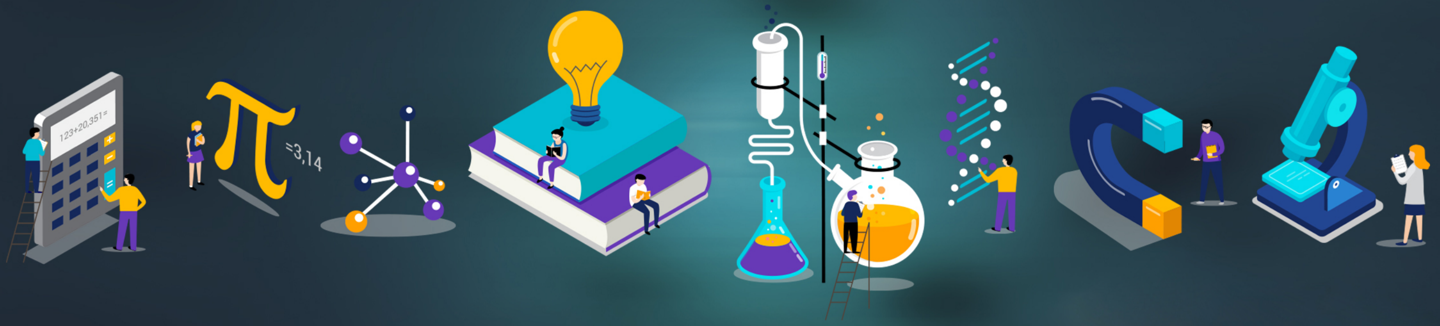


1. CANONICAL TRANSFORMATION

For each problem there may be one particular choice for which all coordinates q_i are cyclic. Then the conjugate momenta p_i are all constant:

$$p_i = \alpha_i$$

Consider a situation in which Hamiltonian is a constant of motion, then

$$H = H(\alpha_1, ...\alpha_n)$$

so the Hamiltonian equations for $\dot{q}_i$ are

$$\dot{q}_i = \frac{\partial H}{\partial \alpha_i} = \omega_i$$

Where ω_i's are functions of ω_i's only $\therefore$ solutions are

$$q_i = \omega_i t + \beta_i$$

where β_i's are constants of integration.

Since the obivious generalized coordinates suggested by the problem will not normally be cyclic, we must have a specific proceedure for transforming from one set of variables to some other set that may be more suitable.

In the Hameltonian formulation the momenta are also independent variable on the same level as the generalized coordinates. The similtaneous transformation of the independent coordinates and momenta q_i, p_i to a new se t Q_i, P_i with (invertible) equation of transformation:

$$Q_i = Q_i(q, p, t)$$

$$P_i = P_i(q, p, t)$$

Which define a point transformation in phase space.

1.1 Jacobian

If u and v are two function of two independent variables x and y then the determinant $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u and v with respect to of x and y which is written as $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$

If u,v and w are three functions of three independent variables x,y and z then the determinant $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$ is called the Jacobian of u,v and w with respect to of x,y and z which is written as $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ or $J\left(\frac{u,v,w}{x,y,z}\right)$

1.1.1 Properties of Jacobian

- If $J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$ and $J' = \frac{\partial(x,y,z)}{\partial(u,v,w)}$ then $JJ' = 1$
- Chain rule for Jacobian if u,v are functions of r,s and r,s functions of x,y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$
- If u_1,u_2,u_3 instead of being given explicitly in terms x_1,x_2,x_3 be connected with them with equations such as
$$f_1(u_1,u_2,u_3,x_1,x_2,x_3) = 0 f_2(u_1,u_2,u_3,x_1,x_2,x_3) = 0 f_3(u_1,u_2,u_3,x_1,x_2,x_3) = 0$$
then $\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = (-1)^3 \frac{\partial(f_1,f_2,f_3)}{\partial(x_1,x_2,x_3)} / \frac{\partial(f_1,f_2,f_3)}{\partial(u_1,u_2,u_3)}$

  $\left((-1)^3\right.$ is for three variable system $)$
- If u_1,u_2,u_3 be functions of x_1,x_2,x_3 then the necessary and sufficient condition for existence of a functional relationship of the form $f_1(u_1,u_2,u_3) = 0$ is $J\left[\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}\right] = 0$

1.2 Canonical Transformation

In Hamiltonian mechanics the transformation should be such that Q and P are canonical coordinates We would like to change variables from the set (a,p) to a new set (Q,P) such that:

1. Determinent of Jacobian matrix of transformation, $\left|\frac{\partial(Q,P)}{\partial(q,p)}\right| = +1$ (This ensures that it is volume and orientation are preserved during transformation)
2. Ensures structure of Hamilton's equation is not changed, so the exist a function $K = K(Q,P,t)$ such that the equation of motion of new set are
$$\dot{Q}_i - \frac{\partial K}{\partial P_i} \quad , \quad \dot{P}_i = \frac{\partial K}{\partial Q_i}$$
K plays role of Hamiltonian in new coordinate set and known as 'kamiltonian'

Note The transformation considered be problem independent that is to say (Q,P) must be canonical coordinates not onlu for some specific mechanical systems, but for all systems of the same number of degrees of freedom.

Hamilton's Priciple of Transformed Coordinates

As Q_i and P_i are canonical variable they must satisfy Hamiltonians principle

$$\delta \int_{t_1}^{t_2} (P_i\dot{Q}_i - K(Q,P,t))dt = 0 \tag{1.1}$$

as of old canonical variables

$$\delta \int_{t_1}^{t_2} (p_i\dot{q}_i - H(Q,P,t))dt = 0 \tag{1.2}$$

$\therefore$ we can say

$$\lambda\left(p_i\dot{q}_i - H\right) = P_i\dot{Q}_i - K + \frac{\partial F}{\partial t}$$

F is a function of phase space coordinates with continuous second derivatives
λ is a constant independent of canonical coordinates and the time and it is related to scale transformation.

For $\lambda = 1$ we have

$$p_i\dot{q}_i - H = P_i\dot{Q}_i - K + \frac{dF}{dt}$$

Which is simply called canonical transformation

Note A transformation of canonical coordinates for which is called extended canonical transformation

Canonical Transformation have following four properties

1. The identity transformation is canonical.
2. If a transformation is canonical, so is its inverse.
3. Two successive canonical transformations (the group "product" operation) define a transformation that is also canonical.
4. The product operation is associative.

1.3 Generating Function

- The term $\frac{df}{dt}$ in canonical transformation contributes to the variation of action integral only at the end points, and will vanish if F is a function of (q, pt) or (Q, P, t)or any mixture of the phase space coordinates since these have zero variation at the end points.
- Through equations of transformation and their inverses F can be expressed partly in terms of the old set of variables and partly of the new.
- F acts as a bridge between the two sets of canonical variables and is called the generating function of transformation.

There are four types of generating function:

1. $F = F_1\left(q_i, Q_i, t\right)$ known as $F_1\left(q_i, Q_i, t\right)$ is F_1 type Generating function.

$$\left(\sum_i p_i\dot{q}_i - H\right) = \left(\sum_i P_i\dot{Q}_i - K\right) + \frac{dF}{dt} \Rightarrow \left(\sum_i p_i\dot{q}_i - H\right) = \left(\sum_i P_i\dot{Q}_i - K\right) + \frac{dF_1}{dt}$$
$$\Rightarrow \left(\sum_i p_i\dot{q}_i - H\right) = \left(\sum_i P_i\dot{Q}_i - K\right) + \sum_i \frac{\partial F_1}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F}{\partial t}$$

Comparing both the coefficient of $\dot{q}_i$ and $\dot{Q}_i$

$$\frac{\partial F_1}{\partial q_i} = p_i \quad \frac{\partial F_1}{\partial Q_i} = -P_i \text{ and } K = H + \frac{\partial F_1}{\partial t}$$

2. $F = F_2\left(q_i, P_i, t\right) - Q_iP_i$ known as $F_2\left(q_i, P_i, t\right)$ is F_2 type Generating function.

$$\left(\sum_i p_i\dot{q}_i - H\right) = \left(\sum_i P_i\dot{Q}_i - K\right) + \frac{dF}{dt}$$
$$\Rightarrow \left(\sum_i p_i\dot{q}_i - H\right) = \left(\sum_i P_i\dot{Q}_i - K\right) - \sum_i \dot{Q}_i P_i - \sum_i Q_i\dot{P}_i - \frac{dF_2}{dt}$$
$$\Rightarrow \left(\sum_i p_i\dot{q}_i - H\right) = \left(\sum_i P_i\dot{Q}_i - K\right) + \sum_i \frac{\partial F_2}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial F_2}{\partial P_i}\dot{P}_i + \frac{\partial F_2}{\partial t} - \sum_i \dot{Q}_i P_i - \sum_i Q_i\dot{P}_i$$

