Generating stationary noise for noise simulation

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started sometime in 1996

Abstract

Stationary noise sources are fundamental in noise analysis. This work introduces a simple new method of generating stationary noise for time-domain simulation, backed by a proof of stationarity and analytic expressions for power spectral density. The new method is useful when stationarity and precisely-known power-spectral-density functions are required for Monte-Carlo noise simulation. Noise predictions using the theory are verified against simulation results.

1 Introduction

Stationary stochastic processes play an important rôle in noise analysis. Thermal noise generated by resistors, for example, is always stationary; so is shot noise (in PN junctions) and flicker noise (in MOSFETs) in devices with DC bias. Even when device biases are time-varying, the noises generated in them are usually modelled as products of stationary stochastic processes and deterministic time-varying waveforms[**?**].

Well-established analysis techniques exist for stationary noise under quiescent biasses [**?**], based on frequency-domain linear time-invariant small-signal analysis. When biases are time-varying, or if noise levels are too large to be considered small perturbations, linear time-invariant analyses do not suffice; Monte-Carlo simulation methods [**?**] are useful for analyzing noise in these situations. Monte-Carlo simulation is also useful for verifying thories and algorithms that have been proposed recently for calculating cyclostationary noise in nonlinear systems [**?**]. For such simulation to produce reliable results, it is important to be able to generate stationary noise with a-priori known power spectral densities.

1.1 Previous Work

Existing methods for generating samples of stationary noise use spectral discretization, auto-regressive moving average (ARMA) models or broken-line processes. Spectral discretization relies on approximating the process as a finite sum of sinusoids with independent random amplitudes[**?**]. A variation on this uses deterministic amplitudes but random phases[**?**]. These methods define a continuous-time waveform and require only a fixed number of random variables to generate each sample for all time. ARMA methods generate discrete-time stationary noise by exciting a linear difference equation with i.i.d random variables[**?**]. By appropriate choice of the difference equation, different power spectral densities can be synthesized.

In this paper, a simple new way of generating stationary noise, useful for time-domain Monte-Carlo simulation, is presented. It is proved that the new technique produces stationary noise and analytic expressions for its power spectral density are presented. Using only a single current source, resistor and capacitor for each noise source, the technique is especially efficient in generating wide-band noise. Simulation results are presented to verify the new method and its power spectral density formulae.

Broken-line process - shifting creates ensemble averaging; filtering - temporal and ensemble averaging. Generalization of ARMA to continuous time. Simple interpretation in the frequency domain of the condition required for averaging to produce stationary noise.

Issues: 1. Stationarity, 2. Whiteness, 3. Ergodicity, 4. ramp-up time to asymptotic stationary value, 5. gaussianness.

Paper organization: The model and theoretical analysis using cyclostationary noise concepts, choice of parameters and numerical calculations, verification against Monte-Carlo, step-by-step procedure for designing your own stationary noise source, conclusion.

2 Generating Stationary White Noise in the Time Domain – Theory

The noise source is generated by linear interpolation of a uniformly spaced sequence of independent Gaussian sources each of variance 1. The spacing between the sources is T.

Consider the autocorrelation function $R(t,\tau) = E[x(t)x(t+\tau)]$. Now $x_1 = x(t) = \left(1 - \frac{t}{T}\right) n_1 + \frac{t}{T} n_2$, where n_1 and n_2 are the independent noise sources at time 0 and T. If $t <= T - \tau$, then $x_2 = \left(1 - \frac{t+\tau}{T}\right) n_1 + \frac{t+\tau}{T} n_2$; if $T - \tau \leq t \leq T$, $x_2 = \left(1 - \frac{t+\tau-T}{T}\right) n_2 + \frac{t+\tau-T}{T} n_3$. Here $t \in [0, T]$. Then

$$R(t,\tau) = \begin{cases} \left(1 - \frac{t}{T}\right)\left(1 - \frac{t+\tau}{T}\right) + \frac{t}{T}\frac{t+\tau}{T} & \text{if } -t \leq \tau \leq T - t; \\ \frac{t}{T}\left(1 - \frac{t+\tau-T}{T}\right) & \text{if } T - t \leq \tau \leq 2T - t; \\ 0 & \text{if } 2T - t \leq \tau; \\ \left(1 - \frac{t}{T}\right)\left(1 + \frac{t+\tau}{T}\right) & \text{if } -T - t \leq \tau \leq -t; \\ 0 & \text{if } \tau \leq -T - t. \end{cases}$$

