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Bellman inequalities of generalized matrix trace

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Abstract: A generalized matrix trace $\tau: M_n(C(\Omega))\to C(\Omega)$ is introduced, Bellman inequality on generalized matrix trace τ is studied. The following inequality is proved:

$$\tau((AB)^k)\leqslant\tau(A^kB^k),$$

where A, B are positive elements in a unital C^*-algebra $M_n(C(\Omega))$ and $k\in\mathbf{N}$. This gives a positive answer to Bellman problem in a more general setting. Some related inequalities on it are also proved by using the knowledge of C^*-algebra.

Keywords: compact Hausdorff space; continuous function; generalized matrix trace; Bellman inequality

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Bellman in [1] posed a matrix trace inequality

$$Tr((AB)^k)\leqslant Tr(A^kB^k),$$

where A and B are positive semi-definite matrices of the same order and k is any natural number. Related works refer to [2~5].

Through out this note, we assume that Ω is a compact Hausdorff space and $C(\Omega)$ is the C^*-algebra of continuous functions on Ω, $C(\Omega)^+$ is the set of all positive elements in $C(\Omega)$. Thus, $f\in C(\Omega)^+$ if and only if $f(\omega)\geqslant 0, \forall\ \omega\in\Omega$. Recall that

$$M_n(C(\Omega))=\{[f_{ij}]_{1\leqslant i,j\leqslant n}\mid f_{ij}\in C(\Omega)\}$$

is an unital C^*-algebra and $A\in M_n(C(\Omega))^+$ means that $A=[f_{ij}]_{1\leqslant i,j\leqslant n}$ is a positive element of $M_n(C(\Omega))$. For every $A=[f_{ij}]_{1\leqslant i,j\leqslant n}\in M_n(C(\Omega))$, $A^*=[\bar{f}_{ji}]_{1\leqslant i,j\leqslant n}$ denotes the adjoint of A. $\mathrm{Her}(M_n(C(\Omega)))$ denotes the set of all Hermitian elements in $M_n(C(\Omega))$, i. e., $A\in\mathrm{Her}(M_n(C(\Omega)))$ if and only if $A=A^*$. $A\geqslant B$ if and only if $A-B\geqslant 0$. $U(M_n(C(\Omega)))$ is the set of all unitary elements in $M_n(C(\Omega))$. $\forall\ \omega\in\Omega$ and $\forall\ A\in M_n(C(\Omega))$ set $A_\omega=[f_{ij}(\omega)]_{1\leqslant i,j\leqslant n}$, then $A_\omega\in M_n(C)$.

Definition 1 A linear mapping $\tau: M_n(C(\Omega))\to C(\Omega)$ is called a generalized matrix trace if it satisfies the following conditions:

(1) $\tau(A)\geqslant 0, \forall\ A\in M_n(C(\Omega))^+$;

(2) $\tau(U^*AU)=\tau(A), \forall\ A\in M_n(C(\Omega)), \forall\ U\in U(M_n(C(\Omega)))$;

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(3) $\tau(A^2)\leqslant[\tau(A)]^2, \forall A\in M_n(C(\Omega))^+$.

In this note, let us assume that $\tau: M_n(C(\Omega))\to C(\Omega)$ is a generalized matrix trace. It is easy to see that $\tau(A^*)=(\tau(A))^*, \forall A\in M_n(C(\Omega))$ (where $(\tau(A))^*$ means that $\overline{\tau(A)}$).

Lemma 1[4] $\forall A,B\in M_n(C(\Omega))^+$, and $\forall k\in\mathbf{N}$, then $A(AB)^k$, $(AB)^kA$, $B(AB)^k$, $(AB)^kB$ are all belong to $M_n(C(\Omega))^+$.

Theorem 1 Let τ be a generalized matrix trace, then

(Ⅰ) $\forall A,B\in M_n(C(\Omega))$ and $\forall k\in\mathbf{N}, \tau((AB)^k)=\tau((BA)^k)$.

(Ⅱ) $\forall A,B\in M_n(C(\Omega))^+$ and $\forall k\in\mathbf{N}, \tau((AB)^k)\geqslant 0$.

(Ⅲ) $\forall A,B\in M_n(C(\Omega)), |\tau(B^*A)|^2\leqslant\tau(A^*A)\tau(B^*B)$.

