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Bursting the bitcoin bubble: Do market prices reflect fundamental bitcoin value?

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ABSTRACT

This paper develops a theoretical model of the bitcoin market and demonstrates that the bitcoin's volatile and explosive price path is a consequence of the Bitcoin protocol's system of supply management. The model implies that the marginal cost of mining the target supply of bitcoins is the fundamental value of the bitcoin since it corresponds to an equilibrium in the Bitcoin protocol and the rent-seeking tournament among miners. The data provide strong empirical evidence of cointegration between the bitcoin's price and the marginal cost of mining the target supply of bitcoins, demonstrating the existence of their long-run equilibrium relationship. Current bubble detection techniques indicate that there is no evidence of explosive departures in the price of the bitcoin from its model-implied fundamental value. Since the raw price data exhibit explosive behavior, the apparent bubbles in the price of the bitcoin can be attributed to its nonstationary market fundamentals.

1. Introduction

In this paper bitcoins are viewed as tradable commodities whose supply is managed by the Bitcoin protocol.[1] It makes no sense to value bitcoins as if they were a stock because the Bitcoin network, as an institution, is not owned by anyone. And while it is a digital currency, it cannot assume a value as do fiat currencies because no government decrees and maintains its value. Bitcoin is most analogous to a commodity such as coffee, which is produced by 'small' farmers who are uncoordinated in their production decisions. While miners use electricity to produce bitcoins (instead of sunshine), the analogy is not far-fetched, as evidenced by the fact that miners demonstrate a strong preference for joining mining pools,[2] which are similar in structure to coffee cooperatives since they are designed to share the risk among their members.[3] Coffee farmers once benefited from the now defunct International Coffee Agreement (ICA), which was a system of quotas that resulted in high and stable prices by organizing the supply of farmers worldwide (Talbot, 2004). Encoded in the Bitcoin protocol is a similar system of supply management that the Bitcoin network sustains by its near-perfect monitoring of the rate of block formation (and thus the quantity of bitcoins supplied) and enforces by regular adjustments in the level of difficulty of mining a block. Bitcoin, however, has the additional feature of being a medium of exchange and a tradable asset with numerous well-developed market exchanges, resulting in a unique class of asset with characteristics that have never before been seen.

To ascertain the value of the bitcoin, this paper develops a theoretical model of the bitcoin market that incorporates the functioning of the Bitcoin protocol and the production of bitcoins by miners. The fundamental value of the bitcoin is defined as the price that implements an equilibrium in the protocol and the rent-seeking tournament among miners. Since the bitcoin is not an income generating asset, it should be valued according to equilibrium market conditions. It is shown that the model-implied fundamental value of the bitcoin is the marginal cost of mining the target supply of bitcoins. Following a permanent demand shock, successive adjustments in the difficulty will cause the market price to approach the fundamental value consistent with the limiting equilibrium. Nearly 5 years of data are used to test whether there is empirical evidence of a long run equilibrium relationship between the price of the bitcoin and the marginal cost of mining the target supply of bitcoins, which would support it as the fundamental value of the

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[1] I follow the convention of capitalizing the word 'bitcoin' when referring to the protocol or network and writing it in lowercase when referring to the unit of currency.

[2] According to hashrate distribution statistics provided by BTC.com, over 95% of the Bitcoin network hashrate can currently be attributed to mining pools. See https://btc.com/stats/pool.

[3] Mining pools enable miners to decrease the variance of their returns by sharing their processing power over a network and splitting the reward according to the amount of work that each has contributed to the probability of finding a block.

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bitcoin. Current bubble detection techniques are then applied to test whether there are explosive departures in the price of the bitcoin from its model-implied fundamental value, which would confirm true bubbles in the price of the bitcoin, or if the apparent bubbles in the price of the bitcoin arise from the market fundamentals themselves. This paper demonstrates that the Bitcoin protocol's difficulty adjustment mechanism gives rise to explosive market fundamentals and hence we should not presume that explosive periods in bitcoin price data are cryptocurrency bubbles.

Asset price bubbles are typically associated with dramatic price increases followed by a collapse. It is important to identify bubbles because asset prices affect the real allocation of an economy and bubbles can raise concerns that spur government intervention (Gronwald, 2021). A common definition is that bubble conditions arise when the price of an asset significantly exceeds its intrinsic or fundamental value. It is challenging to identify bubbles in market data, however, because one needs to know an asset's fundamental value in order to identify a divergence between it and the asset's price. Moreover, econometric tests of asset price bubbles do not do a good job of differentiating between misspecified fundamentals and bubbles (Gurkaynak, 2005). Models in which investors have rational expectations and identical information yield the testable implication that bubbles have to follow an explosive price path (Brunnermeier, 2008). It is this deviation from martingale behavior that permits the identification of the boom phase of a bubble and the subsequent crash, since the efficient market martingale property implies unit root time series dynamic behavior (Phillips & Yu, 2011).

Bitcoin was invented by an unknown individual and launched on January 3, 2009. Cryptocurrencies such as Bitcoin are electronic payment systems that permit transactions to be made with pseudo-anonymity[4] and without middlemen like banks. Bitcoin's price path is notoriously volatile and has evinced a multitude of boom-and-bust cycles over its nearly 15-year lifespan. Until July 2010, the price of a bitcoin was less than 0.01 USD. By the beginning of 2017, the price had risen astoundingly to a stable 1000 USD and by the end of 2017, it had skyrocketed to 19,783 USD. The price has demonstrated numerous peaks and troughs since that time. For instance, it reached just under $29,000 by the end of 2020, a 416% increase from the start of the year, and an all-time high of over $68,000 in November 2021. After June 2022, the price plummeted below $23,000 for the first time since December 2020 and it has rebounded somewhat since that time. While there are numerous historical examples of bubbles, starting as far back as the Dutch tulipmania (1634–7),[5] one is hard-pressed to identify an asset price or an episode of market exuberance that exemplifies the ceaseless deflations and re-inflations that are apparent in the price path of Bitcoin and similar cryptocurrencies.[6]

Mining is the process by which bitcoins are created. Bitcoin miners use electricity to solve complex mathematical puzzles in order to verify the transactions added to the blockchain. Solving this 'proof of work' (PoW) problem requires tremendous computational power and the first miner to succeed (find a correct hash) is rewarded with new bitcoins.[7] The Bitcoin protocol specifies a target that a correct hash must fall below, which implies a level of difficulty for the computational problem. Changes in the target and hence the level of difficulty affect the rate of block formation. An increase (decrease) in difficulty decreases (increases) the probability that a miner will find a correct hash and miners must use more (less) hashpower and thus electricity for the same expected reward. Since the production of a block increases the supply of bitcoins according to the block reward, changes in the level of difficulty also determine the growth rate of the supply of bitcoins over time.

Although the Bitcoin network is managed by peer-to-peer technology without a central authority, it uses the level of difficulty as an instrument to enforce an inflexible system of supply management. The protocol regulates the quantity of bitcoins that are mined by adjusting the level of difficulty every 2016 blocks (approximately every two weeks). There is an interval of time between adjustments in the level of difficulty so that the network can accurately estimate the waiting time to find a block. If the network detects that the time required to find the last 2016 blocks differs from 20,160 min, then the network uses the estimated mining rate to adjust the level of difficulty proportionally in order to target a ten-minute interval between successive blocks mined.[8] This results in a target supply of bitcoins per day equal to the block reward multiplied by 144 blocks.

In Section 2 of the paper, the bitcoin mining industry is modeled with the free entry of miners in response to profits that are created in accordance with the Bitcoin protocol. Since there are no barriers to entry and there is very little heterogeneity among miners, they operate in a market that can be closely approximated by perfect competition (Podhorsky, 2023; Prat & Walter, 2021). For simplicity, there is no secondary market for the bitcoin and all input markets are held constant. Proposition 1 of the paper demonstrates that the marginal cost of mining the target supply of bitcoins naturally defines the fundamental value of the bitcoin since it corresponds to a steady state in the system. When the price of the bitcoin is equal to it, the Bitcoin protocol is in equilibrium since it has no incentive to change the level of difficulty, and there is equilibrium in the rent-seeking tournament among bitcoin miners since their expenditures are equal to their expected rewards.

The model demonstrates how adjustments in the difficulty in response to demand shocks result in exaggerated price movements that are consistent with the volatility and explosive behavior evident in the bitcoin's price path. Since adjustments in the difficulty enforce a vertical supply curve at the target supply of bitcoins, variations in demand are expressed in the price of the bitcoin, resulting in supernormal price volatility. Also, whenever a positive demand shock results in excess demand for the bitcoin, the protocol will increase the difficulty since the mining rate will exceed the target. Because this decreases the supply of bitcoins in the presence of excess demand, the difficulty adjustment will cause the price to jump, where the magnitude depends on the extent of excess demand. While the sudden increase in the price resembles a bubble, it is not a bubble in the theoretical sense since the greater difficulty is incorporated into a higher marginal cost of mining.

Bitcoin is an advantageous choice of asset for studying price behavior since the protocol's rules are clearly stated and the entire population of data pertaining to the functioning of the protocol is available from its blockchain ledger. The Bitcoin blockchain is parsed to obtain daily time series data for the level of difficulty, the number of blocks mined per day and the miners' block rewards and fees. Daily data for the USD price of the bitcoin are obtained from coindesk.com, the USD price of Antminer mining rigs (models S1, S2, S3, S4, S5, S7, S9 and S11) sold on Amazon Marketplace by third party sellers are obtained from Keepa.com and the respective equipment specifications (hash rate and energy efficiency) are obtained directly from Amazon.com. Since 17 March 2014 was the first day that price information on the Antminer S1 rig was tracked by Keepa.com, the data cover the period from 17 March 2014 to 13 January 2019.

[4] Bitcoin addresses are not tied to the identity of their users but since all transactions over the Bitcoin network are completely transparent and traceable, multiple Bitcoin addresses can be clustered together and then associated with a particular user. See Meiklejohn et al. (2013).

[5] See Brunnermeier (2008) and Garber (1989, 1990) and for a discussion of the history of price bubbles. Also see Kyriazis et al. (2020) for a survey of the academic literature concerning the formation of pricing bubbles in digital currency markets.

