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ME564 – Autumn 2025

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HOMEWORK - 01

Excercise 1-1: Compute the derivate of following:

(a) $f(x) = \sin(x^2)$

$\Rightarrow$ using chain rule to solve this composite function
As we know,

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) \quad \text{—— (I)}$$

So, $f(a) = \sin(a)$
& $g(x) = x^2$
$\therefore$ $f'(a) = \cos(a)$ —— (1)
& $g(x) = x^2 \Rightarrow g'(x) = 2x$ —— (2)
Using (1) & (2) in (I), we get

$$f'(x) = f'(g(x)) \cdot g'(x)$$
$$= \cos(x^2) \cdot 2x$$

$\therefore$ $\cancel{f'(x) = 2x\cos(x)^2}$ $\boxed{f'(x) = 2x\cos(x^2)}$

(b) $f(x) = x^x$

$\rightarrow$ Let $y = x^x$ —— (1)
By taking natural logarithm on both sides,

$$\ln(y) = \ln(x^x)$$
$$\therefore \ln(y) = x\ln(x)$$

Differentiate on both sides,

$$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}[x\ln(x)]$$

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using chain rule of left side & product rule on right side, we get:

$$\frac{1}{y}\frac{dy}{dx} = \ln(x) + 1$$

→ Solving for $\frac{dy}{dx}$; multiply both sides by y.

$$\therefore \frac{dy}{dx} = y(\ln(x) + 1)$$

As we know from ① that $y = x^x$

$$\therefore \frac{dy}{dx} = x^x(\ln(x) + 1)$$

$$\therefore \boxed{f'(x) = x^x(\ln(x) + 1)}$$

ⓒ $f(x) = e^{\cos(3x)}\cos(x)$

⇒ As per product rule:

$$(x \cdot y)' = x'y + y'x \quad \text{—— ①}$$

So, let $x = e^{\cos(3x)}$

& $y = \cos(x)$

→ Differentiate each part:

$$x' = e^{\cos(3x)} \cdot (-3\sin(3x)) \qquad \because \frac{d[e^{g(x)}]}{dx} = e^{g(x)} \cdot g'(x)$$

$$x' = -3e^{\cos(3x)}\sin(3x) \quad \text{—— ②}$$

& $y' = -\sin(x)$ —— ③

Now, from ①, ② & ③ we get,

$$\cancel{f'(x)} \; x \quad f'(x) = [-3\sin(3x)e^{\cos(3x)}] \cdot [\frac{\cos(x)}{\cancel{\sin(x)}}] + e^{\cos(3x)}(-\sin(x))$$

$$\therefore \boxed{f'(x) = e^{\cos(3x)}\{[-3\sin(3x)\cdot\cos(x)] - [\sin(x)]\}}$$

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(d) $f(x,y) = \sin(x^2 - y^2)$ w.r.t t, assuming that $x(t)$ & $y(t)$ vary with time.

⇒ Lets assume

$x = x(t)$ & $y = y(t)$ are function of time

Now, by chain rule: (multivariable)

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \quad \text{—— (1)}$$

So, partial derivites are as follows:

$$\frac{\partial f}{\partial x} = \cos(x^2 - y^2) \cdot 2x \longrightarrow \text{by chain rule}$$

$$\frac{\partial f}{\partial y} = -2y \cdot \cos(x^2 - y^2)$$

Using above in eq^n (1), we get:

$$\frac{df}{dt} = \left[2x\cos(x^2-y^2)\right]\frac{dx}{dt} + \left[-2y\cos(x^2-y^2)\right]\frac{dy}{dt}$$

$$\therefore \quad \boxed{\frac{df}{dt} = 2\cos(x^2-y^2)\left[x\frac{dx}{dt} - y\frac{dy}{dt}\right]}$$