If τ is fixed, the autocorrelation is T-periodic in t. For $0 \leq t \leq T$, fix τ at a given value, and rewrite the above to show better the variation with t:

$$R(t,\tau) = \begin{cases} 0 & \text{if } t \leq -T - \tau; \\ \left(1 - \frac{t}{T}\right)\left(1 + \frac{t+\tau}{T}\right) & \text{if } -T - \tau \leq t \leq -\tau; \\ \left(1 - \frac{t}{T}\right)\left(1 - \frac{t+\tau}{T}\right) + \frac{t}{T}\frac{t+\tau}{T} & \text{if } -\tau \leq t \leq T - \tau; \\ \frac{t}{T}\left(1 - \frac{t+\tau-T}{T}\right) & \text{if } T - \tau \leq t \leq 2T - \tau; \\ 0 & \text{if } 2T - \tau \leq t. \end{cases}$$

$R(t,\tau)$ is plotted in Figure 1 ($T=1$).

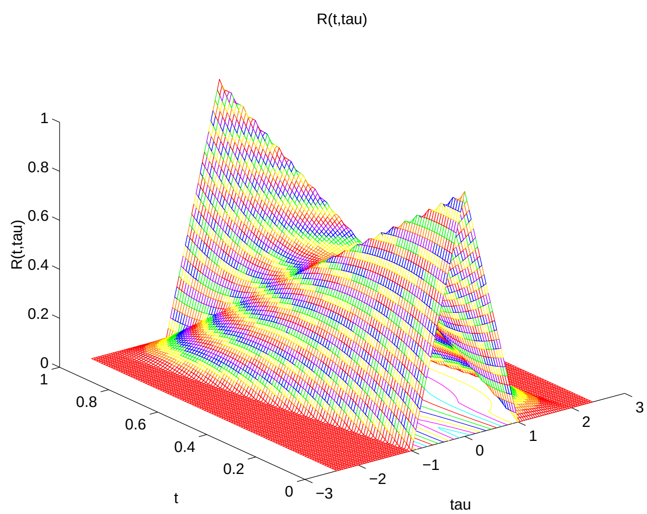

Figure 1:

We want to expand $R(t,\tau)$ in a Fourier series in t. We are interested in particular in the DC component of the Fourier series since it is the stationary component. Hence for a fixed τ, we need to find the average value of $R(t,\tau)$ over $t \in [0,T]$. To do this, we look at $R(t,\tau)$ over segments of τ:

$$R(t,\tau) = \begin{cases} 0 & \text{if } \tau < -2T; \\ \begin{cases} 0 & \text{if } 0 \le t \le -T-\tau; \\ \left(1-\frac{t}{T}\right)\left(1+\frac{t+\tau}{T}\right) & \text{if } -T-\tau \le t \le T. \end{cases} & \text{if } -2T \le \tau \le -T; \\ \begin{cases} \left(1-\frac{t}{T}\right)\left(1+\frac{t+\tau}{T}\right) & \text{if } 0 \le t \le -\tau; \\ \left(1-\frac{t}{T}\right)\left(1-\frac{t+\tau}{T}\right)+\frac{t}{T}\frac{t+\tau}{T} & \text{if } -\tau \le t \le T. \end{cases} & \text{if } -T \le \tau \le 0; \\ \begin{cases} \left(1-\frac{t}{T}\right)\left(1-\frac{t+\tau}{T}\right)+\frac{t}{T}\frac{t+\tau}{T} & \text{if } 0 \le t \le T-\tau; \\ \frac{t}{T}\left(1-\frac{t+\tau-T}{T}\right) & \text{if } T-\tau \le t \le T. \end{cases} & \text{if } 0 \le \tau \le T; \\ \begin{cases} \frac{t}{T}\left(1-\frac{t+\tau-T}{T}\right) & \text{if } 0 \le t \le 2T-\tau; \\ 0 & \text{if } 2T-\tau \le t \le T. \end{cases} & \text{if } T \le \tau \le 2T; \\ 0 & \text{if } \tau \ge 2T. \end{cases}$$