(Ⅳ) $\forall A\in M_n(C(\Omega)), \tau(A^2)\leqslant\tau(A^*A)$.

(Ⅴ) $\forall A,B\in \mathrm{Her}(M_n(C(\Omega)))$, then $\tau(AB)\leqslant(\tau(A^2)\tau(B^2))^{1/2}$.

Proof (Ⅰ) From Lemma 2.2 of [5], we know that the inequality is valid.

(Ⅱ) From Lemma 2.2 of [5], we know that the inequality is valid.

(Ⅲ) Let ω be an arbitrary element of Ω and define $\langle A,B\rangle=\tau(B^*A)(\omega)$. Then we obtain a semi-inner product on $M_n(C(\Omega))$. Thus, $\forall A,B\in M_n(C(\Omega))$, we have $|\langle A,B\rangle|^2\leqslant\langle A,A\rangle\langle B,B\rangle$, i. e., $|\tau(B^*A)(\omega)|^2\leqslant\tau(A^*A)(\omega)\tau(B^*B)(\omega)$, that is,

$$\{|\tau(B^*A)|^2\}(\omega)\leqslant\{\tau(A^*A)\tau(B^*B)\}(\omega).$$

Thus, $|\tau(B^*A)|^2\leqslant\tau(A^*A)\tau(B^*B)$ is valid.

(Ⅳ) Let $A=B^*$, we see from (Ⅰ) and (Ⅲ) that the inequality holds.

(Ⅴ) $\forall A,B\in\mathrm{Her}(M_n(C(\Omega)))$, then $A=A^*, B=B^*$, we see from (Ⅲ) that $|\tau(AB)|^2\leqslant\tau(A^2)\tau(B^2)$. Note that $\tau(AB)$ is selfadjoint in $C(\Omega)$, i. e., a real-valued function on Ω, we have $\tau(AB)\leqslant(\tau(A^2)\tau(B^2))^{1/2}$.

Lemma 2[1] Let A,B be positive semidefinite matrices in $M_n(C)$, then $Tr(BA)\leqslant Tr(B)Tr(A)$.

Theorem 2 Let $A,B\in(M_n(C(\Omega)))^+$, then $\tau(BA)\leqslant\tau(B)\tau(A)$.

Proof $\forall A,B\in(M_n(C(\Omega)))^+$ and $\forall\omega\in\Omega$, A_ω, B_ω are positive semidefinite matrices of order n in $M_n(C)$, then from Lemma 2, we have $Tr(B_\omega A_\omega)\leqslant Tr(B_\omega)Tr(A_\omega)$.

From Theorem 2 in [4], we know that there exists a $\lambda\in C(\Omega)$ with $0\leqslant\lambda\leqslant\lambda^2$ such that $\tau(A)=\lambda Tr(A)$ $(\forall A=[f_{ij}]_{1\leqslant i,j\leqslant n}\in M_n(C(\Omega)))$, where $Tr(A)=\sum_{i=1}^n f_{ii}$.

$$\begin{aligned}[\tau(BA)](\omega)&=\lambda(\omega)[Tr(BA)](\omega)=\lambda(\omega)Tr(B_\omega A_\omega)\leqslant\\ &[\lambda(\omega)]^2Tr(A_\omega)Tr(B_\omega)=\lambda(\omega)[Tr(B)](\omega)\lambda(\omega)[Tr(A)](\omega)=\\ &[\tau(B)](\omega)[\tau(A)](\omega)=[\tau(B)\tau(A)](\omega).\end{aligned}$$

Therefore, $\tau(BA)\leqslant\tau(B)\tau(A)$.

Lemma 3[2] Let A,B be positive semidefinite Hermitian matrices in $M_n(C)$ and $k\in\mathbf{N}$, then $Tr((AB)^k)\leqslant Tr(A^kB^k)$.

Theorem 3 Let $A,B\in M_n(C(\Omega))^+$ and $k\in\mathbf{N}$, then $\tau((AB)^k)\leqslant\tau(A^kB^k)$.

Proof Since $\forall A,B\in M_n(C(\Omega))^+$ and $\forall\omega\in\Omega, A_\omega, B_\omega$ are positive semidefinite Hermitian matrices of order n, then from Lemma 3, we have $Tr((A_\omega B_\omega)^k)\leqslant Tr((A_\omega)^k(B_\omega)^k)$.

