# Modelling the Magnetization of a Spintronic Device as a Probability Density Function

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***Abstract*—We wish to determine the switching dynamics of spintronic devices in the presence of randomness generated by thermal noise in the system containing the device. The switching can be appropriately represented by the Write Error Rate (WER) in different spintronic devices and circuits. In this work, we model this WER as a probability density function and capture its variation in the presence of thermal noise by applying equations of statistical mechanics and spintronics.**

***Index Terms*—Fokker-Planck, LLGS Equation, magnetization**

## I. Introduction

In order to determine the magnetization switching, we model the magnetization as a probability density function over a unit sphere.

We refer to statistical mechanics where we come across the **Fokker-Planck Equation** that is used to express the time evolution of probability density function under the influence of drag force and Brownian motion.

In 1-dimension, the Fokker-Planck equation is given as

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\mu(x,t)p(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[D(x,t)p(x,t)\right] \quad (1)$$

Where, $p(x,t)$ is the probability of finding the particle at a given location $x$ at time $t$, $\mu(x,t)$ is the drift term and $D(x,t)$ is the diffusion constant.

Now, we do our magnetization calculation in 3 dimensions and hence we need to modify the Fokker-Planck equation as follows

$$\frac{\partial}{\partial t}\rho(m) = -\nabla.\left[v(m)\rho(m) - D\nabla\rho(m)\right] \quad (2)$$

Where, $\rho(m)$ is probability density function of magnetization, $v(m)$ is velocity of magnetization (discussed later), and $D$ is diffusion constant (gives spreading of probability)

Now, the velocity term in (2) is given according to the **Landau-Lifshitz-Gilbert-Slonczewski Equation**, more commonly known as the LLGS Equation in spintronics.This equation is widely used to study the time evolution of the magnetization subject to spin torque and thermal noise.

The LLGS Equation in 3-D in given as

$$\left(1+\alpha^2\right)\frac{dm}{dt} = -\mu_0\gamma m \times \left(H_{eff} + \alpha m \times H_{eff}\right) - \beta m \times \left[\epsilon\left(m \times m_p\right) - \epsilon' m_p\right] \quad (3)$$

$H_{eff}$ in (3) is the net effective magnetic field given by the sum of anisotropic field $(H_{anis})$, demagnetizing field $(H_{demag})$, externally applied magnetic field $(H_{ext})$ and VCMA controlled field $(H_{VCMA})$

The origin of various terms in (3) are shown as follows

| Term in expression | Origin |
|---|---|
| $m \times H$ | precession |
| $m \times m \times H$ | damping |
| $m \times m \times m_p$ | Spin Orbit Torque & Spin Transfer Torque |
| $m \times m_p$ | Spin Orbit Torque & Spin Transfer Torque |

We intend to simulate the system by solving (2) and (3) together in spherical co-ordinates, since we assume a unit sphere as our system. Since we want to notice the variation of magnetization only along the surface of the sphere, we ignore the radial component and terms from all vector equations and consider only the $\theta$ and $\phi$ directions.

## II. Method

We first simplify (3) to obtain the following equations

$$\left(\frac{1+\alpha^2}{\mu_0\gamma}\right)\frac{dm_\theta}{dt} = H_{eff}.\left(\alpha\hat{\theta}+\hat{\phi}\right) + \left(\frac{\beta_1+\alpha\beta_2}{\mu_0\gamma}\right)m_p.\hat{\theta} + \left(\frac{\beta_2-\alpha\beta_1}{\mu_0\gamma}\right)m_p.\hat{\phi} \quad (4)$$

$$\left(\frac{1+\alpha^2}{\mu_0\gamma}\right)\frac{dm_\phi}{dt} = H_{eff}.\left(\alpha\hat{\phi}-\hat{\theta}\right) + \left(\frac{\beta_1+\alpha\beta_2}{\mu_0\gamma}\right)m_p.\hat{\phi} - \left(\frac{\beta_2-\alpha\beta_1}{\mu_0\gamma}\right)m_p.\hat{\theta} \quad (5)$$

Where, $\frac{dm_\theta}{dt}$ and $\frac{dm_\phi}{dt}$ correspond to the required velocity terms along the $\theta$ and $\phi$ directions, in (2).

