Ramsey Numbers by Stochastic Algorithms with New Heuristics

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Abstract. In this paper, we are interested in combinatorial problems of graph and hypergraph colouring linked to Ramsey's theorem. We construct correct colourings for the edges of these graphs and hypergraphs, by stochastic optimization algorithms in which the criterion of minimization is the number of monochrome cliques. To avoid local optima, we propose a technique consisting of an enumeration of edge colourings involved in monochrome cliques, as well as a method of simulated annealing. In this way, we are able to improve some of the bounds for the Ramsey numbers. We also introduce cyclic colourings for the hypergraphs to improve the lower bounds of classical ternary Ramsey numbers and we show that cyclic colourings of graphs, introduced by Kalbfleisch in 1966, are equivalent to symmetric Schur partitions.

Key Words: stochastic optimization, cyclic colouring, Ramsey number, Schur number, graph, hypergraph.

Introduction

The success in the resolution of some combinatorial NP–Hard problems by stochastic optimization methods such as the "N–Queens" problem for $N > 3 \times 10^6$ [43, 44] and the salesman problem [29], has spurred research on the use of this type of algorithm for solving other combinatorial problems. In particular, the problem of graph colouring as linked to the Ramsey's theorem. The use of these methods gives promising experimental results for improving some bounds on Ramsey numbers ([9, 10, 11, 12, 13],[24, 25, 26]). Other applications on NP–Hard problems have also been proposed by Selman *et al.* [41, 42], Adorf and Johnson [2], Minton *et al.* [35] and by Gu [20].

In this paper, we are interested in the NP–Hard problem of the evaluation of Ramsey numbers. These numbers are particularly used for calculating the complexity of some algorithms on parallel machines [45]. It has been confirmed in [18] that Ramsey numbers serve to construct the "best" communication network.

Ramsey theory deals with the distribution of subsets of elements of sets. The theorem of Ramsey is a generalization of the "pigeon–hole" principle: suppose that a flock of pigeons flies into a set of pigeonholes to roost. The *pigeonhole principle* (also called the *Dirichlet drawer principle*) states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it. This principle can be applied to other objects besides pigeons and pigeonholes. In general:

if $(k-1)T+1$ objects are distributed in T drawers then at least one of the drawers contains k objects.

We can use Ramsey numbers to solve the following problems:

Problem 1 What is the minimum number of students required in a class to be sure that at least six of them will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Problem 2 What is the least number of area codes needed to guarantee that the 25 million phones in a state have distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form NXX–NXX–$XXXX$, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)

Problem 3 Assume that in a group of eighteen people, each pair of individuals consists of two friends or two enemies. Show that there are either four mutual friends or there are four mutual enemies in the group.

Problem 4 A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

Problem 5 During a month with 30 days a baseball team plays at least 1 game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Problem 6 Show that, if there are 101 people of different heights standing in a line, it is possible to find 11 people in the order they are standing in the line with heights that are either increasing or decreasing and in general, every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

The evaluation of each Ramsey number is a combinatorial NP–Hard problem (see [7] Appendix NP–Complete Problems (Monochromatic Triangle)). So, it is costly to explore the whole space or the search-tree needed to find a solution. Indeed, the search for a solution through the enumeration of all cases leads to procedures of a disastrous efficiency. Thus, to determine these numbers, or to improve their bounds, we propose three stochastic optimization algorithms with colouring based criterion. These algorithms, which are incomplete constraint satisfaction search, are generally methods of rapid descent, completed by enumeration or "backtracking" algorithms when we are "close" to a solution, and methods of simulated annealing to avoid local optima.

To improve the lower bounds of binary Ramsey numbers we use the properties of cyclic colouring of graphs introduced by Kalbfleisch [28] in 1966. We generalize the cyclic colouring for hypergraphs associated with Ramsey numbers.

1 Ramsey's General Theorem

Definition 1. Let X be a set of n elements, and let $\mathcal{P}_h(X)$ be the set of parts with h elements of X, ($|\mathcal{P}_h(X)| = \binom{n}{h}$). The couple $H = (X, \mathcal{P}_h(X))$ constitutes a *complete hypergraph of rank h and order n*. Elements of X and $\mathcal{P}_h(X)$ denote respectively vertices and edges of H. Each edge of H is therefore a set of cardinal h. Such hypergraph is called a *n–clique of rank h* and denoted by K_n^h. We also denote by K_n (or K_n^2), *the simple complete graph of order n.*

Definition 2. Let H be a complete hypergraph of n vertices and rank h, and let $k_1, k_2, \ldots, k_m$ be the positive integers greater than h. A $(k_1, k_2, \ldots, k_m; h)$–*colouring* is an application that attributes to each edge of H a "colour" among a set of m colours. This $(k_1, k_2, \ldots, k_m; h)$–colouring is said to be *correct* if none of the k_i–cliques (i.e., a complete sub–hypergraph of rank h and order k_i) of H has all edges of colour i. In other words, there is no monochrome k_i–clique of colour i and rank h, for $i = 1, 2, \ldots, m$.

Definition 3. We say that the complete hypergraph K_n^h, is $(k_1, k_2, \ldots, k_m; h)$–*colourable*, if we can associate a correct $(k_1, k_2, \ldots, k_m; h)$–colouring with its edges.

To formulate Ramsey's general theorem as a colouring of hypergraph edges we express this theorem in the following manner:

Theorem 4. *Consider m colours $1, 2, \ldots, m$ and $m+1$ positive integers $k_1, k_2, \ldots, k_m$, h with $k_1, k_2, \ldots, k_m \geq h$. There exists a finite integer n ($n < \infty$) such that any complete hypergraph of rank h and order n has no correct $(k_1, k_2, \ldots, k_m; h)$–colouring for its edges.*

The smallest integer n is called the *classical Ramsey number of rank h* in $k_1, k_2, \ldots, k_m$ which is denoted by $R(k_1, k_2, \ldots, k_m; h)$. The existence of this number has been proven by Ramsey [38] and by Ryser [39].

According to this theorem, all $(k_1, k_2, \ldots, k_m; h)$–colourings of the edges of K_R^h, where R is the number $R(k_1, k_2, \ldots, k_m; h)$, contains by definition, a certain number of monochrome k_i–cliques of colour i (for $i = 1, 2, \ldots, m$) and rank h. The minimum number of these monochrome k_i–cliques is called *multiple Ramsey number of rank h* in $k_1, k_2, \ldots, k_m$ which is denoted by $r(k_1, k_2, \ldots, k_m; h)$.

