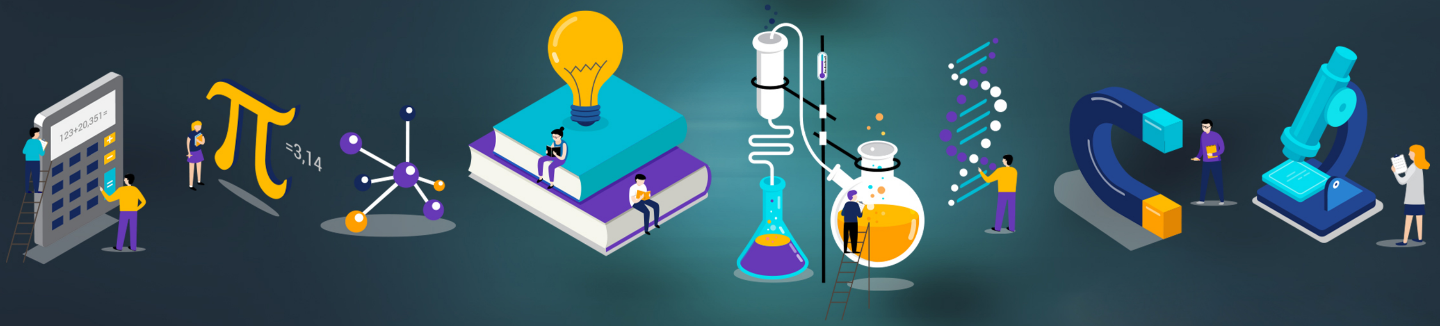


1. Newton’s law and Dynamical system

Newtons laws of motion describes how matter behaves in the universe. These are the three physical laws that, together, laid the foundation for classical mechanics. They describe the relationship between a body and the forces acting upon it, and it’s motion in response to those forces. In describing motion it is essential to specify a reference system . These are the physico-mathematical structures that have the purpose of allowing the objective description, also from a quantitative viewpoint, of physical phenomena that are observed in nature.

Inertial frames of reference

Inertial frames are related by a constant velocity along any axis. An object is at rest in an inertial frame when it’s position in space does not change with time.

Non-Inertial frames of reference

These are the accelerated frame of reference.

It is always possible to find a co-ordinate system with respect to which isolated bodies move uniformly. This is the essence of Newton’s first law of motion.

1.1 First Law

Newton’s first law of motion is the assertion that inertial systems exist. Newton’s first law is part definition and part experimental fact. Isolated bodies move uniformly in inertial systems by virtue of the definition of an inertial system. In contrast, that inertial systems exist is a statement about the physical world.

> **Definition 1.1.1** Every body continues in it’s state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it.
>
> $$\text{If },\ F_{ext} = 0 \Rightarrow V = Constant.$$

After a bus or train starts, the acceleration is often so small we can barely perceive it. We are often startled because it seems as if the station is moving in the opposite direction while we seem to be at rest. Newton’s First Law states that there is no physical way to distinguish between whether we are moving or the station is moving, because there is nearly zero total force acting on the body. Once we reach a constant velocity, our minds dismiss the idea that the ground is moving backwards because we think it is impossible, but there is no actual way for us to distinguish whether the train is moving or the ground is moving.

1.2 Second Law

Unlike first law of motion , the Newton's second law of motion is more quantitative and it is used in calculations involving the motion of particles.

Definition 1.2.1 The second law states that the net force on an object is equal to the rate of change of it's linear momentum $\vec{p}$ in an inertial reference frame,

$$\begin{aligned}\vec{F} &= \frac{d\vec{P}}{dt} = \frac{d(m\vec{v})}{dt}\\ &= m\frac{d(\vec{v})}{dt} + \vec{v}\frac{d(m)}{dt}\\ &= m\frac{d(\vec{v})}{dt} \qquad \text{(Iff mass is a constant)}\end{aligned}$$

The second law can also be stated in terms of an object's acceleration.

A force F on a body of mass m is $F = F_i$, where F_i is the i th applied force. If a is the net acceleration, and a_i the acceleration due to F_i alone, then we have,
(In most of the cases, mass is a constant quantity then we take the assumption that mass is a constant.)

$$\begin{aligned}F &= \Sigma F_i = \Sigma m\vec{a}_i\\ &= m\Sigma\vec{a}_i = m\vec{a} \quad \text{(Can be used only if mass is constant.)} \qquad (1.1)\\ \vec{a} &= \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2} \qquad \text{(Acceleration.)}\\ F &= m\frac{d^2\vec{x}}{dt^2} \qquad (1.2)\end{aligned}$$

$$\boxed{\begin{aligned}F &= ma\\ F &= m\tfrac{d^2\vec{x}}{dt^2}\end{aligned}}$$

In cartesian coordinate system, the components of force, F, can be written as,

$$\boxed{F_x = m\frac{d^2x}{dt^2} \quad ; \quad F_y = m\frac{d^2y}{dt^2} \quad ; \quad F_z = m\frac{d^2z}{dt^2}}$$

The right-hand-side of Newton's Second Law is the product of mass with acceleration. Acceleration is a mathematical description of how the velocity of a body changes. Knowledge of all the forces acting on the body enables us to predict the acceleration. Equation.1.2 is known as the equation of motion. Once we know this equation we may be able to determine the velocity and position of that body at all future times.

1.3 Third Law

The fact that force is necessarily the result of an interaction between two systems is made explicit by Newton's third law.

Definition 1.3.1 Third law states that forces always appear in pairs . If body b exerts force F_a on body a, then there must be a force F_b acting on body b, due to body a, such that

$$F_b = -F_a.$$

There is no such thing as a lone force without a partner.

Newton's third law provides such a test. If the acceleration of a body is the result of an outside force, then somewhere in the universe there must be an equal and opposite force acting on another body. If $\vec{F}_{12}$ be force on an object 1 due to an object 2, and F_{21} be force on 2 due to 1 then,

$$\vec{F}_{12} = -\vec{F}_{21}$$

1.4 Application of Newton's Law of Motion

1.4.1 Free body diagram

1. Mentally divide the system into smaller systems, each of which can be treated as a point mass.
2. Draw force diagram for each mass by considering the body by a point and draw foce vector on the mass for each mas acting on it.
3. Assign coordinates to the system of interest. The coordinate system must be an inertial frame.
4. Assign positive values to all rotations, displacements, velocities, linear velocities, and rotational velocities, and all type of external forces in the problem.
5. write it's constraint equation. (In many problems the objects are constrained to move.)
6. Using all the force equations and constraint equations find every unknowns of the system.

1.5 Contact Force and Field Force

1.5.1 Contact Force

Contact force is a type of force that requires contact to occur. This type of force is responsible for most of the visible interactions that happen between macroscopic collections of matter.

■ **Example 1.1** Pushing a car up a hill , Kicking a ball, Friction, Tension on a string. ■

1.5.2 Field Force

Field force or force field is a vector field that describes a non-contact force that acts on a particle at various positions in space. These forces can be defined as ways of showing a force felt over an area of space. For example, if we hold a compass near a magnetic field, its needle can move according to the magnetic field dimensions. The movement of the needle stops if we go away from the area that is affected by the magnetic field.

■ **Example 1.2** The gravity of gravitational force. Electric field, Magnetic field. ■

1.6 Conservative and Non conservative Forces

1.6.1 Conservative Force

Conservative forces are those forces for which work is done depends only on the initial and final points, while Non-Conservative forces are those forces for which the work is done or the kinetic energy did depend on the other factors such as velocity or the particular path taken by the body.

■ **Example 1.3** Gravitational forces, Magnetic force, Electrostatic force, Elastic spring force, Electric force, ■

1.6.2 Non-Conservative Force

Whenever the work done by a force in moving an object from an initial point to a final point depends on the path, the force is called a non-conservative force. If there is no scalar function V such that $\mathbf{F} = -\nabla V$ [or, equivalently, if $\nabla \times \mathbf{F} \neq 0$], then F is called a non-conservative force feld.

■ **Example 1.4** Frictional force, Air resistance force, The force of gravity ■

1.6.3 External forces and it's directions

Weight

The weight of body is $w = mg$. The weight of the body is always in vertically downward direction.

Normal force

When body of mass touching any surface then surface will exert normal force N. The direction of normal force is perpendicular to plane of surface.

Tension

Force F is exerted on mass m through string . Then any section of string is pulled by two equal and opposite forces. Any one of these forces is called tension. Tension always gives pulling effect. In the figure, the force F is acted on mass m through string so there is tension T in the string giving pulling effect.

Friction

Friction force is force which is responsible to oppose the motion . There is two type of frictional force

a. Static Friction:

When there is not any relative motion between surface and body m then friction identified as static friction which is force is equal to external force acted on mass m. **The direction of frictional force is tangent or parallel to surface**. If μ is coefficient of friction between surface and mass m then maximum value of frictional force is μN.

b. Kinetic Friction:

If there is relative speed between surface and mass m then frictional force is identifying as $f_k = \mu N$ where μ is coefficient of kinetic friction and N is normal force. The direction of f is directed opposite to the motion.

Exercise 1.1 A block of mass ' m ' is pulled with a force F at an angle θ with horizontal. The block does not move on the surface. Calculate force of friction on the block and value of coefficient of friction. ■

Solution:

The block is stationary, therefore horizontal components of F must be balanced by friction force.

$$f_r = F\cos\theta$$

$$\text{For vertical equilibrium,}\quad N + F\sin\theta = mg$$

$$N = mg - F\sin\theta$$

If μ be coefficient of static friction then,

$$f_r \leq \mu N$$

$$F\cos\theta \leq \mu(mg - F\sin\theta)$$

$$\mu \geq \frac{F\cos\theta}{mg - F\sin\theta}$$

1.7 Connected body problems

1.7.1 Pulley Problem (Atwood's machine)

Case-1 : Both the masses are in vertical motion.

