (4.11), (4.13), and $|A_{4,4}| \leq 7$, the bound $2^{\omega(N)} \leq \psi(N)/\sqrt{N}$ from Lemma 4.4, and the trivial bound $\sigma_0(N) \leq N \leq \psi(N)$ to get

$$
\begin{aligned}
\left| \operatorname{Tr} T_4(2k, N) - \frac{k}{6} \psi(N) 4^{k-1} \right| &\leq \frac{1}{12} \psi(N) 4^{k-1} + \frac{61}{6} \cdot 4^{k-1/2} 2^{\omega(N)} + 2\sigma_0(N) + 2^{2k-2} \psi(N) + 7 \\
&\leq \frac{1}{6} \psi(N) 4^{k-1} \left(\frac{1}{2} + 61 + \frac{12}{4^{k-1}} + 6 + \frac{42}{\psi(N) 4^{k-1}} \right) \\
&\leq \frac{1}{6} \psi(N) 4^{k-1} \cdot 93.5,
\end{aligned}
$$

since $N \geq 3$ and $k \geq 1$. If we assume that $k \geq 4$, and thus $A_{4,4} = 0$, the following bound holds:

$$
\begin{aligned}
\left| \operatorname{Tr} T_4(2k, N) - \frac{k}{6} \psi(N) 4^{k-1} \right| &\leq \frac{1}{6} \psi(N) 4^{k-1} \left(\frac{1}{2} + 61 + \frac{12}{4^{k-1}} + 6 \right) \\
&\leq \frac{1}{6} \psi(N) 4^{k-1} \cdot 68,
\end{aligned}
$$

as desired. □

The next proposition implies that the a_2 coefficient in $T_2(N, 2k)(x)$ is nonvanishing for $k \geq 293$ and odd $N > 1$.

Proposition 4.8. *Recall that $s(N,2k) = \dim S(N,2k)$ and $\{\alpha_1, \cdots, \alpha_{s(N,2k)}\}$ is the set of eigenvalues of $T_2(N,2k)$. Then for $k \geq 4$,*

$$\sum_{1\leq i<j\leq s(N,2k)} \alpha_i\alpha_j = \frac{\psi(N)}{12}4^{k-1}(-3k + B_{N,k}),$$

with $|B_{N,k}| \leq 878$.

Proof. Recalling (2.2),

$$\sum_{1\leq i<j\leq s(N,2k)} \alpha_i\alpha_j = \frac{1}{2}\left[(\operatorname{Tr} T_2(N,2k))^2 - \operatorname{Tr} T_4(N,2k)\right] - 2^{2k-2}s(N,2k).$$

We first use the bound from Proposition 4.6 to get an upper bound for $(\operatorname{Tr} T_2(N,2k))^2$:

$$\begin{aligned}(\operatorname{Tr} T_2(N,2k))^2 &\leq (3 + 2^{k+3/2}2^{\omega(N)} + \sigma_0(N))^2\\ &\leq (3 + 7\cdot 2^{k-1/2}\sqrt{N} + 2\sqrt{N})^2\\ &\leq (8\cdot 2^{k-1/2}\sqrt{N})^2\\ &= 4^{k+5/2}N,\end{aligned}$$

which is true as $k \geq 4$ and $N \geq 3$. On the other hand, by Proposition 4.7,

$$-\frac{1}{2}\operatorname{Tr} T_4(N,2k) = -\frac{\psi(N)}{12}4^{k-1}\left(k + E_{N,k}\right),$$

with $|E_{N,k}| \leq 68$. By Proposition 4.5,

$$\begin{aligned}-2^{2k-2}s(N,2k) &= \frac{-2^{2k-2}\psi(N)}{6}\left(k + F_{N,k}\right)\\ &= \frac{-4^{k-1}\psi(N)}{6}\left(k + F_{N,k}\right),\end{aligned}$$

with $|F_{N,k}| \leq 12$. Combining these results, we obtain the following:

$$\begin{aligned}\sum_{1\leq i<j\leq s(N,2k)} \alpha_i\alpha_j =& \frac{1}{2}(\operatorname{Tr} T_2(N,2k))^2 - \frac{\psi(N)}{12}4^{k-1}\left(k + E_{N,k}\right) - \frac{4^{k-1}\psi(N)}{6}(k + F_{N,k})\\ =& -\frac{\psi(N)}{12}4^{k-1}\cdot 3k - \frac{\psi(N)}{12}4^{k-1}(E_{N,k} + 2F_{N,k}) + \frac{1}{2}(\operatorname{Tr} T_2(N,2k))^2.\end{aligned}$$