Obviously, we are able to determine the value of the multiple Ramsey number, $r(k_1, k_2, \ldots, k_m; h)$, only when the classical Ramsey number, $R(k_1, k_2, \ldots, k_m; h)$, is known.

The *multiplicity function* $M_{(n)}(k_1, k_2, \ldots, k_m; h)$ of *rank h* in $k_1, k_2, \ldots, k_m$ is defined as the minimum number of monochrome k_i–cliques of colour i (for $i = 1, 2, \ldots, m$) in the colouring of the edges of K_n^h (i.e., the complete hypergraph of rank h and order $n > R(k_1, k_2, \ldots, k_m; h) + 1$). Obviously, the multiple Ramsey number $r(k_1, k_2, \ldots, k_m; h)$ is equal to $M_{(R)}(k_1, k_2, \ldots, k_m; h)$ where R is the classical Ramsey number $R(k_1, k_2, \ldots, k_m; h)$.

2 Evaluation of Ramsey Numbers

The evaluation of each Ramsey number, in itself, is a combinatorial NP–Hard problem [19][1]. It is not surprising that, classical "backtracking" methods fail to find "good" lower and upper bounds for Ramsey numbers $R(k_1, k_2, \dots, k_m; h)$ even when $h = 2$ (i.e., the hypergraph $H = (X, \mathcal{P}_2(X))$ becomes a simple complete graph $G = (X, E)$ of n vertices K_n, with $n = |X|$), as the complexity of these methods is exponential and the number of trials to get a solution (i.e., the number of possible colourings), with exhaustive search, is very large. So, we have to be satisfied with the improvement of lower and upper bounds of these numbers (see [6], [14, 15, 16], [21, 22], [23], [27], [32, 33] etc.).

In general, to improve lower and upper bounds associated with Ramsey numbers, we look for the greatest integer $n < R(k_1, k_2, \dots, k_m; h)$, where n is the number of vertices of the complete hypergraph of rank h, $H = (X, \mathcal{P}_h(X))$, for which there exists a correct $(k_1, k_2, \dots, k_m; h)$–colouring for its edges. Similarly, to improve upper bounds of classical Ramsey numbers, we look for the smallest integer $n \geq R(k_1, k_2, \dots, k_m; h)$, for which no correct $(k_1, k_2, \dots, k_m; h)$–colouring exists for K_n^h edges.

2.1 Binary Ramsey Numbers

To find the exact value of *the classical binary Ramsey number* $R(k_1, k_2, \dots, k_m; 2)$, it is necessary to construct a correct $(k_1, k_2, \dots, k_m; 2)$–colouring for the K_{R-1} edges, where R is the Ramsey number $R(k_1, \dots, k_m; 2)$, and to prove that no such correct $(k_1, k_2, \dots, k_m; 2)$–colouring exists for the edges of K_R. This is a very difficult task. Indeed, the value of the number $R(k_1, k_2, \dots, k_m; 2)$ increases very "rapidly" when the values of k_i $(i \in \{1, \dots, m\})$ increase. The number of possible colourings of the graph edges associated with $R(k_1, k_2, \dots, k_m; 2)$ is about $m^{(R(\dots)-1)(R(\dots)-2)/2}$. Thus, for $|X| = 16$, $m = 3$ (i.e., three colours) with $k_1 = k_2 = k_3 = 3$ (i.e., a simple case), the number of possible colourings to get a solution is about $3^{120} \simeq 10^{57}$.

In 1947, Erdös [8] proposed a probabilistic method that associates lower bounds with binary Ramsey numbers. In particular, he proved that the Ramsey number $R(34, 34; 2)$ is greater than 10^6. This method gives information about the value of Ramsey numbers but does not indicate how to construct the correct colourings for their associated graphs. Greenwood and Gleason [17], have also given some properties which make it possible to associate upper bounds with

[1] To illustrate the difficulty of the evaluation of Ramsey numbers, the Hungarian mathematician, Paul Erdös, (see [18]) proposed the following anecdote:

> the extra-terrestrials invade the Earth and threaten to annihilate it within a year if humanity does not find the classical Ramsey number $R(5, 5; 2)$. Under the threat, the best mathematicians and the most powerful computers are mobilized, and the catastrophe is avoided. On the other hand, if the extra-terrestrials ask for the classical Ramsey number $R(6, 6; 2)$, the only possibility would be war.

every binary Ramsey number. Abott [1] also proved some inequalities that give lower bounds for binary Ramsey numbers. However, all the values obtained by these inequalities are very far from the exact values of Ramsey numbers. Therefore, it was necessary to seek other methods for improving the bounds associated with Ramsey numbers.

Much work has been done during the last fifty years to improve the bounds associated with binary Ramsey numbers (see Giraud [14, 15, 16], McKay and Radziszowski [32, 33, 34] Exoo [9, 10, 11, 12, 13]). Only eleven numbers have been exactly evaluated, ten of them for $m = h = 2$ (i.e., two colours) and only one for $h = 2, m = 3$ (i.e., three colours) ($R(3,3,3;2) = 17$). Most of these numbers were evaluated by Greenwood-Gleason (1955) and by Kalbfleisch (1966).

2.2 Ternary Ramsey Numbers

The evaluation of *the classical ternary Ramsey numbers*, $R(k_1, k_2, \ldots, k_m; 3)$, is more difficult than that of the binary Ramsey numbers $R(k_1, k_2, \ldots, k_m; 2)$, due to the fact that the number of possible colourings of edges of their associated hypergraphs is very large (i.e., this number is about $m^{\binom{R(\ldots)}{3}}$). In 1966, Kalbfleisch [28] showed that ternary and binary Ramsey numbers are closely linked by the following inequality:

$$R(k_1, k_2; 3) \leq R(x, y; 2) + 1, \text{ with } x = R(k_1 - 1, k_2; 3), y = R(k_1, k_2 - 1; 3).$$

This shows that the first non–trivial ternary Ramsey number, $R(4,4;3)$, is lower than 20. This means that any correct $(4,4;3)$-colouring does not exist for the edges of K^3_{20}. Kalbfleisch also proposed two methods to construct correct (4,4;3)–colourings for the edges of K^3_{11}. However, these two methods have failed in their attempt to construct a correct (4,4;3)–colouring for the edges of K^3_{12}. Therefore, we have $12 \leq R(4,4;3) \leq 19$. In 1969, Giraud [16] proved that the ternary Ramsey number $R(4,4;3)$ is smaller than 15. Also in 1969, Isbell [23] improved the lower bound of this number by 1. Recently (1991), by using Turán numbers, McKay and Radziszowski [34] have shown that any $(4,4;3)$–colouring of the edges of K^3_{13} always contains a monochrome 4–clique of rank 3. They have also found a correct $(4,4;3)$–colouring of the edges of K^3_{12}. Consequently, we have: $R(4,4;3) = 13$.