[6] It is theoretically possible, however, for rational bubbles to periodically collapse to a small nonzero value and then to continue to increase. See Evans (1991).

[7] The current block reward is 6.25 bitcoins.

[8] See Antonopoulos (2017).

Based on the theoretical findings of Section 2, it is hypothesized that the marginal cost of mining the target supply of bitcoins is the fundamental value of the bitcoin and that there are no explosive departures in the price of the bitcoin from it. Section 4 outlines how the price of the bitcoin and the marginal cost of mining the target supply of bitcoins are tested for their cointegration by using the Engle–Granger test (Engle & Granger, 1987). It also outlines how the supremum augmented Dickey–Fuller (SADF) test (Phillips et al., 2011) and the generalized supremum augmented Dickey–Fuller (GSADF) test (Phillips et al., 2015a, 2015b) are applied to test whether the boom-and-bust cycles evident in bitcoin price data are explosive departures from the model-implied fundamental value of the bitcoin. Since asset prices are typically well approximated by a random walk in the absence of bubbles but are characterized by an explosive path during periods of bubbles, these techniques identify bubbles by testing for a mildly explosive departure from a random walk (Homm & Breitung, 2012; Phillips et al., 2015a, 2015b, 2011; Phillips & Yu, 2011).

Section 5 presents the results. It is shown that the price of the bitcoin is cointegrated with the marginal cost of mining the target supply of bitcoins, supporting the marginal cost of mining the target supply of bitcoins as the fundamental value of the bitcoin since the variables share a common long run stochastic trend. It is also shown that, while the raw bitcoin price data demonstrate evidence of explosiveness, the residuals from the regression of the price of the bitcoin on the marginal cost of mining the target supply of bitcoins do not. Since there are no explosive departures in the price of the bitcoin from its model-implied fundamental value, the apparent bubbles in the price of the bitcoin can be attributed to the nonstationarity of its market fundamentals.

There is a rapidly growing economics literature on the topic of Bitcoin and cryptocurrencies. Several papers investigate the question of whether the bitcoin acts as an alternative currency or has properties that resemble those of commodities (or speculative assets). Dyhrberg (2016) uses GARCH models to examine whether bitcoin behaves like a well-known financial asset or as something in between a commodity and a currency by analyzing several aspects of its price volatility. The author demonstrates that most aspects of the bitcoin are similar to a commodity like gold as it reacts to similar variables, possesses similar hedging capabilities, and reacts symmetrically to good and bad news. Baur et al. (2018) find that bitcoin is a hybrid of commodity money and fiat money but that bitcoins are mainly used as a speculative investment and not as an alternative currency and medium of exchange. Symitsi and Chalvatzis (2019) find statistically significant diversification benefits from the inclusion of bitcoin in the portfolios of various asset classes, which are more pronounced for commodities. These papers, on balance, support modeling the bitcoin as a commodity.

Another strand of literature further demonstrates that bitcoin is an asset that is radically different from those belonging to traditional asset classes. Using a LASSO approach, Panagiotidis et al. (2018) investigate the influence of various factors such as stock market returns, exchange rates, gold and oil returns, the Federal Reserve and European Central Bank's rates, and internet trends on bitcoin returns. The authors show that Google search intensity and gold returns are the most important drivers of bitcoin returns. Bianchi (2020) uses a large panel of prices, traded volumes, and market capitalization on 14 actively quoted cryptocurrencies to demonstrate that, except for a mild correlation with gold and crude oil, there is no significant relation between returns on cryptocurrencies and more traditional asset classes. The comprehensive empirical analysis conducted in Liu and Tsyvinski (2021) demonstrates that the mean and standard deviation of returns for cryptocurrencies are an order of magnitude higher than those for traditional asset classes and that cryptocurrencies have no exposure to most common stock market and macroeconomic factors. Dong et al. (2022) find that bitcoin price dynamics are significantly sensitive to investor sentiment and that market sentiment positively comoves with bitcoin prices. Chowdhury and Damianov (2023) demonstrate that crypto price and policy uncertainty indices based on news coverage (Lucey et al., 2022) are associated with the emergence of bubbles in cryptocurrencies. These empirical studies characterize features of bitcoin returns that are consistent with the main premise of this paper: since the protocol uses the difficulty to target a constant supply of bitcoins per day, prices are demand driven and highly sensitive to investor attention and sentiment.

While the bitcoin was introduced in 2009, today there is still very little consensus about how to model the fundamental value of a cryptocurrency. For instance, 56% of a recent panel of experts stated that they were either uncertain (42%) or had no opinion (14%) about whether the fundamental value of the bitcoin is at least \$1000.[9] Using bitcoin price data, Cheah and Fry (2015) find that bitcoin prices contain a substantial speculative bubble component and that the fundamental value of the bitcoin is zero. The authors apply the model of Johansen et al. (2000) that is based on the pricing of a speculative asset that pays no dividends but neither take into consideration the functioning of the Bitcoin protocol nor the production of bitcoins by miners. Prasad (2021) maintains that bitcoin has no intrinsic value since its value comes from scarcity but scarcity itself can hardly be a source of value. In contrast, Bouoiyour et al. (2016) apply the technique of Empirical Mode Decomposition to bitcoin price data and find that long-term fundamentals greater than one year are likely to be the major contributors of Bitcoin price variation. While the method can detect possible hidden features in the price data, it cannot specify which variables comprise the fundamentals. Bolt and Van Oordt (2020) develop a framework that abstracts from cryptocurrency production (mining) and combines an investor's portfolio model with a payment network model to study the exchange rate of a virtual currency. The authors show that the exchange rate of virtual currency is determined by the current value of transactions in virtual currency, the decisions and expectations of forward-looking investors to buy virtual currency and the elements that jointly drive future consumer adoption and merchant acceptance of virtual currency. Biais et al. (2023) develop a model in which agents can trade standard fiat money and a cryptocurrency. The authors show that the price of the cryptocurrency is equal to the present value of the expectation of a pricing kernel that captures the correlation between the marginal utility of consumption and the cryptocurrency price, the risk of hacks, and the sum of the price of the cryptocurrency in the next period and its net transactional benefit. The authors find that the valuation of cryptocurrencies differs from the valuation of stocks since the fundamental value stems from the transactional benefits instead of the firm's dividend. It is for this reason that Gronwald (2021) cautions against testing for cryptocurrency bubbles using procedures based on a theoretical stock price model. For stocks, in the absence of a bubble, the degree of nonstationarity of the asset price is controlled entirely by the dividend series that is believed from empirical evidence to be at most an integrated order 1 process (Phillips & Shi, 2018). Presuming that evidence of price explosiveness implies evidence of a bubble amounts to placing the same restriction on an asset's fundamental value.

Most closely related to this paper are Easley et al. (2019), which develops a game theoretic model to explain the strategic behavior of miners and users, demonstrating that equilibrium in the bitcoin blockchain is a complex balancing of user and miner participation. Hayes (2019) was first to propose the marginal cost of production as a model to value the bitcoin. The author formalizes a pricing model based on the marginal cost of mining new bitcoins, which arises from miners' computational effort that consumes electricity. The author uses the Granger causality test to demonstrate unidirectional causality from the pricing model to the market price. Lambrecht et al. (2021) undertake an experiment to investigate how features associated with the PoW consensus mechanism affect pricing. In the experiment, mining is modeled by an asset cost that is increasing in the cumulative units of the assets that have been generated by the participants. While the

[9] See: https://www.chicagobooth.edu/review/whats-fundamental-value-bitcoin.

authors do not use advanced bubble detection techniques likely due to insufficient data, they find patterns of bubbles and crashes in the price that they causally attribute to mining. Podhorsky (2023) applies the same framework and data as the present paper but the focus is on how the difficulty adjustment mechanism can be incentivized by a tax to decrease the Bitcoin network's electricity costs.

In contrast with other papers in the literature, the present paper develops a model that includes costly mining activity and the functioning of the Bitcoin protocol. While Easley et al. (2019) study how the Bitcoin protocol affects the interaction between miners and users, and thus the determination of fees, the present paper treats fees as exogenous and studies how the protocol affects the interaction between the miners and the buyers of bitcoins, and thus the determination of the price of the bitcoin in the market. It gives special attention to the workings of the difficulty adjustment mechanism and does not assume that the target quantity of bitcoin production always holds. To focus on the supply-side effects of the protocol, the model includes a relatively unstructured demand side of the bitcoin market. It treats network quality and security, in addition to buyer's expectations, and specific use cases for the bitcoin that are mostly unobservable due to the pseudo-anonymity of bitcoin transactions as exogenous factors that may drive the demand for bitcoins. It is shown that because the protocol adjusts the level of difficulty in response to changes in the equilibrium mining rate detected by the network, which depends on both the supply and demand for bitcoins, the difficulty is a sufficient statistic for the demand parameters. Hence the marginal cost of mining the target supply of bitcoins, which depends largely on the network difficulty, thoroughly characterizes the fundamental value of the bitcoin. The framework clearly shows how the protocol's interference in the market results in volatility and apparent bubbles in the price of the bitcoin. The present paper further demonstrates that market data uphold the marginal cost of mining the target supply of bitcoins as the fundamental value of the bitcoin and that the apparent bubbles in the price of the bitcoin arise from a supply side phenomenon that can be attributed to Bitcoin's system of supply management.

2. The model

2.1. Supply

A miner collects new transactions into a block and then hashes the block header to form a 256-bit block hash value. If the value is below a target set by the protocol, which corresponds to a given level of difficulty δ, then other miners will confirm the solution and agree that the block can be added to the blockchain. Because the minimum level of difficulty (equal to 1) requires the hash of the block header to start with 8 hexadecimal zeros, which represents 32 bits, the expected number of hashes per second needed to find a solution is $\delta 2^{32}$, where the difficulty δ is a unitless scaling parameter that is a multiple of the minimum amount of work that any valid block can contain. It follows that the expected waiting time for a miner to find a block (in seconds) is $\frac{\delta 2^{32}}{\phi 10^9}$, where ϕ is the hashrate employed by the miner measured in gigahashes per second.[10] When a miner finds a block, the miner earns the block reward ω (denominated in bitcoins) and may also earn fees f per block (denominated in bitcoins) that senders of bitcoins can include in any transaction to reduce their waiting time.