In the simulation framework that we have developed, we consider only the terms in (4) and (5) that correspond to precession and damping i.e. we ignore the terms involving $m_p$. The equation hence simulated is the LLG equation.

Now, $H_{eff}$ in (4) and (5) is given as

$$H_{eff} = H_{ani} + H_{demag} + H_{ext} + H_{VCMA}$$

With

$$\begin{aligned}
H_{ani} =& \frac{2K_u}{\mu_0 M_s}\cos(\theta)\,\hat{z} \\
H_{VCMA} =& -\frac{2\xi V_{MTJ}}{\mu_0 M_s t_{ox} t_{FL}}\cos(\theta)\,\hat{z} \\
H_{ext} =& H_x\hat{x} + H_y\hat{y} + H_z\hat{z} \\
H_{demag} =& -N_{xx}\sin(\theta)\cos(\phi)\hat{x} - N_{yy}\sin(\theta)\sin(\phi)\hat{y} \\
& - N_{zz}\cos(\theta)\hat{z}
\end{aligned}$$

Where,

Table I
SYMBOLS AND THEIR MEANINGS

| Symbol | Quantity |
|---|---|
| $\mu_0$ | Vacuum permeability |
| $\gamma$ | Gyromagnetic ratio |
| $\alpha$ | Damping constant |
| $\hbar$ | Reduced Planck's constant |
| $M_s$ | Saturation magnetization |
| $t_{ox}$ | Thickness of oxide layer |
| $t_{FL}$ | Thickness of free layer |
| $K_u$ | Bulk anisotropy |
| $N_{xx}$, $N_{yy}$, $N_{zz}$ | Demagnetizing tensor values |
| $V_{MTJ}$ | Voltage across magnetic tunneling junction |

The parameters are chosen as follows

Table II
PARAMETERS USED IN SIMULATION

| Parameter | Value |
|---|---|
| $\mu_0$ | $4\pi \times 10^{-7} NA^{-2}$ |
| $\gamma$ | $1.76 \times 10^{11} s^{-1}T^{-1}$ |
| $\alpha$ | 0.027 |
| $M_s$ | $6.25 \times 10^5 Am^{-1}$ |
| $t_{ox}$ | $2nm$ |
| $t_{FL}$ | $1nm$ |
| $V_{MTJ}$ | $2.1V$ |
| $\xi$ | $32fJV^{-1}m^{-1}$ |

And we also define

$$\zeta = \frac{2K_u}{\mu_0 M_s} - \frac{2\xi V_{MTJ}}{\mu_0 M_s t_{ox} t_{FL}}$$

### A. Time Scale

A problem that we might often encounter while developing simulation frameworks is error due to computation. Generally, all forms of computation are associated with rounding off in decimal places. So for very large numbers involved in computation, we might results that blow up.

Hence, we modify our time scale to ensure that we end up with reasonable values for simulation. To facilitate ease of computation and avoid error during computation, we convert the time scale according to the following relation

$$\tau = \frac{\gamma\mu_0\alpha\zeta}{1+\alpha^2}t \tag{6}$$

We rewrite (4) after taking the dot product with unit vectors as

$$\begin{aligned}
\frac{1+\alpha^2}{\gamma\mu_0}\frac{\partial m_\theta}{\partial t} &= -\alpha\sin(\theta)\cos(\theta)\big(\zeta + \big(H_{demag,x}\cos^2(\phi) \\
&\quad + H_{demag,y}\sin^2(\phi) - H_{demag,z} \\
&+\sin(\theta)\sin(\phi)\cos(\phi)\left(H_{demag,x} - H_{demag,y}\right) - H_z\sin(\theta)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial m_\theta}{\partial \tau} &= -\sin(\theta)\cos(\theta)\bigg(1 + \frac{1}{\zeta}\big(H_{demag,x}\cos^2(\phi) \\
&\quad + H_{demag,y}\sin^2(\phi) - H_{demag,z} \\
&+ \frac{1}{\alpha\zeta}\sin(\theta)\left(\sin(\phi)\cos(\phi)\left(H_{demag,x} - H_{demag,y}\right) - H_z\right)
\end{aligned} \tag{7}$$

