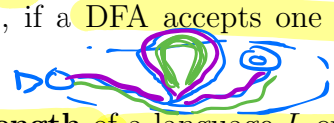
Announcements: Project Part 1 due this week
Review Quiz Questions Week 1-3 available on website

Monday April 18

Extra question: what if a DFA does not have a loop? What is the pumping length of the language recognized by such a DFA?

Recap so far: In DFA, the only memory available is in the states. Automata can only "remember" finitely far in the past and finitely much information, because they can have only finitely many states. If a computation path of a DFA visits the same state more than once, the machine can't tell the difference between the first time and future times it visits this state. Thus, if a DFA accepts one long string, then it must accept (infinitely) many similar strings.



Definition A positive integer p is a **pumping length** of a language L over Σ means that, for each string $s \in \Sigma^*$, if $|s| \geq p$ and $s \in L$, then there are strings x, y, z such that

$s = xyz$

and

loop

$|y| > 0$, for each $i \geq 0$, $xy^iz \in L$, and $|xy| \leq p$.

loop happens quickly

Negation: A positive integer p is **not a pumping length** of a language L over Σ iff

$$\exists s \left(|s| \geq p \wedge s \in L \wedge \forall x \forall y \forall z \left((s = xyz \wedge |y| > 0 \wedge |xy| \leq p) \rightarrow \exists i (i \geq 0 \wedge xy^iz \notin L) \right) \right)$$

Informally: there is a long string in the language that is not pumpable.

Restating **Pumping Lemma**: If L is a regular language, then it has a pumping length.

(pos integer)

Contrapositive: If L has no pumping length, then it is nonregular.

To prove L is nonregular, we try to show it has no pumping length.

The Pumping Lemma *cannot* be used to prove that a language *is* regular.

The Pumping Lemma **can** be used to prove that a language *is not* regular.

Extra practice: Exercise 1.49 in the book.

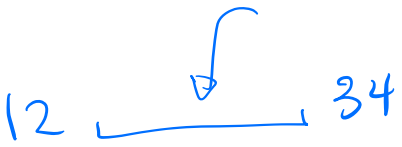
Extra practice: Give general description of the pumping length of a finite set.

Proof strategy: To prove that a language L is **not** regular,

1. Consider an arbitrary positive integer p — fixed but unknown.
2. Prove that p is not a pumping length for L
3. Conclude that L does not have *any* pumping length, and therefore it is not regular.

$L = L(12\Sigma^*34)$ $\quad \Sigma = \{1,2,3,4\}$

regular



$S \in L$ and S is long (length ≥ 5)

$S = \underset{12}{x} \; \underset{}{y} \; \underset{34}{z}$

where xz, xyz, $xyyz$, $xyyyz$, $xyyyyz$ all in L

Balance!

Examples of strings in L
ε 01 0011 000111

Examples of strings not in L
10 0 111111

Example: $\Sigma = \{0, 1\}$, $L = \{0^n 1^n \mid n \geq 0\}$. $n \in \mathbb{Z}$

Fix p an arbitrary positive integer. List strings that are in L and have length greater than or equal to p:

$0^p 1^p$ length $2p$ $0^{2p} 1^{2p}$ length $4p$ $0^{p+1000} 1^{p+1000}$ length $2p+2000$

Goal: show that p is not a pumping length for L.

Pick $s =$ $0^p 1^p$ NOTE: $s \in L$ and $|s| \geq p$ because $|s| = 2p$.

Suppose $s = xyz$ with $|xy| \leq p$ and $|y| > 0$. arbitrary x, y, z satisfying constraints

$m > 0$

$r = p - m - k$

$x = 0^k$ $y = 0^m$ $z = 0^r 1^p$ where

Then when $i =$ 2 , $xy^iz =$

$$xyyz = 0^k 0^m 0^m 0^r 1^p$$
$$= 0^{k+m+m+r} 1^p$$
$$= 0^{p+m} 1^p$$

Show s is not pumpable.

xz
$s = xyz$
$xyyz$
$xyyyz$
$xyyyyz$

Since we've found a string in L that should have been pumpable but wasn't, we've shown that p is not a pumping length for L. But p was arbitrary, so no pos integer is a pumping length for L, so L is not regular ☒

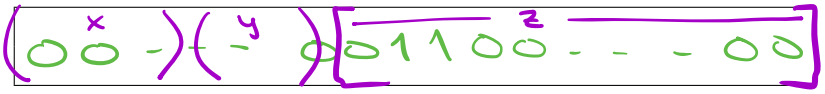
Notice: we could have chosen i to be 0 or 3,105.