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Excercise 1-2: A given mass x of a radioactive element obeys the following differential equation: $\dot{x} = \lambda x$

where $\lambda \Rightarrow$ constant of rate of decay

(a) Solution for $x(t)$ to the differential equation

$\Rightarrow$ Rearranging eq^n & integrating on both sides:

$$\frac{dx}{x} = \lambda dt$$

$$\therefore \int \frac{1}{x} dx = \int \lambda dt$$

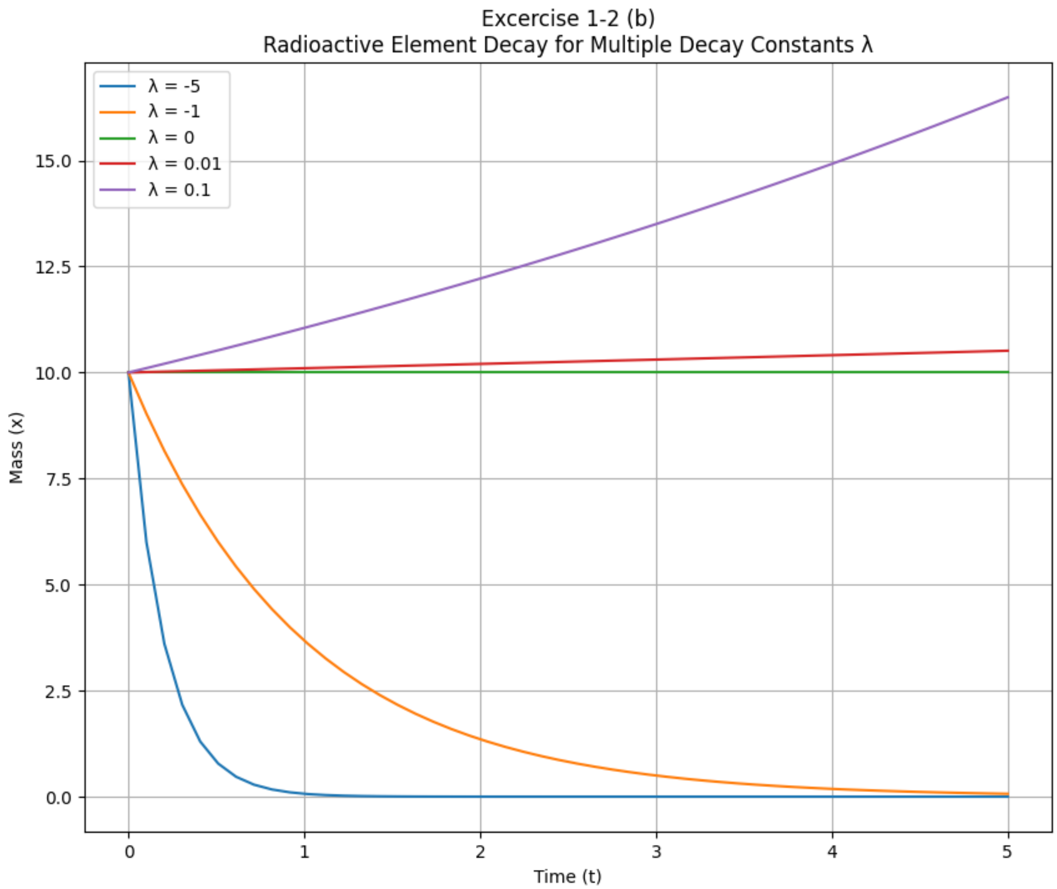
$$\therefore \ln|x| = \lambda t + C$$

Exponentiating on both sides:

$$x(t) = e^{\lambda t + C} = e^{C} e^{\lambda t}$$

Here, $x_0 = e^C$ as the initial mass at $t=0$

$$\therefore \boxed{x(t) = x_0 e^{\lambda t}}$$


Excercise 1-2 (b)
Radioactive Element Decay for Multiple Decay Constants λ
λ = -5
λ = -1
λ = 0
λ = 0.01
λ = 0.1
15.0
12.5
10.0
7.5
5.0
2.5
0.0
Mass (x)
0
1
2
3
4
5
Time (t)

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(c) The half-life T is defined as the time it takes for the material to be reduced to half of it's mass through radioactive decay. The half-life of uranium-238 is 4.468 billion years. What is corresponding value of λ?