The protocol regulates the quantity of bitcoins that are mined by adjusting the difficulty every 2016 blocks. It adjusts the difficulty in such a way that the current network hashrate results in a ten-minute block interval. If the network detects that the time required to find the last 2016 blocks differs from 20,160 min, which is a daily mining rate (blocks per day) that differs from 144, then the level of difficulty will be adjusted as follows:

$$\frac{\delta_2}{\delta_1} = \frac{20{,}160 \text{ min}}{\text{Actual time of last 2016 blocks in minutes}} = \frac{\text{daily mining rate}}{144} \tag{1}$$

where δ_2 is the new level of difficulty and δ_1 is the previous level of difficulty.

There are identical potential entrants (miners) to the bitcoin mining industry. Each miner is risk neutral,[11] knows the rules of the Bitcoin protocol that govern the network, and must pay a fixed cost F (thereafter sunk) to purchase mining equipment in order to enter. Upon entry, a miner's daily expected bitcoin production is

$$x\left(\tau_i\right) = \frac{\omega \tau_i 60^2}{\frac{\delta 2^{32}}{\phi 10^9}} \tag{2}$$

where $\tau_i 60^2$ is the number of seconds spent mining per day. Hence if $\phi 10^9 \tau_i 60^2$ hashes are created by a miner in one day, the expected number of blocks mined is $\frac{\phi 10^9 \tau_i 60^2}{\delta 2^{32}}$ per day, at a reward of ω bitcoins per block. A miner's daily electricity cost is

$$\frac{\phi \xi \tau_i}{1000} p_e \tag{3}$$

where ξ is the energy efficiency of the miner's hardware measured in joules per gigahash (and hence $\phi\xi$ is the power usage measured in joules per second, or watts) and p_e is the dollar price of electricity per kilowatt hour (kWh). It follows from (2) and (3) that a miner's operating profit is linear in τ_i and if the dollar price of a bitcoin (the exchange rate) $p_b > \frac{\delta 2^{32} \xi p_e}{(\omega + f)(1000)60^2 10^9} \equiv \underline{p_b}$, then it is optimal the miner to set $\tau_i = 24$ and 0 otherwise.[12]

Since hashing power scales linearly (doubling the number of miners doubles the network hashrate), the total hashpower of the Bitcoin network is ϕM, where M is the total number of miners who enter the industry. It follows that gross of investment costs, miners' aggregate expected daily profits are given by

$$\Pi = \left[\frac{p_b(\omega + f)60^2 10^9}{\delta 2^{32}} - \frac{\xi p_e}{1000}\right] 24\phi M \tag{4}$$

where it is assumed that $p_b > \underline{p_b}$. Because an increase in the number of miners M increases the network hashrate proportionally, each miner has the same expected profit $\frac{\Pi}{M}$ regardless of the number of entrants. Every 2016 blocks, the level of difficulty δ is adjusted so that the average waiting time to find a block on the network is approximately 10 min (600 s), so that $\frac{\delta 2^{32}}{\phi M 10^9} = 600$ or

$$\delta = \frac{600\phi M 10^9}{2^{32}}. \tag{5}$$

Since the target waiting time to find a block on the network in (5) is encoded in the protocol and known to the potential entrants, it pins down the number of miners M. From (4) and (5) it follows that the expected daily profit for a miner is

$$\begin{aligned}\pi &= \frac{\Pi}{M} - \eta F \\ &= \left[\frac{p_b(\omega + f)60^2}{600\phi M} - \frac{\xi p_e}{1000}\right] 24\phi - \eta F\end{aligned}$$

where η is the daily depreciation rate of the miner's equipment. Since there is free entry to the bitcoin mining industry, miners have zero

[10] There are 10^9 hashes in a gigahash.

[11] Easley et al. (2019) also assumes that miners are risk neutral.

[12] These corner solutions realistically capture the fact that when the price of the bitcoin falls to the point where it is sufficiently low (ie. $p_b < \underline{p_b}$), the miners simply turn off their machines until the price recovers.

expected profits and hence the number of miners per day is given by $\pi = 0$ or

$$M^* = \frac{p_b(\omega + f)\left[\frac{(24)60^2}{600}\right]}{\eta F + \frac{\phi\xi}{1000}(24)\,p_e}. \tag{6}$$

From (6) it is clear that the number of miners is equal to the total dollar value of the block reward and fees, for each of the 144 possible blocks mined, divided by each miner's daily equipment and electricity costs

$$\eta F + \frac{\phi\xi}{1000}(24)\,p_e. \tag{7}$$

While the number of entrants M^* adjusts immediately to changes in the price of a bitcoin p_b, the level of difficulty adjusts only approximately every two weeks while the network learns the network hashrate ϕM^* from observing the average number of blocks mined per day.

The aggregate supply of bitcoins per day X_S is equal to the block reward ω multiplied by the daily mining rate, which is determined by the network hashrate ϕM^* for a given δ. If $p_b > \underline{p_b}$, it follows that

$$\begin{aligned} X_S &= \frac{\omega(24)60^2}{\frac{\delta 2^{32}}{\phi M^* 10^9}} \\ &= \frac{p_b(\omega + f)\left[\frac{(24)60^2\phi 10^9}{\delta 2^{32}}\right]}{\eta F + \frac{\xi\phi}{1000}(24)\,p_e}\overline{X} \end{aligned} \tag{8}$$

where the second line follows from M^* of (6) and

$$\begin{aligned} \overline{X} &= \frac{\omega(24)60^2}{600} \\ &= 144\omega \end{aligned}$$

is the target supply of bitcoins per day since the protocol adjusts δ so that one block is created approximately every 10 min (600 s). From (8) it follows that the supply curve is linear because hashing power scales linearly.

Since the network's choice of the level of difficulty depends on the network hashrate ϕM^*, the equilibrium level of difficulty δ^* will depend on M^*. Hence it follows from substituting (6) into (5) that the equilibrium level of difficulty is

$$\delta^* = \frac{p_b(\omega + f)\left[\frac{(24)60^2\phi 10^9}{2^{32}}\right]}{\eta F + \frac{\phi\xi}{1000}(24)\,p_e}. \tag{9}$$

From (9) it is clear that, for a given price of a bitcoin p_b, the difficulty will increase in response to an increase in the hashrate of miners' equipment ϕ, an improvement in the energy efficiency of the miners' equipment (a decrease in ξ), a decrease in the price of electricity p_e, or an increase in the Bitcoin block reward ω or fees f. It follows from (8) and (9) that we can write $X_S = \frac{\delta^*}{\delta}\overline{X}$ and hence $X_S = \overline{X}$ if and only if $\delta = \delta^*$.

2.2. Demand

There are $i = 1 \dots N$ buyers of bitcoins. Their use case is making a remittance or an anonymous payment. A representative agent i's utility from bitcoin at a given point in time is

$$U_i = u\left(x_i, r, A, S\right)$$

where u is continuous and quasi-concave, x_i is the quantity of bitcoins held by user i, r is the expected one period return from holding bitcoin, A represents the anonymity associated with the transfer of bitcoins and S is the security of the Bitcoin network.

For the purpose of undertaking an empirical analysis, the aggregate daily demand for the bitcoin is specified as a standard constant elasticity of demand function

$$X_D = \beta_0 W^{\beta_1} p_b^{-\epsilon} \tag{10}$$

where β_0 and β_1 are constants, ϵ is the price elasticity of demand, and W includes the determinants of the demand for the bitcoin other than its price.

2.3. Equilibrium

I define a comprehensive equilibrium to be a four-tuple $\left(X^*, p_b^*, \delta^*, M^*\right)$, where the first element is the equilibrium quantity of bitcoins supplied per day, the second element is the equilibrium price, the third element is the equilibrium level of difficulty and the fourth element is the equilibrium number of miners per day, which determines the equilibrium hash rate. A comprehensive equilibrium is the unique solution to the system of equations determined by the zero profit condition obtained from setting (4) equal to the miners' fixed costs, the Bitcoin protocol's specified waiting time to find a block of (5), the supply curve of (8) and the demand curve of (10). In a comprehensive equilibrium, since $X_S\left(p_b^*; \delta^*\right) = X_D\left(p_b^*\right) = X^*$ and $X_S = \overline{X}$ if and only if $\delta = \delta^*$, it follows that $X^* = \overline{X}$. We have seen that since the level of difficulty δ is adjusted only at intervals, an equilibrium in the market $\left(X_S\left(p_b^*; \delta\right) = X_D\left(p_b^*\right) = X^*\right)$ may not occur at the same time as an equilibrium in the protocol ($\delta = \delta^*$, $M = M^*$).

Fig. 1 depicts the aggregate daily supply of bitcoins by miners and the aggregate daily demand for bitcoins by individuals. Starting from an initial comprehensive equilibrium labeled 1 with price p_{b1}, a level of output $X_1 = \overline{X}$, a mass of entrants M_1, and a level of difficulty δ_1, an increase in demand from X_D to X'_D leads to an increase in the price of a bitcoin to p_{b2} and a movement along the supply curve consistent with an increase in the number of entrants to M_2. Because the probability of successfully mining a block is determined by δ_1 and more hashpower ϕM_2 is directed at the network, the quantity of bitcoins supplied increases to $X_2 = X_S\left(p_{b2}; \delta_1\right)$ per day in the market equilibrium labeled 2a. The new equilibrium will be short-lived, however, since the mining rate exceeds the protocol's target mining rate of 144 blocks per day.