Example: $\Sigma = \{0,1\}$, $L = \{ww^{\mathcal{R}} \mid w \in \{0,1\}^*\}$.

Fix p an arbitrary positive integer. ~~List strings that are in L and have length greater than or equal to p~~:

Pick $s =$ $0^p 1 1 0^p$

Suppose $s = xyz$ with $|xy| \leq p$ and $|y| > 0$. Then $x = 0^k$, $y = 0^m$, $z = 0^r 1 1 0^p$ for some $k \in \mathbb{Z}^{\geq 0}$, $m \in \mathbb{Z}^+$, $r \in \mathbb{Z}^{\geq 0}$ with $k+m+r = p$.



Then when $i =$ 0, $xy^iz = xz = 0^k 0^r 1 1 0^p = 0^{p-m} 1 1 0^p$ which is not in L because if split into two pieces, either each piece has single 1 but then different length pieces (since $m > 0$) so not reversals of each other), or one piece has both 1s and the other none, so not reversals of each other.

Example: $\Sigma = \{0,1\}$, $L = \{0^j 1^k \mid j \geq k \geq 0\}$. extra practice

Fix p an arbitrary positive integer. ~~List strings that are in L and have length greater than or equal to p~~:

Pick $s =$

Suppose $s = xyz$ with $|xy| \leq p$ and $|y| > 0$.

Then when $i =$, $xy^iz =$

Example: $\Sigma = \{0,1\}$, $L = \{0^n 1^t 0^n \mid t, n \geq 0\}$. extra practice

Fix p an arbitrary positive integer. List strings that are in L and have length greater than or equal to p:

Pick $s =$

Suppose $s = xyz$ with $|xy| \leq p$ and $|y| > 0$.

Then when $i =$, $xy^iz =$

Review: Week 4 Monday

Recall: Review quizzes based on class material are assigned each day. These quizzes will help you track and confirm your understanding of the concepts and examples we work in class. Quizzes can be submitted on Gradescope as many times (with no penalty) as you like until the quiz deadline: the three quizzes each week are all due on Friday (with no penalty late submission open until Sunday).

Please complete the review quiz questions on Gradescope about pumping lemma and nonregular sets.

Pre class reading for next time: Page 112

office hours today after class in same Zoom room.
Friday's class back in WLH 2001

Wednesday April 20

Language	$s \in L$	$s \notin L$	Is the language regular or nonregular?
$\{a^n b^n \mid 0 \le n \le 5\}$ = {ε, ab, aabb, aaabbb, aaaabbbb, aaaaabbbbb}	aabb	abbb, aaaaabbbbbb	Regular because finite
$\{b^m a^n \mid n \ge 2\}$	bbaa	ba, aabb	Nonregular using pumping lemma.
$\{a^i b^j \mid 0 \le i \le j\}$			
$\{a^i b^j \mid i \ge j+3, j \ge 0\}$			
$\{b^i a^j \mid i \ge 1, j \ge 3\}$			
$\{w \in \{a,b\}^* \mid w = w^{\mathcal{R}}\}$ The set of palindromes			Nonregular using pumping lemma
$\{ww^{\mathcal{R}} \mid w \in \{a,b\}^*\}$	aabbaa (w = aab, $w^{\mathcal{R}}$ = baa)	aabb, abbba	Nonregular

(1) The set of strings of even length that are palindromes

(2) Using pumping lemma
- Consider arbitrary pos int, p.
- WTS it is not a pumping length for $\{ww^{\mathcal{R}} \mid w \in \{a,b\}^*\}$
- By giving a specific value $s = 0^p 1 1 0^p$ and show that s is not pumpable
- Conclude this language has no pumping length so it's not regular.