Consider the arrangement of pulley and blocks as shown in Figure 1.1. The pulley is assumed massless and frictionless and the connecting strings are massless and inextensible.

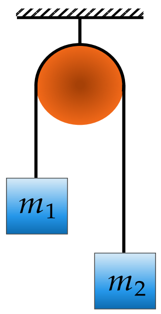


Figure 1.1: Pulley Problem1

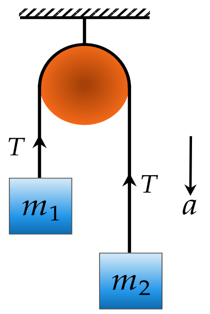


Figure 1.2: Free body diagram(1)

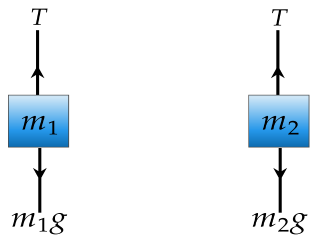


Figure 1.3: Free body diagram(2)

Consider for the masses, $m_2 > m_1$ and they are constant.

Let us write the force equations for m_1 and m_2 . In the case of m_1,

$$T - m_1 g = m_1 a \qquad \text{(Where } T \text{ is the tension on the string.)} \tag{1.3}$$

In the case of m_2,

$$m_2 g - T = m_2 a \tag{1.4}$$

a, the acceleration will be the same for the two masses because the string is continuous. Solving equations.1.3 and 1.4 , we get,

$$(m_2 - m_1)g = (m_1 + m_2)a \tag{1.5}$$

$$a = \frac{m_2 - m_1}{m_1 + m_2} g \tag{1.6}$$

$$T = m_1 \left(g + \frac{m_2 - m_1}{m_1 + m_2} g \right) \tag{1.7}$$

$$= \frac{2m_1 m_2}{m_1 + m_2} g \tag{1.8}$$

Case-2 : One of the masses in vertical motion and the other is in horizontal motion.

Consider the arrangement of pulley and blocks as shown in Figure 1.4 . The pulley is assumed massless and frictionless and the connecting strings are massless and inextensible and the platform is frictionless.

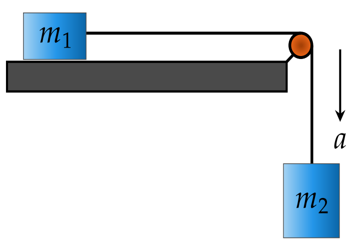


Figure 1.4: Pulley Problem2

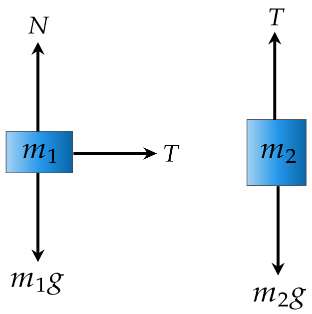


Figure 1.5: Free body diagram

Consider for the masses, $m_2 > m_1$ and they are constant.

Let us write the force equations for m_1 and m_2 . In the case of m_1,

$$T = m_1 a \quad \text{(Since the normal force and the gravitational forces cancells out.)} \tag{1.9}$$

In the case of m_2,

$$m_2g - T = m_2a \tag{1.10}$$

a, the acceleration will be the same for the two masses because the string is continuous. Solving equations.1.9 and 1.10 , we get,

$$m_2g = m_2a + m_1a \tag{1.11}$$

$$a = \frac{m_2}{m_2+m_1}g \tag{1.12}$$

And the tension, T

$$m_2g - 2T = (m_2 - m_1)a \tag{1.13}$$

$$m_2g - 2T = (m_2 - m_1)\frac{m_2}{m_2+m_1}g = m_2\frac{(m_2-m_1)}{m_2+m_1}g \tag{1.14}$$

$$2T = m_2\left[1 - \frac{(m_2-m_1)}{m_2+m_1}\right]g \tag{1.15}$$

$$2T = \frac{2(m_2m_1)}{m_2+m_1}g \tag{1.16}$$

$$T = \frac{(m_1m_2)}{m_2+m_1}g \tag{1.17}$$

Case-2 : One of the masses in vertical motion and the other is in horizontal motion on a wedge.

Consider the arrangement of pulley and blocks as shown in Figure 1.6 . The pulley is assumed massless and frictionless and the connecting strings are massless and inextensible and the platform is frictionless.

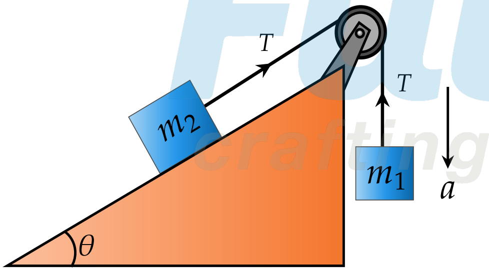


Figure 1.6: Pulley Problem

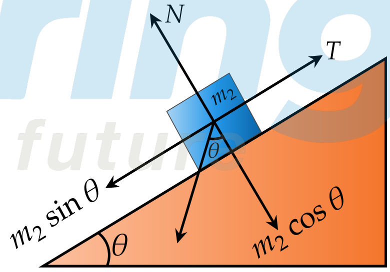


Figure 1.7: Free body diagram

Consider for the masses, $m_2 > m_1$ and they are constant.

Let us write the force equations for m_1 and m_2 . In the case of m_1,

$$m_1g - T = m_1a \tag{1.18}$$

In the case of m_2,

$$m_2g\cos\theta = N \quad \text{(They cancells out.)} \tag{1.19}$$

$$T - m_2g\sin\theta = m_2a \tag{1.20}$$

Solving equations.1.18 and 1.20 , we get,

$$m_1g - m_2g\sin\theta = (m_1+m_2)a \tag{1.21}$$

$$a = \frac{(m_1 - m_2 \sin\theta)g}{(m_1 + m_2)} \tag{1.22}$$

And the tension, T

$$-2T + (m_1 + m_2 \sin\theta)g = (m_1 - m_2)a \tag{1.23}$$

$$-2T + (m_1 + m_2 \sin\theta)g = (m_1 - m_2)\frac{(m_1 - m_2 \sin\theta)g}{(m_1 + m_2)} \tag{1.24}$$

$$-2T = -\frac{2m_1m_2 + 2m_1m_2 \sin\theta}{m_1 + m_2} \tag{1.25}$$

$$T = \frac{m_1m_2(1 + \sin\theta)}{m_1 + m_2} \tag{1.26}$$

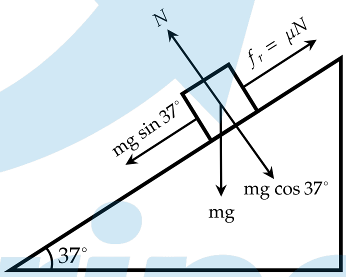
Exercise 1.2 A block of mass 250 gm slides down an incline of inclination 37° with a uniform speed. Find the workdone against the friction as the block slides through 1.0 m ■

Solution:

$$W = F.S$$
$$= FS\cos\theta$$

Since the workdone is against frictional force, The force responsible for the workdone is $mg\sin\theta$

$$W = mg\sin\theta \times 1 \times \cos\theta$$
$$= mg\sin 37 \times 1 \times \cos 0 \quad (\because \text{F and S are parallel})$$
$$= 0.25 \times \times 9.8 \times \sin 37 \times 1$$
$$= 1.5J$$



1.7.2 Block on a string

Consider a mass m whirls with constant speed v at the end of a string of length R. We have to find the force on m in the absence of gravity or friction. The figure.1.8 is shown below.

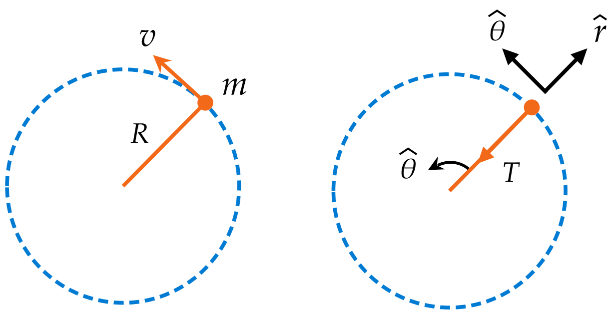


Figure 1.8: Block on string 1

The only force on m is the string force T, which acts toward the center. Here we need to use polar coordinate system.

if r and θ are the polar coordinates of the particle,then the acceleration is,

$$a_{(r,\theta)} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

The radial acceleration,

$$a_r = \ddot{r} - r\dot{\theta}^2 \qquad \text{Where } \dot{\theta} \text{ is the angular velocity.}$$

a_r is directed positive outward.

The tension on the wire is directed towards the origin then,

$$\begin{aligned} T &= -T\hat{r} \\ -T &= ma_r \\ &= m\left(\ddot{r} - r\dot{\theta}^2\right) \end{aligned}$$

Here r is the radius which is a constant. Then,

$$\begin{aligned} \ddot{r} = \ddot{R} &= 0 \text{ and } \dot{\theta} = v/R. \\ \text{Hence, } a_r &= -R(\frac{v}{R})^2 = -v^2/R \\ \text{And , } T &= \frac{mv^2}{R} \end{aligned}$$

Since, the tension is directed towards the origin there is no outward forces on the mass m. When we whirl a pebble at the end of a string, we feel an outward force. However, the force we feel does not act on the pebble, it acts on us. This force is equal in magnitude and opposite in direction to the force with which we pull the pebble, assuming the string's mass to be negligible.