3 Theoretical Results

The theoretical results given in this section are of Greenwood-Gleason [17], Erdös [8] and Abott [1]. These results permit to approximate "roughly" the Ramsey numbers.

Lemma 5. *The Ramsey number $R(k_1, k_2, \ldots, k_m; 2)$ is invariant for any permutation of k_i.*

Lemma 6. *The Ramsey number $R(2, k_2, \ldots, k_m; 2)$ is equal to $R(k_2, \ldots, k_m; 2)$ and, we also have: $R(2, k_2; 2) = R(k_2; 2) = k_2$.*

Lemma 7. *Let G be a $(k_1, k_2, \ldots, k_m; 2)$-colourable graph, for each vertex of G there are at most $R(k_1, k_2, \ldots, k_i - 1, \ldots, k_m; 2) - 1$ edges of colour i leading to the other vertices.*

Lemma 8. *$R(k_1, k_2; 2) \leq R(k_1 - 1, k_2; 2) + R(k_1, k_2 - 1; 2)$; moreover, this inequality is strict if these last two terms are both even integers.*

Lemma 9. *The following inequality gives an upper bound for the classical binary Ramsey number $R(k_1, k_2, \ldots, k_m; 2)$:*

$$R(k_1, k_2, \ldots, k_m; 2) \leq 2 - m + \sum_{i=1}^{m} R(k_1, k_2, \ldots, k_i - 1, \ldots, k_m; 2) \ .$$

Theorem 10 (Abott, 1966). *This theorem gives a lower bound for binary Ramsey numbers.*

$$R(k_1, k_2, \ldots, k_m; 2) \geq 1 + ([R(k_1, \ldots, k_s; 2) - 1] \times [R(k_{s+1}, \ldots, k_m; 2) - 1]) \ .$$

4 Ramsey Numbers and Chromatic Number

To study the binary Ramsey numbers turns out to be equivalent to study the chromatic number of a hypergraph, $\mathcal{X}(H)$, whose concept was introduced in 1966 by Erdös and Hajnal (see [4]). Indeed, let $H = (X, \mathcal{E})$ be a hypergraph of order q with $\mathcal{E} \subseteq \mathcal{P}_q(X)$ where $q = 2, \ldots, h$ (i.e., H is a complete hypergraph or not). A set $S \subset X$ is called *stable* if it contains no edge $E_i \in \mathcal{E}$ of cardinal greater than 1. The *chromatic number* $\mathcal{X}(H)$ is the smallest number of the necessary colours that we can use to colour vertices of H, in such a way that no edge E_i (with $|E_i| > 1$) has its vertices coloured with the same colour. An *m-colouring* of vertices of H (with $m \geq \mathcal{X}(H)$) is a partition of the set of vertices X into m stable sets $\mathcal{S}_1$, $\mathcal{S}_2$, ..., $\mathcal{S}_m$, while a correct $(k_1, k_2, \ldots, k_m; 2)$-colouring of the edges of the simple complete graph $G = (X, \mathcal{P}_2(X))$ is a partition of $\mathcal{P}_2(X)$ into m sets (or m colours) $\mathcal{B}_1$, $\mathcal{B}_2$, ..., $\mathcal{B}_m$, such that for each part A_i of X of cardinal k_i, we have $\mathcal{P}_2(A_i) \not\subset \mathcal{B}_i$. Therefore we can consider a hypergraph of rank k, whose vertices are the edges of the simple complete graph G, and whose edges are the k-cliques of rank 2 of G. This hypergraph will be denoted by G^k. The proposed problem of the evaluation of classical binary Ramsey numbers consists in studying the chromatic number $\mathcal{X}(G^k)$. We then have the following inequality:

$$\mathcal{X}(G^k) \leq R(k_1, k_2, \ldots, k_m; 2) - 1 \text{ (with } k_1 = k_2 = \cdots = k_m) \ .$$

5 Stochastic Optimization Algorithms

In this section, we give the stochastic optimization algorithms that we have used to improve the bounds associated with Ramsey numbers.

The principle of these algorithms consists in minimizing an objective function depending on the problem being considered. For example, to evaluate the Ramsey number $R(k_1, k_2, k_3; 2)$, the objective function is the number of monochrome k_i–cliques of colour i (for $i = 1, 2$ and 3) that are found in the coloured graph associated with this number. The optimization criterion is therefore the number of monochrome k_i–cliques of colour i.

To increase the efficiency of these algorithms, we use a strategy similar to the *"divide and conquer"* technique [30]. The general idea is to decompose the problem in sub–problems, which are less difficult to solve (if this is possible) and then solve each sub–problem independently. The obtained solutions of the sub–problems are "pasted" together to give a solution of the initial problem. Thus, to evaluate the classical Ramsey number $R(k_1, k_2, \ldots, k_m; 2)$ (with $k_m \geq k_{m-1} \geq \cdots \geq k_1$), we eliminate first, in the corresponding coloured graph, all monochrome k_m–cliques of colour m, then all monochrome k_{m-1}–cliques of colour $m-1$ and so on until the elimination of all the monochrome k_1–cliques of colour 1.

We will describe the stochastic algorithms, that we have used, to construct correct colourings of the hypergraphs associated with Ramsey numbers. To simplify the description, we assume that $k_1 = k_2 = \cdots = k_m$.