It is assumed that equilibrium 2a is representative of the daily mining rate during a 2016-block period. As such, the Bitcoin protocol will choose the new level of difficulty $\delta_2 = \delta^*\left(p_{b2}\right)$. It follows from X_S of (8) that the increase in the level of difficulty from δ_1 to δ_2 results in an upward rotation of the supply curve. Referring to Fig. 1, the supply curve rotates upward until $X_S = \overline{X}$ at the price p_{b2}, since p_{b2} gives rise to the network hashrate ϕM_2. The marginal cost of mining has increased because the greater difficulty causes miners to expend more resources on electricity to mine a given number of blocks. Since the price p_{b2} is unchanged, it follows from (6) that an increase in difficulty does not result in an exit of miners from the industry.[13] At the point labeled 2b, the protocol is in equilibrium since the mining rate is equal to its target given the network hashrate ϕM_2, and the network has no further incentive to change the level of difficulty. At 2b there is excess demand, however, which causes the price of a bitcoin to rise to p_{b3} and the number of miners to increase to M_3. At the market equilibrium labeled 3 with price p_{b3}, the demand X'_D is equal to the supply of bitcoins given the new level of difficulty δ_2. While the protocol is no longer in equilibrium, the mining rate is closer to its target than before the increase in difficulty. More pertinently, however, since p_{b3} exceeds p_{b2}, it is clear from Fig. 1 that the decrease in supply due to the greater difficulty results in an exaggerated price response relative to the price that would have prevailed had the difficulty not been adjusted.

Fig. 2 depicts successive positive demand shocks and demonstrates that they result in a rapidly increasing price path that can be mistaken for a bubble despite being based on marginal costs. The analysis of a negative demand shock is analogous and a series of negative demand shocks can be mistaken for a bursting bubble since the equilibrium price will fall rapidly as the difficulty decreases. It follows that the interaction of the Bitcoin protocol with the market can manifest as boom-and-bust cycles in the price of the bitcoin. Unlike a bubble, however, price is equal to marginal cost all the while.

[13] Recall that the number of miners adjusts immediately to changes in the price of a bitcoin p_b and then the difficulty adjusts in turn.

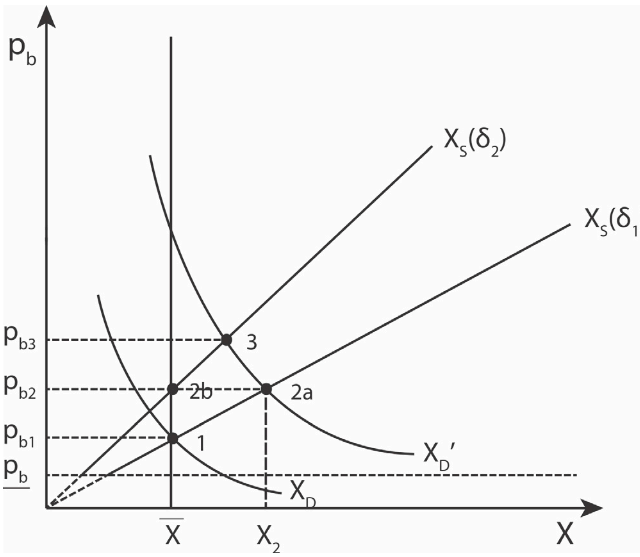

Fig. 1. Bitcoin price adjustment, where X is the quantity of bitcoins, p_b is the price of the bitcoin and δ is the network difficulty.

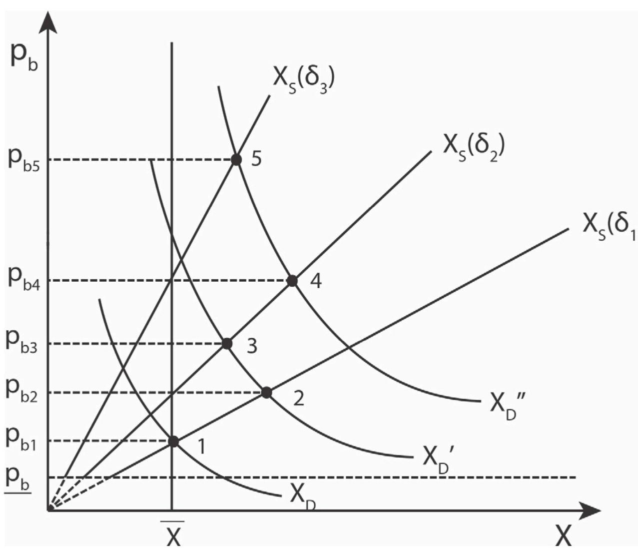

Fig. 2. Successive positive demand shocks, where X is the quantity of bitcoins, p_b is the price of the bitcoin and δ is the network difficulty.

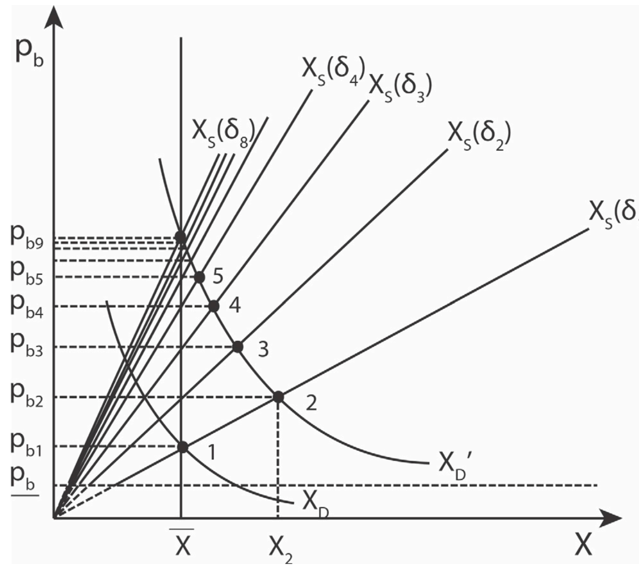

Fig. 3. Limiting bitcoin price adjustment, where X is the quantity of bitcoins, p_b is the price of the bitcoin and δ is the network difficulty.

The fundamental value of a bitcoin p_b^f is defined to be the marginal cost of mining the target supply of bitcoins $\overline{X}$. For a given level of difficulty δ, the fundamental value is given by the inverse supply curve $p_b(X;\delta)$ evaluated at $X = \overline{X}$. As shown in Fig. 3, if a demand shock is permanent, the market price will approach the fundamental value that is consistent with a comprehensive equilibrium since successive adjustments of the difficulty will occur until the mining rate is equal to its target in the limiting comprehensive equilibrium. The protocol automatically maneuvers the market price toward the price consistent with equilibrium in the protocol, a process that is only temporarily disrupted by shocks to demand.

Referring to Fig. 1, it is clear that the fundamental value can be identified by using only the information contained in the supply curve X_S and the target $X = \overline{X}$. The level of difficulty can be viewed as a sufficient statistic for demand, since it is determined by the protocol according to the equilibrium mining rate detected by the network, which depends on both the supply and demand for bitcoins.

The following proposition characterizes the fundamental value of the bitcoin and specifies its relation to the price of the bitcoin.

Proposition 1. *(i) The fundamental value of the bitcoin p_b^f is equal to the miners' dollar costs relative to their expected bitcoin block rewards and fees. (ii) $p_b = p_b^f$ if and only if $\delta = \delta^*$. (iii) $p_b = \left[\frac{\beta_0}{\overline{X}} W^{\beta_1} p_b^f(\delta)\right]^{\frac{1}{1+\epsilon}}$.*

Proof. See the Appendix A. ■

Proposition 1(i) follows from the inverse supply curve derived from X_S of (8), which shows that the marginal cost of producing $\overline{X}$ bitcoins at a given level of difficulty δ is

$$p_b^f(\delta) \equiv p_b\left(\overline{X};\delta\right) = \frac{\left[\eta F + \frac{\xi\phi}{1000}(24)\,p_e\right] M^*}{(\omega + f)\left[\frac{(24)60^2\phi M^* 10^9}{\delta 2^{32}}\right]}. \tag{11}$$

It follows that the marginal cost of mining the target supply of bitcoins is equal to the miners' daily cost of mining, denominated in dollars, relative to their daily expected block rewards and fees, denominated in bitcoins. In other words, the miners' daily expected block rewards and fees, when valued at p_b^f, are equal to their daily mining costs. This is the standard equilibrium in a rent-seeking tournament that has been widely used to model bitcoin mining (Budish, 2018) since the prize in the tournament is dissipated by the expenditures aimed at winning the prize. Part (ii) of the proposition demonstrates that $p_b = p_b^f$ is indicative of an equilibrium in the Bitcoin protocol since it follows from (9) and (11) that $p_b = p_b^f$ if and only if $\delta = \delta^*$. Hence, whenever the bitcoin is valued according to p_b^f, the protocol has no incentive to change the level of difficulty. Lastly, part (iii) of the proposition relates the price of the bitcoin to its fundamental value, for a given level of the difficulty. It follows that the market price of the bitcoin p_b and the fundamental value of the bitcoin p_b^f have a log-linear relationship. Furthermore, whenever the protocol is in equilibrium, since $p_b = p_b^f$, it follows that $p_b = \left[\frac{\beta_0}{\overline{X}} W^{\beta_1}\right]^{\frac{1}{\epsilon}}$. It is clear that since the difficulty adjustment mechanism enforces a vertical supply curve at the target supply of bitcoins $\overline{X}$, prices are demand driven as p_b depends entirely on the determinants of demand in W.

To summarize, this section established that the supply of bitcoins is linear and upward sloping through the origin. An increase (decrease) in the level of difficulty results in an upward (downward) rotation of the supply curve. After 2016 blocks have been mined, if the Bitcoin network detects that the mining rate differs from the target of 144 blocks per day, the protocol will adjust the difficulty so that the existing network hashrate will result in a 10-minute interval between successive blocks mined. Positive demand shocks can result in a rapidly increasing price path that may be mistaken for a bubble despite being supported by the marginal cost of mining. Proposition 1 characterizes the fundamental value of the bitcoin and demonstrates that when the price of the bitcoin is equal to its fundamental value, there is simultaneously equilibrium in the protocol and the rent-seeking tournament among miners. The protocol's imposition of a target supply of bitcoins also results in price volatility, since variations in demand are expressed in the price.

3. Data description

This section describes the data and examines how well the theoretical model in Section 2 fits the data.