1.8 Dynamical systems

We start with a very broad definition of a dynamical system and introduce the key theoretical concepts of phase space and fixed points (or limit cycles). While the definition may not cover all dynamical systems, it does cover a whole variety systems of interest to us. We give examples of systems with very simple dynamics to begin with and prepare ground for the treatment of more complicated systems which may not be analytically solvable.

1.9 Dynamical systems of order n

A dynamical system of order n is defined as follows:

1. The state of the system at any time t is represented by n-real variables

$$\{x_1, x_2, x_3, \ldots, x_n\} \Rightarrow \vec{r}$$

   as coordinates of a vector $\vec{r}$ in an abstract $n-$ dimensional space. We refer to this space as the state space of simply phase space [1] in keeping with its usage in Hamiltonian dynamics. Thus the state of the system at any given time is a point in this phase space.

2. The time evolution of the system, motion is represented by a set of first order equations- the so called "equations of motion":

$$\frac{dx_1}{dt} = v_1\left(x_1, x_2, \ldots, x_n, t\right) \tag{1.27}$$

$$\frac{dx_2}{dt} = v_2\left(x_1, x_2, \ldots, x_n, t\right) \tag{1.28}$$

$$\frac{dx_n}{dt} = v_n(x_1, x_2, \dots, x_n, t) \tag{1.29}$$

or simply

$$\frac{d\vec{r}}{dt} = \vec{v}(\vec{r}, t)$$

where

$$\vec{v} \Rightarrow \{v_1, v_2, \dots, v_n\}$$

is called the velocity function. While we use the notation x and v in analogy with mechanics, they do not always have the usual meaning of position and velocity.

The statements 1 and 2 together define **a dynamical system of order n**.
If the velocity function does not depend on time explicitly, then the system is timeindependent or autonomous. The set of all possible motions in called **phase flow**.
We also require that the solution to be unique which requires $\vec{v}$ to obey certain conditions. Without going to mathematical details, it suffices to say that the solution of the differential equations in (1.5) are unique if the velocity vector $\vec{v}$ is a continuous function of its arguments and at least once differentiable. With the time evolution, the initial state of the system (denoted by a point in the phase space) evolves and follows a continuous trajectory which we shall call a **phase curve** which may be closed or open. Distinct phase curves are obtained when the initial state of the system is specified by a point which is not one of the points on the other trajectory. This leads to an important fact that two distinct trajectories can not intersect in a finite time period. The no intersection of phase space trajectories has to do with the fact that the evolution is deterministic. This is an important concept to which we shall return later.

1.10 First order systems

This is the simplest case of a dynamical system. The equation of motion [2] is given by

$$\frac{dx}{dt} = v(x, t)$$

where v is the velocity function. For any given $v(x,t), x(t)$ is completely determined given $x(t)$ at some $t = t_0$. If, in particular, the system is autonomous, or v is not explicitly dependent on time then the solution can be written as,

$$t - t_0 = \int_{x(t_0)}^{x(t)} \frac{dx'}{v(x')}$$

Thus the solution $x(t)$ depends only on the difference $(t - t_0)$. Thus the time evolution of the system depends entirely on the time elapsed no matter where the origin of time is fixed.

■ **Example 1.5 Radio-activity** A classic example of a dynamical system of first order is the Radio-active decay of a nucleus modelled by the equation

$$\frac{dN}{dt} = -\sigma N$$

where N is number of nuclei present at some time t. The solution of course is well known,

$$N(t) = N_0 \exp\{-\sigma(t - t_0)\}$$

where N_0 is the number of unstable nuclei present at t_0. ■

■ **Example 1.6 Spread of epidemics** Unlike radio-decay, here the growth is usually exponential at least in the initial period. If σ is an explicit function of time, as in the case of diseases one may obtain a power-law growth instead of exponential growth. A case which has been studied in detail is the threat of AIDS which has devastated many parts of Africa and is threatening many other countries like India.

In an effort to make quantitative assessment of the threat, efforts have been made to look at the reliable data compiled by Centres for Disease Control in the USA as a function of time. If I is the number of infected persons in a population of size N, then the rate of change of I may be given by

$$\frac{dI}{dt} \approx \alpha I$$

which gives rise to exponential growth in the initial phases which is usually true in an epidemic. However, it has been observed that in the case of AIDS that the growth shows a cubic dependence on time and not exponential. How is this achieved? Suppose the relative growth rate α is not a constant in time but a decreasing function of time, say,

$$\alpha = m/t$$

where m is a constant. Then the equation may be written as

$$\frac{dI}{dt} \approx mI/t$$

which has a power law solution, namely,

$$I = I_1 t^m + I_0$$

It turns out that for AIDS $m = 3$. ■

■ **Example 1.7 Population growth** A similar exponential growth also occurs in the population growth of various species. When 24 wild rabbits from Europe, not indigenous to Australia, were introduced in Australia they had a disaster on hand. With abundant food with no natural enemies they were in millions within a few years. The impact was so deep and widespread that it was called a national tragedy. Since the birth rate is proportional to the size of the population, one gets an equation of motion

$$\frac{dN}{dt} = \sigma N$$

which is similar to radioactive decay equation, but with a positive sign indicating an exponential growth (σ is positive).

The population problem, however, gets more complicated when considerations such as food and predators are introduced. ■

■ **Example 1.8 a non-linear equation** The equations of motion can, except in simple situations, get extremely complicated and the solutions are often not easy to obtain. However many qualitative features of a dynamical system may be obtained without actually solving the equations of motion. We will illustrate this with an example below- Consider the equation

$$\frac{dx}{dt} = v(x) = -x\left(1 - x^2\right)$$

which is similar to the radioactive decay problem with a nonlinear term added. ■

Let us look at the properties of the velocity function:

- The system is autonomous- no explicit time dependence.
- The velocity function has zeros at $x_k = 0, \pm 1 - x_k$ are the roots of $v(x)$. If the system is at x_k at any time, it will continue to remain there for all times. x_k are therefore called **fixed points**. The system is said to be in **equilibrium** when it is at a fixed point.
- The phase space is one dimensional. The phase flows may be indicated by a set of arrows, for example, pointing left(right) if the sign of $v(x)$ is positive(negative) and whose length is proportional to the magnitude of $v(x)$ as shown in figure below. For reference we have also shown $v(x)$ as a function of x. The x-axis is the one dimensional phase space of the system.

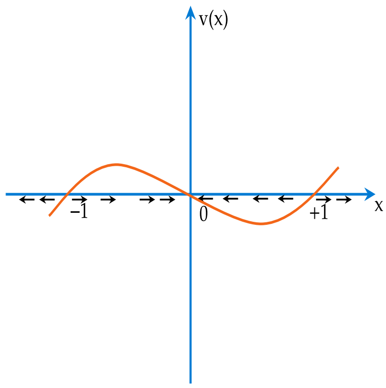


Figure 1.9

Properties of Fixed Points

- The set of zeros of the function $v(x)$ are called the fixed points of the system. The fixed points divide the phase space into several regions.
- x_k is **a stable fixed point or sink** if the flow is directed towards the fixed point, otherwise x_k is an **unstable fixed point or repellor.** In the above example obviously $x_k = 0$ is a stable fixed point where as $x_k = \pm 1$ are unstable. It is easy to see that the system evolves towards stable fixed point and away from an unstable fixed point. That is, if x_0 is a fixed point then define

$$\lambda = \left.\frac{dv}{dx}\right|_{x_0}$$

  If $\lambda < 0$ the fixed point is stable otherwise it is unstable. λ is called the characteristic value or some times called Lyapunov exponent.
- A fixed point may be both stable and unstable, ie., the neighbouring states approach the fixed point on one side but leave from the other side. We call this a **saddle point**. This property leads to the so called structural instability, that is even a small perturbation some time can change the nature of fixed point. For example analyse the system with $v(x) = x^2$.
- The system can not cross a fixed point, by definition. The motion is therefore bounded by the fixed point or fixed points. A system which starts out in the open interval between fixed points remains there for arbitrarily long periods of time. These intervals are therefore called invariant sets. In the example- 4 above the invariant sets are $(-\infty,-1);(-1,0);(0,1);(1,\infty)$

Often the motion may be terminating. The terminating motion happens when at some time t the solution of the differential equation is undefined.

