

## Definitions

| Term                      | Notation  | Example(s) | We say in English ...   |
|---------------------------|---|------------|---|
| sequence                  | $x_1, \dots, x_n$   |            | A sequence $x_1$ to $x_n$   |
| summation                 | $\sum_{i=1}^n x_i$ or $\sum_{i=1}^n x_i$  |            | The sum of the terms of the sequence $x_1$ to $x_n$   |
| all reals                 | $\mathbb{R}$  |            | The (set of all) real numbers (numbers on the number line)  |
| all integers              | $\mathbb{Z}$  |            | The (set of all) integers (whole numbers including negatives, zero, and positives)  |
| all positive integers     | $\mathbb{Z}^+$  |            | The (set of all) strictly positive integers   |
| all natural numbers       | $\mathbb{N}$  |            | The (set of all) natural numbers. <b>Note:</b> we use the convention that 0 is a natural number.  |
| piecewise rule definition | $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ |            | Define $f$ of $x$ to be $x$ when $x$ is nonnegative and to be $-x$ when $x$ is negative   |
| function application      | $f(7)$<br>$f(z)$<br>$f(g(z))$   |            | $f$ of 7 <b>or</b> $f$ applied to 7 <b>or</b> the image of 7 under $f$<br>$f$ of $z$ <b>or</b> $f$ applied to $z$ <b>or</b> the image of $z$ under $f$<br>$f$ of $g$ of $z$ <b>or</b> $f$ applied to the result of $g$ applied to $z$ |
| absolute value            | $ -3 $  |            | The absolute value of $-3$  |
| square root               | $\sqrt{9}$  |            | The non-negative square root of 9   |

# Defining sets

*To define sets:*

To define a set using **roster method**, explicitly list its elements. That is, start with  $\{$  then list elements of the set separated by commas and close with  $\}$ .

To define a set using **set builder definition**, either form “The set of all  $x$  from the universe  $U$  such that  $x$  is ...” by writing

$$\{x \in U \mid \dots x \dots\}$$

or form “the collection of all outputs of some operation when the input ranges over the universe  $U$ ” by writing

$$\{\dots x \dots \mid x \in U\}$$

We use the symbol  $\in$  as “is an element of” to indicate membership in a set.

**Example sets:** For each of the following, identify whether it’s defined using the roster method or set builder notation and give an example element.

$$\{-1, 1\}$$

$$\{0, 0\}$$

$$\{-1, 0, 1\}$$

$$\{(x, x, x) \mid x \in \{-1, 0, 1\}\}$$

$$\{\}$$

$$\{x \in \mathbb{Z} \mid x \geq 0\}$$

$$\{x \in \mathbb{Z} \mid x > 0\}$$

$$\{\text{A, C, U, G}\}$$

$$\{\text{AUG, UAG, UGA, UAA}\}$$

# Least greatest proofs

For a set of numbers  $X$ , how do you formalize “there is a greatest  $X$ ” or “there is a least  $X$ ”?

**Prove or disprove:** There is a least prime number.

**Prove or disprove:** There is a greatest integer.

*Approach 1, De Morgan’s and universal generalization:*

*Approach 2, proof by contradiction:*

*Extra examples:* Prove or disprove that  $\mathbb{N}$ ,  $\mathbb{Q}$  each have a least and a greatest element.

## Gcd definition

**Definition: Greatest common divisor** Let  $a$  and  $b$  be integers, not both zero. The largest integer  $d$  such that  $d$  is a factor of  $a$  and  $d$  is a factor of  $b$  is called the greatest common divisor of  $a$  and  $b$  and is denoted by  $\gcd(a, b)$ .

## Gcd examples

Why do we restrict to the situation where  $a$  and  $b$  are not both zero?