Comparing both the coefficient of $\dot{q}_i$ and $\dot{P}_i$

$$\frac{\partial F_2}{\partial q_i} = p_i,\ \frac{\partial F_2}{\partial P_i} = Q_i \text{ and } K = H + \frac{\partial F_2}{\partial t}$$

3. $F = F_3(Q_i, p_i, t) + q_i p_i$ known as $F_3(Q_i, p_i, t)$ is F_3 type Generating function.

$$\left(\sum_i p_i\dot{q}_i - H\right) = \left(\sum_i P_i\dot{Q}_i - K\right) + \frac{dF}{dt}$$
$$\Rightarrow \left(\sum_i p_i\dot{q}_i - H\right) = \left(\sum_i P_i\dot{Q}_i - K\right) + \frac{dF_3}{dt} + \sum_i \dot{q}_i p_i + \sum_i q_i\dot{p}_i$$
$$\Rightarrow \left(\sum_i p_i\dot{q}_i - H\right) = \left(\sum_i P_i\dot{Q}_i - K\right) + \sum_i \frac{\partial F_3}{\partial Q_i}\dot{Q}_i + \sum_i \frac{\partial F_3}{\partial p_i}\dot{p}_i + \frac{\partial F_3}{\partial t} + \sum_i \dot{q}_i p_i + \sum_i q_i\dot{p}_i$$

Comparing both the coefficient of $\dot{q}_i$ and $\dot{P}_i$

$$\frac{\partial F_3}{\partial Q_i} = -P_i,\ \frac{\partial F_3}{\partial p_i} = -q_i \text{ and } K = H + \frac{\partial F_3}{\partial t}$$

4. $F = F_4(p_i, P_i, t) + q_i p_i - Q_i P_i$ known as $F_3(P_i, p_i, t)$ is F_3 type Generating function.

$$\left(\sum_i p_i\dot{q}_i - H\right) = \left(\sum_i P_i\dot{Q}_i - K\right) + \frac{dF}{dt}$$
$$\Rightarrow \left(\sum_i p_i\dot{q}_i - H\right) = \left(\sum_i P_i\dot{Q}_i - K\right) + \frac{dF_4}{dt} + \sum_i \dot{q}_i p_i + \sum_i q_i\dot{p}_i - \sum_i \dot{Q}_i P_i - \sum_i Q_i\dot{P}_i$$
$$\Rightarrow \left(\sum_i p_i\dot{q}_i - H\right) = \left(\sum_i P_i\dot{Q}_i - K\right) + \sum_i \frac{\partial F_4}{\partial P_i}\dot{P}_i + \sum_i \frac{\partial F_4}{\partial p_i}\dot{p}_i + \frac{\partial F_4}{\partial t} + \sum_i \dot{q}_i p_i + q_i\dot{p}_i - \sum_i \dot{Q}_i P_i - \sum_i Q_i\dot{P}_i$$

Comparing both the coefficient of $\dot{p}_i$ and $\dot{P}_i$

$$\frac{\partial F_4}{\partial p_i} = -q_i,\ \frac{\partial F_4}{\partial P_i} = Q_i \text{ and } K = H + \frac{\partial F_4}{\partial t}$$

Generating Function	Generating Function Derivatives	Trivial Special Case
$F = F_1(q, Q, t)$	$p_i = \frac{\partial F_1}{\partial q_i} \quad P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i, \quad Q_i = p_i, \quad P_i = -q_i$
$F = F_2(q, P, t) - Q_i P_i$	$p_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i, \quad Q_i = q_i, \quad P_i = p_i$
$F = F_3(p, Q, t) + q_i p_i$	$q_i = -\frac{\partial F_3}{\partial p_i} \quad P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i, \quad Q_i = -q_i, \quad P_i = -p_i$
$F = F_4(p, P, t) + q_i p_i - Q_i P_i$	$q_i = -\frac{\partial F_4}{\partial p_i} \quad Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i, \quad Q_i = p_i, \quad P_i = -q_i$

