Week 5 at a glance

Textbook reading: Section 2.2, 2.1.
Before Monday, read Theorem 2.20.

Before Wednesday, read Example 2.18 (page 114).

Before Friday, read Figure 3.1.

For Week 6 Monday: Page 165-166 Introduction to Section 3.1.

We will be learning and practicing to:

- Clearly and unambiguously communicate computational ideas using appropriate formalism. Translate across levels of abstraction.
  - Describe and use models of computation that don't involve state machines.
    - **Use context-free grammars and relate them to languages and pushdown automata.**
  - Use precise notation to formally define the state diagram of a Turing machine
  - Use clear English to describe computations of Turing machines informally.
    - **Design a PDA that recognizes a given language.**
  - Give examples of sets that are context-free (and prove that they are).
    - **State the definition of the class of context-free languages**
    - **Explain the limits of the class of context-free languages**
    - **Identify some context-free sets and some non-context-free sets**
- Know, select and apply appropriate computing knowledge and problem-solving techniques. Reason about computation and systems.
  - Describe and closure properties of classes of languages under certain operations.
    - **Apply a general construction to create a new PDA or CFG from an example one.**
    - **Formalize a general construction from an informal description of it.**
    - **Use general constructions to prove closure properties of the class of context-free langugages.**
    - **Use counterexamples to prove non-closure properties of the class of context-free langugages.**

TODO:

Schedule your Test 1 Attempt 1, Test 2 Attempt 1, Test 1 Attempt 2, and Test 2 Attempt 2 times at PrairieTest (http://us.prairietest.com)

Review Quiz 5 on PrairieLearn (http://us.prairielearn.com), complete by Sunday 11/4/2024

$G = (V, \Sigma, R, S)$ $\quad L(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$

start variable

Monday: Context-free languages

Warmup: Design a CFG to generate the language $\{a^i b^j \mid j \geq i \geq 0\}$ over $\{a,b\}$

$G = (\{S\}, \{a,b\}, R, S)$

where R is given by

$S \to \varepsilon \mid aSb \mid Sb$

1 2 3

Remember: to generate $\{a^n b^n \mid n \geq 0\}$, can use

$(\{S\}, \{a,b\}, \{S \to aSb \mid \varepsilon\}, S)$

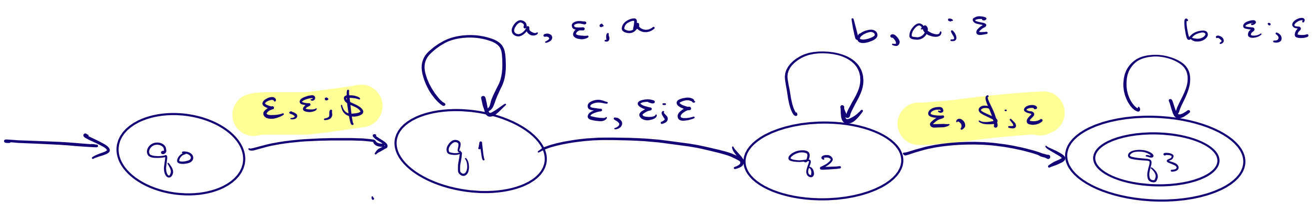
Sample derivation:

$S \overset{1}{\Longrightarrow} \varepsilon$ good b/c $\varepsilon = a^0 b^0$ $i = j = 0$

$S \overset{3}{\Longrightarrow} Sb \overset{3}{\Longrightarrow} Sbb \overset{2}{\Longrightarrow} aSbbb \overset{1}{\Longrightarrow} abbb$

good b/c $abbb = a^1 b^3$ $i = 1$ $j = 3$

Design a PDA to recognize the language $\{a^i b^j \mid j \geq i \geq 0\}$



read and record a's.

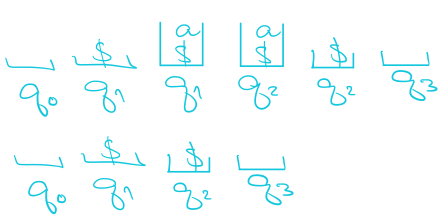
spontaneous relative to input and the stack.

match b's to a's we read

reading more b's is ok.