We rewrite (5) after taking dot product with unit vectors as

$$\begin{aligned}
\frac{1+\alpha^2}{\gamma\mu_0}\frac{\partial m_\phi}{\partial t} &= \sin(\theta)\cos(\theta)\big(\zeta + \big(H_{demag,x}\cos^2(\phi) \\
&\quad + H_{demag,y}\sin^2(\phi) - H_{demag,z} \\
&+\alpha\sin(\theta)\sin(\phi)\cos(\phi)\left(H_{demag,x} - H_{demag,y}\right) + H_z\sin(\theta)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial m_\phi}{\partial \tau} &= \sin(\theta)\cos(\theta)\bigg(\frac{1}{\alpha} + \frac{1}{\alpha\zeta}\big(H_{demag,x}\cos^2(\phi) \\
&\quad + H_{demag,y}\sin^2(\phi) - H_{demag,z} \\
&+ \frac{1}{\zeta}\sin(\theta)\sin(\phi)\cos(\phi)\left(H_{demag,x} - H_{demag,y}\right) + \frac{1}{\alpha\zeta}H_z\sin(\theta)
\end{aligned} \tag{8}$$

Hence, the velocity term in Fokker-Planck Equation is given as follows

$$v(\tau) = \frac{\partial m_\theta}{\partial \tau}\hat{\theta} + \frac{\partial m_\phi}{\partial \tau}\hat{\phi}$$

### B. Fokker-Planck Operator

Let the velocity vector $v = V_\theta\hat{\theta} + V_\phi\hat{\phi}$ (considering time scale conversion)

And we know

$$\nabla\rho = \frac{\partial\rho}{\partial\theta}\hat{\theta} + \frac{1}{\sin(\theta)}\frac{\partial\rho}{\partial\phi}\hat{\phi}$$

Therefore we have

$$\begin{aligned}
\frac{\partial\rho}{\partial\tau} &= -\nabla.\left(\rho v - D_1\nabla\rho\right) \\
&= -\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)V_\theta\rho - D_1\sin(\theta)\frac{\partial\rho}{\partial\theta}\right) \\
&- \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\left(V_\phi\rho - \frac{D_1}{\sin(\theta)}\frac{\partial\rho}{\partial\phi}\right) \\
&= M_1\rho + M_2\rho
\end{aligned}$$

Where,

$$\begin{aligned}
M_1 &= -\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)V_\theta - D_1\sin(\theta)\frac{\partial}{\partial\theta}\right) \\
M_2 &= -\frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\left(V_\phi - \frac{D_1}{\sin(\theta)}\frac{\partial}{\partial\phi}\right)
\end{aligned}$$

The Fokker-Planck operator $M$ is given as $(M_1 + M_2)$, and we can hence the equation to be simulated as

$$\frac{\partial \rho}{\partial t} = M\rho \tag{9}$$

### C. Differential Equation Solving

There are many ways to go about solving (9). Quite a few of them were tried out, namely the forward Euler, Runge-Kutta and Cranck-Nicolson method

In the Cranck-Nicolson method, (9) can be written as

$$\frac{\rho^{N+1} - \rho^N}{\Delta\tau} = \frac{1}{2}\left[M^{N+1}\rho^{N+1} + M^N\rho^N\right]$$

Which gives us,

$$\left[I_{n_1 m_1} - \frac{\Delta\tau}{2}M^{N+1}\right]\rho^{N+1} = \left[I_{n_1 m_1} + \frac{\Delta\tau}{2}M^N\right]\rho^N \tag{10}$$

Now since the operator $M$ is independent of time, we have

$$M^{N+1} = M^N$$

Thus, (10) can be written as

$$\begin{aligned} M_{LHS}.\rho^{N+1} &= M_{RHS}.\rho^N \\ \rho^{N+1} &= \left[inv\left(M_{LHS}\right)M_{RHS}\right].\rho^N \\ \rho^{N+1} &= M'\rho^N \end{aligned}$$

We have $M'$ as the operator which links $\rho$ at $(N+1)^{th}$ instant with that at $N^{th}$ instant. At any instance, we have $\rho^N$ and $M'$ known to us, and so we can calculate $\rho^{N+1}$.