$\Rightarrow$ T = half-life time (time taken to be reduced to half its initial mass.)

$\therefore$ eq^n with decay formula:

$$x(t) = x_0 e^{\lambda t}$$

becomes,

$$\frac{1}{2} x_0 = x_0 e^{\lambda T}$$

$$\therefore \quad \frac{1}{2} = e^{\lambda T}$$

Taking log on both sides:

$$\ln\left(\frac{1}{2}\right) = \lambda T$$

$$-\ln(2) = \lambda T$$

Solving for value λ;

$$\cancel{\lambda = \frac{-T}{\ln(2)}} \qquad \lambda = \frac{-\ln(2)}{T}$$

As, $T_{uranium\,238} = 4.468 \times 10^9$ years $\longrightarrow$ (given)

$$\therefore \boxed{\lambda = \frac{-\ln(2)}{4.468 \times 10^9 \text{ years}} \approx -1.551 \times 10^{-10} \text{ years}^{-1}}$$

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(d) If you start with 50 kg of uranium-238, how long until you only have 5 kg left? If you have 10 kg of uranium-238, how long ago was it 50 kg?

⇒ How long until 5 kg is left from 50 kg?

Here, initial mass $x_0 = 50\,kg$

final mass $x(t) = 5\,kg$

As, $x(t) = x_0 e^{\lambda t}$

$\lambda = -1.551 \times 10^{-10}$ → from previous solution (c)

$\therefore\ 5 = 50\, e^{(-1.551\times 10^{-1})t} \Rightarrow 0.1 = e^{(-1.551\times 10^{-10})t}$

Taking log on both sides,

$$\ln(0.1) = (-1.551\times 10^{-10})\,t$$

$$\therefore\ t \approx \frac{-2.3026}{-1.551\times 10^{-10}} \approx 1.485\times 10^{10}$$

$$\therefore\ \boxed{t \approx 1.485\times 10^{10}\ \text{years}}$$

⇒ How long ago was it 50 kg, if it is 10 kg now?

Here, $x_0 = 50\,kg$ → initial mass

$x(t) = 10\,kg$ → final mass (now)

As, $x(t) = x_0 e^{\lambda t}$

$\therefore\ 10 = 50\, e^{(-1.551\times 10^{-10})t}$

$0.2 = e^{(-1.551\times 10^{-10})t}$

Taking log on both sides:

$$\ln(0.2) = (-1.551\times 10^{-10})\,t$$

$$\therefore\ t \approx = \frac{-1.6094}{-1.551\times 10^{-10}} \approx 1.038\times 10^{10}$$

$\therefore$ Approx. $\boxed{\text{10.38 billion year ago}}$ there was 50 kg of Uranium-238

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Exercise 1-3: compute taylor series expansion by hand for $f(x)$

ⓐ $f(x) = \sin(x)/x$

$\Rightarrow$ As per Maclaren series at basepoint $a=0$

$$f(x) = f(a) + \frac{df}{dx}(x-a) + \frac{d^2f}{dx^2}\frac{(x-a)^2}{2!} + \cdots$$

---- evaluated at $a=0$]

$$\therefore f(x) = f(0) + f'(0)(x-0) + f''(0)\cdot\frac{(x-0)^2}{2!} + \cdots$$

~~As fx = si~~ for $\sin(x)$]

$$f(x) = \sin(0) + \cos(0)(x) + \left(-\sin(0)\times\frac{x^2}{2!}\right) + \left(-\cos(0)\cdot\frac{x^3}{3!}\right)$$

$$= 0 + 1\cdot x + \left(-0\times\frac{x^2}{2!}\right) + \left(-1\cdot\frac{x^3}{3!}\right) + \cdots$$

$$\therefore f(x) \Rightarrow \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

