Master's Thesis - Econometrics & Operations Research
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Testing for Bubble in Cointegrated Systems

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Research Question

1. Can a system wide bubble be detected by applying SADF test on residuals of co-integrated system.
2. Can this method be empirically tested ?

1 Simulation Study

1.1 Data Generating Process (DGP)

1. Simulation of Cointegrated Time Series

To simulate cointegrated time series, we adopt a *common factor approach.* Each individual time series is generated based on a shared stochastic trend, modeled as a random walk. Specifically, we define the common factor process z_t as:

$$z_t = z_{t-1} + e_t, \quad e_t \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2)$$

where:

- z_t denotes the common stochastic trend at time t,
- e_t is a white noise process with zero mean and variance σ^2,
- $\mathcal{N}(0, \sigma^2)$ denotes the normal distribution with mean zero and variance σ^2,
- i.i.d. stands for "independent and identically distributed."

Each time series $y_{i,t}$ is then constructed as a linear function of the common factor z_t with additive noise:

$$y_{i,t} = \beta_i z_t + \varepsilon_{i,t}, \quad \varepsilon_{i,t} \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2)$$

where:

- $y_{i,t}$ represents the i-th time series at time t,
- β_i is the loading coefficient linking the common factor to series i,
- $\varepsilon_{i,t}$ is an idiosyncratic error term, assumed to be i.i.d. Gaussian.

This setup ensures that all $y_{i,t}$ series are cointegrated through their dependence on the common trend z_t.

2. Inducing a Bubble in One of the Time Series

To introduce a bubble in one of the series (say $y_{i,t}$), we modify the data generating process after a specific breakpoint $\tau_{i,1}T$. The series is thus governed by:

$$y_{i,t} = \begin{cases} \beta_i z_t + \varepsilon_{i,t}, & 1 \leq t \leq \tau_{i,1}T \quad \text{(cointegrated normal phase)} \\ (1+\delta_{i,1}) y_{i,t-1} + \varepsilon_{i,t}, & \tau_{i,1}T + 1 \leq t \leq \tau_{i,2}T \quad \text{(explosive bubble phase)} \end{cases}$$

where:

- $\delta_{i,1} > 0$ controls the degree of explosiveness during the bubble period,
- $\tau_{i,1}$ and $\tau_{i,2}$ are fractional breakpoints of the sample size T.

In the *normal phase*, the series follows the cointegrated structure. In the *bubble phase*, the process becomes locally explosive due to the autoregressive coefficient exceeding unity.

1.2 Methodology

We conduct a simulation-based investigation to examine how explosive behavior in one or more time series affects the stability of cointegration and how such transitions can be identified using recursive unit root testing. Our simulation framework is applied under two scenarios: a bivariate system ($N = 2$) and a multivariate system ($N > 2$).

1. Bivariate Setting ($N = 2$)

We first consider a system comprising two time series. For $t = 0, 1, \ldots, T$, we simulate two time series $y_{1,t}$ and $y_{2,t}$ that are cointegrated via a shared stochastic trend, following the data generating process (DGP) described previously. Once the cointegration relationship is established, it is assumed to persist unless disrupted.

To study the impact of a bubble, we introduce an explosive regime in one of the two series during a predefined subperiod $[\tau_1 T, \tau_2 T]$. This structural deviation is implemented by modifying the dynamics of the affected series to exhibit exponential growth. We then apply the Supremum Augmented Dickey-Fuller (SADF) test to the cointegration residuals. The test recursively evaluates whether the residuals exhibit signs of explosiveness. The point at which the test rejects the null hypothesis of a unit root in favor of an explosive alternative is interpreted as the onset of a bubble phase in the system.

2. Multivariate Setting ($N > 2$)

We extend the framework to systems with more than two time series. Again, we simulate N time series, $y_{i,t}$ for $i = 1, \ldots, N$, such that all are initially cointegrated with respect to a common trend. We then induce explosive behavior in an increasing subset of these series, while keeping the remainder stable.

The objective is to identify the minimum proportion of explosive series required to disrupt the cointegration relationship at the system level. For instance, we may start by introducing a bubble in 2 out of 10 series, then 3 out of 10, and so on, incrementally increasing the number of explosive components. At each step, we evaluate whether the system transitions into an explosive regime using cointegration testing followed by right-tailed unit root tests. This approach enables us to determine a critical threshold proportion of explosive series that causes the breakdown of cointegration.

We repeat this procedure across varying values of N, with $N \in \{2, 3, 5, 10, 20, 50, 100\}$, to investigate the sensitivity of the system to size. We also vary the sample length $T \in \{200, 300, 500, 1000\}$ to evaluate the role of time horizon in bubble detection.

For larger systems (i.e., when both N and T are large), we apply Johansen's cointegration test to determine the number of cointegrating relationships present. If multiple relationships are found, we extract the corresponding cointegration residual vectors and apply the Panel SADF (PSADF) test to assess explosiveness at the panel level.

2 Empirical Study

2.1 Method

This empirical analysis investigates whether systems of time series that initially move together—suggesting cointegration—eventually diverge due to the emergence of explosive dynamics in a subset of the series. We propose a two-step recursive testing procedure to systematically detect such transitions.