The data were acquired from numerous sources. The daily average USD price of the bitcoin across major bitcoin exchanges, daily data on the Bitcoin difficulty level, the Bitcoin block reward and fee, and the number of blocks mined per day were acquired by using Blocksci, an open-source software platform for blockchain analysis.[14,15] Daily USD price data for new (unused) Antminer mining rigs (models S1, S2, S3, S4, S5, S7, S9 and S11) sold on Amazon Marketplace by third party sellers, which is accessible from Amazon.com, were acquired by using an API for the Amazon price tracker Keepa.com.[16] The reported price is the lowest of the prices available from the sellers and does not include shipping costs; missing data correspond to periods of time when all sellers are out of stock. The mining rig specifications regarding the hash rate and energy efficiency were obtained directly from Amazon.com and are provided in Table 1. Since several Antminer models can be sold in the Amazon Marketplace at a given point in time, the daily average USD price was constructed by averaging over the prices of all Antminer models that were available for sale on a given day. Similarly, the daily average hashrate and the daily average energy efficiency of the Antminer rigs were constructed by averaging over the gigahashes per second (GHash/s) and the joules per gigahash (Joules/GHash) for all Antminer models that were available for sale on the given day, respectively. Since 17 March 2014 was the first day that price information on the Antminer S1 rig was tracked by Keepa.com, the data cover the period from 17 March 2014 to 13 January 2019. While there are numerous brands of bitcoin mining rigs available on the market, the Antminer rigs are on the technological frontier in terms of their power and energy efficiency and Bitmain's market share is about 70%–80%.[17] The average price of electricity used in mining is conservatively estimated to be 0.05 USD per kWh since Bitmain, which owns one of the world's largest bitcoin mines, was known to be paying just 4 cents per kWh of electricity in Inner Mongolia (de Vries, 2018). Also, the expected lifespan of a mining rig is estimated to be two years, so that the daily depreciation rate is $\frac{1}{730}$, since large companies like Bitmain are constantly working on releasing faster and more efficient models that render their predecessors obsolete.

Fig. 4 depicts the bitcoin price path over the sample period in both levels and logs. While the price had an exponential growth, since the logarithm of the price is approximately linear, numerous boom-and-bust cycles are evident, with the largest boom occurring in late 2017. On 16 December 2017, bitcoin reached its maximum price of 19,343.04 USD. Fig. 5 depicts the daily level of difficulty over the sample period in both levels and logs. It is clear that the level of difficulty had been increasing exponentially until 17 October 2018. After that time, the difficulty predominantly decreased (4 of the 6 remaining difficulty adjustments were decreases). The level of difficulty was adjusted downward only 21 times over the sample period, which is 15.9% of all difficulty adjustments. Fig. 6 depicts the sum of the Bitcoin block reward and fees over the sample period. It is clear that the block reward was halved from 25 bitcoins to 12.5 bitcoins on 9 July 2016 and that fees were much more prevalent throughout 2017 due to congestion in the Bitcoin blockchain. Fig. 7 presents a standard plot and a boxplot of the daily mining rate, where a horizontal line is drawn at the target of 144 blocks. It is clear that the mining rate frequently differed from its target and hence the protocol was frequently not in equilibrium. The daily mining rate reached a minimum of 80 blocks per day (on 11 and 12 November 2017) and a maximum of 216 blocks per day (on 10 December 2015) during the sample period. The mean and median blocks mined per day are 151.6 and 151, respectively, indicating that the daily mining rate typically exceeded the target during the sample period. Fig. 8 depicts the Antminer rig specifications over the sample period. We can see that, on average, Antminer rigs have become more powerful over time, since their hashrate is increasing. While the energy efficiency of the rigs is improving, as the rate of joules per gigahash is decreasing over time, their greater power was large enough to result in greater energy use since the number of watts used (joules per second) was increasing over time. Note that the gaps in the Antminer rig specification data correspond to periods of time when none of the sellers in the Amazon Marketplace had any of the Antminer rigs listed in Table 1 in stock.[18] This demonstrates that there was likely excess demand for mining equipment during this time period, when the price of the bitcoin was quite high (approximately 5500 USD) and rising rapidly. Fig. 9 depicts the average price of Antminer rigs over the sample period and their average price per gigahash per second. We can see that mining equipment costs had generally increased in tandem with the market price of the bitcoin (the correlation between the bitcoin's price and the average price of Antminer rigs is .59). The average price of Antminer rigs relative to their average hashrate measured in gigahashes per second, however, had been steadily decreasing over time and reached a low of 0.03 USD by the end of the sample period. Fig. 10 depicts the miners' daily average equipment and electricity costs as defined in (7) and the proportion of electricity in their daily costs over time.[19] With the exception of late 2017, when mining equipment was extraordinarily costly due to the plausible excess demand, electricity

Table 1
Antminer bitcoin mining machine model specifications for hash rate, efficiency, and power consumption.

	Tracking since	First available	GHash/s	Joules/GHash	Watts
Antminer S1	14-03-17	13-12-30	180	2	360
Antminer S2	14-06-10	14-05-21	1000	1	1000
Antminer S3	14-12-31	14-09-27	441	.83	366
Antminer S4	14-11-18	14-09-25	2000	.725	1450
Antminer S5	14-12-28	14-12-22	1155	.51	590
Antminer S7	15-09-06	15-08-30	4860	.25	1210
Antminer S9	16-11-02	18-01-16	14,000	.098	1372
Antminer S11	18-11-21	18-11-19	20,500	.064	1312

[14] See Kalodner et al. (2017) and https://github.com/citp/BlockSci.

[15] Note that Blocksci utilizes an API for coindesk.com to provide the end of day price of a bitcoin.

[16] The Amazon standard identification numbers (ASIN) that identify the models are: B00I0F4IMI, B00KH9339O, B00NZDBWKG, B00NWHT18 A, B00RCTIY4G, B014OGCP6 W, B01MCZVPFE, and B07KPF2DJJ.

[17] See https://coincentral.com/how-antminer-became-the-best-bitcoin-mining-hardware-in-less-than-two-years/.

[18] These are from 12 October 2017 to 17 October 2017, from 19 October 2017 to 26 October 2017, from 13 November 2017 to 17 November 2017, from 24 November 2017 to 5 December 2017, from 9 December 2017 to 10 December 2017, and from 3 January 2018 to 4 January 2018.

[19] Which, from (7), is $\frac{\frac{\phi\xi}{1000}(24)p_e}{\eta F+\frac{\phi\xi}{1000}(24)p_e}$.

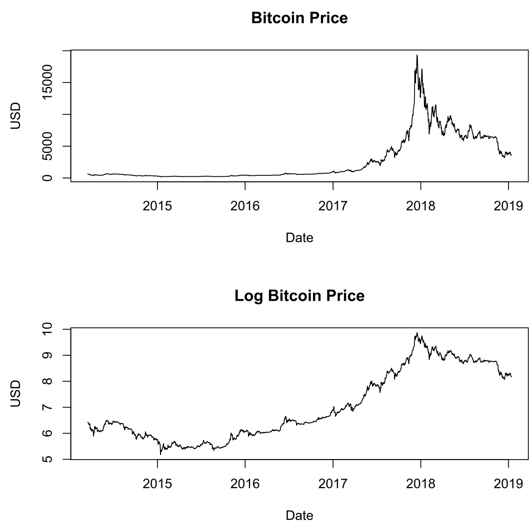

Fig. 4. The bitcoin price in levels and logs.

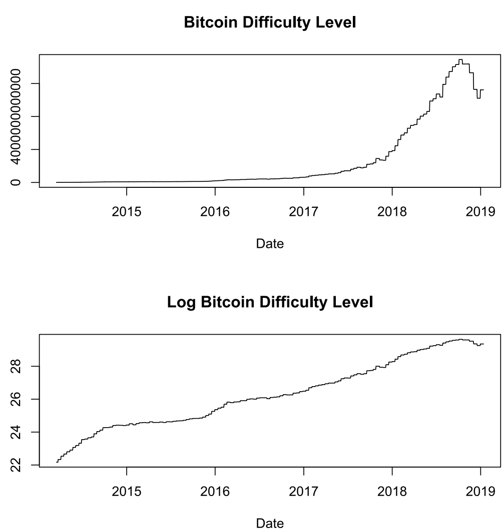

Fig. 5. The Bitcoin network difficulty in levels and logs.

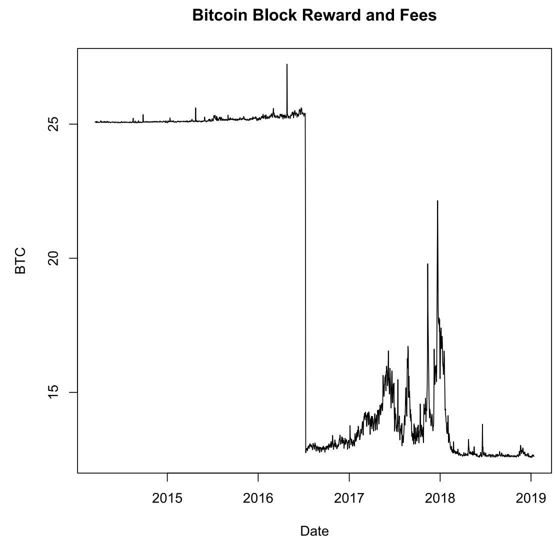

Fig. 6. The bitcoin block rewards and fees.

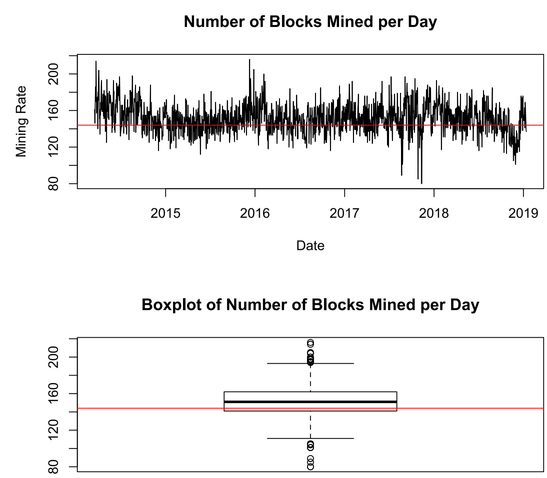

Fig. 7. The daily mining rate (blocks mined per day).

