Computer Science 240
Princeton University, Fall 2022

Take-Home Final Exam Preparation: Additional Problems

This is a list of problems to provide additional practice before the Final exam along with problems in the "*Take-Home Midterm Exam Preparation: Additional Problems*". The problems presented in this document are not necessarily indicative of the type/difficulty of the problems that will be in the exam.

Problem 1

Consider a graph $G = (V, E)$ with chromatic number $\chi(G) = k$.
(A): Consider a k-coloring of the graph G. Show that for each color in the coloring there is a vertex that is adjacent to vertices of every other color.
(B): Show that G has at least k vertices of degree at least $k-1$ each.
(*topic: Graphs, Coloring of graphs*)

Solution:
(A): For contradiction, assume that there is a color, say i, in the k-coloring that has no vertex adjacent to vertices of every other color. Pick any vertex, say v, colored with color i. As a result, there is a color, say j, that is not present in any of the neighbors of v and is different from i. Color v with color j. The coloring remains valid since the only vertex that changed colors is v and its color is not present in any of its neighbors.
We repeat the process above for all vertices that are colored with color i. The coloring derived is valid, because, similarly to above, any vertex that changed a color has neighbors of different colors after the change. Finally, the color i is no longer used, hence the constructed coloring uses $k-1$ colors. However, this implies that $\chi(G) \leq k-1 < k$ which contradicts the fact that the chromatic number of G is k.
Hence, for each color in the k-coloring there is a vertex that is adjacent to vertices of every other color.
(B): Consider a k-coloring of the graph G and consider all colors $c_1, c_2, \ldots c_k$. For each color $c \in \{c_1, c_2, \ldots c_k\}$ pick a vertex colored with the color c that is adjacent to vertices of any other color; let v_c that vertex. Clearly $v_i \neq v_j$ for different colors $i \neq j \in \{c_1, c_2, \ldots c_k\}$. Each of those vertices have degree at least $k-1$, since it has at least $k-1$ neighbors due to part (A). As a result, there are at least k vertices (the vertices $v_{c_1}, v_{c_2}, \ldots, v_{c_k}$) with degree at least $k-1$ each.

Problem 2

A vertex cover of a graph is a subset of its vertices such that every edge of the graph is incident to at least one vertex of that subset. Consider the following problem

Vertex_Cover
Input: A graph $G = (V, E)$ and an integer $k \geq 1$,
Output: 1 if and only if the graph G have a vertex cover of size k. Otherwise it outputs 0.

Furthermore, consider the following problem

Independent_Set
Input: A graph $G = (V, E)$ and an integer $k \geq 1$,
Output: 1 if and only if the graph G have an independent set of size k. Otherwise it outputs 0.

Show the reduction **Independent_Set** $\leq_P$ **Vertex_Cover**.
(*topic: Computational Complexity, Reductions*)

Solution:
Consider an instance of **Independent_Set**, namely a graph $G = (V, E)$ and an integer $k \geq 1$. We are now going to create an instance (G', k') of the **Vertex_Cover** problem as follows:

1. $G' = G$.
2. $k' = |V| - k$.

Now, let's show why this construction demonstrates a valid reduction function. Firstly the construction works in polynomial time in the size of the input. What remains now is to show that the construction maps "yes" instances of **Independent_Set** to "yes" instances of **Vertex_Cover** and vice versa.
($\rightarrow$) Suppose there is an independent set S of size k in the graph G. Then $V \setminus S$ forms a vertex cover in G.
To prove this statement, let's assume that this is not the case and $V \setminus S$ is not a vertex cover in G. Then there exists an edge $e \in E$ such that none of its endpoints, let u, v, belong in $V \setminus S$, namely $u, v \notin (V \setminus S) \Rightarrow u, v \in S$. Since $u, v \in S$ and u, v are incident to an edge, we reached a contradiction.
($\leftarrow$) Suppose that there is a vertex cover S of size $|V| - k$ in G. Then $V \setminus S$ is an independent set of size k in G.
To prove this statement, let's assume that this is not the case and $V \setminus S$ is not an independent set in G. Then there exists a pair of vertices, let u, v, in $V \setminus S$ that are incident to an edge,

namely $(u, v) \in E$. Note that $u, v \in (V \setminus S) \Rightarrow u, v \notin S$, hence we have an edge (u, v) with endpoints u and v that do not belong in the vertex cover S, which is a contradiction. This completes the proof.

Problem 3

In year X (the number X is not given to you), the $100th$ day of the year was a Thursday. In year $X+1$, the 140th day of the year was a Thursday as well. What day of the week was the 180th day of year $X-2$?

Note that a year is either a leap year or a non-leap year. A leap year contains 366 days, while a non-leap year contains 365 days. Furthermore, you can assume that a leap year occurs every four years.

Solution:

If the year X is not a leap year, it contains 365 days. In that case, there are $365-100+140 = 405$ days between the 100th day of year X and the 140th day of year $X+1$. Note that $405 \equiv 6 \mod 7$, hence in that case, if the 100th day of year X is a Thursday, the 140th day of year $X+1$ is a Wednesday. That can't be the case.

Hence the year X must be a leap year. Indeed, for completeness, in that case there are $366 - 100 + 140 = 406$ days between the 100th day of year X and the 140th day of year $X+1$. Note that $406 \equiv 0 \mod 7$, hence in that case, if the 100th day of year X is a Thursday, the 140th day of year $X+1$ is a Thursday as well (which is the case).

Since X is a leap year, $X-2$ and $X-1$ are not leap years. So there are $365+(365-180)+100 = 650$ days between the 100th day of year X and the 180th day of year $X-2$. Note that $650 \equiv 6 \mod 7$, hence in that case, since the 100th day of year X is a Thursday, the 180th day of year $X-2$ is *a Friday.*