Machine Learning - HW 1

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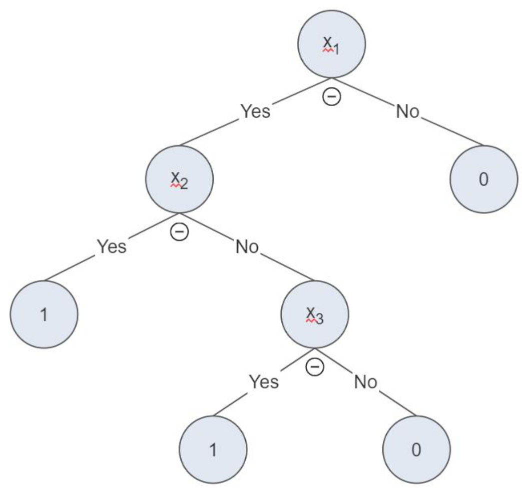
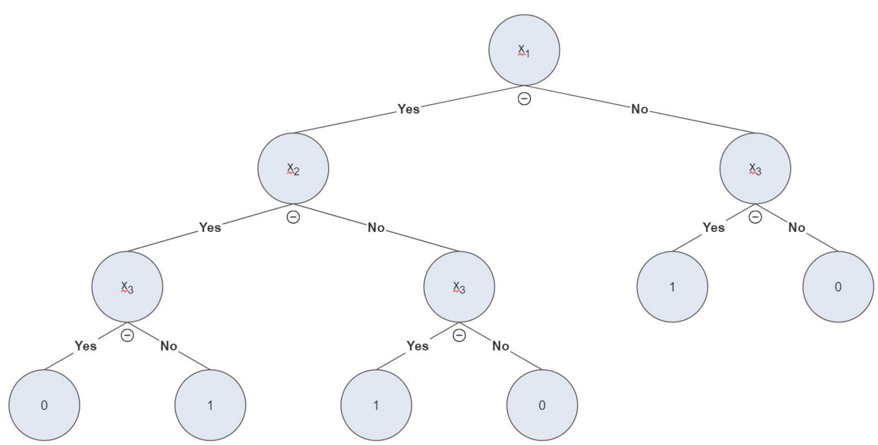
1 Decision Trees

1. Write the following Boolean functions as decision trees.

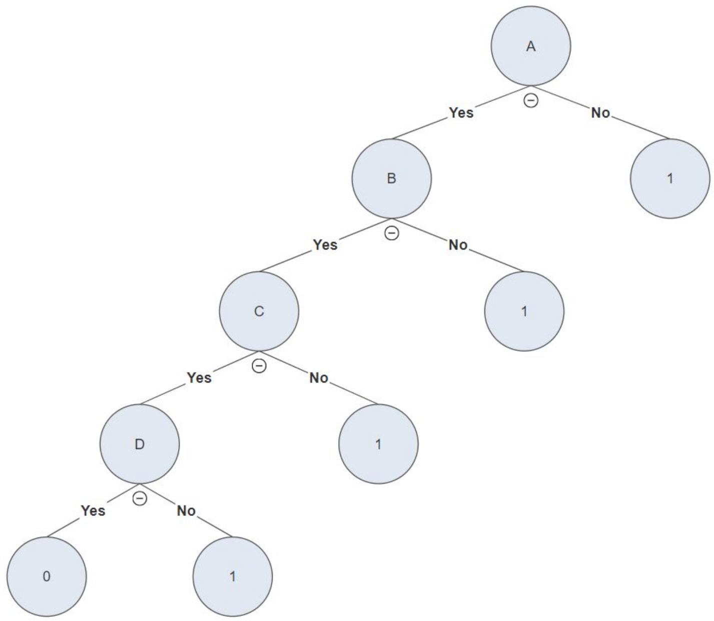
   (a) $(x_1 \wedge x_2) \vee (x_1 \wedge x_3)$



   (b) $(x_1 \wedge x_2)$ xor x_3



(c) $\neg A \vee \neg B \vee \neg C \vee \neg D$



2. (a) How many possible functions are there to map these four features to a boolean decision? How many functions are consistent with the given training dataset?

   Answer: Each of the features can take $2, 2, 3$ and 4 values respectively. Thus there are $2 \times 2 \times 3 \times 4 = 2^{48}$ possible functions. Since we have observed 9 entries of the total 48 in the truth table, there are $2^{48-9} = 2^{39}$ possible functions consistent with given data.

   (b) What is the entropy of the labels in this data?