Consider arbitrary x, y, z satisfying $s = xyz$, $|xy| \le p$, $|y| > 0$, and WTS there is some nonneg int i with $xy^i z \notin \{ww^{\mathcal{R}} \mid w \in \{a,b\}^*\}$

From the constraints on x, y, z we know we can write

$x = 0^k$ $y = 0^m$ $z = 0^r 1 1 0^p$

where $k+m+r = p$, $m > 0$, $k \ge 0$, $r \ge 0$.

Choose $i = 0$ or 2 or . . .

and show $xy^i z \notin \{ww^{\mathcal{R}} \mid w \in \{a,b\}^*\}$

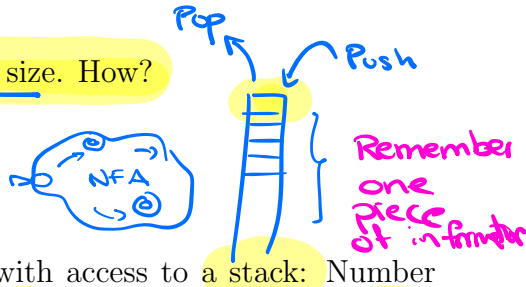
$xyyz = 0^{k+2m} r 110^p = 0^{p+m} 110^p$

To prove a set is nonregular
① Using pumping lemma
② Using closure properties

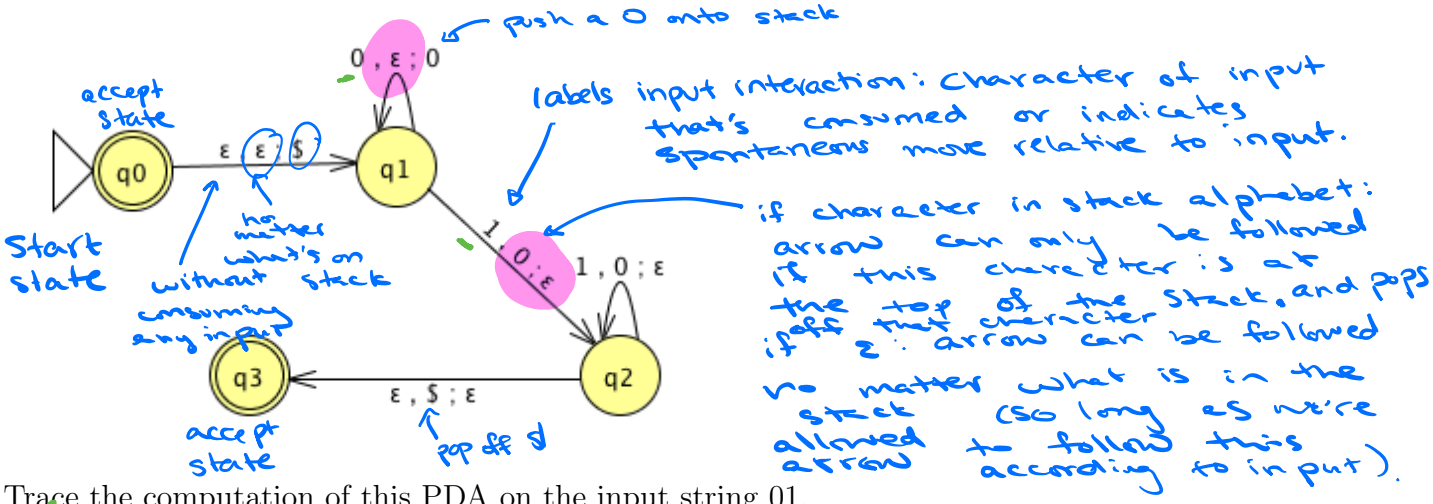
Regular sets are not the end of the story

- Many nice / simple / important sets are not regular — Parsing code
- Limitation of the finite-state automaton model: Can't "count", Can only remember finitely far into the past, Can't backtrack, Must make decisions in "real-time"
- We know actual computers are more powerful than this model...

The **next** model of computation. Idea: allow some memory of unbounded size. How?