Practise set-1

1. Let q and p be the canonical coordinate and momentum of a dynamical system. Which of the following transformations is canonical? 1. $Q_1 = \frac{1}{\sqrt{2}}q^2$ and $P_1 = \frac{1}{\sqrt{2}}p^2$ 2. $Q_2 = \frac{1}{\sqrt{2}}(p+q)$ and $P_2 = \frac{1}{\sqrt{2}}(p-q)$
[NET/JRF (June-2015)]

   a. Neither 1 nor 2 **b.** Both 1 and 2
   c. Only 1 **d.** Only 2

2. Let x denote the position operator and p the canonically conjugate momentum operator of a particle. The commutator
$$\left[\frac{1}{2m}p^2+\beta x^2, \frac{1}{m}p^2+\gamma x^2\right]$$
where β and γ are constants, is zero if
[NET-JRF (Dec-2017)]

   a. $\gamma = \beta$ **b.** $\gamma = 2\beta$
   c. $\gamma = \sqrt{2}\beta$ **d.** $2\gamma = \beta$

3. A Hamiltonian system is described by the canonical coordinate q and canonical momentum p. A new coordinate Q is defined as $Q(t) = q(t+\tau) + p(t+\tau)$, where t is the time and τ is a constant, that is, the new coordinate is a combination of the old coordinate and momentum at a shifted time. The new canonical momentum $P(t)$ can be expressed as
[NET/JRF (June-2017)]

   a. $p(t+\tau) - q(t+\tau)$ **b.** $p(t+\tau) - q(t-\tau)$
   c. $\frac{1}{2}[p(t-\tau) - q(t+\tau)]$ **d.** $\frac{1}{2}[p(t+\tau) - q(t+\tau)]$

4. Let (x, p) be the generalized coordinate and momentum of a Hamiltonian system. If new variables (X, P) are defined by $X = x^\alpha \sinh(\beta p)$ and $P = x^\gamma \cosh(\beta p)$, where α, β and γ are constants, then the conditions for it to be a canonical transformation, are NET/JRF (DEC-2017)

   a. $\alpha = \frac{1}{2\beta}(\beta+1)$ and $\gamma = \frac{1}{2\beta}(\beta-1)$
   b. $\beta = \frac{1}{2\gamma}(\alpha+1)$ and $\gamma = \frac{1}{2\alpha}(\alpha-1)$
   c. $\alpha = \frac{1}{2\beta}(\beta-1)$ and $\gamma = \frac{1}{2\beta}(\beta+1)$
   d. $\beta = \frac{1}{2\gamma}(\alpha-1)$ and $\gamma = \frac{1}{2\alpha}(\alpha+1)$

5. A canonical transformation relates the old coordinates (q, p) to the new ones (Q, P) by the relations $Q = q^2$ and $P = p/2q$. The corresponding time independent generating function is
[NET/JRF (June-2014)]

   a. P/q^2 **b.** q^2P **c.** q^2/P **d.** qP^2

6. A mechanical system is described by the Hamiltonian $H(q, p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$. As a result of the canonical transformation generated by $F(q, Q) = -\frac{Q}{q}$, the Hamiltonian in the new coordinate Q and momentum P becomes
[NET/JRF (DEC-2014)]

   a. $\frac{1}{2m}Q^2P^2 + \frac{m\omega^2}{2}Q^2$ **b.** $\frac{1}{2m}Q^2P^2 + \frac{m\omega^2}{2}P^2$
   c. $\frac{1}{2m}P^2 + \frac{m\omega^2}{2}Q^2$ **d.** $\frac{1}{2m}Q^2P^4 + \frac{m\omega^2}{2}P^{-2}$

7. A canonical transformation $(q,p) \to (Q,P)$ is made through the generating function $F(q,P) = q^2 P$ on the Hamiltonian

$$H(q,p) = \frac{p^2}{2\alpha q^2} + \frac{\beta}{4} q^4$$

where α and β are constants. The equations of motion for (Q,P) are

[NET/JRF (June-2016)]

a. $\dot{Q} = \frac{P}{\alpha}$ and $\dot{P} = -\beta Q$

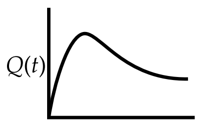
b. $\dot{Q} = \frac{4P}{\alpha}$ and $\dot{P} = \frac{-\beta Q}{2}$

c. $\dot{Q} = \frac{P}{\alpha}$ and $\dot{P} = -\frac{2P^2}{Q} - \beta Q$

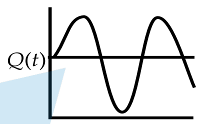
d. $\dot{Q} = \frac{2P}{\alpha}$ and $\dot{P} = -\beta Q$

8. A canonical transformation $(p,q) \to (P,Q)$ is performed on the Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$ via the generating function, $F = \frac{1}{2} m\omega q^2 \cot Q$. If $Q(0) = 0$, which of the following graphs shows schematically the dependence of $Q(t)$ on t ?

