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ME564 - Autumn 2025

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HOMEWORK - 01

Exercise 1-1: Compute the derivate of following:

(a) $f(x) = \sin(x^2)$

⇒ using chain rule to solve this composite function

As we know,

$$\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x) \quad \text{--- (I)}$$

So, $f(a) = \sin(a)$

↳ $g(x) = x^2$

∴ $f'(a) = \cos(a) \quad \text{--- (1)}$

↳ $g(x) = x^2 \Rightarrow g'(x) = 2x \quad \text{--- (2)}$

Using (1) & (2) in (I), we get

$$\begin{aligned} f'(x) &= f'(g(x)) \cdot g'(x) \\ &= \cos(x^2) \cdot 2x \end{aligned}$$

$$\therefore \cancel{f'(x) = 2x \cos(x)^2} \quad \boxed{f'(x) = 2x \cos(x^2)}$$

(b) $f(x) = x^x$

⇒ Let $y = x^x \quad \text{--- (1)}$

By taking natural logarithm on both sides,

$$\ln(y) = \ln(x^x)$$

$$\therefore \ln(y) = x \ln(x)$$

Differentiate on both sides

$$\frac{d}{dx} [\ln(y)] = \frac{d}{dx} [x \ln(x)]$$

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using chain rule of left side & product rule on right side, we get:

$$\frac{1}{y} \frac{dy}{dx} = \ln(x) + 1$$

→ Solving for $\frac{dy}{dx}$; multiply both sides by y .

$$\therefore \frac{dy}{dx} = y(\ln(x) + 1)$$

As we know from (1) that $y = x^x$

$$\therefore \frac{dy}{dx} = x^x (\ln(x) + 1)$$

$$\therefore \boxed{f'(x) = x^x (\ln(x) + 1)}$$

(c) $f(x) = e^{\cos(3x)} \cos(x)$

⇒ As per product rule:

$$(x \cdot y)' = x'y + y'x \quad \text{--- (1)}$$

So, let $x = e^{\cos(3x)}$

$$\& y = \cos(x)$$

→ Differentiate each part:

$$x' = e^{\cos(3x)} \cdot (-3 \sin(3x))$$

$$x' = -3e^{\cos(3x)} \sin(3x) \quad \text{--- (2)}$$

$$\& y' = -\sin(x) \quad \text{--- (3)}$$

Now, from (1), (2) & (3) we get,

$$\cancel{f'(x)} \quad f'(x) = [-3 \sin(3x) e^{\cos(3x)}] \cdot \left[\frac{\cos(x)}{\cancel{\sin(x)}} \right] + e^{\cos(3x)} (-\sin(x))$$

$$\therefore \boxed{f'(x) = e^{\cos(3x)} \left\{ [-3 \sin(3x) \cdot \cos(x)] - [\sin(x)] \right\}}$$

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Q. $f(x, y) = \sin(x^2 - y^2)$ w.r.t t , assuming that $x(t)$ & $y(t)$ vary with time.
 \Rightarrow Lets assume

$x = x(t)$ & $y = y(t)$ are function of time
Now, by chain rule: (multivariable)

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \quad \text{--- (1)}$$

So, partial derivatives are as follows:

$$\frac{\partial f}{\partial x} = \cos(x^2 - y^2) \cdot 2x \quad \text{--- by chain rule}$$

$$\frac{\partial f}{\partial y} = -2y \cdot \cos(x^2 - y^2)$$

Using above in eqⁿ (1), we get:

$$\frac{df}{dt} = [2x \cos(x^2 - y^2)] \frac{dx}{dt} + [-2y \cos(x^2 - y^2)] \frac{dy}{dt}$$

$$\therefore \boxed{\frac{df}{dt} = 2 \cos(x^2 - y^2) \left[x \frac{dx}{dt} - y \frac{dy}{dt} \right]}$$

Exercise 1-2: A given mass x of a radioactive element obeys the following differential equation: $\dot{x} = -\lambda x$ where $\lambda \Rightarrow$ constant of rate of decay

@ Solution for $x(t)$ to the differential equation
 \Rightarrow Rearranging eqⁿ & integrating on both sides:

$$\frac{dx}{x} = -\lambda dt$$

$$\therefore \int \frac{1}{x} dx = \int -\lambda dt$$

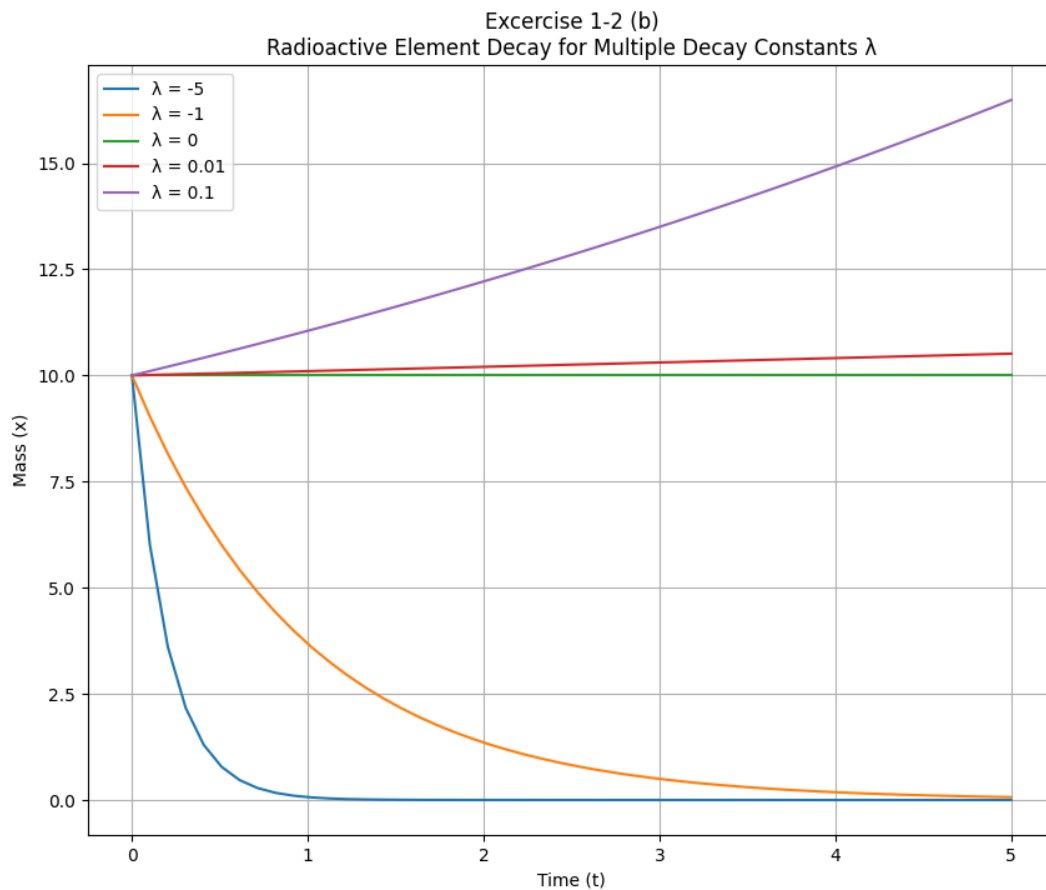
$$\therefore \ln|x| = -\lambda t + C$$

Exponentiating on both sides:

$$x(t) = e^{-\lambda t + C} = e^C e^{-\lambda t}$$

Here, $x_0 = e^C$ as the initial mass at $t=0$

$$\therefore \boxed{x(t) = x_0 e^{-\lambda t}}$$



```
'''
Filename: e_2_b.py
Date Created: 10/09/2025
Created by: Archit Jain
Email: architj@uw.edu
Student No: 2426587
Description: Exercise 1-2: (b)
A given mass x of a radioactive element obeys the following differential equation
in time:
 $\dot{x} = \lambda x$ ,
where  $\lambda$  is a constant describing the rate of decay.
Plot the solution for an initial condition  $x(0) = 10$ 
From time  $t = 0$  to  $t = 5$  for  $\lambda = -5, -1, 0, 0.01, 0.1$ .
Please plot these all on the same figure. Include a legend.
'''

# Importing libraries
import numpy as np          # for arrays data storage
```

```
import matplotlib.pyplot as plt      # for plotting graphs

# Define the function to compute x(t)
def compute_x_t(x_0, t, lambda_const):
    '''
    A function that loops and plot for each lambda_const with graph dim 10x8
    '''
    plt.figure(figsize=(10, 8))
    # looping for lambda in given list
    for  $\lambda$  in lambda_const:
        #  $x(t) = x(0) * e^{(\lambda * t)}$ 
        x_t = x_0 * np.exp( $\lambda$  * t)
        plt.plot(t, x_t, label=f' $\lambda = \{\lambda\}$ ')
    return plt

# Given: Initial condition
x_0 = 10

# Setting a time array from 0->5 with 50 points from t=0 to t=5
t = np.linspace(0, 5, 50)

# Setting list for decay constants  $\lambda$  from the given list:
lambda_const = [-5, -1, 0, 0.01, 0.1]

# Compute x(t) and plot them w.r.t t for different lambda
plt = compute_x_t(x_0, t, lambda_const)

# Adding legends to graph plotted
plt.xlabel('Time (t)')
plt.ylabel('Mass (x)')
plt.title('Exercise 1-2 (b)\nRadioactive Element Decay for Multiple Decay Constants  $\lambda$ ')
plt.legend()

# Show grid to the plot
plt.grid(True)

# Display the graph plotted
plt.show()
```


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© The half-life T is defined as the time it takes for the material to be reduced to half of its mass through radioactive decay. The half-life of Uranium-238 is 4.468 billion years. What is corresponding value of λ ?

$\Rightarrow T = \text{half-life time}$
(time taken to be reduced to half its initial mass.)

