ARCHIT KAILA

BME 548: HOMEWORK 1

Problem 1

$u = \begin{bmatrix} 6 \\ 2 \\ 9 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 3 \\ -5 \\ 2 \end{bmatrix}$

(a) Inner Product: $u^T v$

$$u^T \cdot v = [6 \quad 2 \quad 9 \quad 1] \cdot \begin{bmatrix} 0 \\ 3 \\ -5 \\ 2 \end{bmatrix} = 6\times 0 + 2\times 3 + 9\times(-5) + 1\times 2$$

$$= 0 + 6 + (-45) + 2$$

$$= -37$$

(b) Outer Product: $u \otimes v = uv^T$

$$u \cdot v^T = \underset{4\times 1}{\begin{bmatrix} 6 \\ 2 \\ 9 \\ 1 \end{bmatrix}} \cdot \underset{1\times 4}{[0 \quad 3 \quad -5 \quad 2]} = \underset{4\times}{\begin{bmatrix} 6\times 0 & 6\times 3 & 6\times(-5) & 6\times 2 \\ 2\times 0 & 2\times 3 & 2\times(-5) & 2\times 2 \\ 9\times 0 & 9\times 3 & 9\times(-5) & 9\times 2 \\ 1\times 0 & 1\times 3 & 1\times(-5) & 1\times 2 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 18 & -30 & 12 \\ 0 & 6 & -10 & 4 \\ 0 & 27 & -45 & 18 \\ 0 & 3 & -5 & 2 \end{bmatrix}$$

(c) Hadamard Product: $u \circ v$

$$u \odot v = \begin{bmatrix} 6 \\ 2 \\ 9 \\ 1 \end{bmatrix} \odot \begin{bmatrix} 0 \\ 3 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 6\times 0 \\ 2\times 3 \\ 9\times(-5) \\ 1\times 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ -45 \\ 2 \end{bmatrix}$$

(d) Inner Product: $a^H \cdot b$, where ^{H} denotes conjugate transpose

$$a = \begin{bmatrix} 1 \\ e^{i\pi/3} \\ e^{-i\pi/2} \\ e^{-i\pi/3} \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1e^{i\pi} \\ 2e^{-i\pi/2} \\ 4e^{i\pi/2} \end{bmatrix} \qquad a^H = \begin{bmatrix} 1 & e^{-i\pi/3} & e^{i\pi/2} & e^{i\pi/3} \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ -1 \\ 2e^{-i\pi/2} \\ 4e^{i\pi/2} \end{bmatrix}$$

$$a^H \cdot b = \begin{bmatrix} 1 & e^{-i\pi/3} & e^{i\pi/2} & e^{i\pi/3} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 2e^{-i\pi/2} \\ 4e^{i\pi/2} \end{bmatrix} = (1\times 0) + \left((-1)\times e^{-i\pi/3}\right) + \left(e^{i\pi/2}\times 2e^{-i\pi/2}\right) + \left(e^{i\pi/3}\times 4e^{i\pi/2}\right)$$

$$= 0 - e^{-i\pi/3} + 2e^{i\left(\frac{\pi}{2}-\frac{\pi}{2}\right)} + 4e^{i\left(\frac{\pi}{3}+\frac{\pi}{2}\right)}$$

$$= -e^{-i\pi/3} + 2 + 4e^{i\frac{5\pi}{6}}$$

$$= 2 - e^{-i\frac{\pi}{3}} + 4e^{i\frac{5\pi}{6}}$$

(e) Inner Products: $a^H \cdot a$ & $b^H \cdot b$

$$a^H \cdot a = \begin{bmatrix} 1 & e^{-i\pi/3} & e^{i\pi/2} & e^{i\pi/3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ e^{+i\pi/3} \\ e^{-i\pi/2} \\ e^{-i\pi/3} \end{bmatrix} = (1\times 1) + \left(e^{-i\pi/3}\times e^{+i\pi/3}\right) + \left(e^{i\pi/2}\times e^{-i\pi/2}\right) + \left(e^{i\pi/3}\times e^{-i\pi/3}\right)$$