5.1 Rapid Descent Algorithm

The resolution process of this algorithm (denoted by algorithm 1) starts with the creation of a random colouring $\mathcal{C}$ of the edges of the simple complete graph G of order n which produces a minimum number of monochrome k_i–cliques (for $i =$ 1,2,...,m). In order to accomplish this, we proceed in the following manner: we have $\binom{n}{2}$ edges to colour with m colours. We iteratively select, randomly, an edge and we colour it with the colour that produces a minimum number of monochrome k_i–cliques of colour i (for $i =$ 1,2,...,m). In the resulting colouring, we have, generally, a certain number of monochrome k_i–cliques of colour i, to be calculated, that we will try to delete.

Local Minima. This algorithm cannot find the solution for all random colourings of the edges of the graph G. There are some cases in which whatever the number of undertaken permutations of colour edges is, the algorithm can no longer reduce the number of monochrome k_i–cliques. If, for a given colouring $\mathcal{C}$, we undertake ten thousand changes in the colour of the edges (i.e., empirical value), without being able to decrease the value of $N(\mathcal{C})$, we estimate that we have reached a *local minimum.* In this case, we choose another random colouring $\mathcal{C}'$ and we iterate the process.

In a simplified manner, this algorithm starts from an initial state $(\mathcal{C}_1, \mathcal{F}_1)$ (i.e., that is the random colouring, $\mathcal{C}_1$, of the edges of G and the number of monochrome k_i–cliques $\mathcal{F}_1 \geq 0$ in $\mathcal{C}_1$) and tries to reach (i.e., by changing of the colour of the edges) a final state $(\mathcal{C}_n, \mathcal{F}_n)$, with:

$\mathcal{C}_n$ is the correct colouring of the edges of G and

$\mathcal{F}_n = 0$ is the number of monochrome k_i–cliques in C_n, for all $i \in \{1, 2, \ldots, m\}$.

The local minimum results from the fact that, during the resolution, we reach an intermediate state $(C_i, \mathcal{F}_i > 0)$, in which, the number of monochrome k_i–cliques remains stable whatever the number of changes of the colour of the edges is. In other words, for every colouring C_j obtained by changing the colour of the G edges, we will always have $\mathcal{F}_j \geq \mathcal{F}_i > 0$.

Algorithm 1.

This algorithm tries to construct a correct $(k_1, k_2, \ldots, k_m; 2)$–colouring for the edges of the complete graph G of order n. With this colouring we show that $R(k_1, k_2, \ldots, k_m; 2) > n$. At the beginning, we fix the number of the initial colourings (i.e., the number of different colourings generated to stop the algorithm) as well as the number of consecutive colourings that we accept; this number denoted by NCC corresponds, in fact, to a *local minimum* (i.e., empirical value).

1- *Choose a random colouring C for G edges which produces a minimum number of monochrome k_i–cliques of colour i (for $i = 1, \ldots m$).*
Introduce a counter K and initialize it with zero.
Initialize the successive number of colourings NCC.

2- *If the number $N(C)$ of monochrome k–cliques is null then stop (i.e., a solution is found). Otherwise, set K to zero.*

3- *Choose an edge e that is included in the maximum of monochrome cliques. Introduce a colouring C' that coincides with C except for $C'(e) \neq C(e)$.*
- *If $N(C') < N(C)$ then replace C by C' and go to* 2.
- *If $K = NCC$ then go to* 1.
- *If $N(C') = N(C)$ then replace C by C', increase K by* 1 *and go to* 3.
- *If $N(C') > N(C)$ then increase K by* 1 *and go to* 3.

Fig.1. The algorithm of rapid descent.

5.2 Random Drawing and the Arborescent Method

In this section, we propose another stochastic optimization algorithm for "avoiding" local minima. It consists in combining the rapid descent method with the arborescent method (i.e., a backtracking or a branch and bound method[30]).

This algorithm operates as follows: first, we use a rapid descent algorithm to reduce the size of the search area. As soon as we reach a small enough number of monochrome k_i–cliques, we enumerate all possible colourings of the edges contained in the monochrome k_i–cliques. The enumeration is made by the arborescent or "backtracking" method [3]. At this level, the number of possible colourings to get a solution is relatively small. Indeed, each monochrome k_i–clique is formed by $\binom{k_i}{2}$ edges (i.e., a complete sub–graph of order k_i), then if there remain T monochrome k_i–cliques in the coloured graph, the number of possible colourings is about $\binom{k_i}{2}^T$. Thus, we reduce the size of the search area or paths to cover from $\binom{k_i}{2}^{\binom{n}{2}}$ to $\binom{k_i}{2}^T$ (with $3 \leq T \leq 18$ (i.e., empirical value) and n the order of the complete graph G).

In this algorithm, we can say that the rapid descent method is a pre-treatment method [46]. We use it to get a partial solution of the problem.

Algorithm 2.

1- *Choose a random colouring C for the edges of G which produces the minimum number of monochrome k_i–cliques of colour i (for $i = 1, \ldots m$).*

2- *Minimize the number of monochrome k_i–cliques in G by changing the colour of the edges.*

3- *If a pseudo–local–minimum is reached (i.e. empirical value) then apply the* **enumeration procedure** *after constructing a list U of the edges implied in all the monochrome k_i–cliques.*
Otherwise, stop (i.e., a solution is found).

PROCEDURE enumeration (G, U);
$\{$*C is a variable that takes values* 1, 2, $\cdots$, *m representing the colours* $\}$
BEGIN
IF $U = \emptyset$ **THEN** *stop (i.e., a solution is found);* **OTHERWISE**
  $\{$*Choose an edge* $e \in U$;
  FOR $C \leftarrow 1$ **TO** m **DO**
    $\{$*Allocate the colour C to e;*
    IF *(number–monochrome–cliques (G))* $= 0$ **THEN**
    enumeration $(G, U - \{e\})$;$\}$
  $\}$
END.

Fig.2. The algorithm of stochastic descent and enumeration.

After the construction of the U list, the graph G contains $|U|$ uncoloured edges and zero monochrome k_i–cliques of colour i for all $i \in \{1, 2, \ldots, m\}$.

The edges of the list U are ordered (i.e., the first edge is implied in the maximum of monochrome k_i–cliques).