\therefore eqⁿ with decay formula:

$$x(t) = x_0 e^{\lambda t}$$

becomes,

$$\frac{1}{2} x_0 = x_0 e^{\lambda T}$$

$$\therefore \frac{1}{2} = e^{\lambda T}$$

Taking log on both sides:

$$\ln\left(\frac{1}{2}\right) = \lambda T$$

$$-\ln(2) = \lambda T$$

Solving for value λ ;

$$\lambda = \frac{-\ln(2)}{T}$$

As, Uranium-238 = 4.468×10^9 years \rightarrow (given)

$$\therefore \lambda = \frac{-\ln(2)}{4.468 \times 10^9 \text{ years}} \approx -1.551 \times 10^{-10} \text{ years}^{-1}$$

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(d) If you start with 50 kg of uranium-238, how long until you only have 5 kg left? If you have 10 kg of uranium-238, how long ago was it 50 kg?

⇒ How long until 5 kg is left from 50 kg?

Here, initial mass $x_0 = 50$ kg

final mass $x(t) = 5$ kg

$$\text{As, } x(t) = x_0 e^{\lambda t}$$

$$\lambda = -1.551 \times 10^{-10} \rightarrow \text{from previous solution (c)}$$

$$\therefore 5 = 50 e^{(-1.551 \times 10^{-10})t} \Rightarrow 0.1 = e^{(-1.551 \times 10^{-10})t}$$

Taking log on both sides,

$$\ln(0.1) = (-1.551 \times 10^{-10})t$$

$$\therefore t \approx \frac{-2.3026}{-1.551 \times 10^{-10}} \approx 1.485 \times 10^{10}$$

$$\therefore \boxed{t \approx 1.485 \times 10^{10} \text{ years}}$$

⇒ How long ago was it 50 kg, if it is 10 kg now?

Here, $x_0 = 50$ kg → initial mass

$x(t) = 10$ kg → final mass (now)

$$\text{As, } x(t) = x_0 e^{\lambda t}$$

$$\therefore 10 = 50 e^{(-1.551 \times 10^{-10})t}$$

$$0.2 = e^{(-1.551 \times 10^{-10})t}$$

Taking log on both sides:

$$\ln(0.2) = (-1.551 \times 10^{-10})t$$

$$\therefore t \approx \frac{-1.6094}{-1.551 \times 10^{-10}} \approx 1.038 \times 10^{10}$$

∴ Approx. $\boxed{10.38 \text{ billion year ago}}$ there was 50 kg of uranium-238

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Exercise 1-3: Compute Taylor series expansion by hand for $f(x)$

(a) $f(x) = \sin(x)/x$

⇒ As per Maclaurin series at basepoint $a=0$

$$f(x) = f(a) + \frac{df}{dx}(x-a) + \frac{d^2f}{dx^2} \frac{(x-a)^2}{2!} + \dots$$

--- evaluated at $a=0$ →

$$\therefore f(x) = f(0) + f'(0)(x-0) + \frac{f''(0) \cdot (x-0)^2}{2!} + \dots$$

As $f(x) = \sin(x)/x$ for $\sin(x)$ ⇒

$$f(x) = \sin(0) + \cos(0)(x) + \left(-\sin(0) \times \frac{x^2}{2!} \right) + \left(-\cos(0) \cdot \frac{x^3}{3!} \right)$$

$$= 0 + 1 \cdot x + \left(-0 \times \frac{x^2}{2!} \right) + \left(-1 \cdot \frac{x^3}{3!} \right) + \dots$$

$$\therefore f(x) \Rightarrow \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

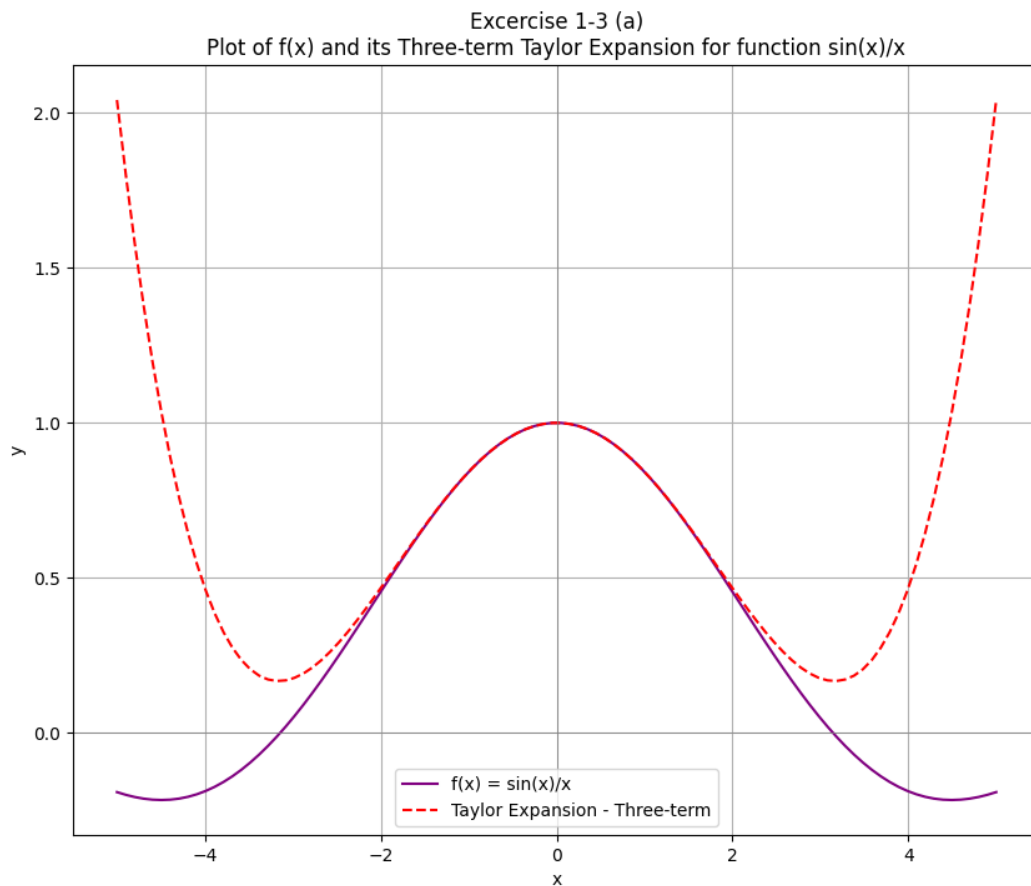
∴ Taylor series for $\frac{\sin(x)}{x}$ is by dividing above by x