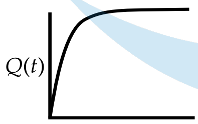
a.



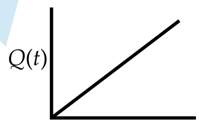
b.



c.



d.



9. The generator of the infinitesimal canonical transformation $q \to q' = (1+\varepsilon)q$ and $p \to p' = (1-\in)p$ is

[NET/JRF (DEC-2019)]

a. $q+p$

b. qp

c. $\frac{1}{2}\left(q^2 - p^2\right)$

d. $\frac{1}{2}\left(q^2 + p^2\right)$

Answer key			
Q.No.	Answer	Q.No.	Answer
1	**d**	2	**b**
3	**d**	4	**c**
5	**b**	6	**d**
7	**b**	8	**d**
9	**b**	10	
11		12	
13		14	
15			

Practise set-2

1. Let p be the momentum conjugate to the generalized coordinate q. If the transformation

$$Q = \sqrt{2}q^m \cos p$$
$$P = \sqrt{2}q^m \sin p$$

is canonical, then $m =$————

[GATE- 2020]

2. Let (p,q) and (P,Q) be two pairs of canonical variables. The transformation

$$Q = q^\alpha \cos(\beta p), P = q^\alpha \sin(\beta p)$$

is canonical for

[GATE- 2011]

a. $\alpha = 2, \beta = \frac{1}{2}$
b. $\alpha = 2, \beta = 2$
c. $\alpha = 1, \beta = 1$
d. $\alpha = \frac{1}{2}, \beta = 2$

3. Given that the linear transformation of a generalized coordinate q and the corresponding momentum $p, Q = q + 4ap, P = q + 2p$ is canonical, the value of the constant a is ————-

[GATE- 2014]

4. For the transformation

$$Q = \sqrt{2qe^{-1+2\alpha}} \cos p, P = \sqrt{2qe^{-\alpha-1}} \sin p$$

(where α is a constant) to be canonical, the value of α is——-

[GATE - 2018]

5. Consider a transformation from one set of generalized coordinate and momentum (q,p) to another set (Q,P) denoted by,

$$Q = pq^s; \quad P = q^r$$

where s and r are constants. The transformation is canonical if

[GATE - 2019]

a. $s = 0$ and $r = 1$
b. $s = 2$ and $r = -1$
c. $s = 0$ and $r = -1$
d. $s = 2$ and $r = 1$

6. If the coordinate q and the momentum p form a canonical pair (q,p), which one of the sets given below also forms a canonical?

[JEST-2012]

a. $(q,-p)$
b. (q^2,p^2)
c. $(p,-q)$
d. $(q^2,-p^2)$

7. (Q_1,Q_2,P_1,P_2) and (q_1,q_2,p_1,p_2) are two sets of canonical coordinates, where Q_i and q_i are the coordinates and P_i and p_i are the corresponding conjugate momenta. If $P_1 = q_2$ and $P_2 = p_1$, then which of the following relations is true?

[JEST-2017]

a. $Q_1 = q_1, Q_2 = p_2$
b. $Q_1 = p_2, Q_2 = q_1$
c. $Q_1 = -p_2, Q_2 = q_1$
d. $Q_1 = q_1, Q_2 = -p_2$

8. If (q,p) is a canonically conjugate pair, which of the following is not a canonically conjugate pair?

[JEST-2018]

a. $\left(q^2, \frac{pq^{-1}}{2}\right)$

b. $\left(p^2, -\frac{qp^{-1}}{2}\right)$

c. $\left(pq^{-1}, -q^2\right)$

d. $\left(f(p) - \frac{q}{f'(p)}\right)$ where $f'(p)$ is the derivative of $f(p)$ with respect to p.

9. Consider the following transformation of the phase space coordinates $(q,p) \to (Q,P)$

$$Q = q^a \cos bp \quad P = q^a \sin bp$$

For what values of a and b will the transformation be canonical?