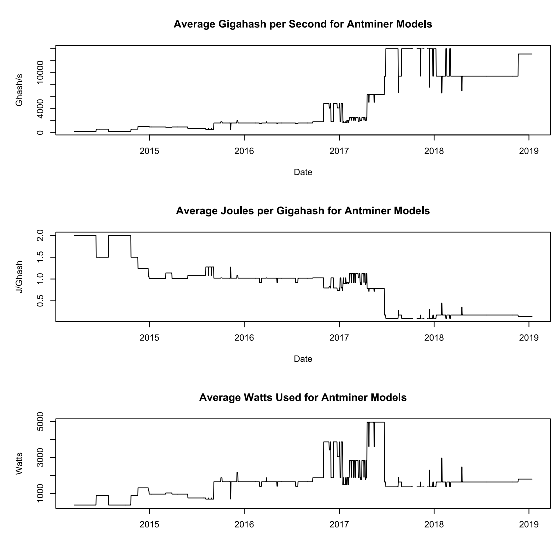

Fig. 8. Average Antminer hash rate, efficiency, and power consumption.

costs were growing as a share of the miners' daily costs. Electricity costs approached 80.2% of daily costs by the end of the sample period due to the increasing energy usage of the mining equipment evident in Fig. 8 and the falling price of mining rigs evident in Fig. 9.

Fig. 11 assesses whether the Bitcoin network adjusted the difficulty according to Eq. (1). The diagram plots the ratio of the new level of difficulty relative to the previous level against the mining rate divided by the target mining rate of 144, where the mining rate is the average number of blocks mined per day during the interval between difficulty adjustments. It is clear that the data are consistent with (1) since the points line up on the 45 degree line and the two variables have a correlation of .99.

Next, the data is used to simulate the fundamental value of the bitcoin p_{bt}^{f} defined in (11) and it is compared with the market price of the bitcoin p_{bt}. Recall that p_{bt}^{f} depends only on supply side parameters, since the difficulty δ can be viewed as a sufficient statistic for demand. This fortuitously permits a straightforward estimation of the fundamental value of the bitcoin without having to directly observe the determinants of demand. The fundamental value of the bitcoin p_{bt}^{f} is strongly related to the difficulty as the correlation between p_{bt}^{f} and δ is .93. Fig. 12 depicts the estimated fundamental value and the actual price of the bitcoin in both levels and logs. The fundamental value tracks variations

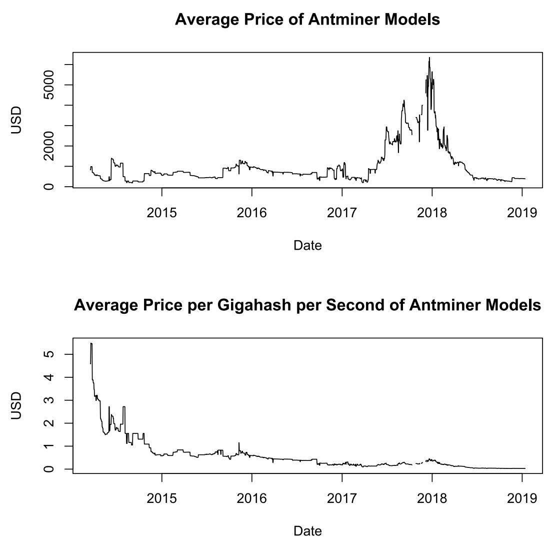

Fig. 9. Average Antminer price and average Antminer price per gigahash.

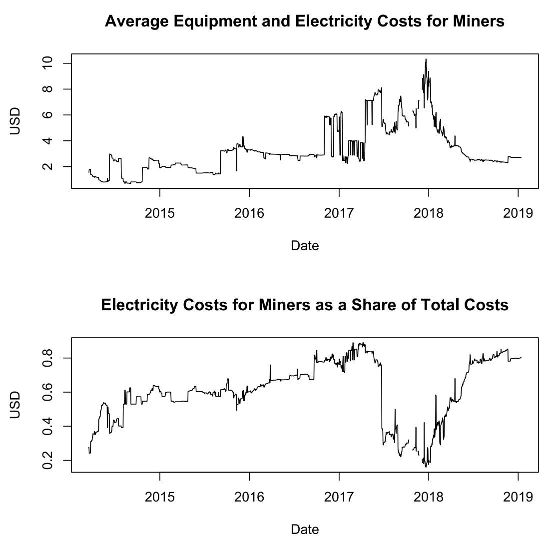

Fig. 10. Average equipment and electricity costs for miners and electricity costs as a share of the total costs.

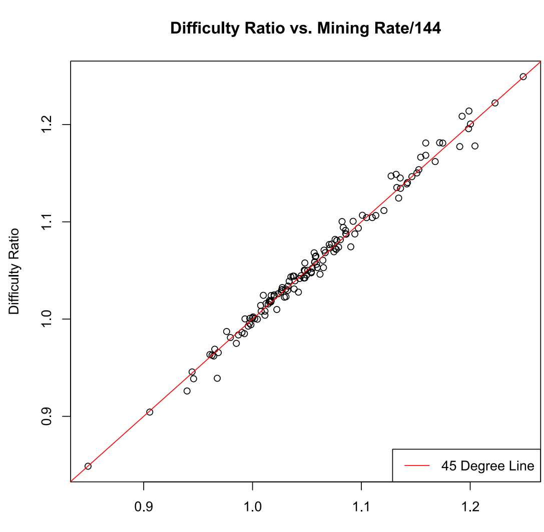

Fig. 11. Difficulty adjustments according to the bitcoin protocol.

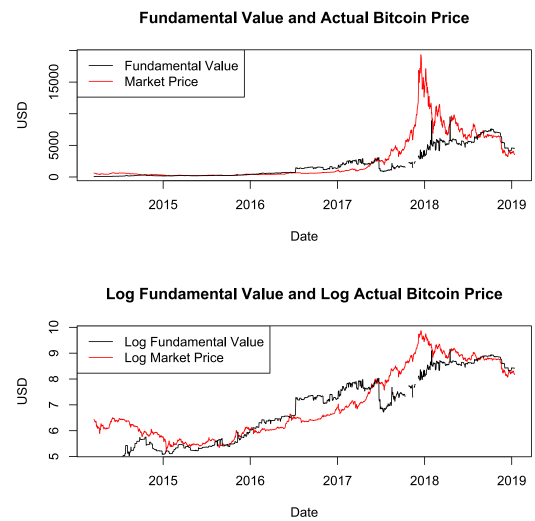

Fig. 12. Simulated model-implied fundamental value and the price of the bitcoin.

in the market price quite well since the correlation between them is .79. The fundamental value rose sharply in July 2016 when the block reward was halved and it also fell sharply in July 2017 because of the introduction of the powerful Antminer S9.[20] The market price of the bitcoin does not exhibit such sharp adjustments likely because both of these events could have been anticipated by participants in the bitcoin market. We can see that the fundamental value was in line with the market price well up to late 2017, after which the actual price exceeded the fundamental value by 100% to 185% until early 2018. While there was a large discrepancy between the market price of the bitcoin and its fundamental value, it likely resulted from barriers to entry for miners that are not wholly captured by the model.[21] As noted above, during this period there were occasions when none of the sellers in the Amazon Marketplace had an Antminer rig in their inventory, indicating that state-of-the-art rigs were generally difficult to acquire.[22] Since the price of the bitcoin had reached historic levels, mining equipment was in short supply during this time, resulting in its extraordinarily high price or complete lack of availability.[23]

In summary, we have seen that the data are largely consistent with the model formulated in Section 2. While there was a discrepancy between the market price of the bitcoin and its fundamental value in late 2017, this likely resulted from the miners' inability to enter the market due to the lack of available equipment. It remains to test whether the deviations in the price of the bitcoin from its fundamental value can be attributed to bubbles.

4. Econometric model

4.1. Testing for cointegration

This section investigates whether the empirical evidence supports Eq. (11) as the fundamental value of the bitcoin. Prior to determining

[20] We can see from the top panel of Fig. 8 that the Antminer S9 became the dominant Antminer rig in the market in July 2017 since the average hashpower of the Antminer rigs approaches 14,000 GHash/s.

[21] While the increase in the price of the bitcoin during this time may have been due to market manipulation (Griffin & Shams, 2020), it would have had no bearing on miners' incentives to enter the market in response to the high prices.

[22] See Footnote 18.

[23] See: https://www.newsbtc.com/tech/popularity-of-mining-has-created-a-worldwide-shortage-of-gpus/ which describes how the cryptocurrency bull run of 2017 created a demand for mining rigs that resulted in a worldwide shortage of computer components.

the stability of their underlying relationship, the time series properties of the price of the bitcoin and the marginal cost of mining the target supply of bitcoins $\overline{X}$ are examined. Two variables are said to be cointegrated if they share a common long run stochastic trend; the presence of cointegration can be interpreted as the existence of a long-run equilibrium relationship between the variables in question (Engle & Granger, 1987). Since the protocol sets the difficulty at regular intervals of time to maintain equilibrium in the protocol, it follows from Proposition 1(ii) that the price of the bitcoin should not persistently deviate from the marginal cost of mining the target supply of bitcoins $\overline{X}$. While cointegration of these time series is sufficient to support the marginal cost of mining the target supply of bitcoins $\overline{X}$ as the fundamental value of the bitcoin, it is not a necessary condition since the price could deviate from its fundamental value due to the presence of bubbles in the price of the bitcoin. Indeed, Diba and Grossman (1988) proposed the use of standard unit root and cointegration tests for stock prices and observable fundamentals to obtain evidence for the existence of explosive rational bubbles.[24]

For a given time t, from Proposition 1(ii) it follows that the equilibrium relationship between the price of the bitcoin p_b and its fundamental value p^f_{bt} can be expressed as

$$p_{bt} = \gamma_0 + \gamma_1 p^f_{bt} + \upsilon_t \tag{12}$$

where the error term υ_t is due to demand shocks in the interim between difficulty adjustments. The differences (or error term) in the cointegration Eq. (12) are interpreted as the unbalance error for each particular point in time. Testing for cointegration determines whether the distance between p_{bt} and p^f_{bt} is stable and evidence of their cointegration would reflect the presence of a long-run equilibrium towards which the economic system converges over time.