Calculate  $\gcd(10, 15)$

Calculate  $\gcd(10, 20)$

## Gcd basic claims

**Claim:** For any integers  $a, b$  (not both zero),  $\gcd(a, b) \geq 1$ .

**Proof:** *Show that 1 is a common factor of any two integers, so since the gcd is the greatest common factor it is greater than or equal to any common factor.*

**Claim:** For any positive integers  $a, b$ ,  $\gcd(a, b) \leq a$  and  $\gcd(a, b) \leq b$ .

**Proof** *Using the definition of gcd and the fact that factors of a positive integer are less than or equal to that integer.*

**Claim:** For any positive integers  $a, b$ , if  $a$  divides  $b$  then  $\gcd(a, b) = a$ .

**Proof** *Using previous claim and definition of gcd.*

**Claim:** For any positive integers  $a, b, c$ , if there is some integer  $q$  such that  $a = bq + c$ ,

$$\gcd(a, b) = \gcd(b, c)$$

**Proof** *Prove that any common divisor of  $a, b$  divides  $c$  and that any common divisor of  $b, c$  divides  $a$ .*

# Gcd lemma relatively prime

**Lemma:** For any integers  $p, q$  (not both zero),  $\gcd\left(\frac{p}{\gcd(p, q)}, \frac{q}{\gcd(p, q)}\right) = 1$ . In other words, can reduce to relatively prime integers by dividing by gcd.

**Proof:**

Let  $x$  be arbitrary positive integer and assume that  $x$  is a factor of each of  $\frac{p}{\gcd(p, q)}$  and  $\frac{q}{\gcd(p, q)}$ . This gives integers  $\alpha, \beta$  such that

$$\alpha x = \frac{p}{\gcd(p, q)} \qquad \beta x = \frac{q}{\gcd(p, q)}$$

Multiplying both sides by the denominator in the RHS:

$$\alpha x \cdot \gcd(p, q) = p \qquad \beta x \cdot \gcd(p, q) = q$$

In other words,  $x \cdot \gcd(p, q)$  is a common divisor of  $p, q$ . By definition of  $\gcd$ , this means

$$x \cdot \gcd(p, q) \leq \gcd(p, q)$$

and since  $\gcd(p, q)$  is positive, this means,  $x \leq 1$ .

## Least greatest proofs

For a set of numbers  $X$ , how do you formalize “there is a greatest  $X$ ” or “there is a least  $X$ ”?

**Prove or disprove:** There is a least prime number.

**Prove or disprove:** There is a greatest integer.

*Approach 1, De Morgan’s and universal generalization:*

*Approach 2, proof by contradiction:*

*Extra examples:* Prove or disprove that  $\mathbb{N}$ ,  $\mathbb{Q}$  each have a least and a greatest element.



## Gcd definition

**Definition: Greatest common divisor** Let  $a$  and  $b$  be integers, not both zero. The largest integer  $d$  such that  $d$  is a factor of  $a$  and  $d$  is a factor of  $b$  is called the greatest common divisor of  $a$  and  $b$  and is denoted by  $\gcd(a, b)$ .

## Gcd examples

Why do we restrict to the situation where  $a$  and  $b$  are not both zero?

Calculate  $\gcd(10, 15)$

Calculate  $\gcd(10, 20)$

## Gcd basic claims

**Claim:** For any integers  $a, b$  (not both zero),  $\gcd(a, b) \geq 1$ .

**Proof:** *Show that 1 is a common factor of any two integers, so since the gcd is the greatest common factor it is greater than or equal to any common factor.*

**Claim:** For any positive integers  $a, b$ ,  $\gcd(a, b) \leq a$  and  $\gcd(a, b) \leq b$ .

**Proof** *Using the definition of gcd and the fact that factors of a positive integer are less than or equal to that integer.*

**Claim:** For any positive integers  $a, b$ , if  $a$  divides  $b$  then  $\gcd(a, b) = a$ .

**Proof** *Using previous claim and definition of gcd.*

**Claim:** For any positive integers  $a, b, c$ , if there is some integer  $q$  such that  $a = bq + c$ ,

$$\gcd(a, b) = \gcd(b, c)$$

**Proof** *Prove that any common divisor of  $a, b$  divides  $c$  and that any common divisor of  $b, c$  divides  $a$ .*

## Gcd lemma relatively prime

**Lemma:** For any integers  $p, q$  (not both zero),  $\gcd\left(\frac{p}{\gcd(p, q)}, \frac{q}{\gcd(p, q)}\right) = 1$ . In other words, can reduce to relatively prime integers by dividing by gcd.