- To generalize regular expressions: **context-free grammars**
- To generalize NFA: **Pushdown automata**, which is like an NFA with access to a stack: Number of states is fixed, number of entries in stack is unbounded. At each step (1) Transition to new state based on current state, letter read, and top letter of stack, then (2) (Possibly) push or pop a letter to (or from) top of stack. Accept a string iff there is some sequence of states and some sequence of stack contents which helps the PDA processes the entire input string and ends in an accepting state.



Trace the computation of this PDA on the input string 01.

① 01 q_0 empty
FOLLOW ARROW
② 01 q_1 $
FOLLOW SELF LOOP
③ 01 q_1 0 $
FOLLOW ARROW
④ 01 q_2 $
FOLLOW ARROW
⑤ 01 q_3 empty

01 is accepted by PDA

Trace the computation of this PDA on the input string 011.

same steps ①-④
011 q_2 $
Can't follow self loop at q_2 because top of stack is $, not 0
FOLLOW ARROW
011 q_3 empty
STUCK! because no outgoing arrows from q_3 but haven't finished processing input string 011

011 is not accepted by this PDA

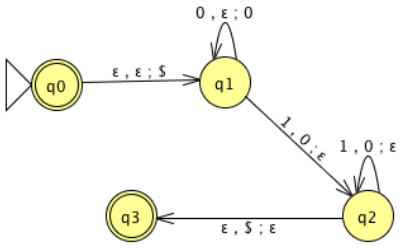
Friday :

Definition A **pushdown automaton** (PDA) is specified by a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q is the finite set of states, Σ is the input alphabet, Γ is the stack alphabet,

$$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to \mathcal{P}(Q \times \Gamma_\varepsilon)$$

is the transition function, $q_0 \in Q$ is the start state, $F \subseteq Q$ is the set of accept states.

Formal definition



Draw the state diagram of a PDA with $\Sigma = \Gamma$.

Draw the state diagram of a PDA with $\Sigma \cap \Gamma = \emptyset$.

A PDA recognizing the set { } can be informally described as:

> Read symbols from the input. As each 0 is read, push it onto the stack. As soon as 1s are seen, pop a 0 off the stack for each 1 read. If the stack becomes empty and there is exactly one 1 left to read, read that 1 and accept the input. If the stack becomes empty and there are either zero or more than one 1s left to read, or if the 1s are finished while the stack still contains 0s, or if any 0s appear in the input following 1s, reject the input.

State diagram for this PDA:

$\Sigma = \{a, b, c\}$. $L = \{a^i b^j c^k \mid i = j \text{ or } i = k,\ i \geq 0, j \geq 0, k \geq 0\}$

Review: Week 4 Wednesday

Please complete the review quiz questions on Gradescope about PDA definitions.

Pre class reading for next time: Page 102

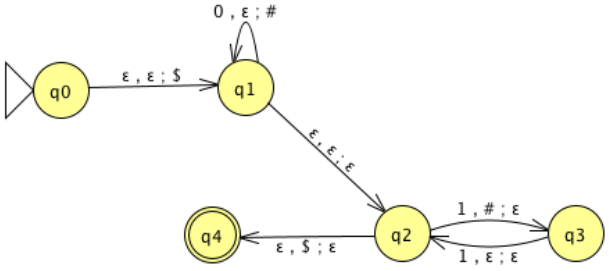
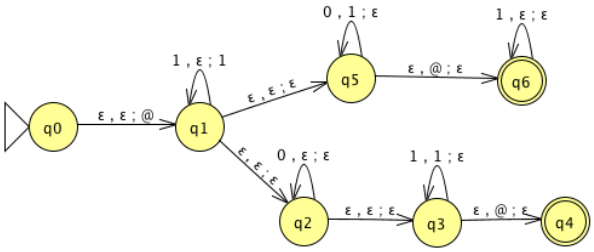
Friday April 22

Consider the state diagram of a PDA with input alphabet Σ and stack alphabet Γ.

Label	means
$a, b \to c$ when $a \in \Sigma$, $b \in \Gamma$, $c \in \Gamma$	
$a, \varepsilon \to c$ when $a \in \Sigma$, $c \in \Gamma$	
$a, b \to \varepsilon$ when $a \in \Sigma$, $b \in \Gamma$	
$a, \varepsilon \to \varepsilon$ when $a \in \Sigma$	

How does the meaning change if a is replaced by ε?