Trace

ab



ε

abb

aba

Theorem 2.20: A language is generated by some context-free grammar if and only if it is recognized by some push-down automaton.

Definition: a language is called **context-free** if it is the language generated by a context-free grammar. The class of all context-free language over a given alphabet Σ is called **CFL**.

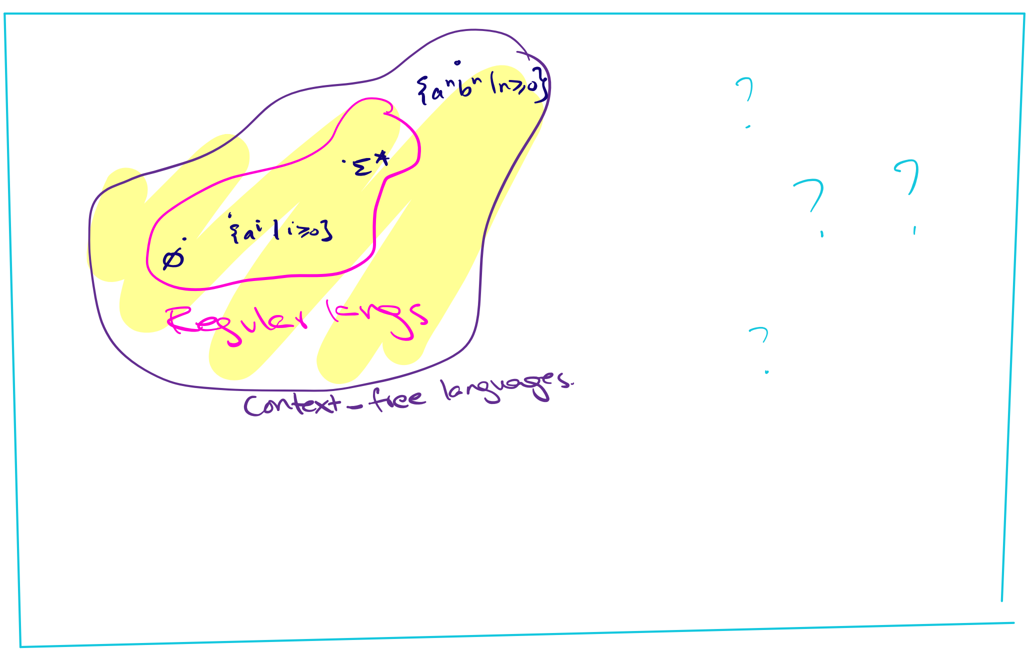
Consequences:

start w/ a NFA recognizing a language. Define PDA simulates it, recognizes the same language.

- Quick proof that every regular language is context free

start w/ DFA recognizing a language. Define CFG generates language

- To prove closure of the class of context-free languages under a given operation, we can choose either of two modes of proof (via CFGs or PDAs) depending on which is easier
- To fully specify a PDA we could give its 6-tuple formal definition or we could give its input alphabet, stack alphabet, and state diagram. An informal description of a PDA is a step-by-step description of how its computations would process input strings; the reader should be able to reconstruct the state diagram or formal definition precisely from such a descripton. The informal description of a PDA can refer to some common modules or subroutines that are computable by PDAs:
  - PDAs can "test for emptiness of stack" without providing details. *How?* We can always push a special end-of-stack symbol, \$, at the start, before processing any input, and then use this symbol as a flag.
  - PDAs can "test for end of input" without providing details. *How?* We can transform a PDA to one where accepting states are only those reachable when there are no more input symbols.



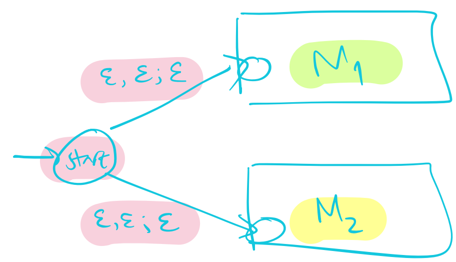
$\mathcal{P}(\Sigma^*)$
the set of all languages over Σ

Suppose L_1 and L_2 are context-free languages over Σ. **Goal**: $L_1 \cup L_2$ is also context-free.

Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define $M = (\ Q,\ \Sigma,\ \Gamma,\ \delta,\ q_{new},\ F_1 \cup F_2\ )$



Assume $Q_1 \cap Q_2 = \emptyset$

and $q_{new} \notin Q_1 \cup Q_2$

Define $Q = Q_1 \cup Q_2 \cup \{q_{new}\}$

$\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$

$\Gamma = \Gamma_1 \cup \Gamma_2$

power set

$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to \mathcal{P}(\ Q \times \Gamma_\varepsilon)$

$$\delta(\ (q, a, b)\ ) = \begin{cases} \{(q_1, \varepsilon), (q_2, \varepsilon)\} & q = q_{new},\ a = \varepsilon,\ b = \varepsilon \\ \delta_1(\ (q, a, b)\ ) & q \in Q_1,\ a \in \Sigma_\varepsilon,\ b \in \Gamma_{1\varepsilon} \\ \delta_2(\ (q_2, a, b)\ ) & q \in Q_2,\ a \in \Sigma_\varepsilon,\ b \in \Gamma_{2\varepsilon} \\ \emptyset & \text{otherwise} \end{cases}$$

Approach 2: with CFGs

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define $G = \left(\ V_1 \cup V_2 \cup \{S\},\ \Sigma,\ R_1 \cup R_2 \cup \{S \to S_1,\ S \to S_2\},\ S\ \right)$

Assume $V_1 \cap V_2 = \emptyset$

and $S \notin V_1 \cup V_2$

$$S \Rightarrow S_1 \Rightarrow \overbrace{\ \cdots\ \Rightarrow}^{\text{like } G_1} w \in L_1$$

$$S \Rightarrow S_2 \Rightarrow \underbrace{\ \cdots\ \Rightarrow}_{\text{like } G_2} w \in L_2$$

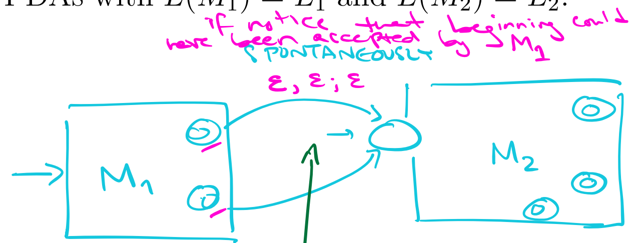
Suppose L_1 and L_2 are context-free languages over Σ. **Goal**: $L_1 \circ L_2$ is also context-free.

Approach 1: with PDAs

Try modifying union construction...

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define $M =$



Details in homework 4.

Simulate M_1 on the beginning of string

intermediate clean-up phase where we empty stack.

Simulate M_2 running on the rest of string. [supposed to start with an empty stack]

Approach 2: with CFGs

Try modifying union construction...

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define $G = (V_1 \cup V_2 \cup \{S\}, \Sigma, R, S)$

$$S \Rightarrow \underbrace{S_1}_{\text{first part can be derived in } G_1} \underbrace{S_2}_{\text{second part can be derived in } G_2}$$

Assume $V_1 \cap V_2 = \emptyset$

$S \notin V_1 \cup V_2$

where $R = R_1 \cup R_2 \cup \{ S \to S_1 S_2 \}$

Wednesday: Context-free and non-context-free languages

Summary

details of algorithms.

Over a fixed alphabet Σ, a language L is **regular**

iff it is described by some regular expression
iff it is recognized by some DFA
iff it is recognized by some NFA

Over a fixed alphabet Σ, a language L is **context-free**

iff it is generated by some CFG
iff it is recognized by some PDA

Fact: Every regular language is a context-free language.

For each L, if there is M is a NFA and $L(M)=L$, then we can build PDA $\tilde{M}$ with same state diagram (ignoring stack) recognizing same language

Fact: There are context-free languages that are not nonregular.

ie. regular

Yes, in fact all regular languages are context-free and there's at least one regular language.

Fact: There are context-free languages that are nonregular.

Yes, witness eg. $\{0^n 1^n \mid n \geq 0\}$.