**Proof:**

Let  $x$  be arbitrary positive integer and assume that  $x$  is a factor of each of  $\frac{p}{\gcd(p, q)}$  and  $\frac{q}{\gcd(p, q)}$ . This gives integers  $\alpha, \beta$  such that

$$\alpha x = \frac{p}{\gcd(p, q)} \qquad \beta x = \frac{q}{\gcd(p, q)}$$

Multiplying both sides by the denominator in the RHS:

$$\alpha x \cdot \gcd(p, q) = p \qquad \beta x \cdot \gcd(p, q) = q$$

In other words,  $x \cdot \gcd(p, q)$  is a common divisor of  $p, q$ . By definition of  $\gcd$ , this means

$$x \cdot \gcd(p, q) \leq \gcd(p, q)$$

and since  $\gcd(p, q)$  is positive, this means,  $x \leq 1$ .

# Definitions

| Term                      | Notation  | Example(s) | We say in English ...   |
|---------------------------|---|------------|---|
| sequence                  | $x_1, \dots, x_n$   |            | A sequence $x_1$ to $x_n$   |
| summation                 | $\sum_{i=1}^n x_i$ or $\sum_{i=1}^n x_i$  |            | The sum of the terms of the sequence $x_1$ to $x_n$   |
| all reals                 | $\mathbb{R}$  |            | The (set of all) real numbers (numbers on the number line)  |
| all integers              | $\mathbb{Z}$  |            | The (set of all) integers (whole numbers including negatives, zero, and positives)  |
| all positive integers     | $\mathbb{Z}^+$  |            | The (set of all) strictly positive integers   |
| all natural numbers       | $\mathbb{N}$  |            | The (set of all) natural numbers. <b>Note:</b> we use the convention that 0 is a natural number.  |
| piecewise rule definition | $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ |            | Define $f$ of $x$ to be $x$ when $x$ is nonnegative and to be $-x$ when $x$ is negative   |
| function application      | $f(7)$<br>$f(z)$<br>$f(g(z))$   |            | $f$ of 7 <b>or</b> $f$ applied to 7 <b>or</b> the image of 7 under $f$<br>$f$ of $z$ <b>or</b> $f$ applied to $z$ <b>or</b> the image of $z$ under $f$<br>$f$ of $g$ of $z$ <b>or</b> $f$ applied to the result of $g$ applied to $z$ |
| absolute value            | $ -3 $  |            | The absolute value of $-3$  |
| square root               | $\sqrt{9}$  |            | The non-negative square root of 9   |

# Defining sets

*To define sets:*

To define a set using **roster method**, explicitly list its elements. That is, start with  $\{$  then list elements of the set separated by commas and close with  $\}$ .

To define a set using **set builder definition**, either form “The set of all  $x$  from the universe  $U$  such that  $x$  is ...” by writing

$$\{x \in U \mid \dots x \dots\}$$

or form “the collection of all outputs of some operation when the input ranges over the universe  $U$ ” by writing

$$\{\dots x \dots \mid x \in U\}$$

We use the symbol  $\in$  as “is an element of” to indicate membership in a set.

**Example sets:** For each of the following, identify whether it’s defined using the roster method or set builder notation and give an example element.

$$\{-1, 1\}$$

$$\{0, 0\}$$

$$\{-1, 0, 1\}$$

$$\{(x, x, x) \mid x \in \{-1, 0, 1\}\}$$

$$\{\}$$

$$\{x \in \mathbb{Z} \mid x \geq 0\}$$

$$\{x \in \mathbb{Z} \mid x > 0\}$$

$$\{\text{A, C, U, G}\}$$

$$\{\text{AUG, UAG, UGA, UAA}\}$$