Computer Assignment 1

Stochastic Processes: The Fundamentals 2024-2025

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Deadline for submission: Monday September 23, 2024 at 23:59 CET.

Instructions:

- The two computer assignments of this course should be done in groups of at least and at most two students. You should sign up for a group through the 'People' Module in Canvas.
- For each assignment, you have to write a short report in which you include at least the results, an interpretation of your results, and a conclusion. Make sure that it is clear what you did and what your interpretation of the results is. There is no maximum to the number of words or pages but please try to be concise.
- Report should be written in English. This means that you also should use English decimal notation in the text and in graphs. Use your spelling checker!
- Figures and graphs should have captions such that they can be read independently from the text in your report.
- Make sure to add your code to the Appendix. We recommend the LaTeX package 'listings'. Without the code we cannot grade your assignment.
- The preferred implementation tools are Python and MATLAB. However, you are not bound to Python or MATLAB and are free to use the programming environment that you like, but we cannot guarantee support for all languages.
- The use of AI-tools is not restricted, but remember to critically evaluate any outcomes.
- Put names and VU student numbers of all group members on the front page.
- Final reports should be submitted electronically on Canvas before 23:59 CET on the deadline day. Assignments submitted after the deadline result in a grade equal to 1.

ENJOY AND GOOD LUCK!!

1 Returns

Download historical data of S&P-500 index levels with a *monthly* frequency (for instance from `fred.stlouisfed.org/series/SP500`). The end date of your sample is August 31, 2024. The number of month ends to cover in your sample depends on your months of birth:
$10\,(\mathrm{m}_1 + \mathrm{m}_2) - \mathrm{m}_1\mathrm{m}_2$, with a minimum of 61 and a maximum of 120.

There are several ways to calculate returns from the level data. The **simple net return** between time t and $t+1$ is defined as:

$$R_{t:t+1} = \frac{S_{t+1}}{S_t} - 1,$$

where S_t stands for the stock price or index level at time t.

Sometimes researchers model the simple net return as:

$$R_{t:t+1} = \mu + \sigma \varepsilon_{t:t+1}, \tag{1}$$

where

$$\varepsilon_{t:t+1} \sim N(0,1).$$

(a) Use the downloaded monthly historical data of the S&P-500 index to provide estimates for μ and σ. Use a 5% significance level to test whether μ differs significantly from zero.

(b) Assume now that the true parameters of model (1) are $\mu = 0.4\%$ and $\sigma = 6\%$. Calculate how many years of return data you would need to find a 95% confidence interval for μ that equals [0.35%, 0.45%].

(c) Verify, by means of a simulation exercise, that your calculation is correct. In your report, describe how you conducted your simulation exercise and provide the relevant numbers from your simulation.

An alternative way to model the S&P-500 index is as follows:

$$S_{t+1} = S_t e^{\left(\tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2\right) + \tilde{\sigma}\varepsilon_{t:t+1}}. \tag{2}$$

(d) Use model (2) and the downloaded monthly historical data of the S&P-500 index to provide estimates for $\tilde{\mu}$ and $\tilde{\sigma}$.

(e) Assume that the true model parameters of model (2) are equal to the parameters estimated in (d). Use the value of the S&P-500 index observed at August 31, 2024 and the model parameters to *calculate* the expected value of the S&P-500 index after 5 years (i.e. after 60 months). Note that this exercise does not require the use of a computer and that you need to derive the expectation analytically (on paper) before calculating the expected value.

(f) Use the estimated parameters of model (2) to simulate 10,000 paths of the S&P-500 index level at a horizon of 5 years (i.e. 60 months). Provide a plot of the distribution of the index levels after 5 years. Did you expect this particular shape of the distribution? Test the distribution for normality and report the result and conclusion of this test.

(g) Consider a digital put option that pays 1 if the S&P-500 index after 5 years is below 5,600 and 0 in all other situations. Use the risk-neutral valuation method to find the no-arbitrage price of a 5-years digital put option in the world of model (2). You may assume that the risk-free interest rate is constant at a level of 0%. Hint: use 10,000 simulations of the S&P-500 index level in a risk-neutral world to approximate the risk-neutral expectation of the digital put payoff. You may assume that the volatility parameter $\tilde{\sigma}$ in the risk-neutral world is the same as in the real world.