The Augmented Dickey Fuller (ADF) test (Dickey & Fuller, 1979) is used to first determine whether p_{bt} and p^f_{bt} are nonstationary time series. It tests the null hypothesis that a unit root is present in a time series sample against the alternative hypothesis of stationarity. Specifically, for a time series y_t that has length T, the ADF test is undertaken by using ordinary least squares (OLS) to estimate the following autoregressive specification

$$\Delta y_t = \alpha + \beta y_{t-1} + \sum_{i=1}^{k} \psi^i \Delta y_{t-i} + \epsilon_t,\ \ \epsilon_t \sim NID\left(0, \sigma^2\right)$$

for a given lag order k, where NID denotes independent and normally distributed. The unit root test is left-tailed and carried out under the null hypothesis $\beta = 0$ against the alternative $\beta < 0$. Once a value for the test statistic $ADF_\beta = \frac{\hat{\beta}}{SE(\hat{\beta})}$ is computed, it is compared to its corresponding critical value; if the calculated test statistic is smaller, then the null hypothesis is rejected and the evidence supports that no unit root is present.

If the ADF test indicates that p_{bt} and p^f_{bt} are individually nonstationary, the Engle–Granger test can then be used to test for their cointegration. The test is based on the intuition that if two variables are cointegrated, then the residuals of their cointegrating regression will be stationary. First, OLS is used to estimate (12) and the residuals $\hat{\upsilon}_t = p_{bt} - \hat{\gamma}_0 - \hat{\gamma}_1 p^f_{bt}$ are computed. Next, the ADF test is used to test for the stationary of υ_t and the test statistic is compared to its corresponding critical values. For the relevant case with more than 500 observations and no trend in the cointegrating regression, the critical values are −3.434, −2.862 and −2.567 for 1%, 5% and 10% levels of significance, respectively (MacKinnon, 1990).

[24] Since bubbles generate an explosive component into the asset price's time series, they argued, the existence of bubbles would result in stock prices and dividends that are not cointegrated.

4.2. Testing for multiple bubbles

This section applies the bubble detection methodology developed in Phillips et al. (2015a, 2015b, 2011) to determine whether the deviations in the market price of the bitcoin from the marginal cost of mining the target supply of bitcoins $\overline{X}$ demonstrate explosive behavior.[25] These methods can detect rational bubbles as well as other bubble-generating mechanisms such as intrinsic bubbles, herd behavior and time-varying discount factor fundamentals.[26] Phillips et al. (2015a, 2015b) extend Phillips et al. (2011), which develops a supremum augmented Dickey–Fuller (SADF) test for the presence of a bubble based on a sequence of forward recursive right-tailed ADF unit root tests, and a dating strategy that identifies points of origin and termination of a bubble based on a backward regression technique. The generalized supremum ADF (GSADF) method developed in Phillips et al. (2015a, 2015b) also relies on recursive right-tailed ADF tests but uses flexible window widths in its implementation. Instead of fixing the starting point of the recursion at the first observation, the GSADF test extends the sample coverage by changing both the starting point and the endpoint of the recursion over a feasible range of flexible windows. This enhanced approach is designed to outperform previous bubble detection methods in detecting explosive behavior whenever multiple bubble episodes occur in the data since it covers more subsamples of the data. It also delivers a consistent dating mechanism whenever multiple bubbles occur.

Specifically, for a time series y_t that has length T, the SADF and GSADF tests are undertaken by using OLS to estimate the following autoregressive specification

$$y_t = \alpha_{r_1,r_2} + \beta_{r_1,r_2} y_{t-1} + \sum_{i=1}^{k} \psi^i_{r_1,r_2} \Delta y_{t-i} + \epsilon_t,\ \ \epsilon_t \sim NID\left(0, \sigma^2_{r_1,r_2}\right) \tag{13}$$

where Eq. (13) is estimated repeatedly using subsets of the sample data. The null hypothesis is that the data contains a unit root $\beta = 1$ and the alternative hypothesis postulates the presence of a mildly explosive autoregressive coefficient $\beta > 1$. If we renormalize the indices of the time series to lie within the interval $[0, 1]$, then the total sample can be indexed by values of r that range from 0 to 1. If r_1 and r_2 are the starting and ending points of a regression sample, the ADF statistic calculated from the sample is the t-statistic for the estimate of β_{r_1,r_2} and is denoted by $ADF^{r_2}_{r_1}$. The SADF statistic is defined as the supremum of the ADF statistics over the range of r_2

$$SADF\left(r_0\right) = \sup_{r_2 \epsilon [r_0, 1]} ADF^{r_2}_0$$

where r_0 is the minimum window size. A bubble occurs if the SADF statistic exceeds its critical value, which is derived from the distribution of the SADF statistic by Monte Carlo Methods.

In contrast with the SADF test, the GSADF test varies the endpoint r_2 from the minimum window size r_0 to 1, and the starting point r_1 also varies from 0 to $r_2 - r_0$. The GSADF statistic is defined as the supremum of the ADF statistics in a double recursion over all feasible ranges of r_1 and r_2

$$GSADF\left(r_0\right) = \sup_{\substack{r_2 \epsilon [r_0, 1] \\ r_1 \epsilon [0, r_2 - r_0]}} ADF^{r_2}_{r_1}.$$

A bubble occurs if the GSADF statistic exceeds its critical value, which is also derived by Monte Carlo Methods. The recommended minimum window size for both the SADF and GSADF tests is $r_0 = .01 + 1.8\sqrt{T}$, which delivers satisfactory size and power performance (Phillips et al., 2015b).

[25] Note that in order to carry out the tests, gaps in the Antminer rig specification and price data during late 2017 were filled by replacing each missing value with the most recent present value prior to it.

[26] See Phillips et al. (2015a), Footnote 5.

Date stamping bubble episodes under the new approach of Phillips et al. (2015a, 2015b) involves constructing a supremum ADF test on a backward expanding sample sequence where the endpoint of each sample is fixed at r_2 and the start point r_1 varies from 0 to $r_2 - r_0$. The estimated origination date of a bubble is defined as the first observation whose backward supremum ADF (BSADF) statistic exceeds its corresponding critical value, which is based on $r_2 T$ observations. The estimated termination date of a bubble is the subsequent observation that exceeds a specified period of time whose BSADF statistic falls below its corresponding critical value. For a bubble to be defined, it is assumed that its duration must exceed one week.[27] The date-stamping strategy may be used as an ex ante real-time dating procedure, whereas the GSADF test is an ex post statistic used for analyzing a given data set for bubble behavior. This approach improves upon the original version of the dating strategy in Phillips et al. (2011), based on a sample sequence of backward ADF (BADF) statistics, where the start point in the real-time analysis is unchanged. It has been shown that the new approach (Phillips et al., 2015a, 2015b) yields a consistent dating mechanism whenever multiple bubbles occur.

The bubble detection methodology of Phillips et al. (2015a, 2015b, 2011) is applied to test for explosive departures in the price of the bitcoin from its model-implied fundamental value. From Proposition 1(iii), it follows that the relation between the market price of the bitcoin p_b and the fundamental value p_b^f is log-linear and is given by

$$\log p_{bt} = \alpha_0 + \alpha_1 \log p_{bt}^f + v_t \tag{14}$$

for daily observations indexed by t. First, OLS is used to estimate (14) and the residuals $\hat{v}_t = \log p_{bt} - \hat{\alpha}_0 - \hat{\alpha}_1 \log p_{bt}^f$ are computed. Since all departures of the price from the marginal cost of mining the target supply of bitcoins $\overline{X}$ must be evident in the residuals $\hat{v}_t$, the SADF and GSADF tests are used to determine whether there is evidence of explosive behavior in them. Both tests are applied for the sake of completeness. While this approach is consistent with the model of Section 2, as a robustness check, the SADF and GSADF tests are also applied to the difference $\log p_{bt} - \log p_{bt}^f$, which does not rely on performing OLS.

5. Results

5.1. Testing for cointegration

The ADF test indicates that both the price of the bitcoin p_{bt} and the fundamental value p_{bt}^f are nonstationary time series. The ADF test on the price of the bitcoin p_{bt} resulted in a test statistic of −2.51 and a p-value of .36, and the ADF test on the fundamental value p_{bt}^f resulted in a test statistic of −1.90 and a p-value of .62, indicating that we cannot reject the null hypothesis of a unit root for both of the time series.[28]

Since the price of the bitcoin p_{bt} and the fundamental value p_{bt}^f are individually nonstationary, the Engle–Granger cointegration test was applied to p_{bt} and p_{bt}^f.[29] It resulted in a test statistic of −2.74 and a p-value of .071, indicating that we can reject the null hypothesis of no cointegration at the 10% level of significance. There is sufficient evidence to demonstrate that the price of the bitcoin shares a long-run equilibrium relationship with marginal cost of mining the target supply of bitcoins, supporting the model set out in Section 2, and its identification of the marginal cost of mining the target supply of bitcoins as the fundamental value of the bitcoin. All results are summarized in Table 2.

Table 2
The ADF and Engle-Granger cointegration test statistics and corresponding p-values for the price and model-implied fundamental value.

	Test statistic	P-value
ADF test on price	−2.51	.36
ADF test on FV	−1.90	.62
Engle-Granger cointegration test	−2.74	.071

5.2. Testing for multiple bubbles

Since the empirical evidence supports p_{bt}^f as the fundamental value of the bitcoin, the summary SADF and GSADF tests were applied to the OLS residuals $\hat{v}_t$ and the difference $\log p_{bt} - \log p_{bt}^f$, in addition to the raw bitcoin price $\log p_{bt}$. Table 3 presents the test statistics and the finite sample critical values obtained from Monte Carlo simulations with 2000 replications of 1764 observations. In performing the ADF regressions and calculating the critical values, the smallest window contained 93 observations of the sample based on the recommended minimum window size $r_0 = .01 + 1.8/\sqrt{1764}$.