$$\therefore \frac{\sin(x)}{x} = \frac{1}{x} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right)$$

$$\boxed{\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots}$$

First three non-zero terms of above series are:

$$P_3(x) = 1 - \frac{x^2}{6} + \frac{x^4}{120}$$



```
'''
Filename: e_3_a.py
Date Created: 10/09/2025
Created by: Archit Jain
Email: architj@uw.edu
Student No: 2426587
Description: Excercise 1-3: (a)
Compute the Taylor series expansion by hand for f (x).
For each function, plot f (x) = sin(x)/x
and the three-term expansion (i.e., the first three nonzero terms) from x = -5 to
x = 5.
'''

# Importing libraries
import math                # for math operations
import numpy as np         # for arrays data storage
import matplotlib.pyplot as plt  # for plotting graphs
```



```
# Define the function f(x)
def compute_f(x):
    '''
    A function that computes f(x) = sin(x)/x for different values of x
    and also handles values where x=0 to be used as 1 to avoid singularity
    errors.
    '''
    return np.where(x == 0, 1, np.sin(x) / x)

# Define the three-term Taylor expansion
def compute_taylor_expansion(x):
    '''
    A function that returns 3-term taylor expansion of f(x) = sin(x)/x
    # P_3(x) = [1 - (x^2/3!) + (x^4/5!)]
    '''
    first_term = 1
    second_term = (x**2) / math.factorial(3)
    third_term = (x**4) / math.factorial(5)
    return first_term - second_term + third_term

# Setting a x array from -5->5 with 100 points from x=-5 to x=5
x_val = np.linspace(-5, 5, 100)

# Calculate y values for f(x)
f_x = compute_f(x_val)
# Calculate the Three-term Taylor expansion
y_val = compute_taylor_expansion(x_val)

# Setting graph dimensions 10x8
plt.figure(figsize=(10, 8))
# Plotting points on graph
plt.plot(x_val, f_x, label='f(x) = sin(x)/x', color='purple')
plt.plot(x_val, y_val, label='Taylor Expansion - Three-term', color='red',
linestyle='--')
# Adding title and legends to graph plotted
plt.xlabel('x')
plt.ylabel('y')
plt.title('Excercise 1-3 (a)\nPlot of f(x) and its Three-term Taylor Expansion
for function sin(x)/x')
plt.legend()
# Show grid to the plot
plt.grid(True)
# Highlighting the origin lines
plt.axvline(0, color='gray', linestyle='-', linewidth=0.5)
plt.axhline(0, color='gray', linestyle='-', linewidth=0.5)
```

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Q. 10. (b) $f(x) = (\cos(x) - 1)/x$
 \Rightarrow The Maclaurin series for $\cos(x)$ at $a=0$ is