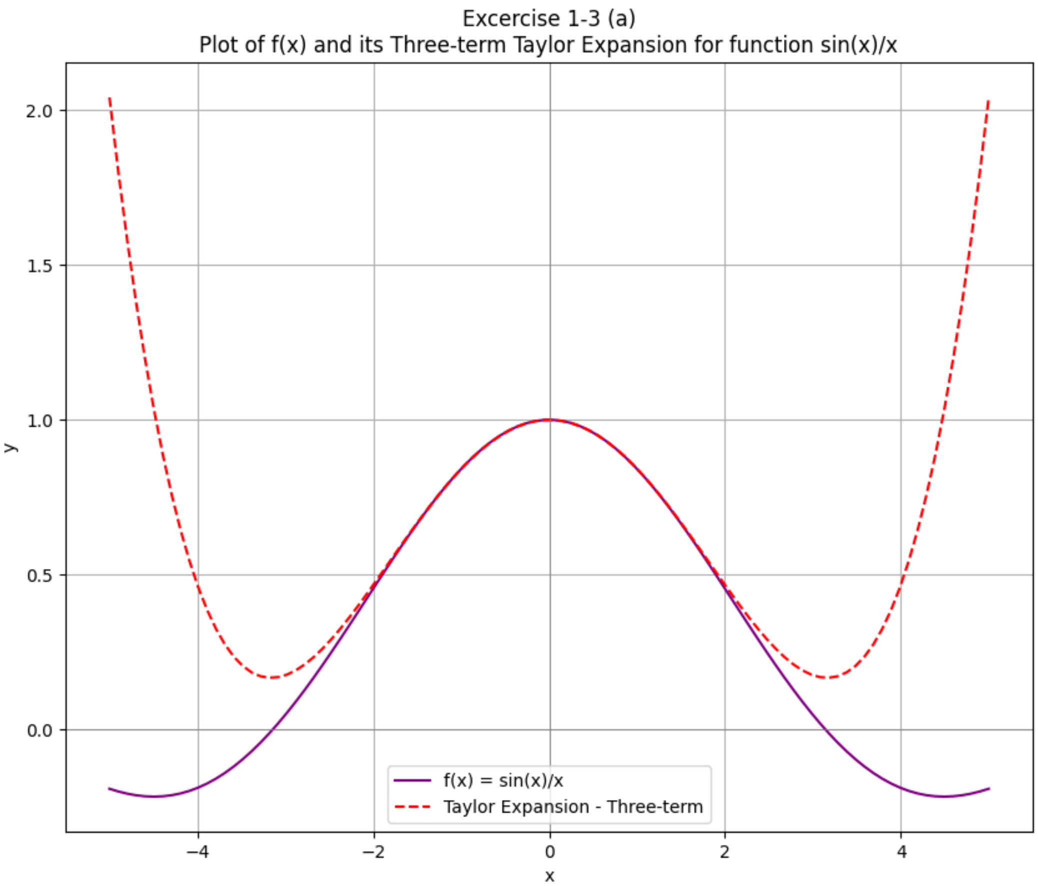
$\therefore$ Taylor series for $\frac{\sin(x)}{x}$ is by dividing above by x

$$\therefore \frac{\sin(x)}{x} = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$\boxed{\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots}$$

First three non-zero terms of above series are:

$$P_3(x) = 1 - \frac{x^3}{6} + \frac{x^4}{120}$$


Excercise 1-3 (a)
Plot of f(x) and its Three-term Taylor Expansion for function sin(x)/x
2.0
1.5
1.0
0.5
0.0
y
−4
−2
0
2
4
x
f(x) = sin(x)/x
Taylor Expansion - Three-term

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(b) $f(x) = (\cos(x) - 1)/x$

$\Rightarrow$ The macleron series for $\cos(x)$ at $a=0$ $\Rightarrow$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