For the PDA state diagrams below, $\Sigma = \{0, 1\}$.

Mathematical description of language	State diagram of PDA recognizing language
	
	
$\{0^i 1^j 0^k \mid i, j, k \geq 0\}$	

Assume $a \in \Sigma$ and L is a language over Σ. Recall that

$$aL = \{aw \mid w \in L\}$$

If $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ is a PDA with $L(M) = L$, a PDA M_1 that recognizes aL is

$$M_1 = (\qquad\qquad\qquad, \Sigma, \Gamma, \delta_1, \qquad\quad, \qquad\quad)$$

with

Review: Week 4 Friday

Please complete the review quiz questions on Gradescope about PDA construction.

(Notes)

Friday April 22

$\emptyset$ · Σ^* · $\{0^n 1^n \mid n \geq 0\}$ · $\{ww^R \mid w \in \Sigma^*\}$ ctble $\quad \mathcal{P}(\Sigma^*)$

Definition A **pushdown automaton** (PDA) is specified by a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q is the finite set of states, Σ is the input alphabet, Γ is the stack alphabet, Gamma stack alphabet

$$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to \mathcal{P}(Q \times \Gamma_\varepsilon)$$

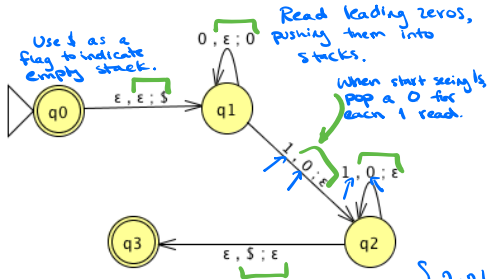
state, input label, stack label; state, actions for stack.

is the transition function, $q_0 \in Q$ is the start state, $F \subseteq Q$ is the set of accept states.

Formal definition

$(\{q_0, q_1, q_2, q_3\}, \{0,1\}, \{\$, 0\}, \delta, q_0, \{q_0, q_3\})$

$$\delta((q,a,b)) = \begin{cases} \{(q_1,\$)\} & \text{if } q = q_0,\ a = \varepsilon,\ b = \varepsilon \\ \{(q_1,0)\} & \text{if } q = q_1,\ a = 0,\ b = \varepsilon \\ \{(q_2,\varepsilon)\} & \text{if } q = q_1,\ a = 1,\ b = 0 \\ \{(q_2,\varepsilon)\} & \text{if } q = q_2,\ a = 1,\ b = 0 \\ \{(q_3,\varepsilon)\} & \text{if } q = q_2,\ a = \varepsilon,\ b = \$ \\ \emptyset & \text{otherwise} \end{cases}$$



The language recognized by this PDA is $\{0^n 1^n \mid n \geq 0\}$

Draw the state diagram of a PDA with $\Sigma = \Gamma$.

$(\{q_0\}, \{0,1\}, \{0,1\}, \delta_{\text{given}}, q_0, \{q_0\})$

The language recognized by this PDA is $\{0,1\}^*$

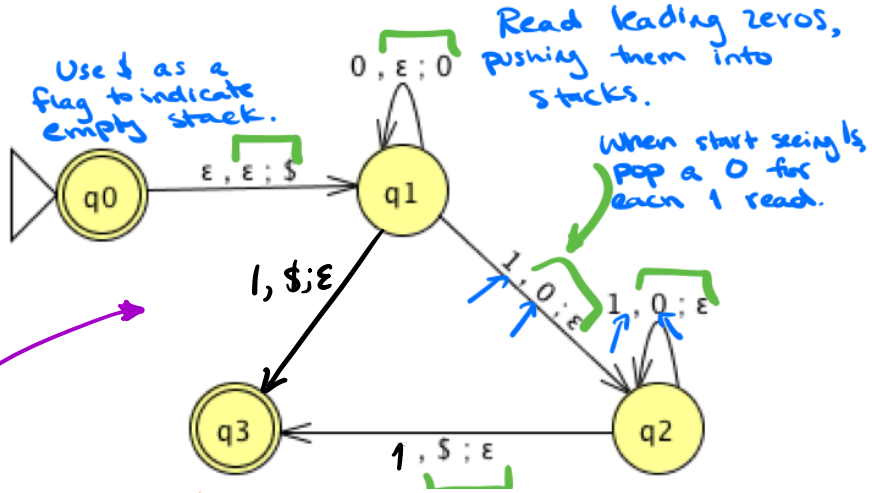
Draw the state diagram of a PDA with $\Sigma \cap \Gamma = \emptyset$.