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

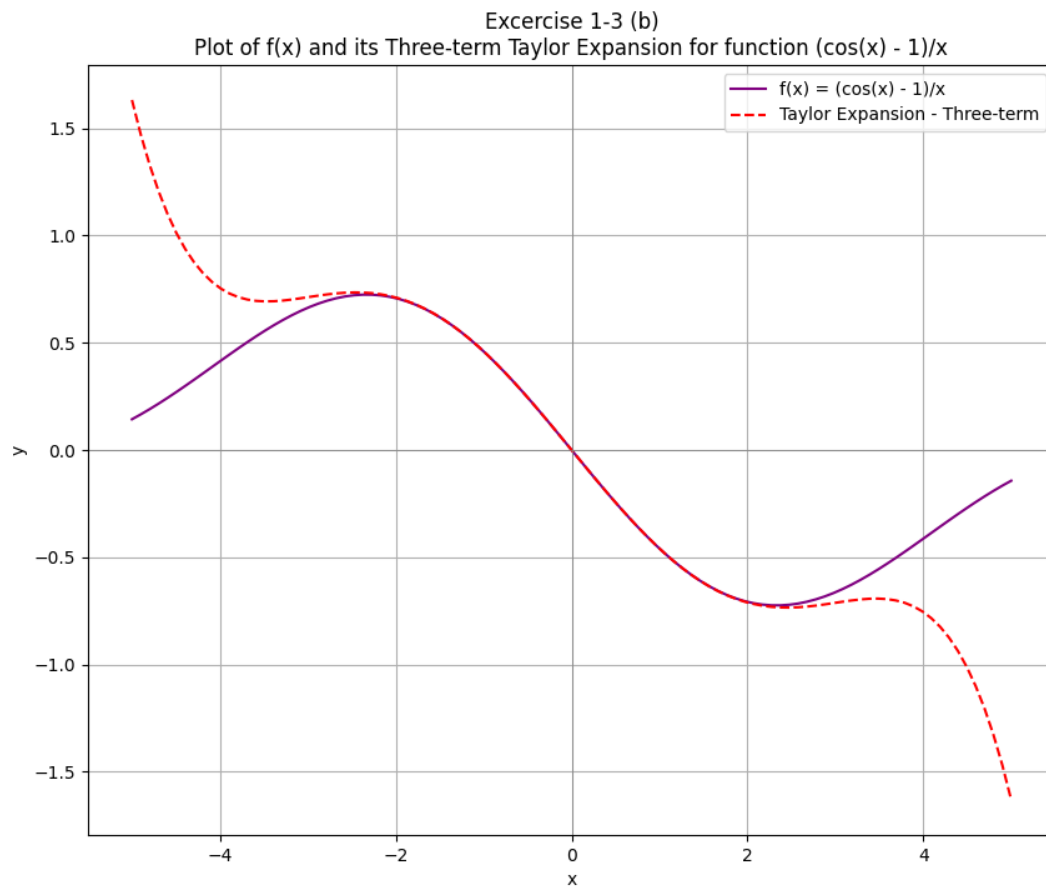
So, to solve for $(\cos(x) - 1)/x$,

$$\frac{(\cos(x) - 1)}{x} = \frac{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) - 1}{x}$$

$$\frac{\cos(x) - 1}{x} = \frac{1}{x} \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right)$$

$$\frac{\cos(x) - 1}{x} = -\frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \dots$$

The first three non-zero terms of above series are:
 $P_3(x) = -\frac{x}{2} + \frac{x^3}{24} - \frac{x^5}{720}$



```
'''
Filename: e_3_b.py
Date Created: 10/09/2025
Created by: Archit Jain
Email: architj@uw.edu
Student No: 2426587
Description: Excercise 1-3: (b)
Compute the Taylor series expansion by hand for  $f(x)$ .
For each function, plot  $f(x) = (\cos(x) - 1)/x$ 
and the three-term expansion (i.e., the first three nonzero terms) from  $x = -5$  to
 $x = 5$ .
'''

# Importing libraries
import math                # for math operations
import numpy as np         # for arrays data storage
import matplotlib.pyplot as plt  # for plotting graphs
```

```
# Define the function f(x)
def compute_f(x):
    '''
    A function that computes  $f(x) = (\cos(x) - 1)/x$  for different values of x
    and also handles values where  $x=0$  to be used as 0
    '''
    if x==0:
        # limit of  $(\cos(x) - 1)/x$  as  $x \rightarrow 0$  is 0
        return 0
    return (np.cos(x) - 1) / x

# Define the three-term Taylor expansion
def compute_taylor_expansion(x):
    '''
    A function that returns 3-term taylor expansion of  $f(x) = (\cos(x) - 1)/x$ 
    #  $P_3(x) = [(-x/2) + (x^3/4!) - (x^5/6!)]$ 
    '''
    first_term = -1*x/2
    second_term = (x**3) / math.factorial(4)
    third_term = (x**5) / math.factorial(6)
    return first_term + second_term - third_term

# Setting a x array from -5->5 with 100 points from x=-5 to x=5
x_val = np.linspace(-5, 5, 100)

# Calculate y values for f(x) for each value of x
f_x = [compute_f(val) for val in x_val]
# Calculate the Three-term Taylor expansion
y_val = compute_taylor_expansion(x_val)

# Setting graph dimensions 10x8
plt.figure(figsize=(10, 8))
# Plotting points on graph
plt.plot(x_val, f_x, label='f(x) = (cos(x) - 1)/x', color='purple')
plt.plot(x_val, y_val, label='Taylor Expansion - Three-term', color='red',
linestyle='--')
# Adding title and legends to graph plotted
plt.xlabel('x')
plt.ylabel('y')
plt.title('Excercise 1-3 (b)\nPlot of f(x) and its Three-term Taylor Expansion
for function (cos(x) - 1)/x')
plt.legend()
# Show grid to the plot
plt.grid(True)
# Highlighting the origin lines
```



```
plt.axvline(0, color='gray', linestyle='-', linewidth=0.5)
plt.axhline(0, color='gray', linestyle='-', linewidth=0.5)
# Display the graph plotted
plt.show()
```

Exercise 1-4: Please compute an analytic expression, by hand, for the real & imaginary parts of the following complex numbers functions.

