

## Wednesday: Computational Problems

The Church-Turing thesis posits that each algorithm can be implemented by some Turing machine.

Describing algorithms (Sipser p. 185) To define a Turing machine, we could give a

- **Formal definition:** the 7-tuple of parameters including set of states, input alphabet, tape alphabet, transition function, start state, accept state, and reject state. This is the low-level programming view that models the logic computation flow in a processor.
- **Implementation-level definition:** English prose that describes the Turing machine head movements relative to contents of tape, and conditions for accepting / rejecting based on those contents. This level describes memory management and implementing data access with data structures.
  - Mention the tape or its contents (e.g. “Scan the tape from left to right until a blank is seen.”)
  - Mention the tape head (e.g. “Return the tape head to the left end of the tape.”)
- **High-level description** of algorithm executed by Turing machine: description of algorithm (precise sequence of instructions), without implementation details of machine. High-level descriptions of Turing machine algorithms are written as indented text within quotation marks. Stages of the algorithm are typically numbered consecutively. The first line specifies the input to the machine, which must be a string.
  - Use other Turing machines as subroutines (e.g. “Run  $M$  on  $w$ ”)
  - Build new machines from existing machines using previously shown results (e.g. “Given NFA  $A$  construct an NFA  $B$  such that  $L(B) = \overline{L(A)}$ ”)
  - Use previously shown conversions and constructions (e.g. “Convert regular expression  $R$  to an NFA  $N$ ”)

### Formatted inputs to Turing machine algorithms

The input to a Turing machine is always a string. The format of the input to a Turing machine can be checked to interpret this string as representing structured data (like a csv file, the formal definition of a DFA, another Turing machine, etc.)

This string may be the encoding of some object or list of objects.

**Notation:**  $\langle O \rangle$  is the string that encodes the object  $O$ .  $\langle O_1, \dots, O_n \rangle$  is the string that encodes the list of objects  $O_1, \dots, O_n$ .

**Assumption:** There are algorithms (Turing machines) that can be called as subroutines to decode the string representations of common objects and interact with these objects as intended (data structures). These algorithms are able to “type-check” and string representations for different data structures are unique.

For example, since there are algorithms to answer each of the following questions, by Church-Turing thesis, there is a Turing machine that accepts exactly those strings for which the answer to the question is “yes”

- Does a string over  $\{0, 1\}$  have even length? Yes.
- Does a string over  $\{0, 1\}$  encode a string of ASCII characters? ☐ Yes
- Does a DFA have a specific number of states?
- Do two NFAs have any state names in common?
- Do two CFGs have the same start variable?

= On input  $x$ .

1. If  $|x|$  is even accept.
2. Otherwise reject.

= On input  $\langle S \rangle$ , where  $S$  is a string of ASCII characters

1. accept

= On input  $\langle D, n \rangle$   $D$  is a DFA,  $n$  is a pos. int.

1. If  $D = (Q, \Sigma, \delta, q_0, F)$  and  $|Q| = n$ , accept
2. Reject

A computational problem is decidable iff language encoding its positive problem instances is decidable.

The computational problem “Does a specific DFA accept a given string?” is encoded by the language

$$\{\text{representations of DFAs } M \text{ and strings } w \text{ such that } w \in L(M)\}$$

$$= \{\langle M, w \rangle \mid M \text{ is a DFA, } w \text{ is a string, } w \in L(M)\}$$

The computational problem “Is the language generated by a CFG empty?” is encoded by the language

$$\{\text{representations of CFGs } G \text{ such that } L(G) = \emptyset\}$$

$$= \{\langle G \rangle \mid G \text{ is a CFG, } L(G) = \emptyset\}$$

The computational problem “Is the given Turing machine a decider?” is encoded by the language

$$\{\text{representations of TMs } M \text{ such that } M \text{ halts on every input}\}$$

$$= \{\langle M \rangle \mid M \text{ is a TM and for each string } w, M \text{ halts on } w\}$$

*Note: writing down the language encoding a computational problem is only the first step in determining if it's recognizable, decidable, or ...*

Deciding a computational problem means building / defining a Turing machine that recognizes the language encoding the computational problem, and that is a decider.