So, to solve for $(\cos(x) - 1)/x$, $\Rightarrow$

$$\frac{(\cos(x)-1)}{x} = \frac{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - 1}{x}$$

$$\frac{\cos(x)-1}{x} = \frac{1}{x}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$\boxed{\frac{\cos(x)-1}{x} = -\frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \cdots}$$

The first three non-zero terms of above series are:

$$P_3(x) = \frac{-x}{2} + \frac{x^3}{24} - \frac{x^5}{720}$$

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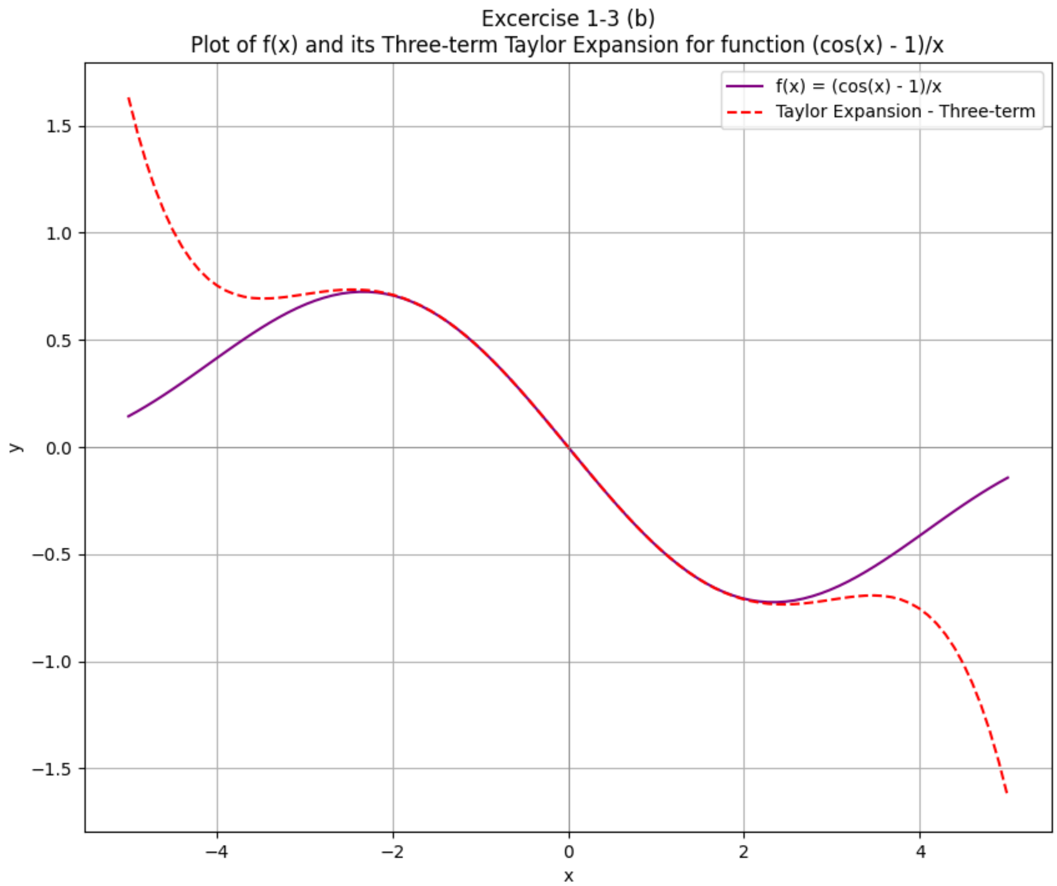

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Excercise 1-4: Please compute an analytic expression, by hand, for the real & imaginary parts of the following complex ~~numbers~~ functions.