(a) $f(t) = e^{it}$

⇒ As per Euler's formula for any real number x :

$$e^{ix} = \cos(x) + i\sin(x)$$

$$f(t) = e^{it} = \cos(t) + i\sin(t)$$

Real part $\Rightarrow \cos(t)$

Imaginary part $\Rightarrow \sin(t)$

(b) $f(t) = e^{(1-i)t}$

$$\Rightarrow f(t) = e^{(1-i)t} = e^{t-it} = e^t \cdot e^{-it}$$

From Euler's formula, we get, (here $x = -t$)

$$\therefore e^{-it} = \cos(-t) + i\sin(-t)$$

$$e^{-it} = \cos(t) - i\sin(t)$$

∴

$$f(t) = e^t \cdot (\cos(t) - i\sin(t))$$

∴

$$f(t) = e^t \cos(t) - i(e^t \sin(t))$$

Real part $\Rightarrow e^t \cos(t)$

Imaginary part $\Rightarrow -e^t \sin(t)$

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$$\textcircled{c} f(t) = e^{(-1-\pi i)t}$$

$$\Rightarrow f(t) = e^{-t-\pi it} = e^{-t} \cdot e^{-\pi it}$$

using Euler's formula,

↓

$$e^{ix} = \cos(x) + i\sin(x)$$

here, $x = -\pi t$

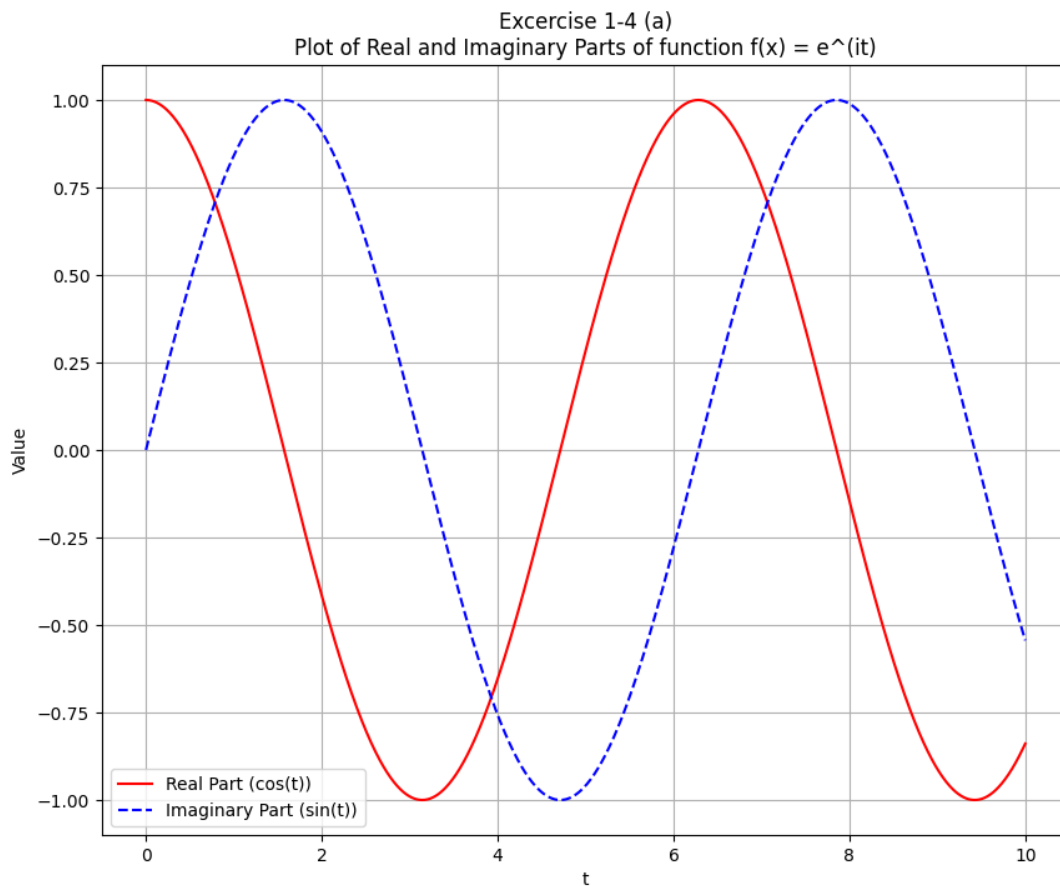
$$\therefore e^{-\pi it} = \cos(-\pi t) + i\sin(-\pi t) = \cos(\pi t) - i\sin(\pi t)$$