Then,

$$\begin{aligned}\left|\sum_{1\leq i<j\leq s(N,2k)} \alpha_i\alpha_j + \frac{\psi(N)}{12}4^{k-1}\cdot 3k\right| &\leq \frac{\psi(N)}{12}4^{k-1}(E_{N,k} + 2F_{N,k}) + \frac{1}{2}(\operatorname{Tr} T_2(N,2k))^2\\ &\leq \frac{\psi(N)}{12}4^{k-1}(68 + 24) + \frac{1}{2}4^{k+5/2}\cdot N\\ &\leq \frac{\psi(N)}{12}4^{k-1}(92 + 6\cdot 4^{7/2})\\ &= \frac{\psi(N)}{12}4^{k-1}\cdot 878.\end{aligned}$$

Thus,

$$\sum_{1\leq i<j\leq s(N,2k)} \alpha_i\alpha_j = \frac{\psi(N)}{12}4^{k-1}(-3k+B_{N,k}),$$

with $|B_{N,k}| \leq 878$. □

4.3. **Asymptotics in** N

We now develop an asymptotic bound on $a_2 = \sum_{i<j}\alpha_i\alpha_j$ in terms of N. Specifically, we show

$$\sum_{1\leq i<j\leq s(N,2k)} \alpha_i\alpha_j = \Omega(\psi(N)) + O(N^{3/4}),$$

where we use Bachmann-Landau notation. Since $\psi(N) \geq N$, our main term will also be $\Omega(N)$ while our error term will be $O(N^{3/4})$. Alongside Proposition 4.8, this shall allow us to assert the nonvanishing of the a_2 coefficient when either k or N is sufficiently large. Since the dimension formula for $s(N, 2k)$ changes considerably when $2k = 2$ from when $2k \geq 4$, we shall consider these two cases separately in our estimates on a_2 in terms of N.

Proposition 4.9. *Let $k = 1$ and $N > 1$ be odd. Let $L_{N,1}$ be defined such that*

$$\sum_{1\leq i<j\leq s(N,2)} \alpha_i\alpha_j = -\frac{1}{8}\psi(N) + L_{N,1}.$$

Then $|\frac{1}{8}\psi(N)| > |L_{N,1}|$ for $N \geq 95701992$.

Proof. Recall from (2.2) that

$$\sum_{1\leq i<j\leq s(N,2)} \alpha_i\alpha_j = \frac{1}{2}\left[(\operatorname{Tr} T_2(N,2))^2 - \operatorname{Tr} T_4(N,2)\right] - 2^{2-2}s(N,2).$$

We need to determine appropriate bounds for the terms which have a factor of $\psi(N)$ and the other terms and to show that the $\psi(N)$ terms are eventually the largest. By (4.3),

$$\begin{aligned}\operatorname{Tr} T_2(N,2) = &-\frac{1}{2}\sum_{t^2<8} P_2(t,2)\sum_n h_w\left(\frac{t^2-8}{n^2}\right)\mu(t,n,2)\\ &-\frac{1}{2}\sum_{d|2}\min(d,2/d)^{2-1}\sum_\tau{}' \varphi(\gcd(\tau,N/\tau)) + A_{4,2},\end{aligned}$$

where $0 \leq A_{4,2} \leq 3$. By (2.7), it is clear that $P_2(t,2) = 1$. Thus,

$$0 \leq \frac{1}{2}\sum_{t^2<8} P_2(t,2)\sum_n h_w\left(\frac{t^2-8}{n^2}\right)\mu(t,n,2) \leq 2^{\omega(N)}(1+\sqrt{7}+\sqrt{2}) < 5.06\cdot 2^{\omega(N)},$$

and by Proposition 4.3,

$$2 \leq \frac{1}{2}\sum_{d|2}\min(d,2/d)^{2-1}\sum_\tau{}' \varphi(\gcd(\tau,N/\tau)) \leq \sigma_0(N).$$

Hence,

$$\begin{aligned}
|\operatorname{Tr} T_2(N,2)| &< 5.06 \cdot 2^{\omega(N)} + \sigma_0(N),\\
(\operatorname{Tr} T_2(N,2))^2 &< (5.06 \cdot 2^{\omega(N)} + \sigma_0(N))^2\\
&< 25.60 \cdot 4^{\omega(N)} + 10.12 \cdot 2^{\omega(N)}\sigma_0(N) + \sigma_0(N)^2.
\end{aligned}$$