5.3 Simulated Annealing Method

The Simulated Annealing method is a widespread technique used to solve combinatorial problems where the search space is vast. It has been proposed in 1953, by statistical physics specialists (Metropolis *et al.* [31]). An application of this method for the salesman problem has been proposed by Kirkpatrick and Gelatt [29]. This method is an extension of the Monte Carlo method [30].

The purpose of this method is to escape from the local minima. There is always an objective function $\mathcal{F}$ to minimize. In our case, it is the number of monochrome k_i–cliques (for $i = 1, 2, \ldots, m$).

Schematically, the principle of this method is the following: from an initial value of T (i.e., a control parameter) and an initial configuration (i.e., a colouring of the edges of G, in our case), S, of the considered problem, we choose arbitrarily another configuration S' (i.e., after changing an edge colour, for example). If the value of the objective function in S', $\mathcal{F}(S')$, is lower than its value in S (i.e., $\mathcal{F}(S') < \mathcal{F}(S)$), this means that if the drawn configuration is better than the current one, S' replaces S and we repeat the process. If this is not the case (i.e., if S' is worse than S) we can however decide to replace S by S'. This decision is made with a certain probability given by:

$$\text{Prob } = 1 - \exp^{-k\times(T-T_0)} \text{ or } \exp^{\frac{-\Delta}{k\times T}} \text{ with } \Delta = \mathcal{F}(S') - \mathcal{F}(S) \ .$$

k being a constant and T being a control parameter that goes down to T_0. For $T = T_0$, only "beneficial" changes, for which we have $\mathcal{F}(S') < \mathcal{F}(S)$, are admitted, while for $T > T_0$, it is permitted to increase the value of the objective function $\mathcal{F}$. These techniques will probably make it possible to avoid a local minimum which is not global.

Algorithm 3.

1- *Initialize T, $T_{\min}$ (i.e., the minimum value of T), T_{red} (i.e., the reduction factor of T), k and the number of iterations N_{itr}.*

2- *Choose a random colouring C for G edges which produces a minimum number of monochrome k_i–cliques of colour i (for $i = 1, \ldots m$).*

3- *Calculate the number $N(C)$ of monochrome k_i–cliques in C.*

REPEAT

$i \leftarrow 0$;

WHILE $(i < N_{\text{itr}})$ **AND** $(N(C) > 0)$ **DO**

- *Choose randomly an edge e;*
- *Introduce a colouring C' that coincides with C except for $C'(e) \neq C(e)$;*
- $\Delta \leftarrow N(C') - N(C)$, $i \leftarrow i + 1$;
- **IF** $(\Delta \leq 0)$ **OR** (**RAND** $<$ **EXP**$(\frac{-\Delta}{k\times T})$) **THEN** *replace C by C';*

END

$T \leftarrow T \times T_{\text{red}}$;

UNTIL$(N(C) = 0)$ **OR** $(T < T_{\min})$

Fig.3. The algorithm of Simulated Annealing.

The **RAND** function is a generator of random numbers between 0.0 and 1.0. In this type of algorithm, several questions may be asked. For example, how can we determine the value of T and $T_{\min}$? How to reduce T ? etc. The values of all these parameters are empirical. However, T is initialized with a value relatively large, which makes it possible to accept all the configurations obtained by "penalizing" permutations of colours of edges. Practically, the value of T is chosen in the following manner: we generate n random colourings (i.e., a thousand, for example), and we calculate the average value of Δ for these colourings. The value of T will be equal to $\frac{-\Delta}{\ln \text{Const}}$ (with $0.12 \leq \text{Const} \leq 0.95$ (empirical values)).

6 New Heuristics to Colour the Hypergraph Edges

We will introduce the cyclic colourings for the hypergraphs to improve the lower bounds associated with classical ternary Ramsey numbers and we show that the cyclic colourings of graphs are equivalent to symmetric Schur partitions [40].

The properties of cyclic colourings of graphs (see [10]) introduced in 1966 by Kalbfleisch [28] are very useful for constructing correct colourings for the edges of the graphs associated with binary Ramsey numbers.

6.1 Cyclic Colouring of Graphs

Definition 11. Let G be a simple complete graph of order n with its vertices labelled by the integers $1, 2, \ldots, n$, and let e be an edge of G. We define the *length* of e, $l(e)$, by the difference between its two extremities. The length $l(e)$ (with $e = \{i, j\}$ and $i < j$) is equal to:

$$\begin{cases} |i-j|, & \text{if } |i-j| < \lfloor \frac{n}{2} \rfloor, \\ i-j+n & \text{otherwise} . \end{cases}$$

this length varies between 1 and $\lfloor \frac{n}{2} \rfloor$.

The lengths of the edges of the simple complete graph of order 5, K_5, are:

$$\begin{aligned} &l(\{1,2\}) = 1,\ l(\{3,4\}) = 1,\ l(\{4,5\}) = 1,\ l(\{1,5\}) = 1,\ l(\{2,3\}) = 1, \\ &l(\{2,4\}) = 2,\ l(\{2,5\}) = 2,\ l(\{1,3\}) = 2,\ l(\{3,5\}) = 2,\ l(\{1,4\}) = 2. \end{aligned}$$

Definition 12. Let G be a $(k_1, k_2, \ldots, k_m; 2)$–colourable graph of order n, We say that G is *Ramsey–regular*, if there exist a correct $(k_1, k_2, \ldots, k_m; 2)$–colouring of the edges of G, in which all edges with the same length are coloured with the same colour. The correct $(k_1, k_2, \ldots, k_m; 2)$–colouring that associates only one colour with each edge of G is called *a correct cyclic $(k_1, k_2, \ldots, k_m; 2)$–colouring.*

According to this definition, the $(3, 3; 2)$–colourable graph of order 5, K_5, is Ramsey–regular. Indeed, to have a correct cyclic $(3, 3; 2)$–colouring for the edges of K_5 it suffices to assign the colour Red to all the edges of length 1 and the colour Blue to those of length 2.