Taking the average over $t \in [0, T]$, we get

$$R_0(\tau) = \begin{cases} 0 & \text{if } \tau < -2T; \\ \frac{1}{T}\left(\frac{12\,\tau\,T^2+8\,T^3+6\,T\tau^2+\tau^3}{6\,T^2}\right) & \text{if } -2T \le \tau \le -T; \\ \frac{1}{T}\left(-\frac{\tau\left(6\,T^2+\tau^2+6\,T\tau\right)}{6\,T^2}+\frac{2\,T^3-\tau^3+3\,\tau\,T^2}{3\,T^2}\right) & \text{if } -T \le \tau \le 0; \\ \frac{1}{T}\left(\frac{2\,T^3-3\,\tau\,T^2+\tau^3}{3\,T^2}+\frac{\tau\left(6\,T^2-6\,T\tau+\tau^2\right)}{6\,T^2}\right) & \text{if } 0 \le \tau \le T; \\ \frac{1}{T}\,\frac{(2\,T-\tau)^3}{6\,T^2} & \text{if } T \le \tau \le 2T; \\ 0 & \text{if } \tau \ge 2T. \end{cases}$$

This function is plotted in Figure 2 ($T = 1$): The stationary power spectral density

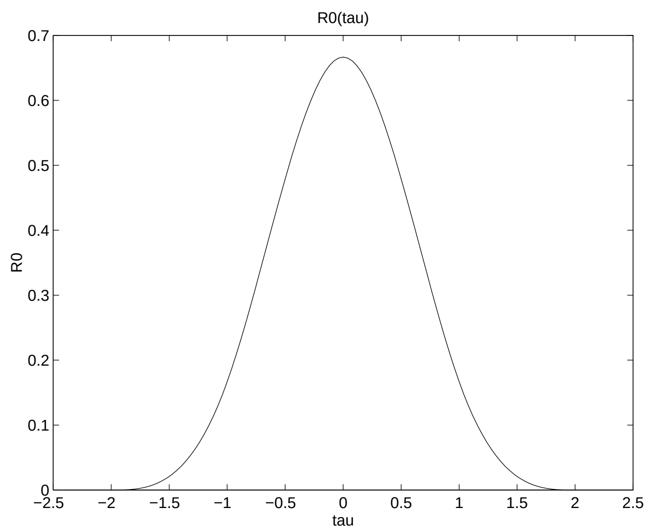

Figure 2:

$S_0(\omega)$ is the Fourier transform of $R_0(\tau)$, given by:

$$S_0(\omega) = -\frac{8\,\cos(T\omega) - 2\,\cos(2\,T\omega) - 6}{\omega^4 T^3}$$

As $\omega \to 0$, $S_0(\omega) \to T$. $S_0(\omega)$ for $T = 1$ns is shown in Figures 3 and 4.

Stationarity can be achieved simply by translating the grid of sample points randomly (uniform distribution over $[0, T]$). Alternatively, if this cyclostationary source is sent through a low-pass filter of bandwidth less that $\frac{1}{2T}$, then all higher-order PSDs $S_i, i \neq 0$ will be completely killed and only S_0 will be passed. Hence the output noise will be stationary, and *if the filter bandwidth is less than* $\frac{1}{100T}$ *or so, the cyclostationary source will look like a stationary white noise source with a flat PSD of* T.