ⓐ $f(t) = e^{it}$

$\Rightarrow$ As per Euler's formula for any real number x:

$$e^{ix} = \cos(x) + i\sin(x)$$

$\therefore$

$$\boxed{\begin{array}{l} f(t) = e^{it} = \cos(t) + i\sin(t) \\ \text{Real part} \Rightarrow \cos(t) \\ \text{Imaginary part} \Rightarrow \sin(t) \end{array}}$$

ⓑ $f(t) = e^{(1-i)t}$

$\Rightarrow$ $f(t) = e^{(1-i)t} = e^{t-it} = e^{t} \cdot e^{-it}$

From Eulers formula, we get, (here $x = -t$)

$\therefore e^{-it} = \cos(-t) + i\sin(-t)$

$e^{-it} = \cos(t) - i\sin(t)$

$\therefore$

$$f(t) = e^{t} \cdot (\cos(t) - i\sin(t))$$

$\therefore$

$$\boxed{\begin{array}{l} f(t) = e^{t}\cos(t) - i(e^{t}\sin(t)) \\ \text{Real part} \Rightarrow e^{t}\cos(t) \\ \text{Imaginary part} \Rightarrow -e^{t}\sin(t) \end{array}}$$

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c) $f(t) = e^{(-1-\pi i)t}$

$\Rightarrow f(t) = e^{-t-\pi i t} = e^{-t} \cdot e^{-\pi i t}$

using Eulers formula,

$$e^{ix} = \cos(x) + i\sin(x)$$

here, $x = -\pi t$

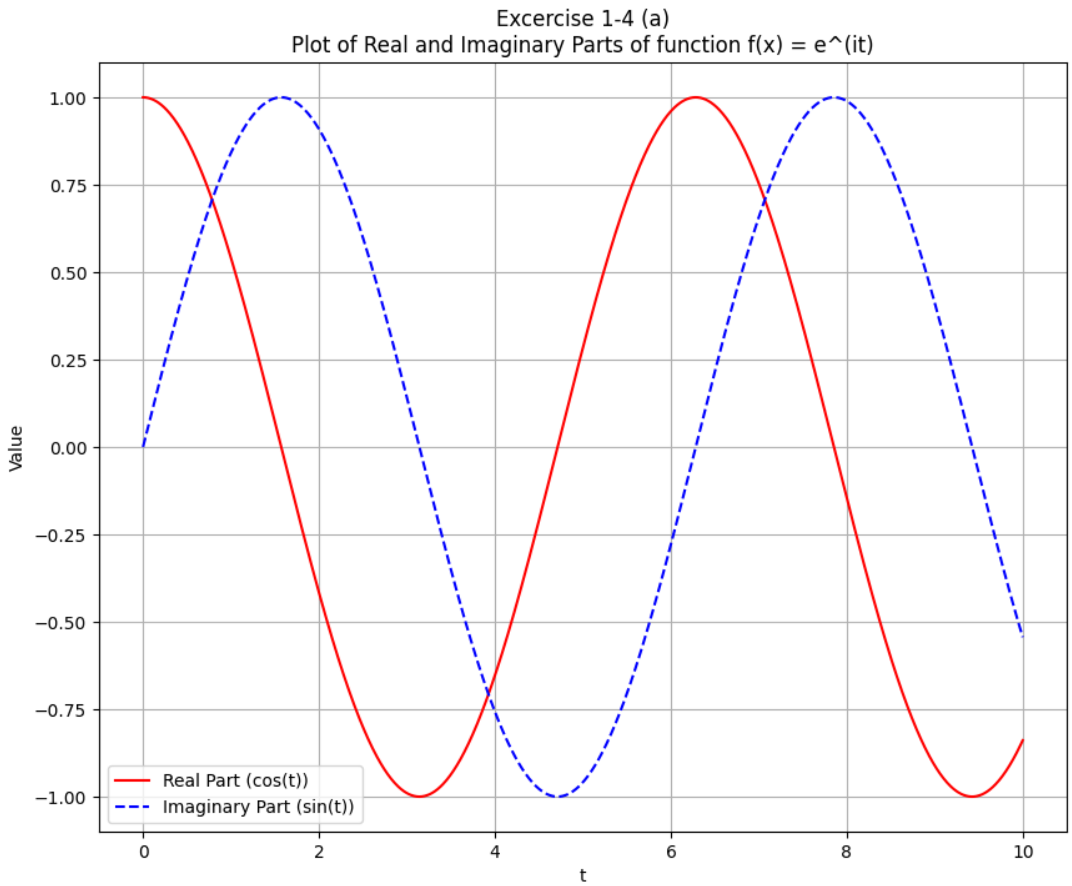
$\therefore e^{-\pi i t} = \cos(-\pi t) + i\sin(-\pi t) = \cos(\pi t) - i\sin(\pi t)$