[JEST-2019]

a. 1,1 **b.** $\frac{1}{2}, \frac{1}{2}$ **c.** $2, \frac{1}{2}$ **d.** $\frac{1}{2}, 2$

10. The Hamiltonian of a classical particle is given by $H(p,q) = \frac{p^2}{2m} + \frac{kq^2}{2}$. Given $F(p,q,t) = \ln(p + im\omega q) - i\alpha\omega t$ is a constant of motion (where $\omega = \sqrt{\frac{k}{m}}$). What is the value of α ?

[JEST-2020]

a. 2π **b.** 0 **c.** 1 **d.** π

Answer key

Q.No.	Answer	Q.No.	Answer
1	**0.5**	2	**d**
3	**0.25**	4	**2**
5	**b**	6	**c**
7	**c**	8	**c**
9	**d**	10	**c**
11		12	
13		14	
15			

Practise set-3

1. A mechanical system is described by the Hamiltonian $H(q,p)=\frac{p^2}{2m}-\frac{1}{2}m\omega^2q^2$. As a result of the canonical transformation generated by $F(q,Q)=-\frac{Q}{q}$, the Hamiltonian in the new coordinate Q and momentum P.

Solution:

$$H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2,\quad F=F_1(q,Q)=-\frac{Q}{q}$$
$$\Rightarrow\frac{\partial F_1}{\partial q}=p\Rightarrow\frac{Q}{q^2}=p$$
$$\Rightarrow\frac{\partial F_1}{\partial Q}=-P\Rightarrow-\frac{1}{q}=-P\Rightarrow q=\frac{1}{P}$$

From equation (a) and (b) $\Rightarrow p=QP^2\quad\because q=\frac{1}{P}$

$$H=\frac{p^2}{2m}-\frac{1}{2}m\omega^2q^2=\frac{Q^2P^4}{2m}-\frac{1}{2}m\omega^2\left(\frac{1}{P^2}\right)=\frac{1}{2m}Q^2P^4-\frac{1}{2}m\omega^2P^{-2}$$

2. The transformation equations between two sets of coordinate are

$$Q=\log\left(1+q^{1/2}\cos p\right)$$
$$P=2\left(1+q^{1/2}\cos p\right)q^{1/2}\sin p$$

(a) Show that Q,P are canonical transform to q,p.
(b) Show that the function that generates this transformation is $F_3=-\left(e^Q-1\right)^2\tan p$

Solution:

$$Q=\log\left(1+q^{\frac{1}{2}}\cos p\right)$$
$$P=2\left(1+q^{\frac{1}{2}}\cos p\right)q^{\frac{1}{2}}\sin p\Rightarrow 2q^{\frac{1}{2}}\sin p+q\sin 2p$$

(a) For canonical transformation: $\frac{\partial Q}{\partial q}\cdot\frac{\partial P}{\partial p}-\frac{\partial Q}{\partial p}\cdot\frac{\partial P}{\partial q}=1$

$$\Rightarrow\frac{\partial Q}{\partial q}=\frac{1}{\left(1+q^{\frac{1}{2}}\cos p\right)}\times\frac{1}{2}q^{-\frac{1}{2}}\cos p,\ \frac{\partial Q}{\partial p}=\frac{q^{\frac{1}{2}}(-\sin p)}{\left(1+q^{\frac{1}{2}}\cos p\right)}$$
$$\frac{\partial P}{\partial p}=2q^{\frac{1}{2}}\cos p+2q\cos 2p,\ \frac{\partial P}{\partial q}=2\times\frac{1}{2}q^{-\frac{1}{2}}\sin p+\sin 2p$$
$$\Rightarrow\frac{\cos p}{2\left(1+q^{\frac{1}{2}}\cos p\right)q^{\frac{1}{2}}}\cdot 2\left(q^{\frac{1}{2}}\cos p+q\cos 2p\right)$$
$$-\frac{\left(-q^{\frac{1}{2}}\sin p\right)}{\left(1+q^{\frac{1}{2}}\cos p\right)}\times\left(q^{-\frac{1}{2}}\sin p+\sin 2p\right)=1$$