$$\therefore f(t) = e^{-t} (\cos(\pi t) - i\sin(\pi t))$$

$$\therefore f(t) = e^{-t} \cos(\pi t) - i(e^{-t} \sin(\pi t))$$

Real part $\rightarrow e^{-t} \cos(\pi t)$

Imaginary part $\rightarrow -e^{-t} \sin(\pi t)$



```
'''
Filename: e_4_a.py
Date Created: 10/09/2025
Created by: Archit Jain
Email: architj@uw.edu
Student No: 2426587
Description: Excercise 1-4: (a)
Please compute an analytic expression, by hand,
for the real and imaginary parts of the following complex functions.
Please also plot these for t = 0 : .01 : 10 where f (t) = e^it
'''

# Importing libraries
import numpy as np          # for arrays data storage
import matplotlib.pyplot as plt  # for plotting graphs

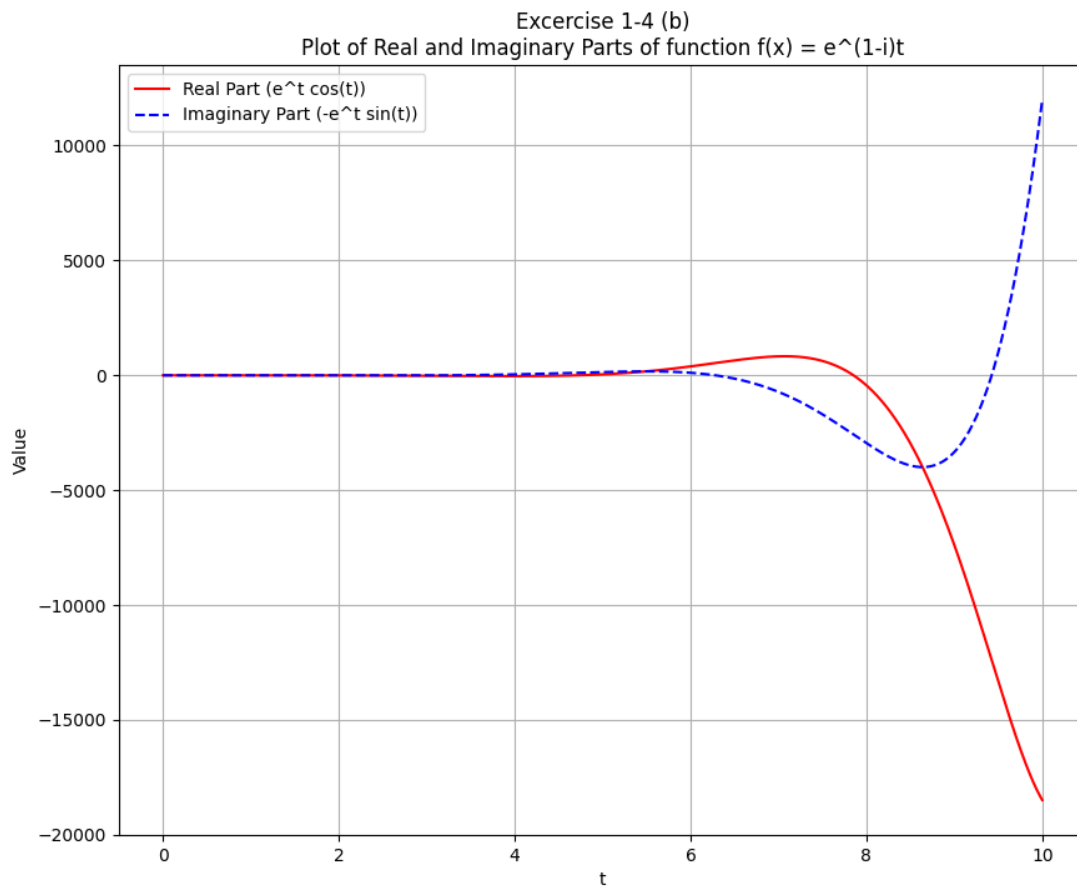
# Setting a range for time from 0->10.01 with interval step size of 0.01
t = np.arange(0, 10.01, 0.01)
```



```
# Calculate the real and imaginary parts as  $f(t) = e^{i \cdot t} = (\cos(t)) + i(\sin(t))$ 
real_part = np.cos(t)
imaginary_part = np.sin(t)

# Setting graph dimensions 10x8
plt.figure(figsize=(10, 8))
# Plotting the real part on graph
plt.plot(t, real_part, label='Real Part (cos(t))', color='red')
# Plotting the imaginary part on graph
plt.plot(t, imaginary_part, label='Imaginary Part (sin(t))', color='blue',
linestyle='--')

# Adding title and legends to graph plotted
plt.xlabel('t')
plt.ylabel('Value')
plt.title('Excercise 1-4 (a)\nPlot of Real and Imaginary Parts of function  $f(x) = e^{i \cdot t}$ ')
plt.legend()
# Show grid to the plot
plt.grid(True)
# Display the graph plotted
plt.show()
```



```
'''
Filename: e_4_b.py
Date Created: 10/09/2025
Created by: Archit Jain
Email: architj@uw.edu
Student No: 2426587
Description: Excercise 1-4: (b)
Please compute an analytic expression, by hand,
for the real and imaginary parts of the following complex functions.
Please also plot these for t = 0 : .01 : 10 where f (t) = e^(1-i)t
'''

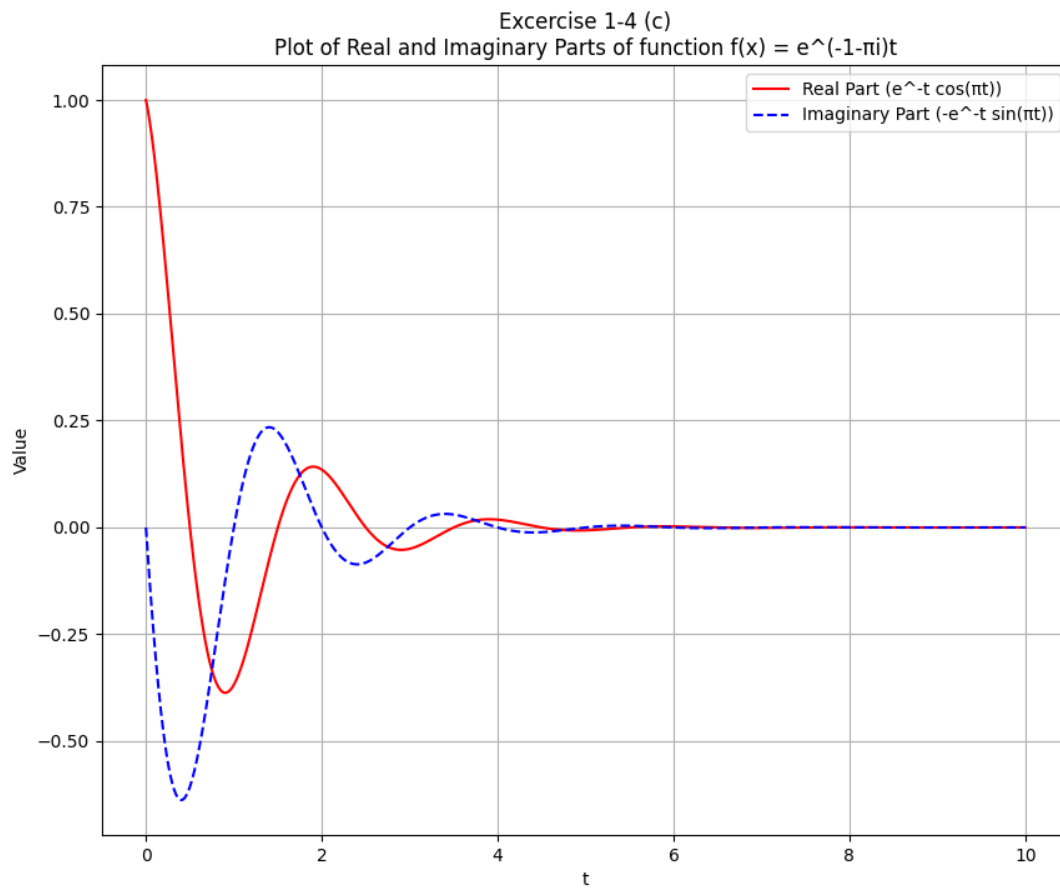
# Importing libraries
import numpy as np          # for arrays data storage
import matplotlib.pyplot as plt  # for plotting graphs

# Setting a range for time from 0->10.01 with interval step size of 0.01
t = np.arange(0, 10.01, 0.01)