We divide the timeframe into smaller steps $\Delta\tau$ to perform the simulation. There can be 2 ways of going about this

- **Uniform time stepping:** $\Delta\tau$ remains constant throughout the required simulation time
- **Adaptive time stepping:** $\Delta\tau$ is varied at each iteration depending upon how convergent the solution is

In the given situation, adaptive time stepping is highly computationally expensive, although more accurate. Hence we resort to uniform time stepping for the simulation.

### D. Simulation conditions

We simulate the Fokker-Planck equation using the Cranck-Nicolson method. Steps followed are as follows:

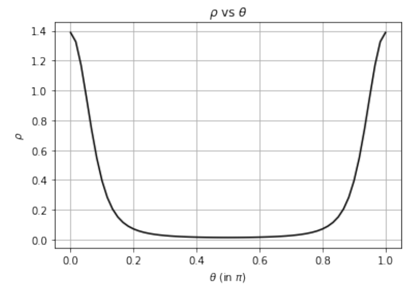
- Create a grid in $\theta$ and $\phi$ directions on a unit sphere
- Write the equations of the matrices $M_1$ and $M_2$ which represent the $\theta$ and $\phi$ operators
- Represent the time-derivative operator $M = M_1 + M_2$
- Convert time scale from $t$ to $\tau$
- Fix the time step in order to calculate the required number of iterations to simulate for the given timeframe
- Use Cranck-Nicolson method for the required number of iterations
- For magnetic field, we consider only anisotropic, VCMA and external field
- External field is considered only along $\hat{z}$ direction

## III. Results

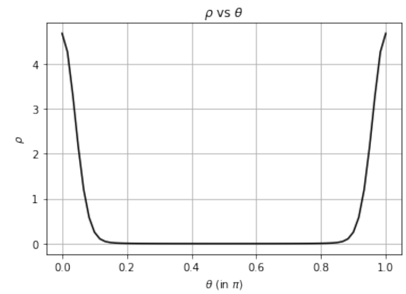
The simulation is first performed considering only anisotropic and VCMA magnetic fields. For VCMA field we take $V_{MTJ} = 2.1V$. After that we consider external magnetic field for simulating.

### A. Without external field

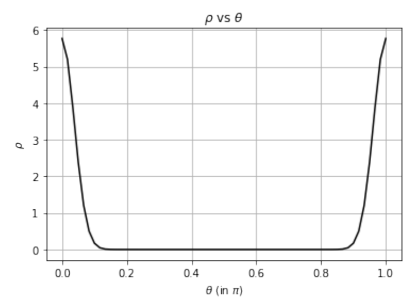
The simulation results (variation of $\rho$ with $\theta$) without external field are shown in Figure 1



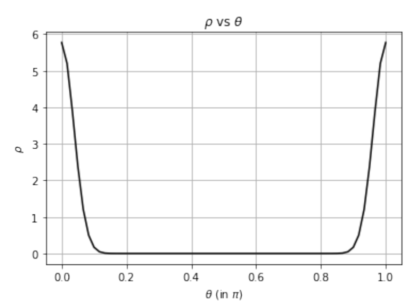

(a) $t = 1ns$




(b) $t = 2ns$




(c) $t = 4ns$



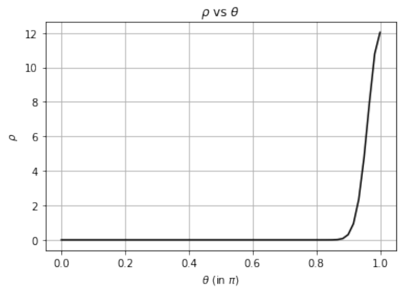

(d) $t = 7ns$

Figure 1. Variation of magnetization while simulating without external field

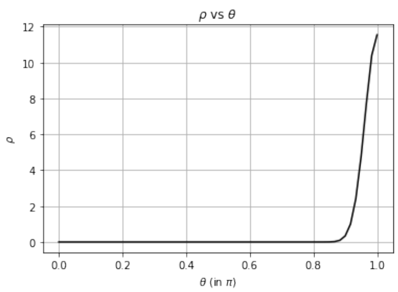
### B. *With external field in $\pm\hat{z}$ direction*

Initially we apply the magnetic field for some time, and then let it undergo relaxation. The magnitude of the external magnetic field is taken as $H_z = -5H_k$ and $H_z = 10H_k$ respectively for 2 instances of simulation.