(b) $F_3=F_3(p,Q,t)$

$$\frac{\partial F_3}{\partial p}=-q,\quad\frac{\partial F_3}{\partial Q}=-P$$

$$Q=\log\left(1+q^{1/2}\cos p\right)\Rightarrow e^Q=1+q^{1/2}\cos p$$
$$\frac{e^Q-1}{\cos p}=q^{1/2}\Rightarrow q=\left(\frac{e^Q-1}{\cos p}\right)^2$$
$$P=2\left(1+q^{1/2}\cos p\right)q^{1/2}\sin p\Rightarrow P=2e^Qq^{1/2}\sin p\Rightarrow 2e^Q\left(e^Q-1\right)\tan p$$
$$\frac{\partial F_3}{\partial p}=-q=-\frac{\left(e^Q-1\right)^2}{\cos^2 p}\Rightarrow F_3=-\int\left(e^Q-1\right)^2\sec^2 p\,dp$$
$$F_3=-\left(e^Q-1\right)^2\tan p+f_1(Q)........(A)$$
$$\frac{\partial F_3}{\partial Q}=-P=-2\left(e^{2Q}-e^Q\right)\tan p......(B)$$

Equating A and B

$$f_1(Q)=0,\quad f_2(p)=-\tan p$$
$$\text{So } F_3=-\left(e^Q-1\right)^2\tan p$$

3. Prove that the transformation is canonical.
(i) $Q_1=q_1, P_1=p_1-2p_2$,
(ii) $Q_2=p_2P_2=-2q_1-q_2$

Solution:

$$\frac{\partial Q_1}{\partial q_1}\frac{\partial P_1}{\partial p_1}-\frac{\partial Q_1}{\partial p_1}\frac{\partial P_1}{\partial q_1}+\frac{\partial Q_1}{\partial q_2}\frac{\partial P_1}{\partial p_2}-\frac{\partial Q_1}{\partial p_2}\frac{\partial P_1}{\partial q_2}=1\times 1-0=1$$
$$\text{and }\frac{\partial Q_1}{\partial q_1}\frac{\partial P_1}{\partial p_1}-\frac{\partial Q_1}{\partial p_1}\frac{\partial P_1}{\partial q_1}+\frac{\partial Q_2}{\partial q_2}\frac{\partial P_2}{\partial p_2}-\frac{\partial Q}{\partial p_2}\frac{\partial P_2}{\partial q_2}=0-(-1)=1$$

4. The Hamiltonian of harmonic oscillator is given by $H=\frac{1}{2m}\left(p^2+m^2\omega^2q^2\right)$

a. If generating function is defined as $F_1=\frac{m\omega q^2}{2}\cot Q$ then find canonical transformation. From the use of canonical transformation $H=\frac{1}{2m}\left(p^2+m^2\omega^2q^2\right)$

b. Find new Hamiltonian $K(Q,P,t)$

c. Plot Q vs t

d. Plot P vs t

e. Plot phase space between P and Q

Solution:

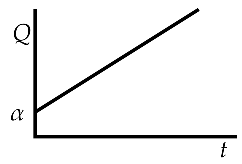
$$\text{(a) } F_1=\frac{m\omega q^2}{2}\cot Q,\ \frac{\partial F_1}{\partial q}=p,\ \frac{\partial F_1}{\partial Q}=-P$$
$$m\omega q\cdot\cot Q=p,-\frac{m\omega q^2}{2}\operatorname{cosec}^2 Q=-P$$
$$q^2=\frac{2P}{m\omega}\frac{1}{\operatorname{cosec}^2 Q}$$
$$q=\sqrt{\frac{2P}{m\omega}}\sin Q,\ p=m\omega\sqrt{\frac{2P}{m\omega}}\cot Q\sin Q$$
$$p=\sqrt{2mP\omega}\cos Q$$
$$\text{(b)}\quad H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2$$

$$K = H + \frac{\partial F_1}{\partial t} \Rightarrow \frac{\partial F_1}{\partial t} = 0, K = H = \frac{1}{2m}(\sqrt{2Pm\omega}\cos Q)^2 + \frac{1}{2}m\omega^2\left(\sqrt{\frac{2P}{m\omega}}\right)^2 \sin^2 Q$$

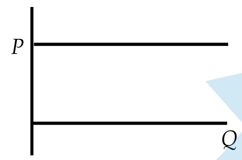
$$K = \omega P$$

(c) $\frac{\partial K}{\partial Q} = -\dot{P} \Rightarrow \dot{P} = 0, \frac{dP}{dt} = 0, P = c$

$\frac{\partial K}{\partial P} = \dot{Q} = \omega, Q = \omega t + \alpha$ where α is constant, can be find with initial condition



(d)



where $P = \frac{E}{\omega}$

(e)

5. If generating function is given by $F(q,P) = q^2 P$ and Hamiltonian of the system is given by $H(q,p) = \frac{p^2}{2\alpha q^2} + \frac{\beta q^4}{4}$ where α and β are constants, then find the equation of motion in term of Q, P.