$$= 1 + e^{i\left(\frac{\pi}{3}-\frac{\pi}{3}\right)} + e^{i\left(\frac{\pi}{2}-\frac{\pi}{2}\right)} + e^{i\left(\frac{\pi}{3}-\frac{\pi}{3}\right)}$$

$$= 1 + 1 + 1 + 1 = 4$$

$$b^H \cdot b = \begin{bmatrix} 0 & e^{-i\pi} & 2e^{i\pi/2} & 4e^{-i\pi/2} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ e^{i\pi} \\ 2e^{-i\pi/2} \\ 4e^{i\pi/2} \end{bmatrix} = (0\times 0) + \left(e^{-i\pi}\times e^{i\pi}\right) + \left(2e^{i\pi/2}\times 2e^{-i\pi/2}\right) + \left(4e^{-i\frac{\pi}{2}}\times 4e^{i\pi/2}\right)$$

$$= 0 + e^{i(\pi-\pi)} + \overset{4}{\cancel{2}}e^{i\left(\frac{\pi}{2}-\frac{\pi}{2}\right)} + \overset{16}{\cancel{4}}e^{i\left(\frac{\pi}{2}-\frac{\pi}{2}\right)}$$

$$= 0 + e^{0} + \overset{4}{\cancel{2}}e^{0} + \overset{16}{\cancel{4}}e^{0}$$

$$= 0 + 1 + \overset{4}{\cancel{2}} + \overset{16}{\cancel{4}} = \cancel{7}\ 21$$

(f) Hadamard Product $a \circ a^*$

a^* – complex conjugate

$$a \circ a^* = \begin{bmatrix} 1 \\ e^{i\pi/3} \\ e^{-i\pi/2} \\ e^{-i\pi/3} \end{bmatrix} \odot \begin{bmatrix} 1 \\ e^{-i\pi/3} \\ e^{i\pi/2} \\ e^{i\pi/3} \end{bmatrix} = \begin{bmatrix} 1 \times 1 \\ e^{i\pi/3} \times e^{-i\pi/3} \\ e^{-i\pi/2} \times e^{i\pi/2} \\ e^{-i\pi/3} \times e^{i\pi/3} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{i\left(\frac{\pi}{3}-\frac{\pi}{3}\right)} \\ e^{i\left(\frac{\pi}{2}-\frac{\pi}{2}\right)} \\ e^{i\left(\frac{\pi}{3}-\frac{\pi}{3}\right)} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 8 & 3 \\ 7 & 4 & 1 \\ 5 & 9 & 1 \end{bmatrix}$$

Write expression to calculate avg of 1st column using matrix operations

Let's say $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

- To calculate average of 1st column of I we can multiply I with x (Matrix Vector Product)
- This will give us the 1st column of Matrix I
- we can then take the sum of all elements in this column vector & divide it by its shape.

$$I.x = \begin{bmatrix} 1 & 8 & 3 \\ 7 & 4 & 1 \\ 5 & 9 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 5 \end{bmatrix}$$

$$\text{Average} = \frac{\text{SUM}(I.x)}{\text{SHAPE}(I.x)} = \frac{13}{3} = 4.33$$

Now repeat this after vectorizing I

- Flattened I :

$i = [1\ 8\ 3\ 7\ 4\ 1\ 5\ 9\ 1]$

- Now to get the same result using Flattened I, we can use the following column vector x

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad i.x = [1\ 8\ 3\ 7\ 4\ 1\ 5\ 9\ 1] . \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 1 + 7 + 5 = 13$$