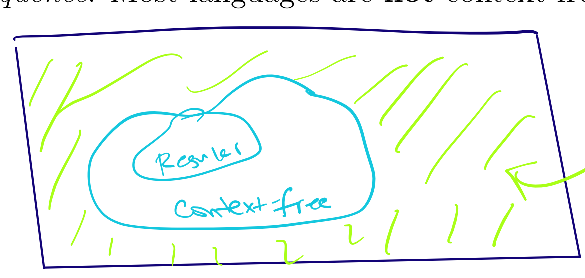
Fact: There are countably many regular languages.

Because regular expressions are strings.

Fact: There are countably inifnitely many context-free languages.

Because CFGs can be represented as strings

Consequence: Most languages are **not** context-free!

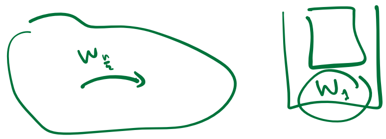


uncountably many languages over Σ that are not context free

Sneak Peek: if we replace stack with a queue then get model of computation that is as powerful as Turing machine!

Examples of non-context-free languages

$W = W_1 W_2 \cdots W_n$



"reading n" destroys it

$\{a^n b^n c^n \mid 0 \leq n, n \in \mathbb{Z}\}$

$\{a^i b^j c^k \mid 0 \leq i \leq j \leq k, i \in \mathbb{Z}, j \in \mathbb{Z}, k \in \mathbb{Z}\}$

$\{ww \mid w \in \{0,1\}^*\}$

Context-free

Compare to $\{w0u \mid w \in \{0,1\}^*, u \in \{0,1\}^*, |w| = |u|\}$

Compare to $\{ww^R \mid w \in \{0,1\}^*, R \text{ is reverse}\}$.

(Sipser Ex 2.36, Ex 2.37, 2.38)

There is a Pumping Lemma for CFL that can be used to prove a specific language is non-context-free: If A is a context-free language, there is a number p where, if s is any string in A of length at least p, then s may be divided into five pieces $s = uvxyz$ where (1) for each $i \geq 0$, $uv^ixy^iz \in A$, (2) $|uv| > 0$, (3) $|vxy| \leq p$. *We will not go into the details of the proof or application of Pumping Lemma for CFLs this quarter.*

Recall: A set X is said to be **closed** under an operation OP if, for any elements in X, applying OP to them gives an element in X.

Extra Practice:

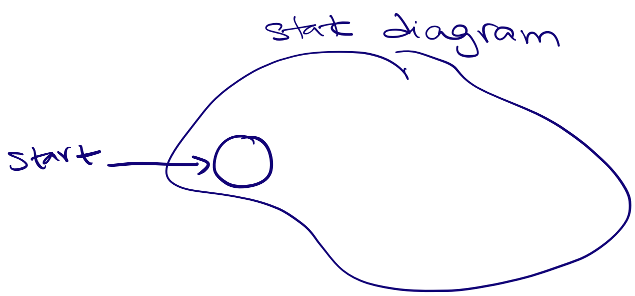
True/False	Closure claim
True	The set of integers is closed under multiplication. $\forall x \forall y\ (\ (x \in \mathbb{Z} \wedge y \in \mathbb{Z}) \to xy \in \mathbb{Z}\ )$
True	For each set A, the power set of A is closed under intersection. $\forall A_1 \forall A_2\ (\ (A_1 \in \mathcal{P}(A) \wedge A_2 \in \mathcal{P}(A) \in \mathbb{Z}) \to A_1 \cap A_2 \in \mathcal{P}(A)\ )$
True	The class of regular languages over Σ is closed under complementation.
True	The class of regular languages over Σ is closed under union.
True	The class of regular languages over Σ is closed under intersection.
True	The class of regular languages over Σ is closed under concatenation.
True	The class of regular languages over Σ is closed under Kleene star.
FALSE	The class of context-free languages over Σ is closed under complementation.
True	The class of context-free languages over Σ is closed under union.
FALSE	The class of context-free languages over Σ is closed under intersection.
True	The class of context-free languages over Σ is closed under concatenation.
True	The class of context-free languages over Σ is closed under Kleene star.