1.11 Second order systems

If the nonlinear differential equation under investigation is of second order and does not contain the independent variable time t explicitly, then it is possible to extract some information regarding the properties of the solution by a geometrical method without actually going into tedious solution of the equation. The dynamical systems whose equations of motion do not involve time t explicitly are called 'autonomous systems'. The equations of motion of second order autonomous systems have the general form

$$\ddot{x} + g(\dot{x}) + f(x) = 0$$

If the velocity $(\dot{x})$ of system is treated as other independent variable and denoted by $y(=\dot{x})$. then equation (1) may be reduced to two first order differential equations as

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -[g(y) + f(x)]$$

Equations represent special case of more general system

$$\frac{dx}{dt} = P(x,y)$$
$$\frac{dy}{dt} = Q(x,y)$$

where $P(x,y)$ and $Q(x,y)$ are the functions of variables x and y.
Thus Phase space is the superposition of position and momentum space and in phase space the position and velocity coordinated are treated on equal footings.

$$\frac{dy}{dx} = -\frac{g(y)+f(x)}{y}, y \neq 0$$

This differential equation defines a definite curve in the phase space.This curve is called phase tajectory or simply trajectory of the system in the phase space.The phase trajectory is defined by general equations

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$$

1.11.1 Illustration of phase trajectories

1-Linear harmonic oscillator
The differential equation of linear harmonic oscillator is

$$m\frac{d^2x}{dt^2} + kx = 0, \frac{dx}{dt} = y$$

These equations may be expressed as

$$\frac{dy}{dt} + \frac{k}{m} = 0, \frac{dx}{dt} = y$$

Then

$$\frac{dy}{dx} = -\frac{kx}{my}$$

or

$$mydy + kxdx = 0$$

Integrating

$$\frac{my^2}{2} + \frac{kx^2}{2} = C$$

The arbitrary constant C is determining by requiring that $x = x_0$ at y=0;This fixes the total energy of the oscillator and C is given by

$$C = 0 + \frac{kx_0^2}{2} = E$$

Now the equation become

$$my^2 + kx^2 = 2E$$

$$\implies \frac{x^2}{2E/k} + \frac{y^2}{2E/m} = 1$$

This equation has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

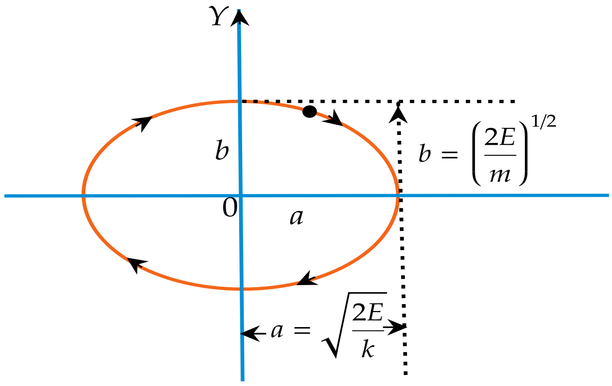


Figure 1.10

This is well known equation of ellipse with center at the origin,semi-major axis $a = \sqrt{\frac{2E}{k}}$ and semi minor axis $b = \sqrt{\frac{2E}{m}}$
The phase trajectory of a differential equation is a definite ellipse for a given value of total energy E.Different ellipse in a phase plane corresponds to different vauesof total energy of the system.The equation $y = \dot{x}$ indicates that the representative point P(x,y) traverse the phase trajectory in clockwise sense.The centre of any ellipse x=0,y=0 is a singular point of the system.This singular point is enclosed by all trajectories.A singular of this type which is enclosed by all trajectories and approched by none is called a **vortex point**.The vortex point is position of stable equilibium of the system.
If $x = x_0, y = y_0$ at t=0 the time integral of the differential equation gives

$$x = \frac{y_0}{\omega} \sin \omega t + x_0 \cos \omega t$$
$$y = y_0 \cos \omega t - \omega x_0 \sin \omega t$$

With $\omega = \sqrt{\frac{k}{m}}$.These are parametric equation of the ellipse.
The period of revolution of representative point

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

2-A Periodic Motion
Consider a particle of mass m being repulsed from the origin along X axis by a force proportional to its distance from the origin.
$F \infty x$ or $F = kx$, where k is a constant.
From Newton's second law $F = m\frac{d^2x}{dt^2}$, therefore equation of motlon becomes
$\therefore \quad m\frac{d^2x}{dt^2} = kx$

$$\text{or } \frac{d^2x}{dt^2} = \frac{k}{m}x \text{ or } \frac{d^2x}{dt^2} = a^2x \tag{1.30}$$

where $\quad a^2 = \frac{k}{m}$
Equation (1.30) is equivalent to two differential equations of first order as

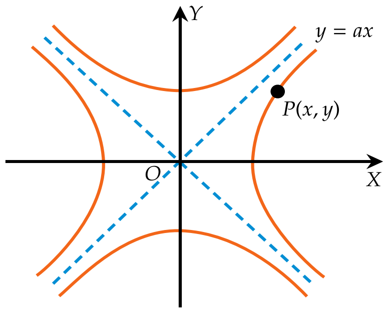
$$y = \frac{dx}{dt}, \quad \frac{dy}{dt} = a^2x$$

Dividing these equations the phase trajectories are given by the equations

$$\frac{dy}{dx} = \frac{a^2x}{y}$$
$$\Rightarrow \quad ydy - a^2xdx = 0$$

Integrating we get

$$y^2 - a^2x^2 = C \tag{1.31}$$



where C is a constant. This equation represents a hyperbola in the phase plane. Different values of C correspond to different hyperbolas, whose two asymptotes are

$$y = \pm ax.$$

The origin O is a singular point of a special type called the saddle point. In this case, there are two special trajectories that pass through the singular point (for which $C = 0$). The motion of the representative point $P(x, y)$ along the trajectories that approach the saddle point is asymptotic with time t. The trajectories in the neighbourhood of a saddle point represent the possible motions that occur in the neighbourhood of a point of unstable equilibrium.

3-Motion of a Damped Oscillator

The differential equation of a damped harmonic oscillator with coefficient of damping ' b ' is" given by

$$\frac{md^2x}{dt^2} = -b\frac{dx}{dt} - kx$$

where k is spring factor. This equation may be written as

$$\begin{array}{l} \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \\ \frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega_0^2 x = 0 \end{array}$$

or

$$\text{where} \quad 2r = \frac{b}{m} \text{ and } \omega_0^2 = \frac{k}{m} \tag{1.32}$$

This equation is equivalent to two differential equations of first order given by and

$$\left.\begin{array}{l} y = \frac{dx}{dt} \\ \frac{dy}{dt} = -\left(2ry + \omega_0^2 x\right) \end{array}\right\}$$

Dividing these equations, we get the equation of phase trajectory as

$$\frac{dy}{dx} = -\frac{2ry + \omega_0{}^2 x}{y}$$

Case (i) Underdamped case

If $r^2 < \omega_0{}^2$, the motion of particle is damped oscillatory motion. Substituting $\omega = \sqrt{\omega_0{}^2 - r^2}$
The general solution of equation (1.32) is

$$x = Ae^{-rt}\cos(\omega t + \phi)$$

$$y = \dot{x} = -Ae^{-rt}[r\cos(\omega t + \phi) + \omega \sin(\omega t + \phi)\}$$

where A and ϕ are arbitrary constants.

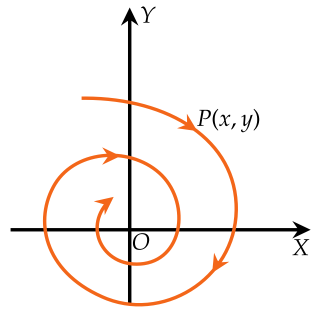


Figure 1.11

These equations represent family of spirals; one spiral is shown in Figure . The origin O is a singular point called the focal point. The representative point $P(x,y)$ spirals to approach the focal point at origin in an asymptotic manner. In this case the focal point at origin is a point of stable cquilibrium.
The representative point spirals to approach the focal point in an infinite number of times about it, so we may conclude that it does not approach O with a definite direction.
Case (ii) Overdamped Motion.
If damping is large, then $r^2 > \omega_0^2$ and motion is not oscillatory and is called overdamped motion.
Let

$$\omega = \sqrt{r^2 - \omega_0^2}$$

In this case the general solution of equation (1.32) is expressed as

$$\text{and} \qquad \left.\begin{aligned} x(t) &= Ae^{-\pi}\cosh(\omega t + \phi) \\ y(t) = \hat{x}(t) &= Ae^{-rt}[\omega \sinh(\omega t + \phi) - r\cosh(\omega t + \phi)] \end{aligned}\right\} \qquad (1.33)$$

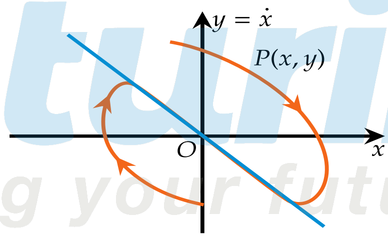


Figure 1.12

where A and ϕ are constants. Equations (1.33) represents the phase trajectories of overdamped oscillator. A typical trajectory is shown in Figure the origin O is a singular point. The phase trajectories are such that the representative point $P(x,y)$ approaches towards the singular point with a definite direction; such a singular point is called a node or nodal point. The nodal point O is the position of stable equilibrium. The four types of singular points are listed in the following table :

Name of singular point	Type of motion	Type of equilibrium	Approach
1. Vortex point	Oscillatory	Stable	none
2. Saddle point	Aperiodic	Unstable	only along asymptotes
3. Focal point	Damped oscillatory	Stable	with no definite direction
4. Nodal point	Aperiodic	Stable	with a definite direction

■ **Example 1.9 Falling body in a gravitational field**
Let x denote the height at some time t. The force equation may be written as two first order equations:

$$\frac{dx}{dt} = v_x(x,y) = y$$
$$\frac{dy}{dt} = v_y(x,y) = -g$$

where g is the acceleration due to gravity. Thus the velocity field is given by,

$$\vec{v} = (y, -g)$$

Since g is never zero there are no fixed points in this system. The equation of the phase curve and its solution is

$$\frac{dy}{dx} = -\frac{g}{y} \Rightarrow x = x_0 - y^2/2g$$

where x_0 is the height at $t = 0$. The phase curves are therefore parabolas as shown in the figure below.