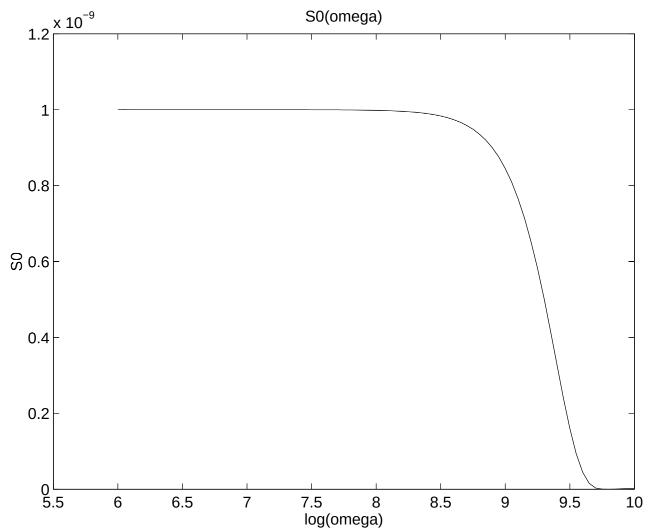

Figure 3:

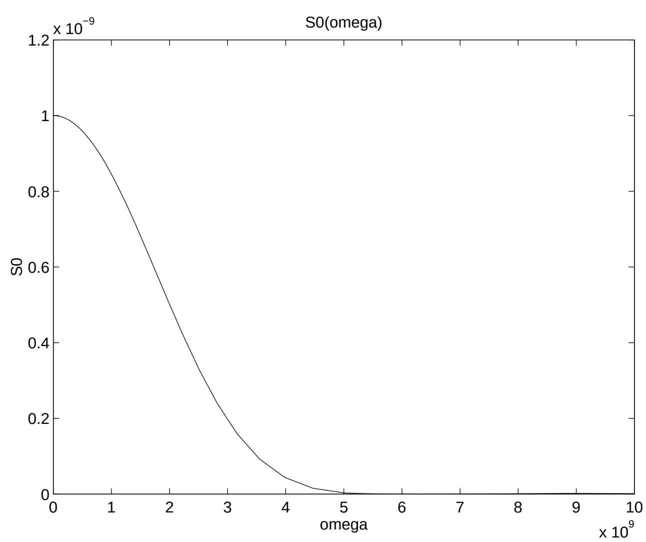

Figure 4:

3 Numerical Calculations

For a current source of unit variance feeding a resistor in parallel with a capacitor, we have $H(s) = \frac{R}{1+sRC}$, $H(f) = \frac{R}{1+j2\pi fRC}$. $\int |H(f)|^2\, df = \frac{R}{2\pi C} \tan^{-1}(2\pi RCf)$, hence $\int_{-\infty}^{\infty} |H(f)|^2\, df = \frac{R}{2C}$. Thus the variance of the voltage generated is approximately $\frac{RT}{2C}$ - this is an upper bound.

More accurately, we have

$$\begin{aligned} R_0(0) &= \int_{-\infty}^{\infty} |H(f)|^2\, S_0(f)\, df \\ &= -\int_{-\infty}^{\infty} \left| \frac{R}{1+j2\pi fRC} \right|^2 \frac{8\,\cos(2\pi fT) - 2\,\cos(4\pi fT) - 6}{(2\pi f)^4 T^3}\, df \\ &= -\int_{-\infty}^{\infty} \frac{R^2}{1+4\pi^2 f^2 R^2 C^2} \frac{8\,\cos(2\pi fT) - 2\,\cos(4\pi fT) - 6}{(2\pi f)^4 T^3}\, df \qquad (1) \end{aligned}$$

This integral is difficult to evaluate analytically (i.e., maple has trouble) so we have to resort to numerical integration for given values of R, C and T. For the mixer and filter combination circuit in doc/noise/montecarlo/mixerfilter, the following values are used:

$$R = 1\Omega, \qquad C = 5\mu F, \qquad T = 1\mu s$$

The RC pole is at $\frac{200}{2\pi}$KHz. The square of the transfer function magnitude is shown in Fig. 5. The integrand of Equation 1 is shown in Figure **??**. When integrated numerically