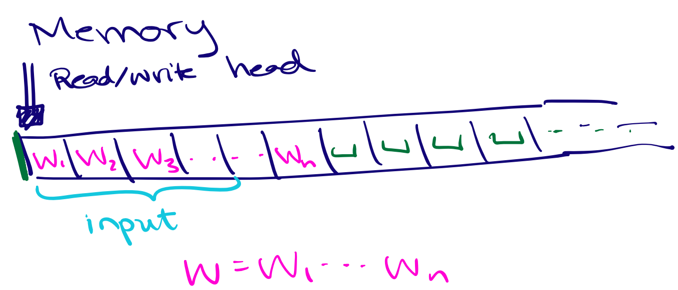
Friday: Turing machines

We are ready to introduce a formal model that will capture a notion of general purpose computation.

- *Similar to DFA, NFA, PDA*: input will be an arbitrary string over a fixed alphabet.
- *Different from NFA, PDA*: machine is deterministic.
- *Different from DFA, NFA, PDA*: read-write head can move both to the left and to the right, and can extend to the right past the original input.
- *Similar to DFA, NFA, PDA*: transition function drives computation one step at a time by moving within a finite set of states, always starting at designated start state.
- *Different from DFA, NFA, PDA*: the special states for rejecting and accepting take effect immediately.

(See more details: Sipser p. 166)





set of states   tape alphabet

Formally: a Turing machine is $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where δ is the **transition function**

input alphabet

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$$

The **computation** of M on a string w over Σ is:

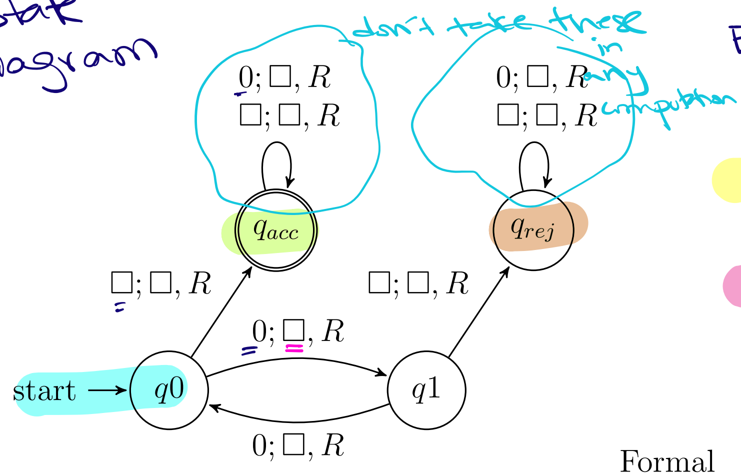
- Read/write head starts at leftmost position on tape.
- Input string is written on $|w|$-many leftmost cells of tape, rest of the tape cells have the blank symbol. **Tape alphabet** is Γ with $\llcorner\!\lrcorner \in \Gamma$ and $\Sigma \subseteq \Gamma$. The blank symbol $\llcorner\!\lrcorner \notin \Sigma$.
- Given current state of machine and current symbol being read at the tape head, the machine transitions to next state, writes a symbol to the current position of the tape head (overwriting existing symbol), and moves the tape head L or R (if possible).

  CONVENTION: if scanning leftmost position and transition says "go left", we stay.
- Computation ends **if and when** machine enters either the accept or the reject state. This is called **halting**. Note: $q_{accept} \neq q_{reject}$.

The **language recognized by the Turing machine** M, is $L(M) = \{w \in \Sigma^* \mid w \text{ is accepted by } M\}$, which is defined as

$$\{w \in \Sigma^* \mid \text{computation of } M \text{ on } w \text{ halts after entering the accept state}\}$$

State diagram



Formal definition: $(Q, \{0\}, \{0, ␣\}, \delta, q_0, q_{acc}, q_{rej})$

$Q = \{q_0, q_1, q_{acc}, q_{rej}\}$

Sample computation: input 000

$q0 \downarrow$						
0	0	0	␣	␣	␣	␣
	$q_1 \downarrow$					
␣	0	0	␣	␣	␣	␣
		$q_0 \downarrow$				
␣	␣	0	␣	␣	␣	␣
			$q_1 \downarrow$			
␣	␣	␣	␣	␣	␣	␣
				$q_{rej} \downarrow$		
␣	␣	␣	␣	␣	␣	␣