Following Proposition 4.7, one obtains

$$\begin{aligned}
\operatorname{Tr} T_4(N,2) =& \frac{2-1}{12}\psi(N)4^{1-1}\\
&- \frac{1}{2}\sum_{t^2<16} P_2(t,4)\sum_n h_w\left(\frac{t^2-16}{n^2}\right)\mu(t,n,4)\\
&- \frac{1}{2}\sum_{d|4}\min(d,4/d)^{2-1}\sum_{\tau}{}' \varphi(\gcd(\tau, N/\tau)) + A_{4,4},
\end{aligned}$$

where $A_{4,4} \leq 7$. Since $P_2(t,4) = 1$,

$$\frac{1}{2}\sum_{t^2<16} P_2(t,4)\sum_n h_w\left(\frac{t^2-16}{n^2}\right)\mu(t,n,4) \leq 2^{\omega(N)}\left(\sqrt{7} + 2\sqrt{15} + \frac{8}{3}\sqrt{3} + 3\right) < 18.02 \cdot 2^{\omega(N)},$$

and as in Proposition 4.7,

$$\frac{1}{2}\sum_{d|4}\min(d,4/d)^{2-1}\sum_{\tau}{}' \varphi(\gcd(\tau, N/\tau) \leq 2\sigma_0(N) + 2^{2-2}2^{\omega(N)}\sqrt{N}.$$

Also, recall from (4.2) that

$$s(N,2) = 1 + \frac{\psi(N)}{12} - \varepsilon_2/4 - \varepsilon_3/3 - \varepsilon_\infty/2,$$

where $0 \leq \varepsilon_2, \varepsilon_3 \leq 2^{\omega(N)}$ and $0 \leq \varepsilon_\infty \leq 2^{\omega(N)}\sqrt{N}$.

Thus, the part of $\sum_{1\leq i<j\leq s(N,2)} \alpha_i\alpha_j$ that has a factor of $\psi(N)$ is

$$-\frac{1}{2}\left(\frac{2-1}{12}\psi(N)4^{1-1}\right) - 2^{2-2}\frac{\psi(N)}{12} = -\frac{\psi(N)}{24} - \frac{\psi(N)}{12} = -\frac{\psi(N)}{8} \leq -\frac{N+1}{8}. \tag{4.14}$$

By Lemma 4.2, we have $\sigma_0(N) \leq 9N^{1/4}$ and $2^{\omega(N)} \leq 11N^{1/4} = 5.5N^{1/4}$. So

$$\begin{aligned}
L_{N,1} =& \sum_{1\leq i<j\leq s(N,2)} \alpha_i\alpha_j + \frac{1}{8}\psi(N)\\
&< \frac{1}{2}\left(25.60 \cdot 2^{2\omega(N)} + 10.12 \cdot 2^{\omega(N)}\sigma_0(N) + \sigma_0(N)^2\right)\\
&+ \frac{1}{2}\left(7 + 18.02 \cdot 2^{\omega(N)} + 2\sigma_0(N) + 4^{1-1}2^{\omega(N)}\sqrt{N}\right) + 1 + \frac{2^{\omega(N)}}{4} + \frac{2^{\omega(N)}}{3} + \frac{2^{\omega(N)}\sqrt{N}}{2}\\
&< 4.5 + 61.76N^{1/4} + 678.23N^{1/2} + 5.5N^{3/4}.
\end{aligned} \tag{4.15}$$

Notice that the equation (4.14) is asymptotic in N and (4.15) is asymptotic in $N^{3/4}$. Hence, eventually the equation (4.14) will be of greater absolute value than equation (4.15). Using a computer algebra system, one can verify that this is true for $N \geq 95701992$. We used Mathematica to complete this check. $\square$

Proposition 4.10. *Let $k \geq 2$ and $N > 1$ be odd. Let $L_{N,k}$ be defined such that*

$$\sum_{1 \leq i < j \leq s(N,2k)} \alpha_i \alpha_j = k4^{k-1} \left(\left(\frac{1}{8k} - \frac{1}{4} \right) \psi(N) + L_{N,k} \right).$$

Then $\left| \left(\frac{1}{8k} - \frac{1}{4} \right) \psi(N) \right| > |L_{N,k}|$ for $N \geq 5832838$.