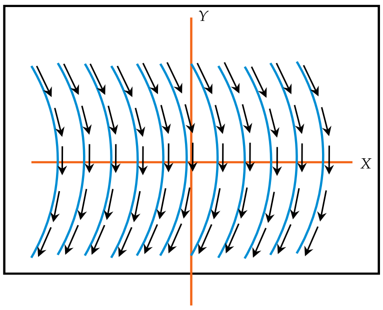


Figure 1.13

Different phase curves correspond to different initial conditions. The phase flows are characterised by the arrows of length and orientation given by

$$|\vec{v}| = \sqrt{y^2 + g^2}, \quad \tan(\theta) = -g/y$$

■

1.12 Linear stability analysis

We shall confine our analysis to 2 nd order autonomous systems, for example, motion of a particle in one dimension or systems with one degree of freedom (dof). The motivation, as we shall see below, for a general stability analysis is to derive the local and where possible global properties of a dynamical system qualitatively, that is, without actually solving the evolution equations. For an autonomous system of order 2 we have

$$\frac{dx}{dt} = v_x(x, y)$$
$$\frac{dy}{dt} = v_y(x, y)$$

where (x, y) define the state of the system in the phase space. We shall assume that $\vec{v}$ is some function of x, y which may be non-linear and therefore have many roots. The fixed point is defined through

$$v_x(x_k, y_k) = 0 = v_y(x_k, y_k)$$

and there may be many solutions. The nature of fixed points and the phase flows in the neighbourhood will be determined by the derivatives evaluated at the fixed point. Consider one such fixed point (x_0, y_0). The stability around this fixed point may be obtained by giving a small displacement around the fixed point:

$$x(t) = x_0 + \delta x(t)$$
$$y(t) = y_0 + \delta y(t)$$

Now Taylor expand the velocity function $\vec{v}$ around the fixed point.

$$v_x(x, y) = v_x(x_0, y_0) + \left.\frac{\partial v_x}{\partial x}\right|_{x_0, y_0} \delta x + \left.\frac{\partial v_x}{\partial y}\right|_{x_0, y_0} \delta y + \dots$$
$$v_y(x, y) = v_y(x_0, y_0) + \left.\frac{\partial v_y}{\partial x}\right|_{x_0, y_0} \delta x + \left.\frac{\partial v_y}{\partial y}\right|_{x_0, y_0} \delta y + \dots$$

By definition the first term is zero since (x_0, y_0) is a fixed point. For an infinitesimal variation in $(\delta x, \delta y)$ we may linearise the equations of motion-

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} d\delta x/dt \\ d\delta y/dt \end{pmatrix} = \begin{pmatrix} V_{xx} & V_{xy} \\ V_{yx} & V_{yy} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

or in a short form, and shifting the origin to (x_0, y_0),

$$\frac{d\delta\vec{r}}{dt} = A\delta\vec{r}$$

where A is a 2×2 matrix given by,

$$A = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} \end{pmatrix}_{x_0, y_0}$$

The local stability analysis is best done in the eigenbasis or in some other convenient basis which we shall call the Standard Basis.

If the system is already linear the above analysis is globally, not just locally, valid since

$$\frac{dx}{dt} = v_x(x,y) = ax + by$$
$$\frac{dy}{dt} = v_y(x,y) = cx + dy$$

and the matrix A is given by,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

However, we need not restrict the analysis only to linear systems. Consider the change of basis

$$M \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} \delta X \\ \delta Y \end{pmatrix}$$

such that

$$B = MAM^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where λ_i are the eigenvalues of the stability matrix given by

$$\lambda_1 = \frac{1}{2}\left(\tau - \sqrt{\tau^2 - 4\delta}\right)$$
$$\lambda_2 = \frac{1}{2}\left(\tau + \sqrt{\tau^2 - 4\delta}\right)$$

where

$$\tau = V_{xx} + V_{yy}$$
$$\delta = V_{xx}V_{yy} - V_{xy}V_{yx}$$

are respectively the trace and the determinant of the stability matrix. The eigenvalues are real or complex depending on whether $\tau^2 \geq 4\delta$ or $\tau^2 < 4\delta$. We shall consider these cases separately.

In the transformed coordinates the linearised equations of motion and their solutions in the neighbourhood of the fixed point are given by

$$\delta\dot{X} = \lambda_1 \delta X \Rightarrow \delta X(t) = C_1 \exp(\lambda_1 t)$$
$$\delta\dot{Y} = \lambda_2 \delta Y \Rightarrow \delta Y(t) = C_2 \exp(\lambda_2 t)$$

The equation for the phase curve is

$$(\delta X/C_1)^{\lambda_2} = (\delta Y/C_2)^{\lambda_1}$$

1.12.1 Classification of the fixed points

We use the properties of the eigenvalues to classify the fixed points. While in the first order systems motion either moves towards or away from the fixed point, the second order systems are richer in the sense there is much more variety in the nature of fixed points.

1. **Stable Node, $\lambda_1, \lambda_2 < 0$**

$$\delta X, \delta Y \to 0$$

as $t \to \infty$
If in particular $\lambda_1 = \lambda_2 < 0$ and the equation is separable, $A = \lambda I$, it is called a stable star. (See Figure). If not a change of basis may be induced such that the matrix

$$B = \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix}$$

or equivalently $c = 0$ and $b \neq 0$. In this case

$$\delta \dot{X} = \lambda \delta X \Rightarrow \delta X(t) = C_1 \exp(\lambda t)$$
$$\delta \dot{Y} = c\delta X + \lambda \delta Y \Rightarrow \delta Y(t) = (C_2 + C_1 ct)\exp(\lambda t)$$

For $\lambda < 0$ this is called an improper node.

2. **Unstable Node, $\lambda_1, \lambda_2 > 0$.**

$$\delta X, \delta Y \to \infty$$

as $t \to \infty$. If in particular $\lambda_1 = \lambda_2 > 0$ and $A = \lambda I$ it is called a unstable star. (See Figure).

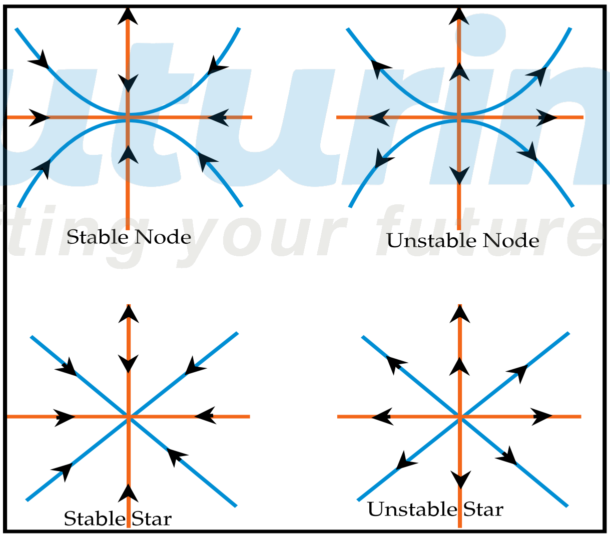


Figure 1.14

3. **Hyperbolic fixed point, $\lambda_1 > 0, \lambda_2 < 0$**

$$|\delta X| \to \infty, |\delta Y| \to 0$$

as $t \to \infty$. This fixed point is also called a saddle point. These are rather special. If in particular the system has only one fixed point then the saddle point divides the phase space into four quadrants, each of which is an invariant subspace. That is a trajectory starting in one of the quadrants remains confined to the same quadrant for all times. These regions are separated by trajectories heading towards or away from the fixed point. The set of points along these trajectories are called invariant manifolds.

Next consider the case when the eigenvalues are complex.

4. **Stable spiral fixed point,** $\lambda_1 = -\alpha + i\beta, \lambda_2 = -\alpha - i\beta$ **where** $\alpha, \beta > 0$ Correspondingly we have,

$$\delta X = C_1 e^{-\alpha t + i\beta t}, \quad \delta Y = C_2 e^{-\alpha t - i\beta t}$$

By a change of basis the solutions may be written as

$$\delta X' = e^{-\alpha t}\left(C_1 \cos\theta t + C_2 \sin\theta t\right)$$
$$\delta Y' = e^{-\alpha t}\left(-C_1 \sin\theta t + C_2 \cos\theta t\right)$$

The fixed point is called the spiral fixed point by looking at the behaviour of the real and imaginary parts as shown in the figure or the solutions given above explicitly in terms of the rotation angle θ. It is stable since for large times the system tends towards the fixed point.

5. **Unstable spiral fixed point,** $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$ **where** $\alpha, \beta > 0$ Correspondingly we have,

$$\delta X = C_1 e^{\alpha t + i\beta t}, \quad \delta Y = C_2 e^{\alpha t - i\beta t}$$

The fixed point is unstable since for large times the system moves away from the fixed point.

6. **Elliptic fixed point,** $\lambda_1 = i\omega = -\lambda_2$ Correspondingly we have,

$$\delta X = C_1 e^{i\omega t}, \quad \delta Y = C_2 e^{-i\omega t}$$

The system is confined to ellipses around the fixed point- each ellipse corresponds to a given initial condition.

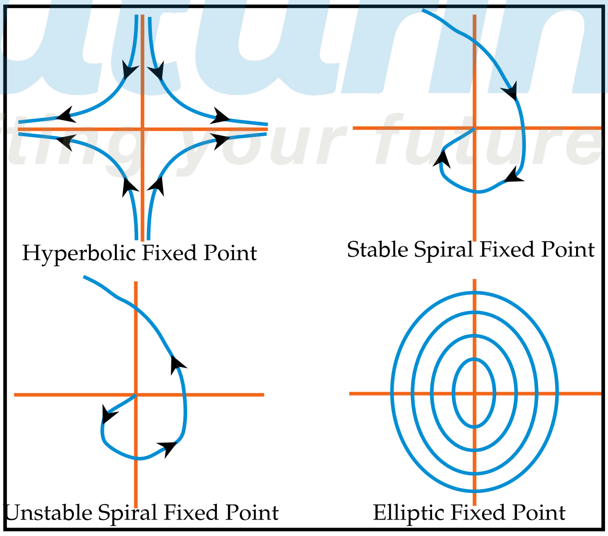


Figure 1.15

1.12.2 Limit cycle

In first-order systems all motions tend to fixed points or to infinity. In second order systems, in addition to fixed points, the system may also exhibit **a limit cycle** which is a closed trajectory encircling the fixed point. The trajectories within and outside tend towards the limiting trajectory as time $t \to \infty$. Such a behaviour occurs typically in systems that are oscillatory. A rigorous analysis of limit cycles, their occurrence and stability is beyond the scope of these lectures.