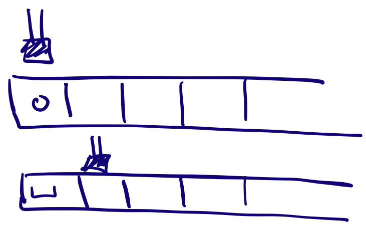
configuration

Formal

$\delta: Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$

in state q_0,

transition to q_1



M rejects 000

The language recognized by this machine is ...

$L(M) = \{w \in \{0\}^* \mid w \text{ has even length}\}$

because TM moves read/write head to the right as we scan 0s (and overwrite them with ␣), keeping track of parity of number of 0s read. When we get to first ␣ symbol, if number of 0s was even, accept; if ~~odd~~, reject.

Describing Turing machines (Sipser p. 185) To define a Turing machine, we could give a

0s and 1s machine code

- **Formal definition**: the 7-tuple of parameters including set of states, input alphabet, tape alphabet, transition function, start state, accept state, and reject state; or,

assembly language

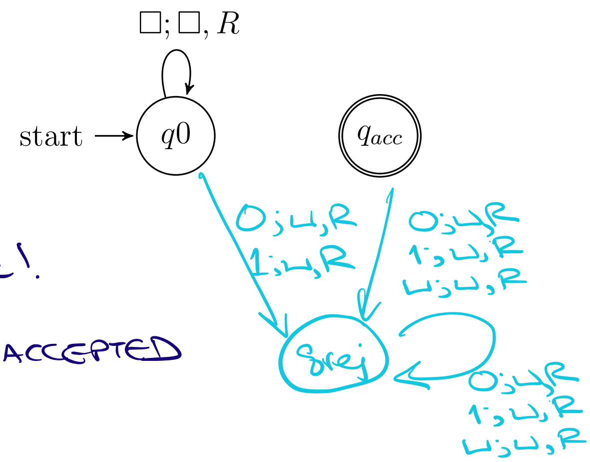
- **Implementation-level definition**: English prose that describes the Turing machine head movements relative to contents of tape, and conditions for accepting / rejecting based on those contents.

Python Java

- **High-level description**: description of algorithm (precise sequence of instructions), without implementation details of machine. As part of this description, can "call" and run another TM as a subroutine.

Fix $\Sigma = \{0, 1\}$, $\Gamma = \{0, 1, \llcorner\!\lrcorner\}$ for the Turing machines with the following state diagrams:

CONVENTIONS



Example of string accepted: None!

Example of string rejected: 0, 1

Example of string NEITHER ACCEPTED NOR REJECTED ε

Implementation-level description

Scan to the right until we see the first 0 or 1, at which point we reject.

High-level description

On input x:

1. If $x = \varepsilon$, go to step 1.
2. otherwise, reject.

Example of string accepted: None

Example of string rejected: 0, 1, ε, 00010

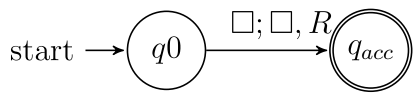
Implementation-level description

Reject (immediately).

High-level description

On input x:

1. Reject.



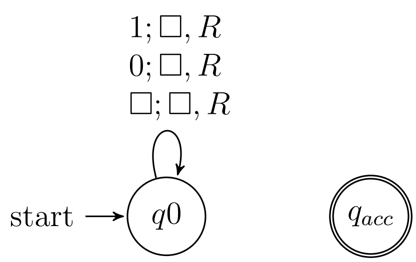
Example of string accepted: ε
Example of string rejected: 0, 1, 1001

Implementation-level description

If first tape symbol is blank, accept. Otherwise, reject.

High-level description

On input x:
1. If $x = \varepsilon$, accept.
2. Otherwise, reject.



Example of string accepted: None.
Example of string rejected: None.

Implementation-level description

Scan tape left to right, erasing each cell in turn.

write blank symbol

High-level description

On input x:
1. Go to step 1