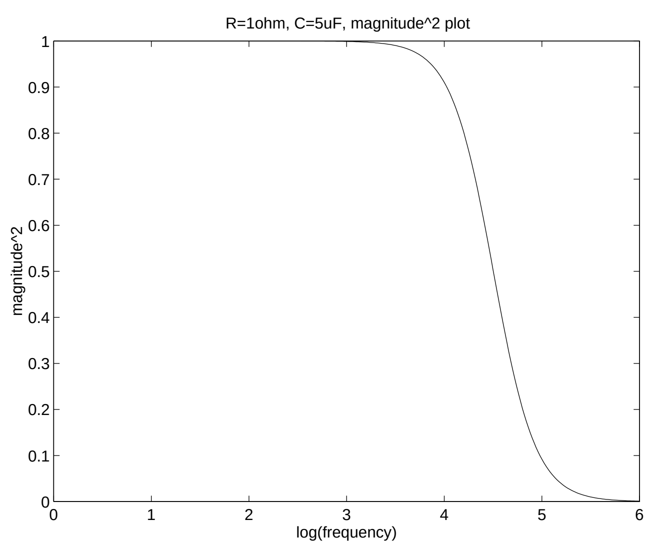

Figure 5:

over the interval $[-10^7, 10^7]$ (using maple), we get a result of 0.0912 for $R_0(0)$, the stationary variance. The upper-bound $\frac{RT}{2C}$ equals 0.1; hence we see that in this case, the rolloff of $S_0(\omega)$ makes a difference of about 10% to the variance.

4 Verification against Monte-Carlo Simulations

Brute-force Monte-Carlo simulations were performed in order the verify the above predictions. The Monte-Carlo simulations were designed to estimate, as a function of time, the variance of the output noise generated by passing a linearly interpolated comb of i.i.d gaussian random variables of unit variance, with comb spacing T, through an RC filter. Two basic approaches were taken to simulating this system: 1. solving the differential equation numerically using a general-purpose numerical integration method, and 2. using the fact that the differential equation for a RC network driven by a PWL current source can be integrated analytically, and applying the formulae thus obtained. The values of R, C and T were the same as in Section 3.

The first approach, solving the differential equation numerically, was tried using the internal circuit simulator ADVICE. A sample size of 10,000 input waveforms was chosen initially, each extending to a length of 5 RC time-constants (200us). This translated to an execution time of about 1 month on a Sun SparcStation 20; this being impractical, the sample size was reduced to 1000 and the simulations were completed in three days. The outputs were postprocessed to obtain the noise variances. The variances of the PWL current source and the filtered output voltage, plotted over a period $30T$ at the end of the simulation, are shown in Figures 6 and 7. The cyclostationary change of the variance of the input current source, between 1.0 and 0.5, can be seen in Figure 6; it is also evident, from Figure 7, that no such cyclostationary variation exists for filtered output voltage variance, which wanders between approximately 0.086 and 0.1 on a scale much coarser than T=1us. (The circuit and results for this experiment are in doc/noise/montecarlo/mixerfilter-advice.)

Since the 1000 samples in the experiment above were not enough to verify conclusively the theoretically predicted output noise variance of 0.0912 (from Section 3), 10000 samples were tried next. To avoid the excessive computation time needed by ADVICE, a custom program was written in C/C++ to perform numerical integration of the circuit's differential equation. A fourth-order Runge-Kutta method (using the Numerical Recipes routine odeint) was used. The simulation took about 10 hours of computer time. The variances of the input current and output filtered voltage, as functions of time, are shown in Figures 8, 9 and 10. It can be seen that the output variance is stationary, at a value between about 0.088 and 0.095, or centered around 0.0915, close to the predicted value of 0.0912.

To obtain a more precise estimate, the number of sample waveforms was increased to 100000. In order to reduce the simulation time and increase accuracy of results, the second approach mentioned above was tried. A program based on analytic expressions for the ODE solution when excited with a PWL source, was used (the code is in doc/noise/montecarlo/rc-in-c++). This program took about 30 minutes to execute.