The simulation results (variation of $\rho$ with $\theta$) without external field are shown

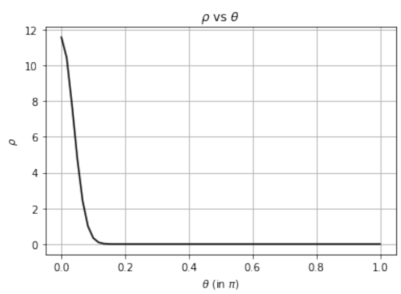



(a) $H_z = -5H_k$ , $t = 1ns$

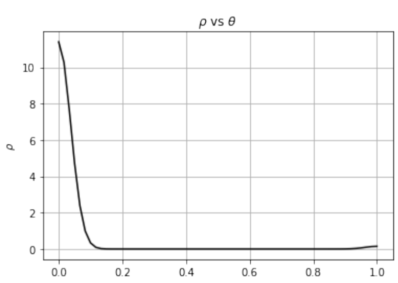



(b) $H_z = -5H_k$ , $t = 5ns$

Figure 2. Variation of magnetization while simulating with $H_{ext} = -5H_k\hat{z}$




(a) $H_z = 10H_k$ , $t = 2ns$




(b) $H_z = 10H_k$ , $t = 10ns$

Figure 3. Variation of magnetization while simulating with $H_{ext} = 10H_k\hat{z}$

The results that are obtained under various simulation conditions can be summarized qualitatively as follows:

Table III
QUALITATIVE SUMMARY OF $\rho(m)$ VS $\theta$ OBTAINED ON SIMULATING UNDER VARIOUS CONDITIONS

| External magnetic field | Observation of probability |
|---|---|
| $H_{ext} = 0$ | Maxima at 2 poles |
| $H_{ext} = -5H_k\hat{z}$ | Maxima at south pole of sphere |
| $H_{ext} = 10H_k\hat{z}$ | Maxima at north pole of sphere |

## IV. CODE

The code that is used to perform the simulations shown in the report, is uploaded at: FokkerPlanck_LLG

## V. DISCUSSION

The results obtained via simulating the code developed, are tallied with those that are obtained on the existing simulator. We see that they 2 sets of results are congruous. This leads to the belief that our code might be correct under the conditions that were tested.

However, we must test the code under more such conditions to claim its correctness.

That said, there are however a few **caveats** in the code that is developed in the process of building the simulation framework.

- At each iteration in the Cranck-Nicolson method to perform simulation for stipulated time, we take modulus of $\rho$ to ensure non-negative value of probability
- We normalize $\rho$ at each step in iteration to preserve integral (surface integral of probability density function over sphere = 1) else it diverges as we perform increasing number of iterations in the Cranck-Nicolson method
- Grid spacing as follows:
  $61$ points in $\theta$ direction, $120$ points in $\phi$ direction
  This is done as computations involving finer meshing could not be run on the local machine due to memory issues.
- On addition of $H_{demag}$ we get results that are incoherent with those observed in the existing simulator

Hence there is definitely further scope of improvement in the existing code to build a framework that can completely solve (2) and (3) on a unit sphere. Some changes that could be brought about are as follows

- Increase accuracy of simulation by increasing the number of divisions in $\theta$ and $\phi$
- Check the addition of $H_{demag}$ in LLG Equation to completely take into accord the effects of damping and precession
- Add the current-induced torque effects (SOT and STT) to simulate the LLGS Equation that can capture effect of current on magnetization
- If possible, can try out adaptive time stepping method to perform iterations

## VI. Conclusion

In all, we have developed the foundation of a simulation framework that can simulate the Fokker-Planck Equation and obtain Write Error Rate in a spintronic device, from scratch. The code developed is used to perform simulations under various conditions, and some of them give coherent results with the existing simulator (shown earlier in the report). Finally, the effect of demagnetizing magnetic field and effects of current-induced torques remain yet to be added to the picture, which would then become complete.

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