Computer Assignment 2

Stochastic Processes: The Fundamentals 2024-2025

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September 24, 2024

Deadline for submission: Monday October 14, 2024 at 23:59 CET.

Instructions:

- The two computer assignments of this course should be done in groups of at least and at most two students. You should sign up for a group through the 'People' Module in Canvas.
- For each assignment, you have to write a short report in which you include at least the results, an interpretation of your results, and a conclusion. Make sure that it is clear what you did and what your interpretation of the results is. There is no maximum to the number of words or pages but please try to be concise.
- Report should be written in English. This means that you also should use English decimal notation in the text and in graphs. Use your spelling checker!
- Figures and graphs should have captions such that they can be read independently from the text in your report.
- Make sure to add your code to the Appendix. We recommend the LaTeX package 'listings'. Without the code we cannot grade your assignment.
- The preferred implementation tools are Python and MATLAB. However, you are not bound to Python or MATLAB and are free to use the programming environment that you like, but we cannot guarantee support for all languages.
- The use of AI-tools is not restricted, but remember to critically evaluate any outcomes.
- Put names and VU student numbers of all group members on the front page.
- Final reports should be submitted electronically on Canvas before 23:59 CET on the deadline day. Assignments submitted after the deadline result in a grade equal to 1.

ENJOY AND GOOD LUCK!!

1 Monte Carlo Simulation

A Geometric Brownian Motion is described by the following stochastic differential equation (SDE):

$$
\begin{aligned}
dS(t) &= rS(t)dt + \sigma S(t)dW(t) \\
S(0) &= S_0,
\end{aligned}
$$

where W is a Brownian Motion under the risk-neutral probability measure.

Let the expiry time of an option be T, and let

$$
\begin{aligned}
N &= \frac{T}{\Delta t} \\
S_n &= S(n\Delta t).
\end{aligned}
$$

Then, given an initial price S_0, M realizations of the path of a risky asset are generated using the algorithm (Euler method):

$$S_{n+1} = S_n + S_n(r\Delta t + \sigma\sqrt{\Delta t}\varphi),$$

where n indicates time and φ is a normally distributed random variable with mean zero and unit variance.

We want to price a European call option with a maturity of 3 months and a strike price equal to USD 5,600. The annualized 3-month interest rate is equal to 4% (with quarterly compounding). You may use the sample volatility of the log-returns that you constructed in Computer Assignment 1 as an estimate of the volatility of monthly log-returns of S. Furthermore, use the value of the S&P-500 index on August 31, 2024 for the starting value S_0.

The price of an option can be calculated by computing the discounted value of the average payoff, i.e.

$$V(S_0) = e^{-rT}\frac{\sum_{m=1}^{M} \text{payoff}^m(S_N)}{M},$$

where r is a continuously compounded interest rate, N is the total number of steps and M is the number of paths.

(a) Use the binomial tree method to calculate the price of a European call option with maturity 3 months and strike price USD 5,600. Use 300 steps in the tree. Compare your result with the Black-Scholes value of the call option.

(b) Use the Euler method to calculate the price of a European call option for different choices of M: 100, 500, 1000, 5000 and 10,000. Choose Δt equal to 1 month (that is, the total number of steps N is 3). Compare the results with your solutions to (a).

Please note that you are not allowed to use programming packages for Monte Carlo simulations. In other words, you are asked to perform Monte Carlo simulation from scratch.

(c) Now choose Δt to be equal to 1 trading day. Assume that there are 63 trading days per quarter. Use the Euler method to calculate the option price for different choices of M: 100, 500, 1000, 5000, and 10,000. Compare the results to the results of (b) and to the Black-Scholes value of the call option.

For special cases of constant coefficients, we can avoid time stepping errors for a Geometric Brownian Motion, as we can integrate the stochastic differential equation exactly to get:

$$S_T = S_0 e^{(r-0.5\sigma^2)T+\sigma\sqrt{T}\varphi}.$$

(d) Generate 100 independent draws from the standard normal distribution. Use the formula provided to calculate S_T for each of the 100 draws. Then calculate the value of the call option. Repeat the exercise for 500, 1000, 5000, and 10,000 draws.

(e) Compare the results of (d) to the Black & Scholes formula. In this exercise, we have tried to approximate the Black-Scholes prices in several different ways. Is the approach in (d) the most accurate approach?