Proof. Again, recall from (2.2) that

$$\sum_{1 \leq i < j \leq s(N,2k)} \alpha_i \alpha_j = \frac{1}{2} \left[(\operatorname{Tr} T_2(N,2k))^2 - \operatorname{Tr} T_4(N,2k) \right] - 2^{2k-2} s(N,2k).$$

Using Proposition 4.6,

$$|\operatorname{Tr} T_2(N,2k)| \leq 3 + 2^{k+3/2} 2^{\omega(N)} + \sigma_0(N).$$

By inspection of (4.3), $A_{4,2} = 0$ since $k \geq 2$. In Proposition 4.6, $A_{4,2}$ shows up directly in the bound as 3, so

$$|\operatorname{Tr} T_2(N,2k)| \leq 2^{k+3/2} 2^{\omega(N)} + \sigma_0(N), \text{ and}$$
$$(\operatorname{Tr} T_2(N,2k))^2 \leq 4^{k+3/2} 4^{\omega(N)} + 2^{k+5/2} 2^{\omega(N)} \sigma_0(N) + \sigma_0(N)^2.$$

Using (4.9), $A_{2,4}$ and $A_{3,4}$ from Proposition 4.7, equations (4.10), and (4.12), we have

$$\operatorname{Tr} T_4(N,2k) = \frac{k}{6} \psi(N) 4^{k-1} - \frac{1}{12} \psi(N) 4^{k-1} + A_{2,4} + A_{3,4}.$$

Since $k \geq 2$, $A_{4,4} = 0$.

In this case, the dimension formula as written in (4.7) is

$$s(N,2k) = \frac{2k\psi(N)}{12} + \left(\left\lfloor \frac{k}{2} \right\rfloor - \frac{k}{2} + \frac{1}{4} \right) \varepsilon_2 + \left(\left\lfloor \frac{2k}{3} \right\rfloor - \frac{2k}{3} + \frac{1}{3} \right) \varepsilon_3 - \frac{\varepsilon_\infty}{2} - \frac{\psi(N)}{12}.$$

From all of this, we write

$$\begin{aligned}
\sum_{1 \leq i < j \leq s(N,2k)} \alpha_i \alpha_j =& \frac{1}{2} \left[(\operatorname{Tr} T_2(N,2k))^2 - \operatorname{Tr} T_4(N,2k) \right] - 2^{2k-2} s(N,2k) \\
=& \frac{1}{2} (\operatorname{Tr} T_2(N,2k))^2 - \frac{k\psi(N)}{12} 4^{k-1} + \frac{\psi(N)}{24} 4^{k-1} - \frac{1}{2} A_{2,4} - \frac{1}{2} A_{3,4} \\
& - 4^{k-1} \left(\frac{k\psi(N)}{6} + \left(\left\lfloor \frac{k}{2} \right\rfloor - \frac{k}{2} + \frac{1}{4} \right) \varepsilon_2 \right) \\
& - 4^{k-1} \left(\left(\left\lfloor \frac{2k}{3} \right\rfloor - \frac{2k}{3} + \frac{1}{3} \right) \varepsilon_3 - \frac{\varepsilon_\infty}{2} - \frac{\psi(N)}{12} \right) \\
=& k4^{k-1} \left(\frac{1}{2k4^{k-1}} (\operatorname{Tr} T_2(N,2k))^2 - \frac{\psi(N)}{12} + \frac{\psi(N)}{24k} - \frac{1}{2k4^{k-1}} (A_{2,4} + A_{3,4}) \right) \\
& + k4^{k-1} \left(-\frac{\psi(N)}{6} - \frac{1}{k} \left(\left\lfloor \frac{k}{2} \right\rfloor - \frac{k}{2} + \frac{1}{4} \right) \varepsilon_2 \right)
\end{aligned}$$

$$+ k4^{k-1}\left(-\frac{1}{k}\left(\left\lfloor\frac{2k}{3}\right\rfloor - \frac{2k}{3} + \frac{1}{3}\right)\varepsilon_3 + \frac{\varepsilon_\infty}{2k} + \frac{\psi(N)}{12k}\right). \tag{4.16}$$

Then, we combine all terms which involve $\psi(N)$, noting that $A_{2,4}, A_{3,4}, \varepsilon_2, \varepsilon_3$ and ε_∞ do not have any terms on the order of $\psi(N)$,

$$\frac{-\psi(N)}{12} + \frac{\psi(N)}{24k} - \frac{\psi(N)}{6} + \frac{\psi(N)}{12k} = \left(\frac{-1}{4} + \frac{1}{8k}\right)\psi(N). \tag{4.17}$$