- Next we can divide this result by sum of column vector x

$$\text{Average} = \frac{13}{\text{SUM}(x)} = \frac{13}{3} = 4.33$$

<u>Problem 2</u> $u = [1, -1]$ $V = [1, 2, 3, 4, 5, 6]$

(a) Discreate Convolution $(W = u * V)$ Flipped $u = [-1, 1]$

K	-1	0	1	2	3	4	5	6
V[K]		1	2	3	4	5	6	
u[-K]	-1	1						
u[1-K]		-1	1					
u[2-K]			-1	1				
u(3-K)				-1	1			
u(4-K)					-1	1		
u[5-K]						-1	1	
u[6-K]							-1	1
	1	1	1	1	1	1	-6	

$W = V * u = [1\ 1\ 1\ 1\ 1\ 1\ -6\ ]$

(b) $W = U.v$ - Matrix $U \in R^{7\times 6}$ holds blur kernel

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

- To write a blur kernel u as a matrix, we can write it as a 7×6 Banded matrix
- (A matrix similar to diagonal matrix where most energy is concentrated towards diagonal)
- Every row of U corresponds to a shifted u.
- This blur kernel is helping us take a weighted avg of neighbouring pixels

(C) Deconvolution

We can determine the Deconvolution matrix D by calculating the pseudo inverse of our convolation matrix U

$\hookrightarrow (U^T.U)^{-1}.U^T = D$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}_{7\times 6} \quad U^T = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}_{6\times 7}$$

$$(U^T.U) = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}_{6\times 6}$$

$$(U^T.U)^{-1} = \frac{1}{7}\begin{bmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 10 & 8 & 6 & 4 & 2 \\ 4 & 8 & 12 & 9 & 6 & 3 \\ 3 & 6 & 9 & 12 & 8 & 4 \\ 2 & 4 & 6 & 8 & 10 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}_{6\times 6}$$

$$D = (U^T.U)^{-1}.U^T = \frac{1}{7}\begin{bmatrix} 6 & -1 & -1 & -1 & -1 & -1 & -1 \\ 5 & 5 & -2 & -2 & -2 & -2 & -2 \\ 4 & 4 & 4 & -3 & -3 & -3 & -3 \\ 3 & 3 & 3 & 3 & -4 & -4 & -4 \\ 2 & 2 & 2 & 2 & 2 & -5 & -5 \\ 1 & 1 & 1 & 1 & 1 & 1 & -6 \end{bmatrix}_{6\times 7}$$

The matrix D is inversing the effects of convolution caused with matrix U

(d) Bonus: Is $D.D^T$ invertable?

Yes, this is an invertible matrix. D is 6×7 & D^T is 7×6 so $D.D^T$ is a 6×6 matrix. It's determinant is nonzero & its rank is 6, hence it is invertable.
This result is like a pseudo inverse of D & can help us undo the convolution effect of the matrix.

Problem 3 Proof for FT theorem $F[u(x) * v(x)] = F[u(x)] . F[v(x)]$

Defination of Fourier Transform: $F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-2\pi i f x} dx$ — ①

Defination of convolution: $u(x) * v(x) = \int_{-\infty}^{\infty} u(\tau) v(x-\tau) d\tau = f(x)$ — ②

Let's consider LHS of our eqⁿ & substitute ② in ①

$$F[u(x) * v(x)] = \int_{-\infty}^{\infty} [u(x) * v(x)] e^{-2\pi i f x} dx$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} u(\tau) v(x-\tau) d\tau \right) e^{-2\pi i f x} dx \longrightarrow \text{changing order of integration}$$

$$= \int_{-\infty}^{\infty} u(\tau) \int_{-\infty}^{\infty} v(x-\tau) e^{-2\pi i f x} dx \, d\tau$$

Let $(x-\tau) = x'$ & $dx = dx'$
so $x = x' + \tau$

$$\therefore = \int_{-\infty}^{\infty} u(\tau) \int_{-\infty}^{\infty} v(x') e^{-2\pi i f (x'+\tau)} dx' \, d\tau$$

$$= \int_{-\infty}^{\infty} u(\tau) \int_{-\infty}^{\infty} (v(x') e^{-2\pi i f x'} dx') (e^{-2\pi i f \tau} d\tau).$$

$$= \int_{-\infty}^{\infty} u(\tau) e^{-2\pi i f \tau} \left[\int_{-\infty}^{\infty} v(x') e^{-2\pi i f x'} dx' \right] d\tau.$$