Property of non Ramsey–Regular Graphs. The non Ramsey–regular graphs have an interesting property: if there exists a $(k_1, \ldots, k_m\ ; 2)$–colourable graph K_n, then there exists a $(k_1, k_2, \ldots, k_m; 2)$–colourable graph K_{n-1}. To have a correct $(k_1, k_2, \ldots, k_m; 2)$–colouring for the edges of K_{n-1}, it suffices to delete a vertex of K_n with all the edges that connect it to the other vertices. However, this is not always true for Ramsey–regular graphs. That is to say, it is possible that there exists a correct cyclic $(k_1, k_2, \ldots, k_m; 2)$–colouring for the edges of K_n, which is not the case for the edges of K_{n-1}; for example, the case of K_{10} that has a correct cyclic $(3, 3, 3; 2)$–colouring for its edges (i.e., easy to find) while K_9 has no such cyclic $(3, 3, 3; 2)$–colouring. This leads to the following theorem:

Theorem 13. *Given m colours and an integer $i > 2$, the simple complete graph of order $3i$ has no correct cyclic $(\underbrace{3, 3, \ldots, 3}_{m}; 2)$–colouring for its edges.*

Proof of theorem. The simple complete graph K_n (with $n = 3i$) contains always a simple complete sub–graph $G = (X, \mathcal{P}_2(X))$ of order 3, with $X = \{x_1, x_2, x_3\}$ and $l(e_1 = \{x_1, x_2\}) = l(e_2 = \{x_2, x_3\}) = l(e_3 = \{x_1, x_3\}) = i$. Indeed, we can take for example, $x_1 = 1$, $x_2 = 1 + i$ and $x_3 = 1 + 2i$. It is clear that $l(e_1) = l(e_2) = i$. The length of e_3 cannot be equal to $|1 - 2i - 1|$ because $2i$ is strictly greater than $\lfloor \frac{n}{2} \rfloor$ and consequently, $l(e_3) = n + 1 - 1 - 2i = i$. The edges of G have therefore the same length and thus they can be coloured with the same colour. So, G forms a monochrome 3–clique of rank 2. Consequently, K_n has no correct cyclic $(\underbrace{3, 3, \ldots, 3}_{m}; 2)$–colouring for its edges. □

According to this theorem, the graphs K_9,K_{12},K_{15},K_{21},...,K_{63} etc. have no correct cyclic $(3, \ldots, 3; 2)$–colouring for their edges whatever the number of colours m is.

6.2 The Cyclic Colouring and the Symmetric Schur Partition

In this section we show the relation between the cyclic colouring of graphs associated with binary Ramsey numbers of the form $R(3, 3, \ldots, 3; 2)$ and the symmetric Schur partition. We can generalize this relation for all Ramsey numbers of rank h, $R(k_1, k_2, \ldots, k_m; h)$.

Definition 14. Let $E = \{1, \ldots, n\}$, a *Schur $(H_1, H_2, \ldots, H_k)$–partition*, is a partition of E into k classes $H_1, \ldots, H_k$, in such a way that if two elements i and j of E belong to the class H_l, then their sum m (if $m \leq n$) does not belong to this class. Schur's lemma [40] asserts that there exists a finite set E of $S(k)$ positive elements that has a $(H_1, H_2, \ldots, H_k)$–partition and for any set of n' elements with $n' > S(k)$, this $(H_1, H_2, \ldots, H_k)$–partition does not exist. The integer $S(k)$ is called *the Schur number of k classes.*

In other words, if we colour the elements of E ($|E| > S(k)$) with k colours ($k > 0$), then there exist always three elements, x,y and z, coloured with the same colour such that: $x + y = z$.

Definition 15. The Schur $(H_1, H_2, \ldots, H_k)$–partition is called *symmetric* iff for all pairs (i, j) of E of sum $n+1$ (with $|E| = n$) we have i and j belong to the same class.

The symmetric Schur $(H_1, H_2, H_3, \ldots, H_k)$–partition and the correct cyclic $(\underbrace{3,3,3,\ldots,3}_{m};2)$–colouring of the edges of the simple complete graphs are equivalent. In other words, finding a correct $(3,3,\ldots,3;2)$–cyclic colouring for the edges of K_n, is the same problem as finding a symmetric Schur $(H_1, H_2, \ldots, H_k)$–partition of the set E of $n-1$ elements.

Example 1. Let E be a set of 13 elements, $E = \{1, 2, \ldots, 13\}$. A symmetric Schur (H_1, H_2, H_3)–partition of E is the following:

$$H_1 = \{1,4,7,10,13\},\ H_2 = \{5,6,8,9\} \text{ and } H_3 = \{2,3,11,12\}.$$

It is known that there is no symmetric Schur (H_1, H_2, H_3)–partition if $|E| = 14$ because $S(3) = 14$ (see [19, 9]). So, there is no correct cyclic $(3,3,3;2)$–colouring for all the edges of the simple complete graph of order $n > 14$.

The correct cyclic $(3,3,3;2)$–colouring of the edges of K_{14} is obtained by colouring the edges of length 1, 4 and 7 ($\{1,4,7\} \in H_1$) with the colour Red, those of length 5 and 6 ($\{5,6\} \in H_2$) with the colour Blue and finally those of length 2 and 3 ($\{2,3\} \in H_3$) with the colour Green. In the same way, we can easily construct the symmetric Schur $(H_1, H_2, \ldots, H_m)$–partition from classes of Ramsey–regular graphs.

To improve the lower bound associated with classical binary Ramsey numbers of the form $R(3,3,\ldots,3;2)$, we can use the corresponding symmetric Schur $(H_1, H_2, \ldots, H_k)$–partition. This leads to the following theorem:

Theorem 16. *All simple complete graphs of order $n > 45$ have no correct cyclic $(3,3,3,3;2)$–colouring for their edges.*

Proof of theorem. We have shown above the equivalence between the cyclic colouring and the symmetric Schur partition. The Schur number $S(4)$ is exactly equal to 44 (see [19]). This implies that there exists no symmetric Schur (H_1, H_2, H_3, H_4)–partition for any set of n elements with $n > 44$. Consequently, there is no correct cyclic $(3,3,3,3;2)$–colouring for the edges of K_n with $n > 45$. □

The improvement of the lower bound associated with the Ramsey number $R(3,3,3,3;2)$ becomes then a very difficult problem due to the fact that its associated graph has no correct cyclic $(3,3,3,3;2)$–colouring. However, in 1972, Chung [5] constructed a correct $(3,3,3,3;2)$–colouring (not cyclic) for the edges of K_{50}. So, we have $R(3,3,3,3;2) > 50$. We mention that this bound, 50, has not been improved yet. Exoo [9] has used a Schur $(H_1, H_2, H_3, H_4, H_5)$–partition to prove that $R(3,3,3,3,3;2) > 162$.