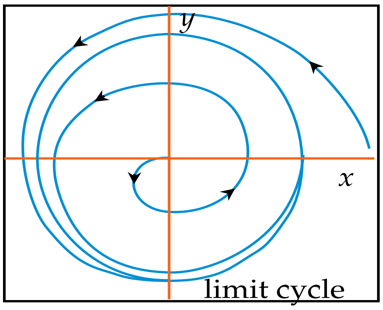


Figure 1.16

However, it may be noted that they occur in systems where the long-term motion of the system is limited to some finite region in the state space. Because of the no intersection theorem, it is entirely reasonable to assume that in this bounded region the trajectory either approaches a fixed point or a closed trajectory (cycle) as $t \to \infty$. Indeed this is the content of the famous **Poincare-Bendixson Theorem.**

1.13 Application to Commonly-Encountered Systems

1.13.1 Simple Pendulum

In this example, because the Hamiltonian is constant and equal to the energy of the system, the easiest way to generate a phase plot is to derive the Hamiltonian in terms of p and q. The Hamiltonian of the simple pendulum illustrated in Figure 1, consisting of a mass m suspended by a massless string of length l is given by:

$$H(\theta, p) = T + V = \frac{p^2}{2ml^2} - mgl\cos\theta$$

In this particular case, it's obvious that $\theta = q$, the generalized coordinate and the Hamiltonian is timeindependent and is equal to the total energy of the pendulum system.
Figure 2 is the phase-space representation of the motion of the pendulum for four different values of the Hamiltonian or energy of the system.

The innermost (green) curve or trajectory represents the motion of the pendulum for the lowest of the four energy levels. The pendulum has sufficient energy to swing through an angle of approximately $\pm\frac{\pi}{2}$ and obviously has the lowest maximum value of momentum. As time advances, the phase point representing the instantaneous state of the pendulum system repeatedly traces out the green trajectory, one period represented by a single orbit. The blue trajectory represents the motion of the pendulum with slightly higher energy than the green trajectory as demonstrated by the higher maximum value for the momentum and θ. The purple trajectory represents a higher energy than the blue or green and shows the swing angle θ approaching $\pm\pi$, the maximum swing angle while still retaining 'back and forth' motion. The orange, outermost trajectory represents the highest of the four energy levels and clearly demonstrates that the pendulum is now rotating continuously in one direction (no longer exhibiting 'back and forth' motion) by virtue of the open trajectory. An obvious implication of the open trajectory is that the system's momentum never falls to zero and continues in the same direction without limit.

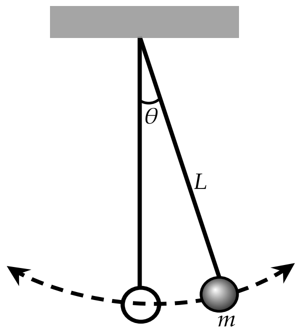


Figure 1 : Simple pendulum

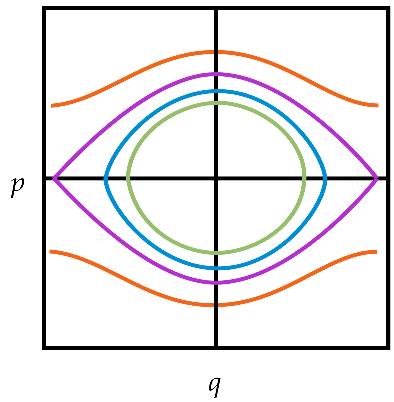
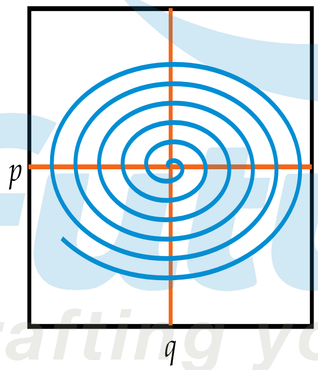


Figure 2 : The simple pendulum represented in phase – space

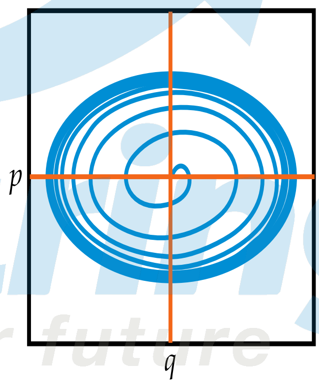
Figure 1.17

The phase-space trajectory that represents the motion of the pendulum at the limit where the motion changes from 'back and forth' to continuous rotation is called the separatrix. The purple trajectory in figure 2 is very close in energy to the separatrix and is extremely close to it in shape.

Note Driven, damped SHO at resonance.



Driven, damped SHO at resonance.
Initial transient motion only



Driven, damped SHO at resonance.
Given enough time to reach limit cycle.

Figure 1.18

1.13.2 Damped Mass on a Spring

In this example, a mass m attached to the free end of a spring with spring constant k is subject to a damping force γ as shown in figure 4 . A mass on an ideal spring exhibits simple harmonic oscillations and is described by the following differential equation:

$$m\frac{d^2x}{dt^2}+kx=0$$

When the oscillations are subject to a damping force, the motion is described by:

$$m\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+kx=0$$

Solving this differential equation provides us with the generalized coordinate $x(t)=q$. The generalized momentum can then be determined from the derivative of $x(t)$.

$$p=m\dot{x}(t)$$

The obvious difference between this example and the simple pendulum is that p and q are both timedependent. Solving the differential equation for the damped motion to find q and then differentiating to obtain p, the phase-plot shown in figure 5 can be generated.

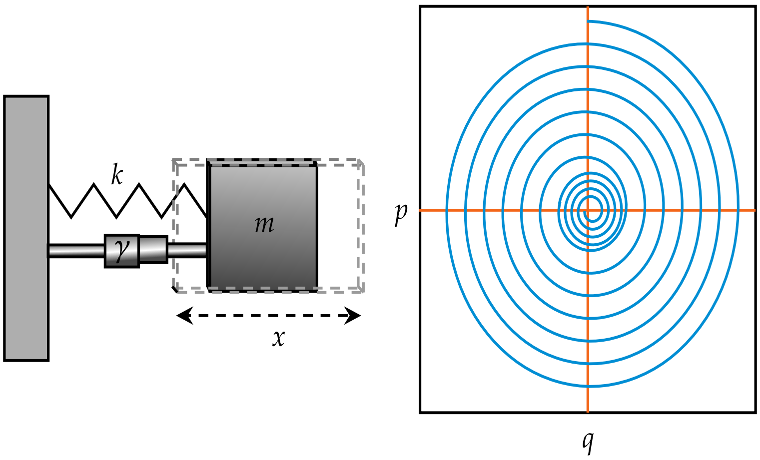


Figure 1.19: figure 4 and 5

The phase plot clearly shows how the momentum and displacement diminish with time resulting in a dramatic spiral trajectory. Without the damping force, the trajectory would be a closed ovoid similar to that for the simple pendulum with small oscillations. Figure 5 also shows how the damping force is reduced with diminishing momentum, a fact that is easily implied from the velocity term of the differential equation $\gamma\frac{dx}{dt}$.

1.14 How to draw a phase curve:

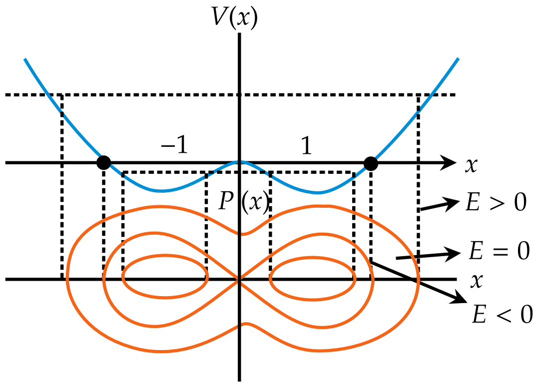
Step 1: Draw a curve of potential $U(x) \quad V_S \quad x$, where $U(x)$ as vertical axis and x as horizontal axis.
Step 2: Just below of potential $U(x)$ vs x curve, draw momentum $P(x)$ as vertical axis and x as horizontal axis.
Step 3: For different values of constant energy in $U(x)Vsx$ draw the trend of $P(x)$ vs x in all classical allowed region.
Step 4: Use sign convention as mention above.

■ **Example 1.10** If potential in one dimension is given by $V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$ then plot the phase curve ie curve between momentum p_x as function of x for all possible range of energy E. ■

Solution: To plot phase curve first one should plot potential $(V \text{ vs } x)$, then on the same axis one should plot momentum with common x axis.

We can check how momentum is changing with position keeping in mind how potential is changing with position.

One will plot the phase curve by assuming that if the potential is increasing, then kinetic energy will be decreasing and if the potential is decreasing, then kinetic energy will be increasing because total energy will always remain constant. One should plot the phase curve for different range of energy. For example in the given potential, there are three range of energy.
For example in the given potential, there are three range of energy.
Case 1: If $-\frac{1}{4} < E < 0$ the particle has motion about stable equilibrium point $x = 1, -1$ the motion is bounded.
Case 2: If $0 < E < \infty$ the particle has motion about unstable equilibrium point $x = 0$ the motion is bounded.
Case 3: At $E = 0$ the particle can be landed exactly at unstable equilibrium point which is nature of transition from case 1 to case 2 .
figure



■ **Example 1.11** If potential in one dimension is given by $V(x) = -kx^2$ then plot the phase curve i.e. curve between momentum p_x as function of x for all possible range of energy E. ■

Solution: To plot phase curve first one should plot potential (V vs x), then on the same axis one should plot momentum with common x axis .