Using the bound $\psi(N) \geq N+1$ and (4.17),

$$\left|\left(\frac{-1}{4} + \frac{1}{8k}\right)\psi(N)\right| \geq \left|\left(\frac{-1}{4} + \frac{1}{8k}\right)(N+1)\right| = \left(\frac{1}{4} - \frac{1}{8k}\right)(N+1). \tag{4.18}$$

We now consider $L_{N,k}$. Note that since $k \geq 2$, $A_{4,4}$ is zero. Then, using bounds for $2^{\omega(N)}$ and $\sigma_0(N)$ given in Lemma 4.2, and the bounds for the coefficients of ε_2 and ε_3 given in (4.8) in conjunction with equation (4.16),

$$\begin{aligned}
|L_{N,k}| =& \frac{1}{k4^{k-1}}\left|\sum_{1\leq i<j\leq s(N,2k)} \alpha_i\alpha_j - \left(\frac{1}{8k} - \frac{1}{4}\right)\psi(N)\right| \\
=& \Bigg|\left(\frac{1}{2k4^{k-1}}\left(\operatorname{Tr} T_2(N,2k)\right)^2 - \frac{1}{2k4^{k-1}}(A_{2,4} + A_{3,4})\right) \\
& - \frac{1}{k}\left(\left\lfloor\frac{k}{2}\right\rfloor - \frac{k}{2} + \frac{1}{4}\right)\varepsilon_2 - \frac{1}{k}\left(\left\lfloor\frac{2k}{3}\right\rfloor - \frac{2k}{3} + \frac{1}{3}\right)\varepsilon_3 + \frac{\varepsilon_\infty}{2k}\Bigg| \\
\leq& \frac{1}{2k4^{k-1}}\left(\operatorname{Tr} T_2(N,2k)\right)^2 + \left|\frac{A_{2,4}}{2k4^{k-1}}\right| + \left|\frac{A_{3,4}}{2k4^{k-1}}\right| + \frac{\varepsilon_2}{4k} + \frac{\varepsilon_3}{3k} + \frac{\varepsilon_\infty}{2k} \\
\leq& \frac{1}{2k4^{k-1}}\left(4^{k+3/2}4^{\omega(N)} + 2^{k+5/2}2^{\omega(N)}\sigma_0(N) + \sigma_0(N)^2\right) + \frac{1}{2k4^{k-1}}\left(\frac{61}{6}\cdot 4^{k-1/2}2^{\omega(N)}\right) \\
& + \frac{1}{2k4^{k-1}}(2\sigma_0(N) + 4^{k-1}2^{\omega(N)}\sqrt{N}) + \frac{1}{4k}2^{\omega(N)} + \frac{1}{3k}2^{\omega(N)} + \frac{1}{2k}2^{\omega(N)}\sqrt{N} \\
\leq& \frac{1}{2k4^{k-1}}\left(\frac{11}{2}N^{1/4}\right)^2 + \frac{2^{7/2-k}}{k}(9N^{1/4})\left(\frac{11}{2}N^{1/4}\right) + \frac{(9N^{1/4})^2}{2k4^{k-1}} + \frac{61}{6k}\left(\frac{11}{2}N^{1/4}\right) + \frac{9N^{1/4}}{km^{k-1}} \\
& + \frac{11N^{3/4}}{4k} + \frac{11N^{1/4}}{8k} + \frac{11N^{1/4}}{6k} + \frac{11N^{3/4}}{4k} \\
=& \left(\frac{11}{8k} + \frac{11}{6k} + \frac{9}{k\cdot 4^{k-1}} + \frac{671}{12k}\right)N^{1/4} + \left(\frac{484}{k} + \frac{2^{5/2-k}\cdot 99}{k} + \frac{81}{2k\cdot 4^{k-1}}\right)N^{1/2} + \frac{11}{2k}\cdot N^{3/4}.
\end{aligned} \tag{4.19}$$

Since the bound for $L_{N,k}$ increases on the order of $N^{3/4}$, and the lower bound for $\psi(N)$ is on the order of N, it suffices to check numerically where the last intersection of (4.19) and (4.18) occurs to determine for what values of N the trace is nonvanishing. Using Mathematica, it is simple to check that for any specified k. For example, in the specific case of $k = 2$, by (4.19),

$$|L_{N,2}| \leq \left(\frac{11}{16} + \frac{11}{12} + \frac{9}{8} + \frac{671}{24}\right) N^{1/4} + \left(242 + \frac{99}{\sqrt{2}} + \frac{81}{16}\right) N^{1/2} + \frac{11}{4} N^{3/4}.$$

Then combining this with (4.18) when $k = 2$ and checking with Mathematica, we find

$$\frac{3}{16}(N+1) > |L_{N,2}|,$$

for $N \geq 5832838$. For a few values of k, this process and the results of Propostition 4.9 produce Table 4.3. Here, for k greater than or equal to the value in the first row and N greater than or equal to the value in the second row, $|(\frac{1}{8k} - \frac{1}{4})\psi(N)| > |L_{N,k}|$.