(f) Compute the price of an Asian call option that depends on the average value of the S&P500 index over the last month. Consider again a maturity of 3 months and a strike price of USD 5,600. First, use the Euler method with $M = 10{,}000$ paths and Δt equal to 1 trading day to simulate independent draws of the S&P500 index over time. Then, use the discounted value method to price the Asian call option. Compare the price of the Asian call option with the price of the European call option in (c) and reflect on the difference.

Hint: The payoff of an Asian option depends on the arithmetic average value of the S&P500 over the last month, compared to a European option that only depends on the value of the S&P500 at maturity. Hence, the payoff of an Asian option for path m after N steps is

$$\text{payoff}^m(S) = \max\left(\left(\frac{1}{n}\sum_{i=N-n}^{N} S_i^m\right) - K,\, 0\right),$$

where the number of steps N is 63 days and the number of days n in the last month is 21.

2 Dynamic Hedging

In the previous exercise, we used the risk-neutral valuation method to approximate the value of a call option in the Black-Scholes world. In this course, we have learned that we can also employ the replication method to derive the no-arbitrage price of a derivative contract. This method was explicitly illustrated in a binomial tree setting. In this exercise we are going to apply that method in a perfect Black-Scholes world.

We choose the practical situation in which a trader sells a European call option on the S&P-500-index with maturity 3 months and strike price USD 5,600 to an institutional investor. We assume the absence of dividends and a perfect Black-Scholes world. The contract size of the call option contract is 100 and the trader sells 10 contracts. The annualized 3-month interest rate is equal to 4% (with quarterly compounding). You may use the sample volatility of the log-returns that you constructed in Computer Assignment 1 as an estimate of the volatility of monthly log-returns of S.

N.B.: The initial value of the S&P500 index at trade date is now assumed to be USD 5,600.

(a) Calculate the amount of money the trader receives from the institutional investor. As we live in a perfect Black-Scholes world, you can use the Black-Scholes option pricing formula.

The trader has a considerable short position in the call option on his book, i.e. if the underlying value goes up, the trader will be unhappy. So, the trader decides to set up a hedge portfolio. The purpose of this hedge portfolio is to minimize the risks of the short position in the call option. To be more precise: the trader wants to replicate a **long position in the call option**. If the replication strategy works, the value of the replicating portfolio cancels out against the value of the option at its maturity date.

(b) Set up the dynamic hedging strategy in which the hedge portfolio is adjusted weekly (use 13 weeks). Generate 5,000 paths for the S&P-500 index (for the avoidance of doubt: each path has 13 weeks). Evaluate the value development of both the hedging strategy and the call option. Provide a histogram of the P&L of the trader after 3 months. Hence, this is the P&L for the total position of shorting the call option and hedging this with a portfolio of stocks and cash. As we live in a perfect Black-Scholes world, you can use the Black-Scholes delta (which is $N(d_1)$) to set up and adjust the replicating portfolio.

Note 1:

Replication takes place in the real world, that is, under the real-world probability measure $\mathbb{P}$. Therefore, you need a model for the underlying value, the S&P-500-index under $\mathbb{P}$:

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dW(t) \\ S(0) &= S_0. \end{aligned}$$

You can assume that the per annum growth rate of the S&P-500 index, μ, is equal to 6%.

Note 2:

The option Greeks (delta, gamma, vega, rho, theta) measure the sensitivity of an option price for changes in a variable or a parameter. Delta, for instance, measures the sensitivity of an option price to a change in the underlying value of the option. Mathematically, delta is calculated as:

$$\delta = \frac{\partial C}{\partial S}.$$

If we have an analytical option pricing formula available, as in the Black-Scholes model, then we can derive δ by calculating this partial derivative. This turns out to be $N(d_1)$, where N stands for the cumulative distribution function of the standard normal distribution and

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

(c) Repeat exercise (b) but now with a monthly hedging frequency. Comment on the difference between the result of this exercise and (b).

(d) Repeat exercise (b) but now with a daily hedging frequency. Comment on the difference between the result of this exercise and (b) and (c).

(e) Repeat exercise (d) but now with $\mu = 10\%$ instead of $\mu = 6\%$. Comment on the difference between the result of this exercise and (d).

(f) Suppose now that the option is priced in the market at an **annualized** volatility that is 5%-points higher than the real-world volatility that you used to construct the hedge strategy. What is the impact of the higher option price, compared to the results of part (e)?

(g) The situation described in (f), i.e., options being more expensive than expected given the volatility of the underlying, has already been apparent in financial markets for more than three decades. This seems counterintuitive. What is a possible explanation for this empirical phenomenon?