<sup>1</sup>An introduction to ASCII is available on the w3 tutorial [here](#).

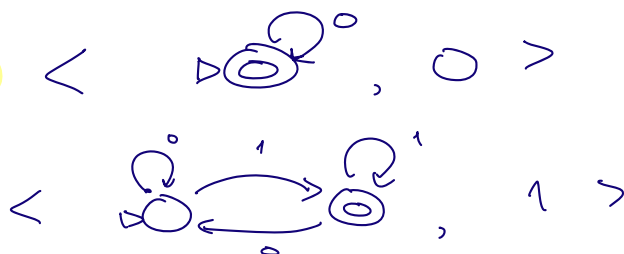
# Friday: Decidable Problems about Regular Languages

Some classes of computational problems will help us understand the differences between the machine models we've been studying. (Sipser Section 4.1)

Goal: Come up w/ a specific example of undecidable language.

Acceptance problem	Does _____ accept _____?
...for DFA	$A_{DFA} \quad \{\langle B, w \rangle \mid B \text{ is a DFA that accepts input string } w\}$
...for NFA	$A_{NFA} \quad \{\langle B, w \rangle \mid B \text{ is a NFA that accepts input string } w\}$
...for regular expressions	$A_{REX} \quad \{\langle R, w \rangle \mid R \text{ is a regular expression that generates input string } w\}$
...for CFG	$A_{CFG} \quad \{\langle G, w \rangle \mid G \text{ is a context-free grammar that generates input string } w\}$
...for PDA	$A_{PDA} \quad \{\langle B, w \rangle \mid B \text{ is a PDA that accepts input string } w\}$
Language emptiness testing	Does _____ <sup>not</sup> accept anything at all?
...for DFA	$E_{DFA} \quad \{\langle A \rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$
...for NFA	$E_{NFA} \quad \{\langle A \rangle \mid A \text{ is a NFA and } L(A) = \emptyset\}$
...for regular expressions	$E_{REX} \quad \{\langle R \rangle \mid R \text{ is a regular expression and } L(R) = \emptyset\}$
...for CFG	$E_{CFG} \quad \{\langle G \rangle \mid G \text{ is a context-free grammar and } L(G) = \emptyset\}$
...for PDA	$E_{PDA} \quad \{\langle A \rangle \mid A \text{ is a PDA and } L(A) = \emptyset\}$
Language equality testing	
...for DFA	$EQ_{DFA} \quad \{\langle A, B \rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$
...for NFA	$EQ_{NFA} \quad \{\langle A, B \rangle \mid A \text{ and } B \text{ are NFAs and } L(A) = L(B)\}$
...for regular expressions	$EQ_{REX} \quad \{\langle R, R' \rangle \mid R \text{ and } R' \text{ are regular expressions and } L(R) = L(R')\}$
...for CFG	$EQ_{CFG} \quad \{\langle G, G' \rangle \mid G \text{ and } G' \text{ are CFGs and } L(G) = L(G')\}$
...for PDA	$EQ_{PDA} \quad \{\langle A, B \rangle \mid A \text{ and } B \text{ are PDAs and } L(A) = L(B)\}$