TABLE 4.3. Nonvanishing for k and N values

$k \geq$	1	2	3	6	12	20	50	75
$N \geq$	95701992	5832838	1434659	197837	38686	23605	1826	791

For a more complete set of values, see [4]. □

We were able to do a computer check using Sage on the remaining cases not covered by Propositions 4.8, 4.9, and 4.10. For code used see [4]. Below, Table 4.4 is a table of all odd levels above level one and all even weights for which the a_2 coefficient of the Hecke polynomial of $T_2(N, 2k)$ vanishes. It is of note that when $s(N, 2k) \geq 2$, the coefficient a_2 only vanishes in levels 33, 37, and 57 and weight 2. With these computations and the results of Propositions 4.8, 4.9 and 4.10, the proof of Theorem 1.3 is now complete.

TABLE 4.4. Levels and weights for which a_2 vanishes

Level	Weights	Level	Weight	Level	Weight	Level	Weight
3	2, 4, 6, 8	11	2	19	2	33	2
5	2, 4, 6	13	2	21	2	37	2
7	2, 4	15	2	25	2	49	2
9	2, 4	17	2	27	2	57	2

5. CONCLUDING REMARKS

The methodology of this paper is suitable for application to a wider range of scenarios than those covered here. One instance is to show the nonvanishing of the a_2 coefficient of Hecke polynomials acting on the new subspace $S(N, 2k)^{\text{new}}$. Following a similar procedure as Section 4, we showed that the a_2 coefficient of the Hecke polynomial associated with $T_2(N, 2k)^{\text{new}}$, $N > 1$ odd, is nonvanishing when either N or k is sufficiently large. We first expressed the a_2 coefficient of the $T_m(N,2k)^{\text{new}}$ operator on the new subspace exactly as in (4.16). Then, following [15, (53–54)], we expressed the trace of Hecke operators on the new subspace and the dimension of the new subspace as Dirichlet convolutions of the arithmetic function $\mu * \mu$ with the trace of Hecke operators on the old subspace and the dimension of the old subspace, respectively.

Extrapolating bounds on the trace of the new subspace and dimension of the new subspace from the bounds in Section 4 requires that we pick up a factor of $4^{\omega(N)}$ on each term associated with the new part. As in Section 4, we deduced that $\frac{a_2}{4^{k-1}}$ has a main term on the order of Nk, and thus is nonvanishing in the limit as either k or N goes to infinity. Nevertheless, formulating an explicit lower bound beyond which nonvanishing occurs is difficult, due to the computational load of finding a sufficiently small constant in the divisor bound $\sigma_0(N) = O(N^\varepsilon)$ when the desired ε is small. Note that $\limsup_{N\to\infty} \frac{2^{\omega(N)}}{\sigma_0(N)} = 1$, for which reason this bound is pertinent.

As another example, one can demonstrate the nonvanishing of the second coefficient of the Hecke polynomial associated with $T_p(N,2k)$ for p prime and arbitrary level and weight following a similar procedure to Section 4. We have proven for level one with sufficiently large weight that the second coefficient of the mth Hecke polynomial is nonvanishing, and we showed that the a_2 coefficient of the Hecke polynomial associated with $T_2(N,2k)$ is nonvanishing for all but finitely many cases. These two results suggest a similar trend for general level and any Hecke operator coprime to that level.

Further investigation can be done to show the nonvanishing of other coefficients of Hecke polynomials. From the results of this work, one would expect that by applying a similar method to the even indexed coefficients of the Hecke polynomials, a_{2j}, one could show their nonvanishing. Due to this and our other results, we propose the following conjecture on other coefficients of Hecke polynomials, (at least) when $N = 1$.

Conjecture 5.1. Suppose $m \geq 1$. Recall that the Hecke polynomial of $T_m(1,2k)$ is given by

$$T_m(1,2k)(x) = \sum_{n=0}^{d_{2k}} (-1)^n a_n x^{d_{2k}-n}.$$

Then the nth coefficient a_n does not vanish when $d_{2k} \geq n \geq 1$.

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