Solution:

$$F_2 = q^2 P$$

$$\frac{\partial F_2}{\partial q} = 2qP = p, \frac{\partial F_2}{\partial P} = q^2 = Q, q = \sqrt{Q}, p = 2\sqrt{Q}P$$

$$K = \frac{p^2}{2\alpha q^2} + \frac{\beta \cdot q^4}{4} = \frac{4QP^2}{2\alpha Q} + \frac{\beta Q^2}{4} \Rightarrow K = \frac{2P^2}{\alpha} + \frac{\beta Q^2}{4}$$

$$\frac{\partial K}{\partial P} = \dot{Q} \Rightarrow \frac{4P}{\alpha} = \dot{Q}$$

$$\frac{\partial K}{\partial Q} = -\dot{P} \Rightarrow \frac{\beta Q}{2} = -\dot{P}$$

6. If Lagrangian of the system is given by $L = \frac{1}{2}m\dot{x}^2 + m\left(\dot{y}^2 + \dot{z}^2\right) - \frac{1}{2}kx^2 - \frac{1}{2}k(y+z)^2$

a. Write down Hamiltonian of system.

b. If $L_z = xp_y - yp_x$, find $\frac{dL_z}{dt}$

c. If $L_x = yp_z - zp_y$, find $\frac{dL_x}{dt}$

d. If $L_y = zp_x - xp_z$, find $\frac{dL_y}{dt}$

e. Find $\frac{dp_x}{dt}$ and $\frac{dp_y}{dt}$

Solution:

(a) $$L=\frac{1}{2}m\dot{x}^2+m\left(\dot{y}^2+\dot{z}^2\right)-\frac{1}{2}kx^2-\frac{1}{2}k(y+z)^2$$

$$H=\frac{p_x^2}{2m}+\frac{p_y^2}{4m}+\frac{p_z^2}{4m}+\frac{1}{2}kx^2+\frac{1}{2}k(y+z)^2$$

(b) $$L_z=xp_y-yp_x$$

$$\frac{dL_z}{dt}=[L_zH]+\frac{\partial L_z}{\partial t},\frac{\partial L_z}{\partial t}=0$$

$$[L_z,H]=[xp_y-yp_x,H]=x\,[p_y,H]+[x,H]p_y-y\,[p_x,H]-[y,H]p_x$$

$$=x\left[p_y,\frac{1}{2}k(y+z)^2\right]+\left[x,\frac{p_x^2}{2m}\right]p_y-y\left[p_x,\frac{1}{2}kx^2\right]-\left[y,\frac{p_y^2}{4m}\right]p_x$$

$$=x(-k(y+z))+\frac{p_xp_y}{m}+kyx-\frac{p_xp_y}{2m}$$

$$[L_z,H]=\frac{p_yp_x}{2m}-kxz,\frac{dL_z}{dt}=\frac{p_xp_y}{2m}-kxz$$

(c) $$[L_x,H]=[yp_z-zp_y,H]=[yp_z,H]-[zp_y,H]$$

$$=[y,H]p_z+y\,[p_z,H]-[z,H]p_y-z\,[p_y,H]$$

$$=\left[y,\frac{p_y^2}{4m}\right]p_z+y\left[p_z,\frac{1}{2}k(y+z)^2\right]-\left[z,\frac{p_z^2}{4m}\right]p_y-z\left[p_y,\frac{1}{2}k(y+z)^2\right]$$

$$=\frac{p_y}{2m}p_z+y[-k(y+z)]-\frac{p_z}{2m}p_y+kz(y+z)=-ky^2-kzy+kzy+kz^2$$

$$[L_x,H]=k\left[z^2-y^2\right],\frac{dL_x}{dt}=k\left[z^2-y^2\right]$$

(d) $$\frac{dL_y}{dt}=-\frac{p_xp_z}{2m}+kxy$$

(e) $$\frac{\partial H}{\partial x}=-\dot{p}_x,kx=-\dot{p}_x=\frac{-dp_x}{dt}\Rightarrow\frac{dp_x}{dt}=-kx$$

$$\frac{\partial H}{\partial y}=-\dot{p}_y,\frac{2k}{2}(y+z)=-\dot{p}_y,\frac{dp_y}{dt}=-k(y+z)$$