   Answer: There are 5 points labelled "yes" and 4 labelled "no".

$$\begin{aligned} entropy &= -p_+ \log_2(p_+) - p_- \log_2(p_-) \\ &= -\frac{5}{9} \log_2 \frac{5}{9} - \frac{4}{9} \log_2 \frac{4}{9} \\ &= 0.9911 \end{aligned}$$

   (c) What is the information gain of each of the features?

   Answer:

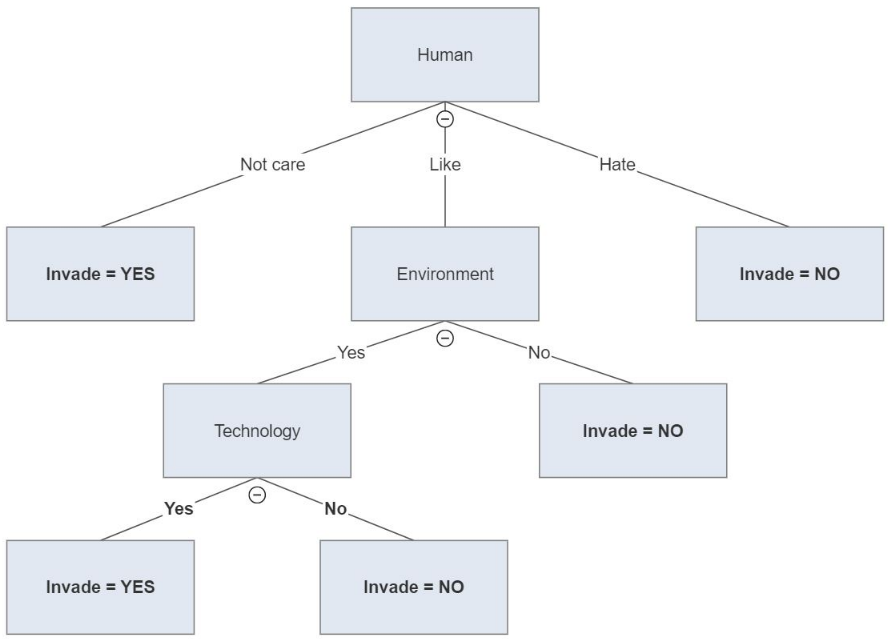
Feature	Value	Invade=Yes	Invade=No	Fraction of data	Entropy $\times$ fraction	Info gain
Technology	Yes	4/6	2/6	6/9	0.612	0.073
	No	1/3	2/3	3/9	0.306	
Environment	Yes	4/5	1/5	5/9	0.401	0.229
	No	1/4	3/4	4/9	0.361	
Human	Not care	4/4	0/4	4/9	0.000	**0.6305**
	Like	1/4	3/4	4/9	0.361	
	Hate	0/1	1/1	1/9	0.000	
Distance	1	1/2	1/2	2/9	0.222	0.046
	2	1/1	0/1	1/9	0.111	
	3	2/3	1/3	3/9	0.306	
	4	1/3	2/3	3/9	0.306	

(d) Which attribute will you use to construct the root of the tree using the ID3 algorithm?

Answer: Since *Human* attribute has the highest information gain, we use it to create the root node for the ID3 algorithm

(e) Using the root that you selected in the previous question, construct a decision tree that represents the data.

Answer:



(f) Suppose you are given three more examples. Use your decision tree to predict the label for each example. Also report the accuracy of the classifier that you have learned.

Answer: The prediction matches 2/3 of the ground truth, hence accuracy is 67.7%.

Technology	Environment	Human	Distance	Invade?	Prediction
Yes	Yes	Like	2	No	Yes
No	No	Hate	3	No	No
Yes	Yes	Lkie	4	Yes	Yes

3. (a) Using the $MajorityError$ measure, calculate the information gain for the four features respectively. Use 3 significant digits.

Answer: The majority error at the root node is $1 - 5/9 = 0.444$.

Feature	Value	Invade=Yes	Invade=No	Fraction of data	Majority Error × fraction	Info gain
Technology	Yes	4/6	2/6	6/9	0.222	0.111
	No	1/3	2/3	3/9	0.111	
Environment	Yes	4/5	1/5	5/9	0.111	0.222
	No	1/4	3/4	4/9	0.111	
Human	Not care	4/4	0/4	4/9	0.000	**0.333**
	Like	1/4	3/4	4/9	0.111	
	Hate	0/1	1/1	1/9	0.000	
Distance	1	1/2	1/2	2/9	0.111	0.111
	2	1/1	0/1	1/9	0.000	
	3	2/3	1/3	3/9	0.111	
	4	1/3	2/3	3/9	0.111	

(b) According to your results in the last question, which attribute should be the root for the decision tree? Do these two measures (entropy and majority error) lead to the same tree?