6.3 Cyclic Colouring of Hypergraphs

We introduce in this section, the cyclic colouring of the hypergraphs associated with the classical ternary Ramsey numbers.

Definition 17. Let H be a complete hypergraph of rank 3 and order n, $H = (X, \mathcal{P}_3(X))$ and let E be an edge of H defined as a set of 3 vertices x_i, x_j, x_k of X. We define the *length* of E, $L(E)$, by:

$$L(E) = l(\{x_i, x_j\}) + l(\{x_i, x_k\}) + l(\{x_j, x_k\}) \ .$$

The lengths of the edges of the complete hypergraph K_5^3 are:

$L(\{1,2,3\}) = 4$, $L(\{1,2,5\}) = 4$, $L(\{3,4,5\}) = 4$, $L(\{2,3,4\}) = 4$, $L(\{1,4,5\}) = 4$,
$L(\{1,3,5\}) = 5$, $L(\{1,2,4\}) = 5$, $L(\{2,4,5\}) = 5$, $L(\{2,3,5\}) = 5$, $L(\{1,3,4\}) = 5$.

Definition 18. Let H be a $(k_1, k_2, \ldots, k_m; 3)$–colourable hypergraph of rank 3 and order n. We say that H is *Ramsey–regular* if all the edges with the same length can be coloured with the same colour. The correct colouring that associates only one colour with each edge of a Ramsey–regular hypergraph, is called *a correct cyclic* $(k_1, k_2, \ldots, k_m; 3)$*-colouring.*

Proposition 19. *A complete hypergraph of order* 5 *and rank* 3, K_5^3, *has a correct cyclic* $(4, 4; 3)$*-colouring for its edges.*

Proof of proposition. A complete hypergraph K_5^3 contains five complete sub–hypergraphs of rank 3 and order 4. These sub–hypergraphs are exactly:

$H_1 = (X_1, \mathcal{P}_3(X_1))$ with $X_1 = \{1,2,3,4\}$, $H_2 = (X_2, \mathcal{P}_3(X_2))$ with $X_2 = \{1,2,3,5\}$,
$H_3 = (X_3, \mathcal{P}_3(X_3))$ with $X_3 = \{1,2,4,5\}$, $H_4 = (X_4, \mathcal{P}_3(X_4))$ with $X_4 = \{1,3,4,5\}$,
$H_5 = (X_5, \mathcal{P}_3(X_5))$ with $X_5 = \{2,3,4,5\}$.

Each H_i contains two edges of different lengths. Indeed, H_1 contains the two edges $e_i = \{1,2,3\}$ and $e_j = \{1,2,4\}$ with $L\{e_i\} = 4$ and $L\{e_j\} = 5$; and in the same way, H_2 contains $e_i = \{1,2,3\}$ and $e_j = \{1,3,5\}$, H_3 contains $e_i = \{1,2,5\}$ and $e_j = \{1,2,4\}$, H_4 contains $e_i = \{1,2,3\}$ and $e_j = \{1,2,4\}$, H_5 contains $e_i = \{1,2,3\}$ and $e_j = \{1,2,4\}$ with $L\{e_i\} = 4$ and $L\{e_j\} = 5$.

To associate a correct cyclic $(4,4;3)$–colouring with the edges of K_5^3, it suffices to assign to the edges of length 4 the Red colour, and to those of length 5 the Blue colour. □

The following is a list of Ramsey–regular hypergraphs $(4,6;3)$–colourable and $(5,6;3)$–colourable.

Table 1. Correct cyclic colourings of the edges of certain Ramsey hypergraphs.

$(4,6;3)$–colourable	K_{17}^3	K_{19}^3
colour 1	4, 8, 17	4, 10, 19
colour 2	6, 10, 12, 14, 16	6, 8, 12, 14, 16, 18

$(5,6;3)$–colourable	K_{23}^3	K_{25}^3	K_{28}^3
colour 1	6, 8, 14, 20, 22	6, 8, 14, 20, 22, 24	8, 14, 20, 22, 24, 26
colour 2	4, 10, 12, 16, 18, 23	4, 10, 12, 16, 18, 25	4, 6, 10, 12, 16, 18, 28

The above tables show that $R(4,6;3) > 19$ and $R(5,6;3) > 28$. We can define, in the same way, the length of the edges of all complete hypergraphs of any order h. We conjecture that:

Conjecture 20. *Given $m+1$ positive integers, k_1,k_2,..,k_m,h with $k_i > h$ for all $i \in \{1,2,\ldots,m\}$, there exists a positive integer n, such that all complete hypergraphs of rank h and order n have a correct cyclic $(k_1,k_2,\ldots,k_m;h)$-colouring for their edges.*

Property of non Ramsey–Regular Hypergraphs. The non Ramsey–regular hypergraphs have the same characteristics as the non Ramsey–regular graphs. That is to say, if there exist a $(k_1,k_2,\ldots,k_m;3)$–colourable hypergraph K_n^3, then there exists a $(k_1,k_2,\ldots,k_m;3)$–colourable hypergraph K_{n-1}^3. However, this is not true for the Ramsey–regular hypergraphs. Indeed, the hypergraph K_5^3 has a correct cyclic $(4,4;3)$–colouring, while K_6^3 has no cyclic colouring. In fact, K_6^3 contains a complete sub–hypergraph H' of rank 3, constituted by vertices 2, 3, 5 and 6. We can easily verify that all the edges of H' have the same length 6 and, by definition, they will be coloured with the same colour. This leads to the following theorem:

Theorem 21. *Given m colours and an integer i greater than 3, the complete hypergraph of rank 3 and order $n = 4i$, K_n^3, has no correct cyclic $(\underbrace{4,\ldots,4}_{m};3)$-colouring for its edges.*