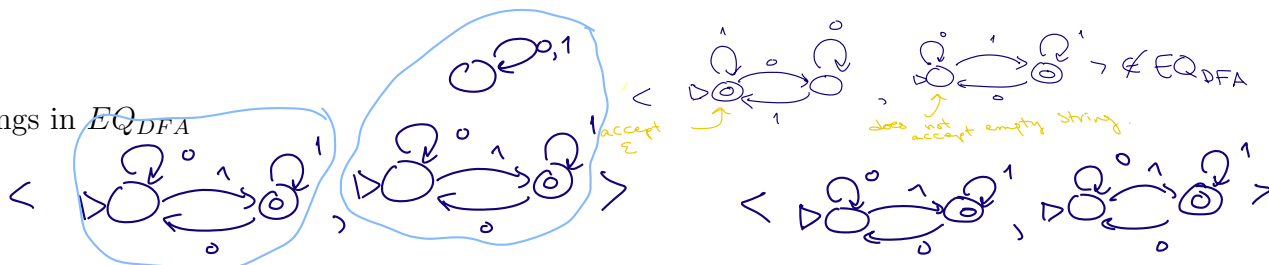
Example strings in  $A_{DFA}$



Example strings in  $E_{DFA}$



Example strings in  $EQ_{DFA}$



Aren't writing strings explicitly. Use  $\langle 0, \dots, 0_n \rangle$

$M_1 =$  "On input  $\langle M, w \rangle$ , where  $M$  is a DFA and  $w$  is a string:

0. Type check encoding to check input is correct type. If not, reject.
1. Simulate  $M$  on input  $w$  (by keeping track of states in  $M$ , transition function of  $M$ , etc.)
2. If the simulation ends in an accept state of  $M$ , accept. If it ends in a non-accept state of  $M$ , reject.

finitely many steps  
comps of DFAs take  
finitely many steps.

finitely many steps.

What is  $L(M_1)$ ?

$$\begin{aligned} L(M_1) &= \{ x \mid M_1 \text{ accept } x \} \\ &= \{ x \mid x = \langle M, w \rangle \text{ where } M \text{ is a DFA} \\ &\quad \text{w string } w \text{ accepted by } M \} \\ &= A_{DFA} \end{aligned}$$

$M_1$  witnesses  $A_{DFA}$  is recognizable

Is  $M_1$  a decider?

Yes, by tracing through each step of high level description for arbitrary input.

$M_1$  witnesses  $A_{DFA}$  is decidable!

Alternate description: Sometimes omit step 0 from listing and do implicit type check.

Synonyms: "Simulate", "run", "call".

Assume: strings encoding different type objects are different

True / False:  $A_{REG} = A_{NFA} = A_{DFA}$

True / False:  $A_{REG} \cap A_{NFA} = \emptyset$ ,  $A_{REG} \cap A_{DFA} = \emptyset$ ,  $A_{DFA} \cap A_{NFA} = \emptyset$

A Turing machine that decides  $A_{NFA}$  is:

On input  $\langle M, w \rangle$  where  $M$  is NFA,  $w$  string

1. Convert  $M$  to equivalent DFA, call it  $M_0$  (using macro state construction).
2. Run  $M_0$  on input  $\langle M_0, w \rangle$
3. If accepts, accept; if rejects, reject "

A Turing machine that decides  $A_{REG}$  is:

[extra practice]

$E_{DFA} = \{ \langle A \rangle \mid A \text{ is a DFA and } L(A) = \emptyset \}$ . True/False: A Turing machine that decides  $E_{DFA}$  is

$M_2 =$  "On input  $\langle M \rangle$  where  $M$  is a DFA,

1. For integer  $i = 1, 2, \dots$
  2. Let  $s_i$  be the  $i$ th string over the alphabet of  $M$  (ordered in string order).
  3. Run  $M$  on input  $s_i$ .
  4. If  $M$  accepts, reject. If  $M$  rejects, increment  $i$  and keep going."
- $s_i \in L(M) \Rightarrow L(M) \neq \emptyset$

Choose the correct option to help fill in the blank so that  $M_2$  recognizes  $E_{DFA}$

- A. accepts
- B. rejects
- C. loop for ever

D. We can't fill in the blank in any way to make this work

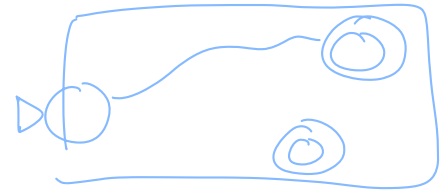
Why?  $L(M_2) = \emptyset$

$$L(M_3) = E_{DFA}$$

and  $M_3$  is a decider!