Answer: The highest information gain is obtained for *Human* attribute, hence this should be chosen as the root for the decision tree. Yes, both the measures lead to the same tree. Though the exact values of information gain are different for majority error and entropy, the relation between value is still same.

2 Linear Classifier

1. Write a linear classifier that correctly classifies the given dataset.

   Answer: Assume that the linear classifier has weights w_1, w_2, w_3, w_4 and bias b. Substituting the datapoints in the equation for the linear classifier we get:

$$\begin{aligned} w_1 + w_3 + w_4 + b &\geq 0 \\ w_2 + w_4 + b &\geq 0 \\ w_3 + b &< 0 \end{aligned}$$

   By inspection, we find that $w_1 = w_2 = w_3$, $b = -1$ and $w_4 = 1$, all the three inequalities above are satisfied.

   So a linear classifier for the given data is $\mathbf{w} = [0, 0, 0, 1]$, $b = -1$

2. Suppose the dataset below is an extension of the above dataset. Check if your classifier from the previous question correctly classifies the dataset. Report its accuracy.

x1	x2	x3	x4	o	$sgn(\mathbf{w}^\mathbf{T}\mathbf{x} + b)$
0	0	0	1	1	1
0	0	1	1	1	1
0	0	0	0	-1	-1
1	0	1	0	1	-1
1	1	0	0	1	-1
1	1	1	1	1	1
1	1	1	0	1	-1

   The linear classifier given correctly classifies 3 of the 7 given inputs. Hence its accuracy is approximately 42.7%.

3. Given the remaining missing data points of the above dataset in the table below, find a linear classifier that correctly classifies the whole dataset (all three tables together)

x_1	x_2	x_3	x_4	o
1	0	1	1	1
0	1	0	1	1
0	0	1	0	-1
0	0	0	1	1
0	0	1	1	1
0	0	0	0	-1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1
1	1	1	0	1
0	1	0	0	-1
0	1	1	0	-1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	1	0	1	1

Answer:

The complete truth table is given below. All features and target values are boolean. We assume that our linear classifier is a manifestation of a boolean function of $\{x_!, x_2, x_3, x_4\}$. Looking at all rows where the output is 1 (the normal form of rows with 0 leads to $\vee$ with zero, which does not factor in the final expression), we get the following disjunctive normal form:

$$
\begin{aligned}
(x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4)\vee \\
(\neg x_1 \wedge x_2 \wedge \neg x_3 \wedge x_4)\vee \\
(\neg x_1 \wedge \neg x_2 \wedge x_3 \neg \wedge x_4)\vee \\
(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4)\vee \\
(x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4)\vee \\
(x_1 \wedge x_2 \wedge \neg x_3 \wedge \neg x_4)\vee \\
(x_1 \wedge x_2 \wedge x_3 \wedge x_4)\vee \\
(x_1 \wedge x_2 \wedge x_3 \wedge \neg x_4)\vee \\
(\neg x_1 \wedge x_2 \wedge x_3 \wedge x_4)\vee \\
(x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4)\vee \\
(x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4)\vee \\
(x_1 \wedge x_2 \wedge \neg x_3 \wedge x_4)\vee
\end{aligned}
$$

Using $(x \wedge y) \vee (x \wedge \neg y) = x$, we simplify the above expression repeatedly.

This simplification yields a final expression of $\mathbf{x_1} \vee \mathbf{x_4}$. (The same simplification could be achieved using Karnaugh maps as well).

The corresponding linear classifier for this is $\mathbf{w} = [1, 0, 0, 1]$ and $b = 0$.

3 Experiments

1. **Implementation**

   (a) Discuss what approaches or choices you had to make during the implementation of decision tree.
   Answer: The choice of implementation language was Python (3.5) for me. The implementation had the following stages:

   - Reading the dataset into a python list

- Converting the strings in the data to feature vectors: This was accomplished by keep a list of functions of length d and then populating the `numpy` array of size `n` $\times$ `d` by calling each of the `d` functions on the name string, where `n` is the number of datapoints. The labels are also stored in an array of size `d` as `[1,0]`.
- Creating the decision tree using ID3 algorithm: Every node in the tree is an object of class `DecisionNode`. Every `DecisionNode` object has attributes:
  - `attr_index`: which stores which attribute this node checks for, `-1` if leaf node
  - `prediction`: prediction if this is a leaf node, `None` otherwise
  - `branches`: a `dict` where keys are possible attribute values for `attr_index`, and values are the subtree returned by ID3. Empty dict for leaf nodes The tree is built by recursive calls of ID3 as outlined in the lecture slides.