From Theorem 2 in [4], we know that there exists a $\lambda\in C(\Omega)$ with $0\leqslant\lambda\leqslant\lambda^2$ such that $\tau(A)=\lambda Tr(A)$ $(\forall A=[f_{ij}]_{1\leqslant i,j\leqslant n}\in M_n(C(\Omega)))$, where $Tr(A)=\sum_{i=1}^n f_{ii}$.

$$\begin{aligned}[\tau((AB)^k)](\omega)&=[\lambda Tr(AB)^k)](\omega)=\lambda(\omega)Tr((A_\omega B_\omega)^k)\leqslant\\ &\lambda(\omega)Tr((A_\omega)^k(B_\omega)^k)=\lambda(\omega)[Tr((A)^k)(B^k))](\omega)=[\tau(A^kB^k)](\omega).\end{aligned}$$

Hence, $\tau((AB)^k)\leqslant\tau(A^kB^k)$.

Lemma 4[2] Let A,B be Hermitian matrices in $M_n(C)$ and $k\in\mathbf{N}$, then $Tr((AB)^{2k})\leqslant Tr(A^{2k}B^{2k})$.

Theorem 4 Let $A,B\in \mathrm{Her}(M_n(C(\Omega)))$ and $k\in\mathbf{N}$, then $\tau((AB)^{2k})\leqslant\tau(A^{2k}B^{2k})$.

Proof Since $\forall\ A,B\in \mathrm{Her}(M_n(C(\Omega)))$ and $\forall\ \omega\in\Omega$, A_ω, B_ω are Hermitian matrices of order n, then from Lemma 4, we have $Tr((A_\omega B_\omega)^{2k})\leqslant Tr(A_\omega^{2k}B_\omega^{2k})$.

From Theorem 2 in [4], we know that there exists a $\lambda\in C(\Omega)$ with $0\leqslant\lambda\leqslant\lambda^2$ such that $\tau(A)=\lambda Tr(A)$ ($\forall\ A=[f_{ij}]_{1\leqslant i,j\leqslant n}\in M_n(C(\Omega))$), where $Tr(A)=\sum_{i=1}^{n}f_{ii}$.

$$[\tau(AB)^{2k}](\omega)=[\lambda Tr((AB)^{2k})](\omega)=\lambda(\omega)Tr((A_\omega B_\omega)^{2k})\leqslant$$
$$\lambda(\omega)Tr((A_\omega)^{2k}(B_\omega)^{2k})=\lambda(\omega)[Tr(A^{2k}B^{2k})](\omega)=[\tau(A^{2k}B^{2k})](\omega).$$

This shows that $\tau((AB)^{2k})\leqslant\tau(A^{2k}B^{2k})$.

Reference:

[1] BELLMAN R. Some inequalities for positive matrices general inequalities[J]. Birkhanses Verlag, 1980, (2): 89-90.

[2] YAN Xi-bo. Bellman inequality about the trace of matrices[J]. Journal of Xinjiang University, 1991, 8 (4): 34-36.

[3] LONG Yong-long. Generalization of Bellman-inequalities and a problem of Bellman[J]. Journal of Central China Normal University, 1991, 25(1): 8-12.

[4] CAO H X, XU Z B, ZHANG J H, et al. Matrix-trace on C^*-algebra $M_n(A)$[J]. Linear Algebra and its Application, 2002, 345: 255-260.

[5] CAO Huai-xin, ZHANG Jian-Hua, LI Wei-huai. Some inequalities for generalized trace[J]. submitted to Math Anal and Appl, 2001.

广义矩阵迹的贝尔曼不等式

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摘要：引入了广义矩阵迹 $\tau:M_n(C(\Omega))\to C(\Omega)$，讨论了在广义矩阵迹下的贝尔曼不等式. 证明了 C^*-代数 $M_n(C(\Omega))$ 中任意两个正元 A,B 及 $\forall\ k\in\mathbf{N}$，有 $\tau((AB)^k)\leqslant\tau(A^kB^k)$，这便在更一般的框架下给出了 Bellman 问题的一个肯定回答。同时还利用 C^*-代数证明了其它一些相关不等式.

关键词：紧致 Hausdorff 空间；连续函数；广义矩阵迹；Bellman 不等式