Proof of theorem. Let H be a complete hypergraph of rank 3 and order n, and let i be a positive integer greater than 3, with $n = 4i$. The hypergraph H contains always a complete sub–hypergraph of rank 3 and order 4, $H' = (X, \mathcal{P}_3(X))$ with $X = \{x_1,x_2,x_3,x_4\}$ and $l(e_1 = \{x_1,x_2\}) = l(e_2 = \{x_2,x_3\}) = l(e_3 = \{x_3,x_4\}) = l(e_4 = \{x_1,x_4\}) = i$. Indeed, we can take for example, $x_1 = 1$, $x_2 = 1+i$, $x_3 = 1+2i$ and $x_4 = 1+3i$. This gives: $l(e_1) = l(e_2) = l(e_3) = i$. The length l of e_4 cannot be equal to $|1-1-3i|$ because $3i$ is strictly greater than $\lfloor \frac{n}{2} \rfloor$. Consequently, $l(e_4) = 1+4i-1-3i = i$. We can verify that the lengths of $e_5 = \{x_1,x_3\}$ and $e_6 = \{x_2,x_4\}$ are both equal to $2i$. All the edges of H' have the same length $(4i)$ and by definition, they will be coloured with the same colour. So, H' constitutes a monochrome 4–clique of rank 3. Consequently, there is no correct cyclic $(\underbrace{4,4,\ldots,4}_{m};3)$–colouring for the edges of the hypergraph H. □

According to this theorem, the hypergraphs K_4^3, K_8^3, K_{12}^3,K_{16}^3, K_{20}^3, K_{24}^3, K_{28}^3, K_{32}^3, K_{36}^3 and K_{40}^3 etc. have no correct cyclic $(4,\ldots,4;3)$–colouring for their edges.

Finally, we conjecture that:

Conjecture 22. *Given m colours and two positive integers i and k with $i > k-1$; then all complete hypergraphs of order $k \times i$ and rank $k-1$ have no correct cyclic $(\underbrace{k,k,\ldots,k}_{m};k-1)$-colouring for their edges.*

Our previous two theorems (i.e., Theorem 13 and Theorem 21), assert that this conjecture is true for $h = 2$ and 3. We can also verify it for $h = 4$ by following exactly the same reasoning. We can, for example, take $x_1 = 1$, $x_2 = 1 + i$, $x_3 = 1 + 2i$, $x_4 = 1 + 3i$ and $x_5 = 1 + 4i$, as being vertices of the complete sub–hypergraph of order 5 and rank 4, H', and we prove that all the edges of H' have the same length, they will therefore be coloured with the same colour.

Results on Classical and Multiple Ramsey Numbers

To evaluate or to approximate the numbers of Ramsey, we apply our stochastic optimization algorithms (i.e., the rapid descent algorithm, the rapid descent and enumeration algorithm and the simulated annealing algorithm) and we use the cyclic colouring heuristics. We compare our results with those given in [1], [10], [18] and [36, 37]. The following tables illustrate our results. The symbol '*' is used to indicate new bounds on Ramsey numbers.

Table 2. Some new lower bounds for Ramsey numbers.

$R(k_1, k_2, k_3; 2)$	R(3,3,4;2)	R(3,3,5;2)	R(3,3,6;2)	R(3,3,7;2)	R(3,3,8;2)	R(3,3,9;2)
Theoretical bounds of Abott	≥ 17	≥ 27	≥ 35	≥ 45	≥ 55	≥ 71
Stochastic algorithms	≥ 30	≥ 45	$\geq 54^*$	$\geq 65^*$	$\geq 75^*$	$\geq 85^*$

Table 3. Two new upper bounds for $M_{(n)}(k_1, k_2; 2)$.

$M_{(n)}(k_1, k_2; 2)$	$M_{(16)}(3,5;2)$	$M_{(17)}(3,5;2)$	$M_{(18)}(3,6;2)$	$M_{(19)}(3,6;2)$	$M_{(18)}(4,4;2)$
Lower bounds of Exoo	≤ 13	≤ 18	≤ 2	≤ 5	≤ 9
Stochastic algorithms	$\leq 12^*$	$\leq 16^*$	≤ 2	≤ 5	≤ 9

Table 4. Some new upper bounds for $M_{(n)}(k_1, k_2; 2)$.

$M_{(28)}(3,8;2) \leq 2$	$M_{(17)}(3,3,3;2) \leq 13^*$
$M_{(29)}(3,8;2) \leq 9^*$	$M_{(18)}(3,3,3;2) \leq 19^*$
$M_{(30)}(3,8;2) \leq 20^*$	$M_{(19)}(3,3,3;2) \leq 24^*$
$M_{(31)}(3,8;2) \leq 29^*$	$M_{(20)}(3,3,3;2) \leq 32^*$
$M_{(14)}(4,4;3) \leq 14^*$	$M_{(15)}(4,4;3) \leq 27^*$

Table 5. New upper bounds for multiple Ramsey numbers.

$r(3,3,3;2) \leq 13^*$	$r(4,4;3) \leq 4^*$	$r(3,8;2) \leq 2^*$

Conclusion

These algorithms not only make it possible to find almost all known Ramsey numbers but also to improve considerably some bounds for those which remain unknown. Generally, they are capable to associate better bounds with Ramsey numbers. However, to construct a correct colourings, it is often necessary to produce a large number of initial colourings. This procedure requires a non negligible calculation time (i.e., about 15 CPU hours on a SUN SPARC machine for each Ramsey number, in averge), therefore, we are thinking about running them on a parallel machine (i.e., a Connection Machine: CM2 or CM5) hoping that this will improve their performance and make it possible to obtain new bounds on binary Ramsey numbers of two colours $R(k_1, k_2; 2)$ that are not yet evaluated.

We have also shown the existence of the cyclic colouring of hypergraphs associated with ternary Ramsey numbers. Thus, we can associate with each Ramsey number a complete hypergraph and construct a correct colouring by using the properties of the cyclic colouring that we have introduced. These new heuristics are very useful for an improvement of the bounds on ternary Ramsey numbers.

The advantage of our methods, despite their incompleteness, resides in their capacity to solve some difficult combinatorial problems of Ramsey theory,.which cannot be solved by classical resolution methods.

We mention that, in the case of the associated graphs possessing many correct colourings, the algorithm of rapid descent (i.e., Algorithm 1) seems to be the best adapted one. The other two algorithms (i.e., Algorithm 2, Algorithm 3), and especially the last one (i.e., Simulated Annealing algorithm) are used in the opposite case.

At the present, methods of stochastic optimization seem to be those that give best results, especially in problems where the search space is particularly large.

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