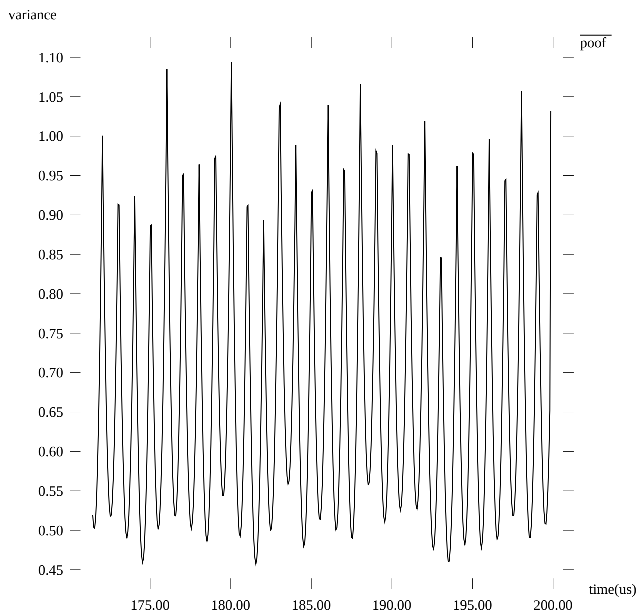

Figure 6: 1000 ADVICE simulations: variance of input current source wrt time

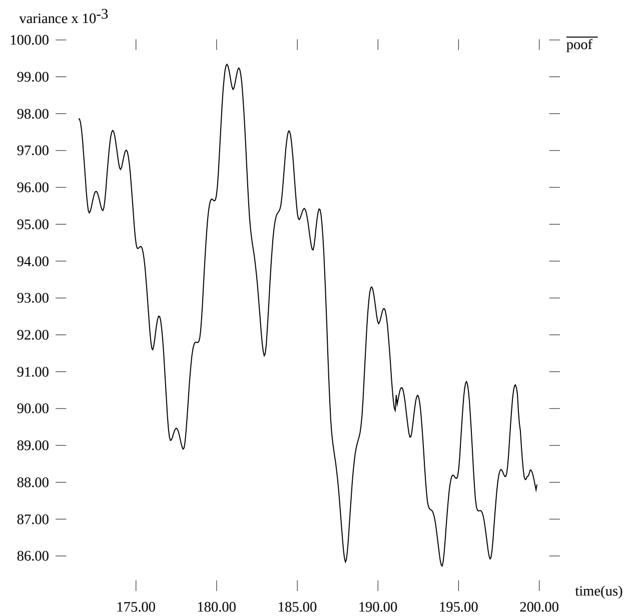

Figure 7: 1000 ADVICE simulations: variance of RC-filtered output voltage wrt time