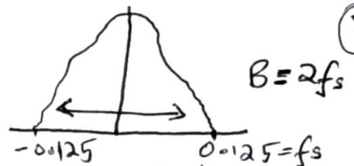
$$= \int_{-\infty}^{\infty} u(\tau) e^{-2\pi i f \tau} d\tau \int_{-\infty}^{\infty} v(x') e^{-2\pi i f x'} dx'$$

Using property of definite integrals

$$= F[u(x)] . F[v(x)]$$

= RHS Hence Proved.

Problem 4



(a) Optical field $U(x,y)$

Max spatial frequency $= 0.125\ \mu m^{-1}$

Image sensor is a cone function with pixel pitch $= 10\ \mu m$

Can we recreate the original field from detected field.

According to Sampling theoram if continuous signal has a frequency of no move than B Hz then we can recreate this signal if the samples are drawn at atleast 2B samples per second

- So Max Frequency of continuous signal $= 0.125\ \mu m^{-1}$
- So sampling should be done at atleast $2 \times 0.125 = 0.25\ \mu m^{-1}$
- Distance b/w centers of Adjacent pixels $= 10\ \mu m$

$\therefore$ it Samples at the rate of $\frac{1}{10}\ \mu m$

$\therefore$ sampling frequency $= 0.1\ \mu m^{-1}$

Since sampling rate $<$ twice the frequency of continuous signal.

$\boxed{\text{Hence original signal cannot be reconstructed.}}$

If pixel pitch $= 6\ \mu m$

then sampling frequency $= \frac{1}{6} = 0.16$

$0.16 < 0.25$

$\boxed{\therefore \text{ original signal cannot be reconstructed}}$

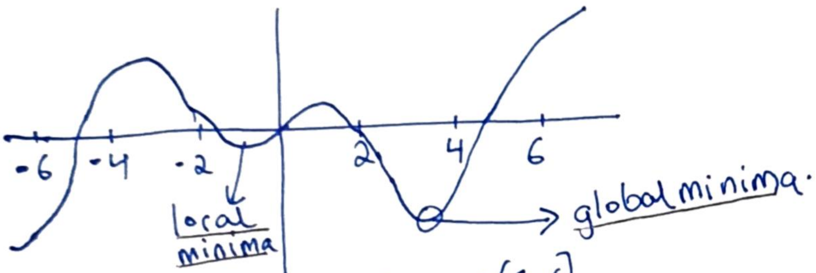
If pixel pitch $= 4\ \mu m$

then sampling frequency $= \frac{1}{4} = 0.25$

$0.25 \leq 0.25$

$\boxed{\therefore \text{ original signal can be reconstructed}}$

(b) $f(x) = x\cos(x) \quad x \in [-6, 6]$ - Local & global minima?
- good value
- bad value



$f(x) = x\cos(x) \quad x \in [6, 6]$

- we can find the minimas by taking a derivative of the function & finding critical points where derivative = 0
- 2nd derivative can be used to find if a critical point is a local minima or global minima.

- Gradient Descent does not always gaurentee to find a global minima
  It is possible to get stuck on a local minima in case the initial value is very far away from the global minima.
- A good value for initial state would be close to 3, some where b/w 2 & 5 as it would start close to the global minima
- A bad value for initial state could be close to 0, somewhere b/w -3 & 0 which is far away from global minima & it might get stuck in the local minima. The convergence would also be slow.

(c) Linear eq^n $f(x) = \text{sign}(W^T x)$ $\quad W = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad X = (1, x_1, x_2)^T$

$$\text{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

F(x) seprates our data into 2 classes → +1, -1

The line which seprates our classes is the decision boundary

eq^n of the line $W^T x = 0$

So $\begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow 1 - 2x_1 + x_2 = 0$

$\Rightarrow \boxed{2x_1 - x_2 - 1 = 0}$