Extra Practice

A PDA recognizing the set $\{0^n 1^{n+1} \mid n \in \mathbb{Z}, n \geq 0\}$ can be informally described as:

> Read symbols from the input. As each 0 is read, push it onto the stack. As soon as 1s are seen, pop a 0 off the stack for each 1 read. If the stack becomes empty and there is exactly one 1 left to read, read that 1 and accept the input. If the stack becomes empty and there are either zero or more than one 1s left to read, or if the 1s are finished while the stack still contains 0s, or if any 0s appear in the input following 1s, reject the input.

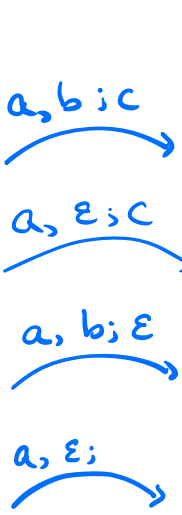
State diagram for this PDA:



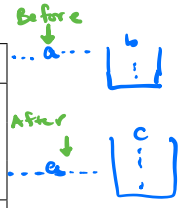
for edge case 1

to add 1 after balanced 0s and 1s

Consider the state diagram of a PDA with input alphabet Σ and stack alphabet Γ.



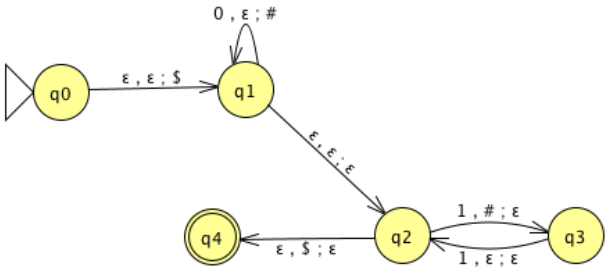
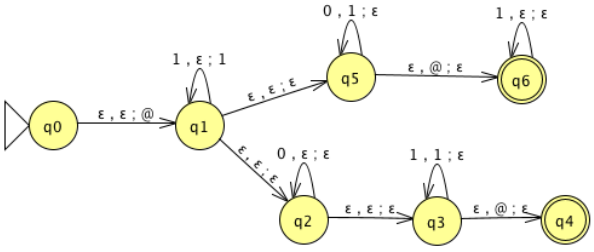
Label	means
$a, b; c$ when $a \in \Sigma$, $b \in \Gamma$, $c \in \Gamma$	if the next symbol to read is a and the current top of stack is b, I can follow this arrow and consume the a, pop the b, and push a c to the top of the stack
$a, \varepsilon; c$ when $a \in \Sigma$, $c \in \Gamma$	if the next symbol to read is a without looking at/caring about what's at the top of the stack, I can follow this arrow and consume the a, pushing a c to the top of the stack.
$a, b; \varepsilon$ when $a \in \Sigma$, $b \in \Gamma$	if the next symbol to read is a and the current top of stack is b, I can follow this arrow and consume the a, pop the b, and not push anything new to the stack.
$a, \varepsilon; \varepsilon$ when $a \in \Sigma$	if the next symbol to read is a I can follow this arrow and consume this a and not touch the stack.



How does the meaning change if a is replaced by ε? ✱

Note: alternate notation is to replace ; with →

For the PDA state diagrams below, $\Sigma = \{0, 1\}$.

Mathematical description of language	State diagram of PDA recognizing language
Extra practice	
Extra practice	
$\{0^i 1^j 0^k \mid i, j, k \geq 0\}$	Extra practice

Review: Week 4 Friday

Please complete the review quiz questions on Gradescope about PDA construction.