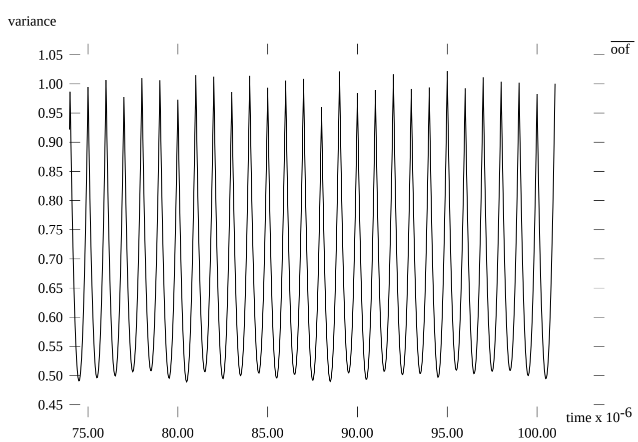

Figure 8: 10000 `odeint` simulations: variance of input current source wrt time

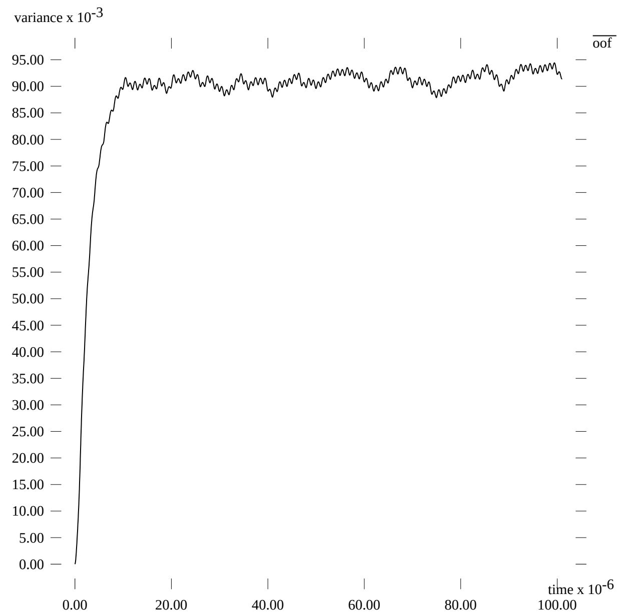

Figure 9: 10000 odeint simulations: variance of RC-filtered output voltage wrt time

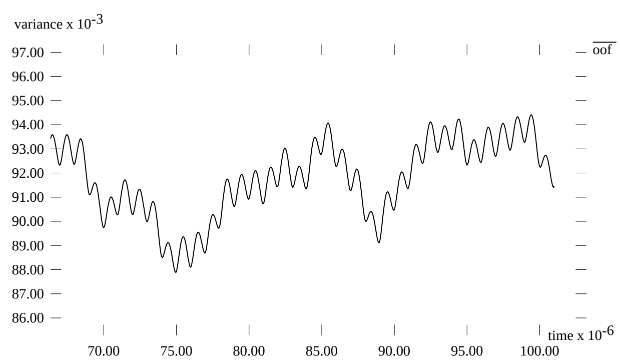

Figure 10: 10000 odeint simulations: variance of RC-filtered output voltage wrt time (detail)

The input and output variances are shown in Figures 11 and 12. The output variance is again stationary, between about 0.090 and 0.0925, or centered around 0.0915, again close to the predicted value of 0.0912.

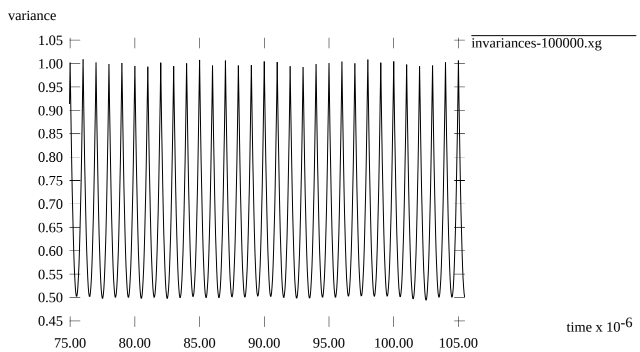

Figure 11: 100000 analytic simulations: variance of input current source wrt time

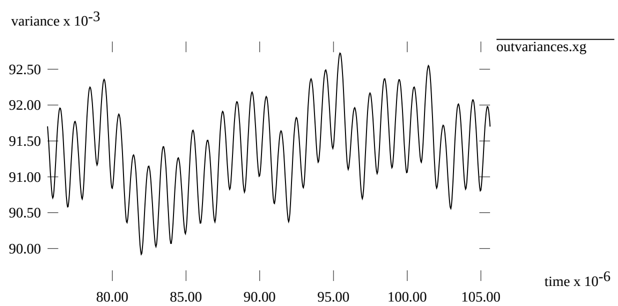

Figure 12: 100000 analytic simulations: variance of RC-filtered output voltage wrt time

The above Monte-Carlo experiments verify the stationarity of the output noise and provide a figure for the stationary noise variance that is consistent with the theory of the previous sections. Another important property of stationary processes is *ergodicity*, i.e., the condition that time-averages of each sample of the stationary process equal ensemble averages at any time. To verify ergodicity, simulations using the analytic formulae were carried out for a length of about 40000 RC time-constants and the average power at the output of the filter computed[1]. Values for this average power from 5 runs (5 samples from the ensemble) were 0.09128, 0.09124, 0.09114, 0.09110 and 0.09104,

[1] Note that a simulation of many times the RC time constant, i.e., the time-period of correlation in the process, is necessary to obtain results meaningful for ergodicity.

matching closely the value predicted by theory in Section 3 and the ensemble-average variance obtained from Monte-Carlo simulation.

5 Conclusion