For the price data, the SADF and GSADF statistics are 3.7 and 3.9, respectively, which both exceed their 1% right-tailed critical values ($3.7 > 2.2$ and $3.9 > 2.9$). This provides strong evidence of explosive subperiods in the bitcoin price data. The top panel of Fig. 13 depicts the sequence of BADF statistics and the corresponding 95% and 99% critical values obtained from Monte Carlo simulations with 2000 replications for each observation of interest, for the price series. The top panel of Fig. 14 depicts the analogous information for the BSADF sequence. From Fig. 13 it is clear that there is one identified period of explosive behavior for the price, when the recursive ADF statistic exceeds the 95% critical value sequence, that is greater than one week long (2017-05-01 to 2018-11-15). From Fig. 14, there are seven identified periods of explosive behavior for the price time series, when the recursive SADF statistic exceeds the 95% critical value sequence, that are greater than one week long (2014-09-28 to 2014-10-08; 2015-01-13 to 2015-01-20; 2015-11-02 to 2015-11-09; 2016-05-28 to 2016-06-21; 2016-12-22 to 2017-01-05; 2017-03-01 to 2018-05-22; and 2018-11-19 to 2018-12-16). It is clear from comparing Figs. 13 and 14 that the recursive SADF test statistic is more sensitive since, at the 5% level of significance, it identifies periods of explosive behavior that are not detected by the recursive ADF test statistic.

Next, the summary SADF and GSADF tests are applied to the residuals $\hat{v}_t$. The SADF and GSADF statistics are −.08 and 1.52, respectively, which fall well below their 10% right-tailed critical values ($-.08 < 1.3$) and ($1.5 < 2.2$). The middle panels of Figs. 13 and 14 depict the backward ADF sequence and the backward SADF sequence, respectively, for the residuals $\hat{v}_t$. Both demonstrate that there is no evidence of an explosive departure in the price of the bitcoin from the marginal cost of mining the target supply of bitcoins $\overline{X}$. Since the raw price data indicates the presence of bubbles while the residuals $\hat{v}_t$ do not, from (14) it follows that we can attribute the apparent bubbles in the price of the bitcoin to the fundamental value p_{bt}^f. Evidence of nonstationarity in the price alone does not establish the existence of bubbles since it can be attributed to the nonstationarity of a variable in market fundamentals. Because p_{bt}^f is highly correlated with the difficulty, the empirical evidence supports that the apparent bubbles in the price of the bitcoin arise from a supply side phenomenon that can be attributed to Bitcoin's difficulty adjustment mechanism.

Finally, as a robustness check, the summary SADF and GSADF tests are applied to the difference $\log p_{bt} - \log p_{bt}^f$. The SADF and GSADF statistics are −.15 and .97, respectively, which fall even further below their 10% right-tailed critical values ($-.15 < 1.3$) and ($.97 < 2.2$). The bottom panels of Figs. 13 and 14 depict the backward ADF sequence and the backward SADF sequence, respectively, for the difference $\log p_{bt} - \log p_{bt}^f$. Both tests confirm that there is no evidence of an explosive departure in the price of the bitcoin from its model-implied fundamental value.

[27] This is conservative as Phillips et al. (2015a) suggest that one may wish to impose a period of time equal to one year.

[28] The lag length k was chosen by the rule of thumb $\mathrm{trunc}(T-1)^{\frac{1}{3}}$ (Said & Dickey, 1984).

[29] Note that the same lag length as for the ADF tests on p_{bt} and p_{bt}^f was used (see Footnote 28).

Table 3
The SADF and GSADF test statistics and their respective critical values for the log price, residuals from regressing the log price on the log model-implied fundamental value, and the difference between the log price and the log model-implied fundamental value.

		Test statistic	Critical values		
			90%	95%	99%
Log price	SADF	3.725	1.3064	1.5806	2.1779
	GSADF	3.8879	2.2066	2.3842	2.8776
Residuals	SADF	–.077541	1.3064	1.5806	2.1779
	GSADF	1.5273	2.2066	2.3842	2.8776
Log price – Log FV	SADF	–.15092	1.3064	1.5806	2.1779
	GSADF	.96727	2.2066	2.3842	2.8776

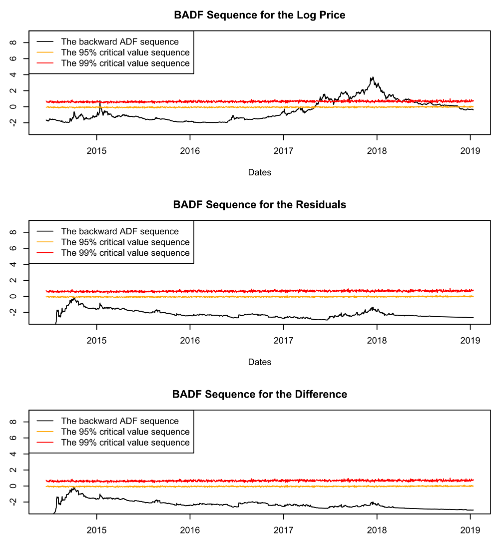

Fig. 13. BADF sequences for the log price, residuals from regressing the log price on the log model-implied fundamental value, and the difference between the log price and the log model-implied fundamental value.

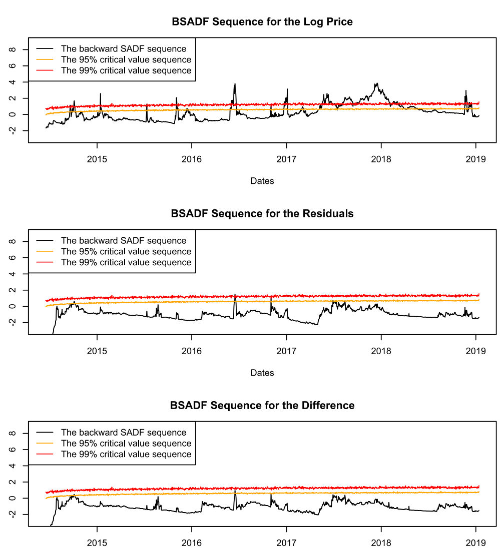

Fig. 14. BSADF sequences for the log price, residuals from regressing the log price on the log model-implied fundamental value, and the difference between the log price and the log model-implied fundamental value.

6. Conclusion

This paper develops a theoretical model of the bitcoin market that incorporates the production of bitcoins by miners and the functioning of the Bitcoin protocol. The model clearly demonstrates that because the bitcoin protocol targets a given quantity of bitcoins supplied per day, variations in demand are expressed in the price of the bitcoin, resulting in supernormal price volatility. Also, whenever the mining rate exceeds the target, since the protocol responds by increasing the difficulty, it effectively works against the market's self-correcting mechanism by decreasing supply in the presence of excess demand. (Analogously, whenever the mining rate falls short of the target, the protocol responds by decreasing the difficulty, which increases supply in the presence of excess supply.) It follows that the intervention of the protocol in the market can result in excess volatility and boom and bust phases that resemble bubble conditions despite that prices reflect fundamental bitcoin value.

A key implication of the model is that the fundamental value of the bitcoin is the marginal cost of mining the target supply of bitcoins. Since the bitcoin is not an income generating asset, it should be valued according to equilibrium market conditions. The framework reveals that when the price of the bitcoin is equal to the marginal cost of mining the target supply of bitcoins, there is equilibrium in the protocol and the rent seeking tournament among bitcoin miners. Also, following a permanent demand shock, the protocol's successive adjustments in the difficulty will cause the market price to approach the fundamental value consistent with the limiting equilibrium.

Nearly 5 years of market data demonstrate that the price of the bitcoin and the marginal cost of mining the target supply of bitcoins are cointegrated time series, which empirically supports the marginal cost of mining the target supply of bitcoins as the fundamental value of the bitcoin. Current bubble detection techniques demonstrate that while the price of the bitcoin exhibits explosive subperiods, there is no evidence of an explosive departure in the price of the bitcoin from the model-implied fundamental value. Because the residuals from regressing the price of the bitcoin on the marginal cost of mining the target supply of bitcoins do not indicate explosive behavior, it follows that we can attribute the apparent bubbles in the price of the bitcoin to the market fundamentals themselves. Furthermore, since they are highly correlated with the difficulty, the empirical evidence supports that the apparent bubbles in the price of the bitcoin can be attributed to Bitcoin's difficulty adjustment mechanism. The bitcoin price path should not be de facto considered bubbly but rather as having price dynamics that arise from the Bitcoin protocol's interference in the market.

While the evidence in this paper shows that the price of the bitcoin and the marginal cost of mining the target supply of bitcoins are cointegrated, further econometric work could analyze the dynamic process of convergence in terms of its degree and timing. This paper has established the important role of a cryptocurrency's protocol in its price determination. While the framework in this paper could be applied to any PoW cryptocurrency, it remains to be understood how other consensus mechanisms influence the price dynamics of the cryptocurrencies that use them. Furthermore, within and across types of consensus mechanisms, little is known about how the specific parameter choices encoded in the protocols influence the cryptocurrencies' exchange rates. I leave these important questions for future research.

Data availability

The authors do not have permission to share data.

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Appendix A

Proof of Proposition 1. (i) From the supply curve X_S of (8), it follows that the marginal cost of producing $\overline{X}$ bitcoins at a given level of difficulty δ is

$$p_b^f \equiv p_b\left(\overline{X};\delta\right) = \frac{\left[\eta F + \frac{\xi\phi}{1000}(24)\,p_e\right]M^*}{(\omega+f)\left[\frac{(24)60^2\phi M^* 10^9}{\delta 2^{32}}\right]}$$

where $p_b(X;\delta)$ is the inverse supply curve. (ii) From (9) it follows that

$$\frac{\delta^*}{\delta} = \frac{p_b(\omega+f)\left[\frac{(24)60^2\phi M^* 10^9}{\delta 2^{32}}\right]}{\left[\eta F + \frac{\phi\xi}{1000}(24)\,p_e\right]M^*}$$

and hence $\delta = \delta^*$ if and only if $p_b = \frac{\left[\eta F + \frac{\xi\phi}{1000}(24)p_e\right]M^*}{(\omega+f)\left[\frac{(24)60^2\phi M^* 10^9}{\delta 2^{32}}\right]} = p_b^f$.(iii) Follows directly from solving (8) and (10) for p_b.

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