We can check how momentum is changing with position keeping in mind how potential is changing for a given value of energy. For given value of potential the phase curve is hyperbolic as shown in equation $E = \frac{p_x^2}{2m} - kx^2 \Rightarrow \frac{p_x^2}{2mE} - \frac{x^2}{E/k} = 1$

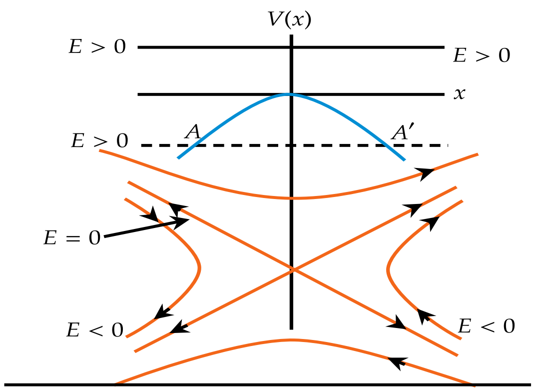
One will plot the phase curve by assuming that if the potential is increasing, then kinetic energy will be decreasing and if potential is decreasing then kinetic energy will be increasing, because total energy will always remain constant. One should plot the phase curve for different range of energy. For example in this potential there are three range of energy

Case 1: $E < 0$, the particle will come from $-\infty$. As it is approaches the potential its kinetic energy as well as momentum decreases finally became zero at turning point A and turn back towards $-\infty$ with increasing kinetic energy and momentum.

Same trend will also follow when particle approaching the potential from $x = \infty$, for turning point A'.

Case 2: $E > 0$, the particle will come from $x = -\infty$. As it approaches the potential. its kinetic energy as well as momentum decreases till $x = 0$. As it crosses $x = 0$ and move towards $x = \infty$, again kinetic energy as well as momentum increases and same trend will be followed, when particle approaches the potential to $x = \infty$.

Case 3: $E = 0$, the particle can reach at $x = 0$, which is unstable equilibrium point and the phase curve will also be separated between $E < 0$ and $E > 0$, identified as separatix. $E = \frac{p_x^2}{2m} - kx^2$ for $E = 0 \Rightarrow p_x \propto \pm x$ which is straight line.



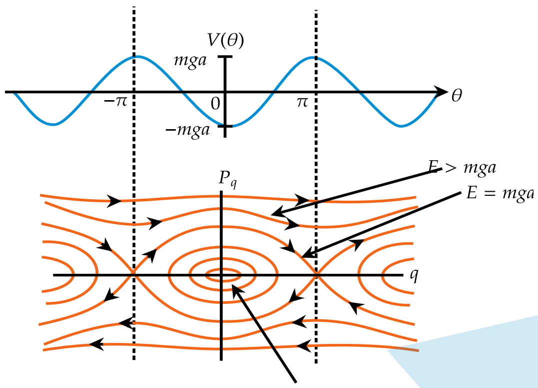
■ **Example 1.12** The energy of simple pendulum is given by $E = \frac{p_\theta^2}{2ma^2} - mga\cos\theta$, where p_θ angular momentum and $-mga\cos\theta$ is potential energy. ■

Solution: One will plot the phase curve by assuming that if the potential energy is increasing, then kinetic energy will be decreasing and if the potential energy is decreasing then kinetic energy will be increasing, because total energy will always remain constant. One should plot the phase curve for different range of energy. For example in this potential there are three range of energy. The stable equilibrium point is $\theta = 0$. $\theta = -\pi$ and $\theta = \pi$ are unstable equilibrium points.

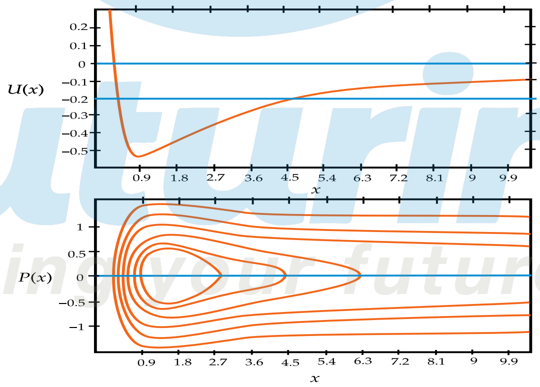
Case 1: For energy $-mga < E < mga$ particle is bounded about stable equilibrium point .so phase curve is periodic.

Case 2: For energy $E > mga$ motion will become unbounded and phase curve will be a periodic. Liberation will take place.

Case 3: For energy $E = mga$ particle will reach at unstable equilibrium point it also separate two type of motion (mention in case 1 and case 2) identified as separatix.

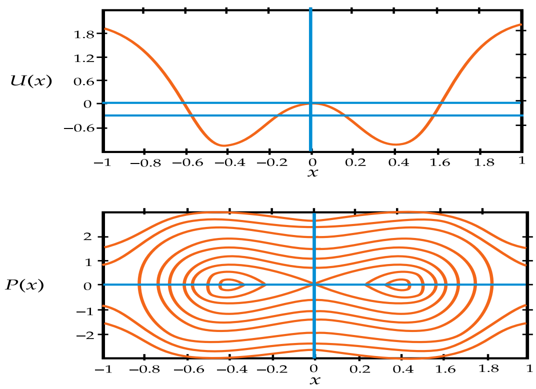


■ **Example 1.13** Phase curves for the Kepler effective potential $U(x) = -x^{-1} + \frac{1}{2}x^{-2}$ ■



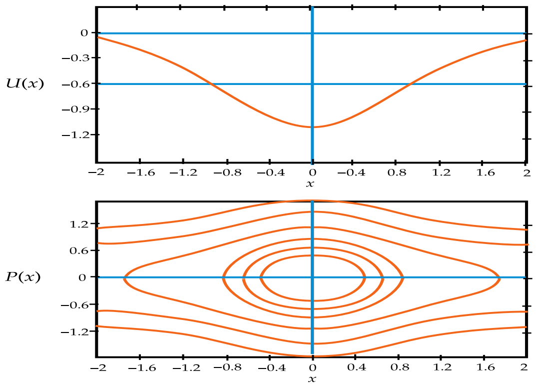
Solution:

■ **Example 1.14** Phase space trajecteries for double well potential ■



Solution:

■ **Example 1.15** Phase curves for the potential $U(x) = -\operatorname{sech}^2(x)$ ■



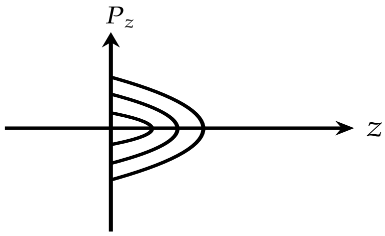
Solution:

Practice set 1

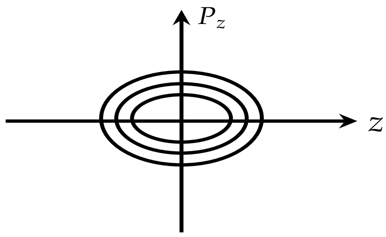
1. The trajectory on the zp_z - plane (phase-space trajectory) of a ball bouncing perfectly elastically off a hard surface at $z = 0$ is given by approximately by (neglect friction):

[NET JUNE 2011]

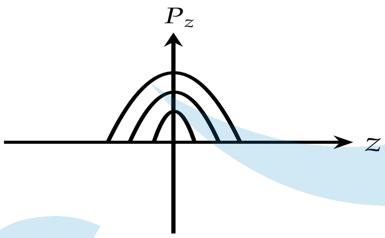
A.



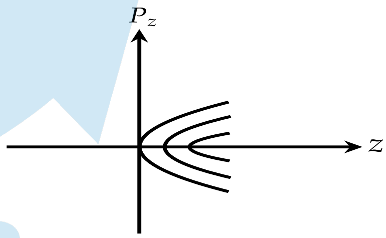
B.



C.



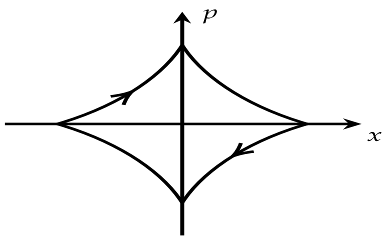
D.



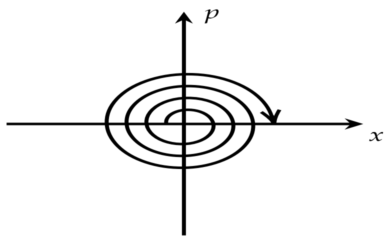
2. The bob of a simple pendulum, which undergoes small oscillations, is immersed in water. Which of the following figures best represents the phase space diagram for the pendulum?

[NET JUNE 2012]

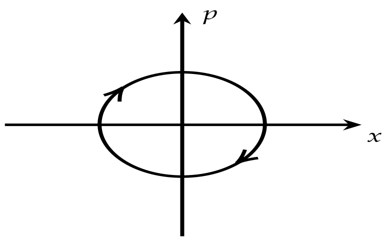
A.



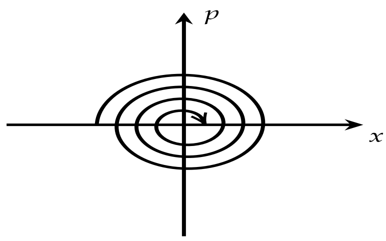
B.



C.