  Conflicting inputs where the feature vector is same but labels are different are handled by setting prediction by the last conflicting example seen in the order of input data.
- Performing prediction from the decision tree: On the test data, we just traverse the tree based on a `attr_index` of the current node and follow the appropriate `branch` based on attribute value.

(b) Suggest at least 4 other features you could have extracted from this dataset.

Answer: More features could be:

- Whether the ASCII sum of characters in the name is even?
- If the absolute difference in lengths of first and last name is less than 4?
- If the number of distinct characters in name is greater than 10?
- If the number of distinct vowels is more than 3?

(c) Report the error of your decision tree on the **Dataset/training.data** file.

Answer: 91.4%

(d) Report the error of your decision tree on the **Dataset/test.data** file.

Answer: 92.8%

(e) Report the maximum depth of your decision tree.

Answer: 23 (due to 23 features, full grown tree)

2. **Limiting Depth**

(a) Run 4-fold cross-validation using the specified files. Experiment with depths in the set $\{1, 2, 3, 4, 5, 10, 15, 20\}$, reporting the cross-validation accuracy and standard deviation for each depth. Explicity specify which depth should be chosen as the best, and explain why.

Answer: A depth of 5 works best as observed by the 4-fold cross validation performed on the training data. This also makes intuitive sense, very deep trees tend t overfit the training data, restricting the depth forces the tree to generalize within a constrain of limited depth and can be seen as form of regularization.

(b) Using the depth with the greatest cross-validation accuracy from your experiments: train your decision tree on the **Dataset/training.data** file. Report the accuracy of your decision tree on the **Dataset/test.data** file.

Answer: 93.03%

(c) Discuss the performance of the depth limited tree as compared to the full decision tree. Do you think limiting depth is a good idea? Why?

Answer: Accuracy = 91.89%.

Limiting depth is indeed a good idea. Very deep trees, though having more expressive power (they can represent boolean functions of length proportional to the depth of the tree), but this also leads to the tree 'memorizing' the labels of the input, and hence is highly likely to overfit the training data. From the cross validation table in the code, we see that increasing depth first gives rise to increase in accuracy but after a certain depth the accuracy starts to drop.

4 Decision Lists

Using the hint, we try to find a weight vector $\mathbf{w}$ and a bias term b that is equivalent to the 1-decision list. Let $\mathbf{w}^T = [w_1, w_2, \ldots, w_n]$ be the linear classifier for a 1-decision list of length n. We also replace each negated variable $\neg x_i$ by $z_i = \neg x_i$. The boolean vector is $\mathbf{x} = [x_1, x_2, \ldots, z_i, \ldots, x_n]$. Hence the linear classifier is given by $sgn(\mathbf{w^T x} + b)$. We modify sgn function in an equivalent way for the convenience of proof(this definition allows us to use $b = 0$).

$$sgn(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Observation: If the first node evaluates to false, the entire decision list returns false. This means that w_1 should "dominate" all $w_i, i \neq 1$. If x_1 evaluates to TRUE, we can follow the same process with x_2 as the starting node of the decision list. This idea of recursively evaluating decision lists, and the notion that the starting weight in each recursive call should "dominate" all other weights leads us to conclude that $w_1, w_2, \ldots, w_n$ should be sorted in decreasing order and the notion of "domination" gives the conclusion that $w_1 \geq \sum_{i=2}^{n} w_i$.

Claim: $w_i = 2^{n-i}$ and $b = 0$ gives a linear classifier for a n length decision list defined as above.

Inductive proof: We prove by induction on length n of decision list.

- *Base case* $n = 1$: $\mathbf{w} = [w_1] = [2]$. Holds trivially.
- *Inductive case.* Let's say the $w = [2^{n-1}, 2^{n-2}, \ldots, 2^1]$ is a correct linear classifier for an $n - 1$ length 1-decision list $x = [x_2, x_3, \ldots, x_n]$. Consider the n length decision list $x = [x_1, x_2, \ldots, x_n]$. There are two cases for possible values of x_1.
  - x_1 = FALSE: In this case the evaluation of decision list just calls the $n - 1$ length decision list, and it's correctness is given by the inductive hypothesis.
  - x_1 = TRUE: In this case, we have to show that if $w_1 \geq w_2 + w_3 \ldots w_n$ (which happens in the worst case where x_1 = TRUE and x_i = FALSE, $\forall i \in \{2, 3, \ldots n\}$), which is true since $2^k \geq \sum_{i=1}^{k-1} 2^i$.

Thus the linear classifier for 1-decision list is $\mathbf{w^T x}$, $\mathbf{w} = [2^n, 2^{n-1}, \ldots, 2^1], b = 0$.