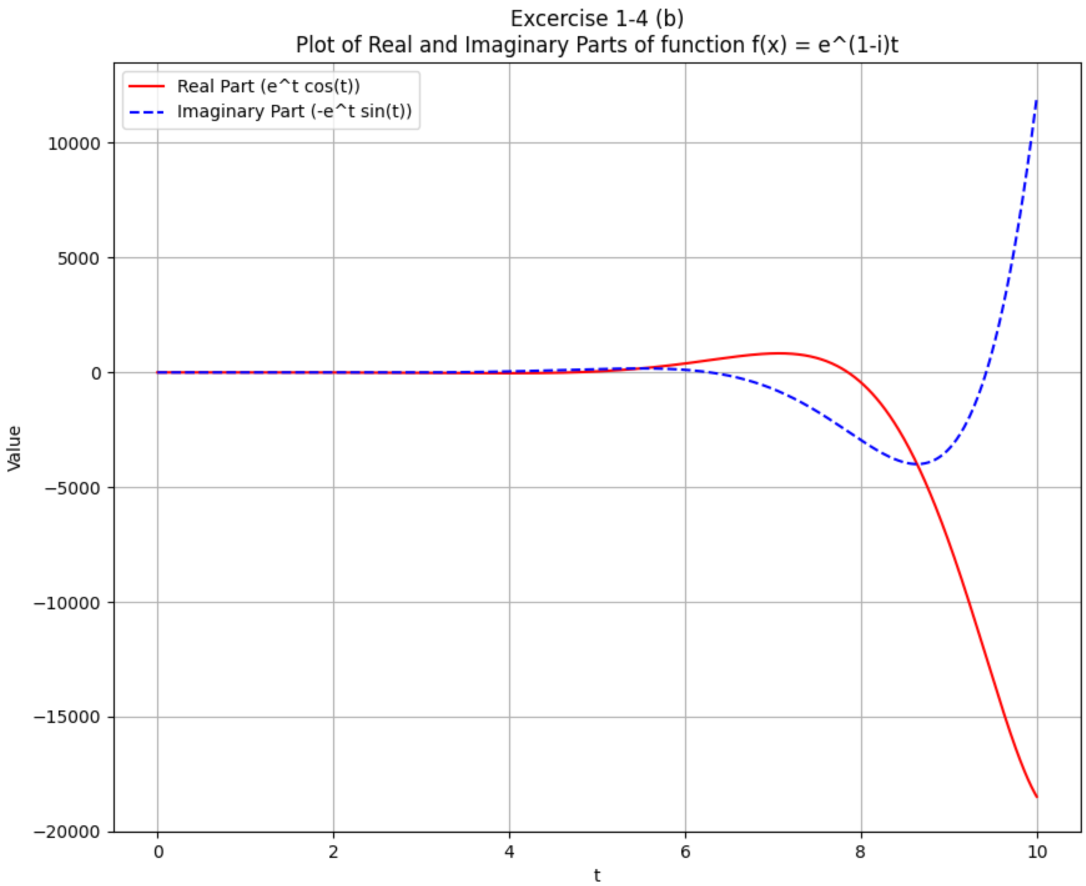
$\therefore f(t) = e^{-t}(\cos(\pi t) - i\sin(\pi t))$