Step 1: Recursive Cointegration Testing

Let $\mathbf{y}_t = [y_{1,t}, y_{2,t}, \ldots, y_{N,t}]^\top$ denote a vector of N time series observed over T time points. We begin by testing whether the residuals $u_t = \boldsymbol{\beta}^\top \mathbf{y}_t$ from the cointegrating relationship are stationary, i.e., $u_t \sim I(0)$, over expanding sub-samples $t = 0$ to $t = n$, where $n = 10, 11, \ldots, T$. This step is repeated until the test fails, indicating a potential breakdown in cointegration.

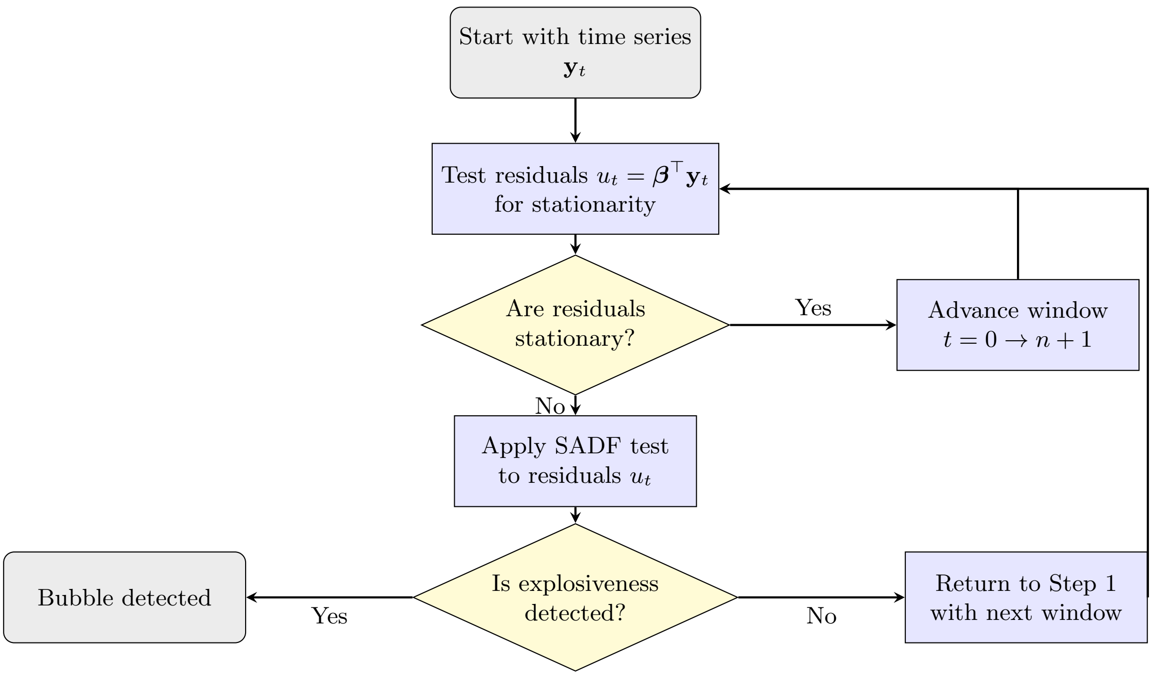
Step 2: Bubble Detection Using the SADF Test

Once the residuals are found to be non-stationary, we apply the Supremum Augmented Dickey-Fuller (SADF) test to the cointegration residuals to detect explosive behavior. If the SADF test does not confirm explosiveness, the procedure returns to Step 1, testing the next window.

The recursive loop continues until either (a) explosive dynamics are statistically detected in the residuals, or (b) residuals revert to a stationary process. This enables us to assess whether the cointegrated system has entered a bubble regime.

The length of each test window is tailored to the frequency of the data. For weekly series, a window of 8–10 weeks is recommended (covering approximately two months), while for monthly data, a window of 2–3 periods is generally sufficient for reliable detection.

2.1 Method Flow Diagram



2.1 Examples

1. Agricultural Grains (2022 – Ukraine War) During the onset of the Russia-Ukraine conflict in 2022, global grain markets experienced divergent behavior. While wheat, corn, and edible oil prices surged (often doubling), rice prices remained relatively flat. This decoupling within a previously cohesive commodity group signals a structural divergence in market dynamics. The event illustrates a situation where the cointegration among major cereals breaks due to exogenous geopolitical shocks, with a subset of the system entering a bubble-like phase.

2. Eurozone Sovereign Bonds (2010–2012 – Debt Crisis) In the European sovereign debt crisis, yields on bonds issued by peripheral countries such as Spain, Italy, Greece, Portugal, and Ireland rose sharply, reflecting market fears of default. In contrast, core economies including Germany, the Netherlands, and Austria saw yields decline due to safe-haven demand. This asymmetric yield behavior led to a breakdown in the cointegration previously observed across Eurozone bond markets, with part of the system transitioning into an explosive regime while others remained anchored.

3. Tech/Growth Stocks (2023 – AI Boom) Amid the 2023 surge in generative AI developments, major tech companies—particularly NVIDIA, Meta, Microsoft, Google, and Apple—experienced rapid price acceleration, marking a clear departure from the broader tech sector. Smaller or non-AI-exposed firms such as PayPal and Zoom did not exhibit similar gains. The divergence within a highly interconnected sector reflects a scenario where a dominant subgroup enters a bubble regime, while the remainder of the system does not, resulting in a breakdown of sector-wide cointegration.