$M_3 =$  "On input  $\langle M \rangle$  where  $M$  is a DFA,

1. Mark the start state of  $M$ .
2. Repeat until no new states get marked:
3. Loop over the states of  $M$ .
4. Mark any unmarked state that has an incoming edge from a marked state.
5. If no accept state of  $A$  is marked, accept; otherwise, reject".



BFS  
over  
state  
diagram

no path from start  
state to an accept state

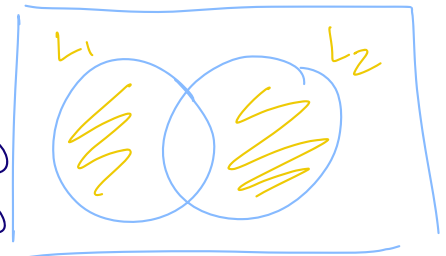
To build a Turing machine that decides  $EQ_{DFA}$ , notice that

$$L_1 = L_2 \text{ iff } ((L_1 \cap \bar{L}_2) \cup (L_2 \cap \bar{L}_1)) = \emptyset$$

There are no elements that are in one set and not the other

$M_{EQDFA} =$  "On input  $\langle M_1, M_2 \rangle$  where  $M_1, M_2$  are both DFA

1. Build DFA  $C_1$  with  $L(C_1) = \bar{L}(M_1)$
2. Build DFA  $C_2$  with  $L(C_2) = \bar{L}(M_2)$
3. Build DFA  $D_1$  with  
 $L(D_1) = L(M_1) \cap L(C_2)$
4. Build DFA  $D_2$  with  
 $L(D_2) = L(C_1) \cap L(M_2)$
5. Build DFA  $D$  with  
 $L(D) = L(D_1) \cup L(D_2)$



6. Run  $M_3$  on  $\langle D \rangle$ . Accept if accepts, reject if rejects.

symmetric  
difference  
of  $L(M_1)$   
and  $L(M_2)$

**Summary:** We can use the decision procedures (Turing machines) of decidable problems as subroutines in other algorithms. For example, we have subroutines for deciding each of  $A_{DFA}$ ,  $E_{DFA}$ ,  $EQ_{DFA}$ . We can also use algorithms for known constructions as subroutines in other algorithms. For example, we have subroutines for: counting the number of states in a state diagram, counting the number of characters in an alphabet, converting DFA to a DFA recognizing the complement of the original language or a DFA recognizing the Kleene star of the original language, constructing a DFA or NFA from two DFA or NFA so that we have a machine recognizing the language of the union (or intersection, concatenation) of the languages of the original machines; converting regular expressions to equivalent DFA; converting DFA to equivalent regular expressions, etc.

## Week 7 at a glance

**Textbook reading: Sections 3.3, 4.1**

*No lecture on Monday in observance of Presidents Day*

*For Wednesday:* Example 3.23 on page 185

*For Friday:* Decidable problems concerning regular languages, Sipser pages 194-196.

*For Monday of Week 8:* An undecidable language, Sipser pages 207-209.

### Make sure you can:

- Use and design automata both formally and informally, including DFA, NFA, PDA, TM.
  - Describe the language recognized by a Turing machine
  - Determine if a Turing machine is a decider
- Explain the Church-Turing thesis and use it to operationalize the notion of algorithm
  - State and use the Church-Turing thesis
  - Use high-level descriptions to define and trace Turing machines
  - Apply dovetailing in high-level definitions of machines
- Translate a decision problem to a set of strings coding the problem.
  - Connect languages and computational problems
  - Describe and use the encoding of objects as inputs to Turing machines
  - Describe common computational problems with respect to DFA, NFA, regular expressions, PDA, and context-free grammars.
  - Give high-level descriptions of Turing machines that decide common computational problems with respect to DFA, NFA, regular expressions, PDA, and context-free grammars.
  - Give examples of decidable problems.
  - Trace high-level descriptions of algorithms for computational problems
- Classify the computational complexity of a set of strings by determining whether it is regular, context-free, decidable, or recognizable.
- Give examples of sets that are regular, context-free, decidable, or recognizable.

### TODO:

Review quizzes based on class material each day.

Project due this Thursday.

Homework assignment 4 due next Thursday.