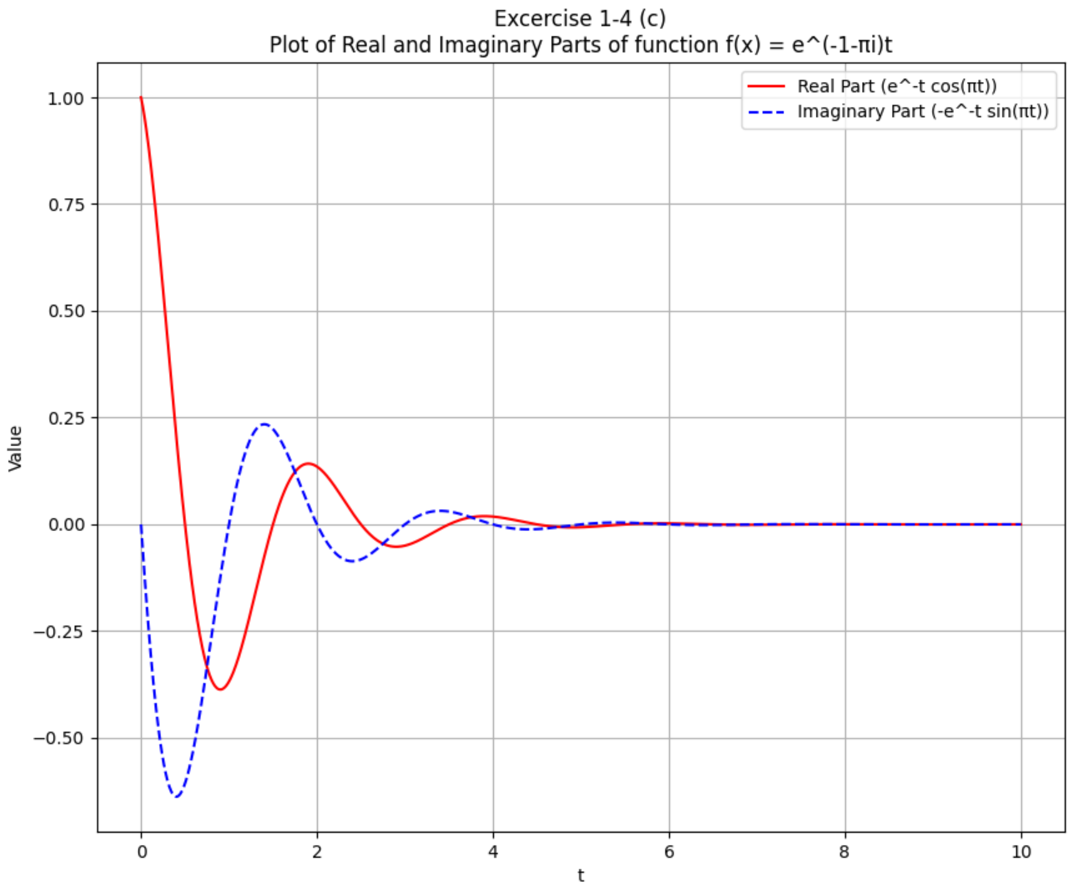
$\therefore$ $f(t) = e^{-t}\cos(\pi t) - i(e^{-t}\sin(\pi t))$

Real part $\rightarrow e^{-t}\cos(\pi t)$

Imaginary part $\rightarrow -e^{-t}\sin(\pi t)$


Excercise 1-4 (a)
Plot of Real and Imaginary Parts of function f(x) = e^(it)
Value
t
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
2
4
6
8
10
Real Part (cos(t))
Imaginary Part (sin(t))


Excercise 1-4 (b)
Plot of Real and Imaginary Parts of function f(x) = e^(1-i)t
Real Part (e^t cos(t))
Imaginary Part (-e^t sin(t))
10000
5000
0
−5000
−10000
−15000
−20000
Value
0
2
4
6
8
10
t


Excercise 1-4 (c)
Plot of Real and Imaginary Parts of function f(x) = e^(-1-πi)t
Real Part (e^-t cos(πt))
Imaginary Part (-e^-t sin(πt))
Value
t
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
0
2
4
6
8
10