```

```
# Calculate the real and imaginary parts as  $f(t) = e^{(1-i)t} = (e^t \cdot \cos(t)) - i(e^t \cdot \sin(t))$ 
real_part = np.exp(t) * np.cos(t)
imaginary_part = -np.exp(t) * np.sin(t)

# Setting graph dimensions 10x8
plt.figure(figsize=(10, 8))
# Plotting the real part on graph
plt.plot(t, real_part, label='Real Part ( $e^t \cos(t)$ )', color='red')
# Plotting the imaginary part on graph
plt.plot(t, imaginary_part, label='Imaginary Part ( $-e^t \sin(t)$ )', color='blue',
linestyle='--')

# Adding title and legends to graph plotted
plt.xlabel('t')
plt.ylabel('Value')
plt.title('Excercise 1-4 (b)\nPlot of Real and Imaginary Parts of function  $f(x) = e^{(1-i)t}$ ')
plt.legend()
# Show grid to the plot
plt.grid(True)
# Display the graph plotted
plt.show()
```

```
'''
Filename: e_4_c.py
Date Created: 10/09/2025
Created by: Archit Jain
Email: architj@uw.edu
Student No: 2426587
Description: Excercise 1-4: (c)
Please compute an analytic expression, by hand,
for the real and imaginary parts of the following complex functions.
Please also plot these for t = 0 : .01 : 10 where f (t) = e^(-1-pi)i)t
'''

# Importing libraries
import numpy as np          # for arrays data storage
import matplotlib.pyplot as plt  # for plotting graphs

# Setting a range for time from 0->10.01 with interval step size of 0.01
t = np.arange(0, 10.01, 0.01)
```

```
# Calculate the real and imaginary parts as  $f(t) = e^{(-1-\pi i)t} = (e^{-t} * \cos(\pi t)) - i(e^{-t} * \sin(\pi t))$ 
real_part = np.exp(-t) * np.cos(np.pi*t)
imaginary_part = -np.exp(-t) * np.sin(np.pi*t)

# Setting graph dimensions 10x8
plt.figure(figsize=(10, 8))
# Plotting the real part on graph
plt.plot(t, real_part, label='Real Part ( $e^{-t} \cos(\pi t)$ )', color='red')
# Plotting the imaginary part on graph
plt.plot(t, imaginary_part, label='Imaginary Part ( $-e^{-t} \sin(\pi t)$ )', color='blue',
linestyle='--')

# Adding title and legends to graph plotted
plt.xlabel('t')
plt.ylabel('Value')
plt.title('Excercise 1-4 (c)\nPlot of Real and Imaginary Parts of function  $f(x) = e^{(-1-\pi i)t}$ ')
plt.legend()
# Show grid to the plot
plt.grid(True)
# Display the graph plotted
plt.show()
```