(h) In the same spirit as the previous question, consider a digital call option that pays 1 if the S&P-500 index after 5 years is above 5,600 and 0 in all other situations. Use the risk-neutral valuation method to find the no-arbitrage price of a 5-years digital call option in the world of model (2). You again may assume that the risk-free interest rate is constant at a level of 0%. Sum the no-arbitrage prices of the digital put option (as calculated in (g)) and the digital call option and explain why the result makes sense.

The assignment continues on the next page.

2 Binomial Trees

A commonly used approach to compute the price of an option is the so-called binomial tree method. In this approach, option prices are computed through a well-known backward induction scheme, which was explained in one of the first lectures and can be found in all option pricing literature.

Consider a European call option on a non-dividend-paying stock with a maturity of 3 months. Suppose that the underlying of this option is the S&P-500 index. The strike price of the call option equals 5,600. Let the 3-months per annum interest rate be equal to 4% (with quarterly compounding). Remark: 4% per annum with quarterly compounding means that the interest over 3 months is 1%.

The first step in applying the binomial tree approach for option pricing is to construct the nodes of the tree. In this exercise we will use a step size of one month. Since we want to price an option with a maturity of 3 months, we need to build a 3-step binomial tree for the S&P-500 index. Starting value of the tree is the S&P-500 index level of August 31, 2024.

The tree will have two nodes after the first time step. We want to choose these nodes in such a way that the variance of the 1-month simple net return in the tree equals the sample variance of simple net returns on the S&P-500 index (using the time series of the previous exercise). The expected monthly simple net return in the tree should be equal to 0.60%. You may assume that the real-world probability p of an upward movement equals 56%.

Construct your tree without the use of a pre-programmed package. You can of course confirm your results with the results from a package.

(a) Using the information above, construct the nodes after the first time step such that the requirements are satisfied. Apply the up and down factors of the first step to the second and third step such that the complete tree with nodes emerges. Provide the resulting tree. Finally, demonstrate that the nodes after the first time step satisfy the given requirements.

(b) Use the tree that you constructed in (a) and the interest rate assumption to calculate the risk-neutral probability q of an upward movement in the tree (see lecture slides for details). Use this risk-neutral probability to calculate the price of a European call option with maturity 3 months and strike price 5,600.

(c) Calculate the price of the European call option with the same characteristics as in (b) by applying the famous Black-Scholes option pricing formula. If you compare the value with (b) you will observe a difference. What could be a reason for this difference?

Now, we are going to increase the number of steps in the tree. Since we still want to price a 3-month option, this implies that the time between two consecutive steps decreases. However, the variance of the stock price after one month should remain the same (roughly). One way to accomplish this is to adjust the scale parameters u and d.

(d) Study the convergence of the binomial tree method for the call option in the previous question (strike 5,600 and maturity 3 months) by increasing the number of steps in the tree (3, 10, 100, 1000, and 10,000 steps). Illustrate the convergence by means of a graph that shows the option value versus the number of steps. Make sure that you also include the Black-Scholes option value (calculated in (c)) in the graph.

(e) Using the binomial tree with 10,000 steps, plot the probability distribution of the outcomes (i.e. the nodes in the tree) at maturity, under both the real-world probability measure as under the risk-neutral probability measure. Use this plot to argue qualitatively what the sign of the expected excess return (i.e. in excess of the risk-free return) of a 3-month European call option should be in this binomial tree model.

(f) Change the code such that it can compute the price of a European put option with strike price 5,600 and maturity 3 months. Do the same as in (d): provide a graph that shows the option value as a function of the number of steps and also include the Black-Scholes value of the put option. Verify whether the put-call parity holds.

(g) Now suppose that the option is American. Change the code such that it can handle early exercise opportunities. What are the values of the American put and call (strike 5,600 and maturity 3 months) for the same initial parameters? Comment on the difference in value between the European put option and the American put option.