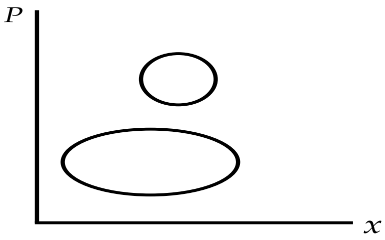
D.



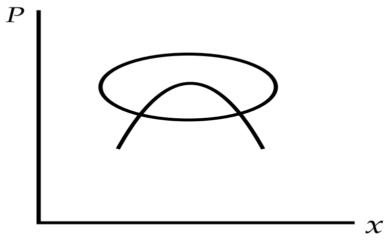
3. Which of the following set of phase-space trajectories is not possible for a particle obeying Hamilton's equations of motion?

[NET DEC 2012]

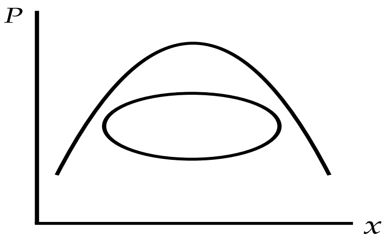
A.



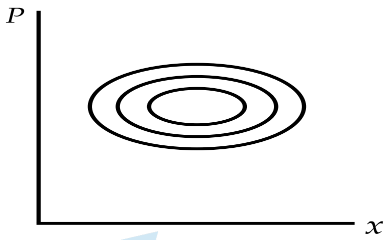
B.



C.



D.



4. The Hamiltonian of a classical particle moving in one dimension is $H = \frac{p^2}{2m} + \alpha q^4$ where α is a positive constant and p and q are its momentum and position respectively. Given that its total energy $E \leq E_0$ the available volume of phase space depends on E_0 as

[NET DEC 2014]

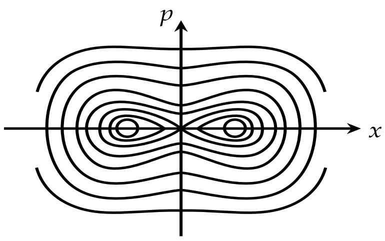
A. $E_0^{3/4}$ **B.** E_0

C. $\sqrt{E_0}$ **D.** is independent of E_0

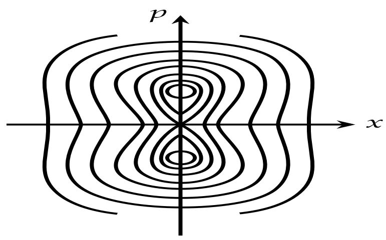
5. Which of the following figures is a schematic representation of the phase space trajectories (i.e., contours of constant energy) of a particle moving in a one-dimensional potential $V(x) = \frac{-1}{2}x^2 + \frac{1}{4}x^4$

[NET JUNE 2015]

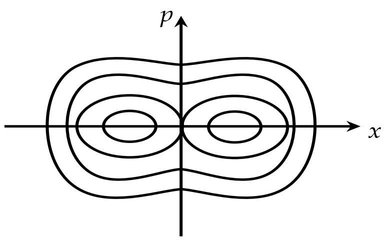
A.



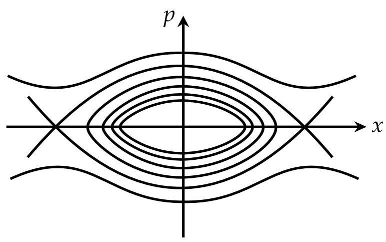
B.



C.



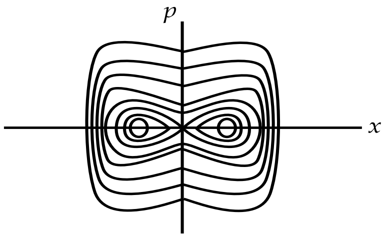
D.



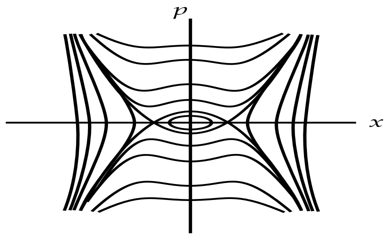
6. A particle moves in one dimension in a potential $V(x) = -k^2x^4 + \omega^2x^2$ where k and ω are constants. Which of the following curves best describes the trajectories of this system in phase space?

[NET DEC 2017]

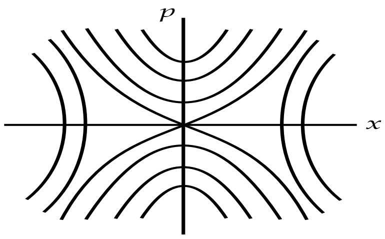
A.



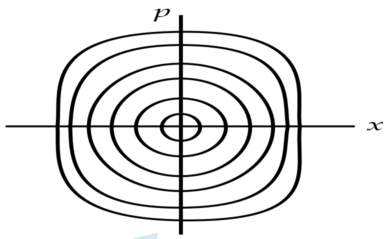
B.



C.



D.



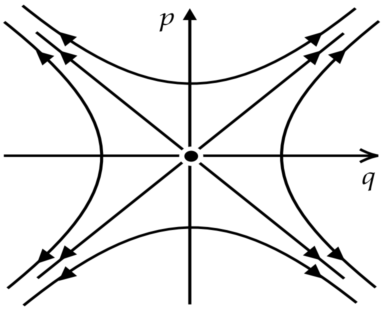
Answer key			
Q.No.	Answer	Q.No.	Answer
1	**a**	2	**d**
3	**b**	4	**a**
5	**a**	6	**c**

Practice set 2

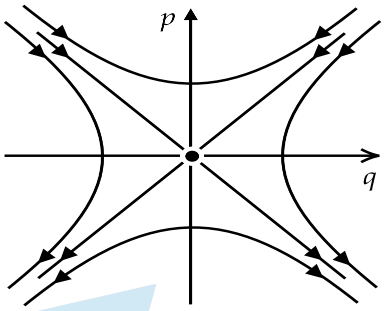
1. The Hamiltonian of particle of mass m is given by $H = \frac{p^2}{2m} - \frac{\alpha q^2}{2}$. Which one of the following figure describes the motion of the particle in phase space?

[GATE 2014]

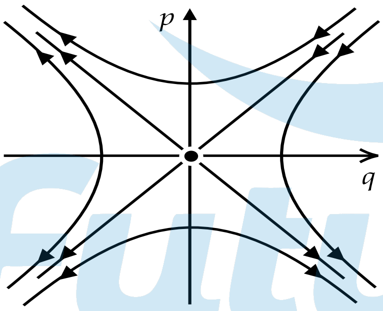
A.



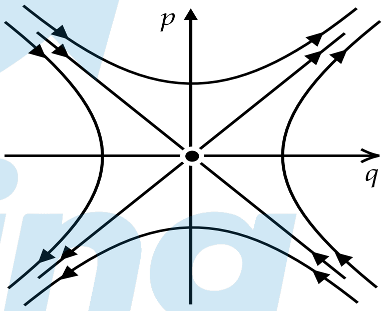
B.



C.



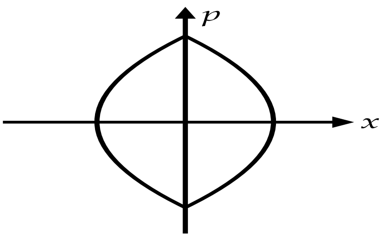
D.



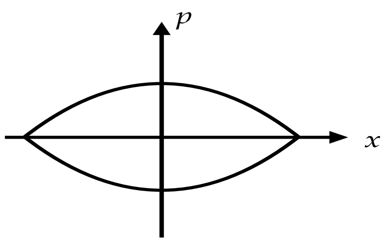
2. A particle moves in one dimension under a potential $V(x) = \alpha|x|$ with some non-zero total energy. Which one of the following best describes the particle trajectory in the phase space?

[GATE 2018]

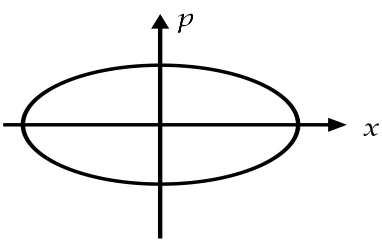
A.



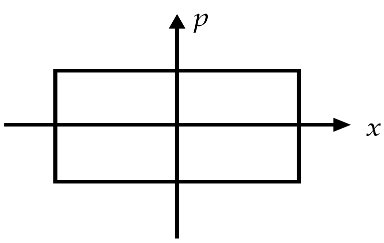
B.



C.



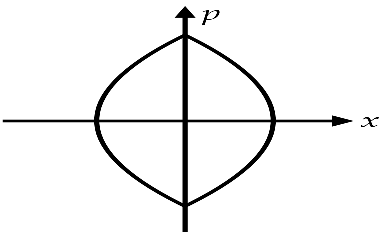
D.



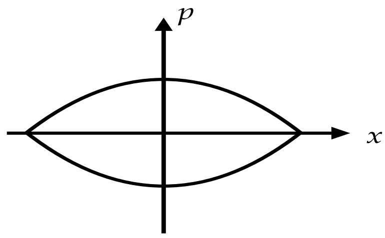
3. A particle moves in one dimension under a potential $V(x) = \alpha|x|$ with some non-zero total energy. Which one of the following best describes the particle trajectory in the phase space?

[GATE 2018]

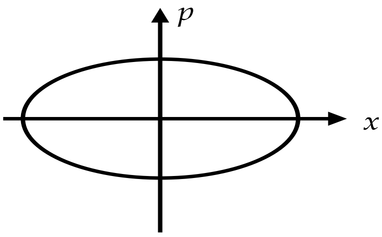
A.



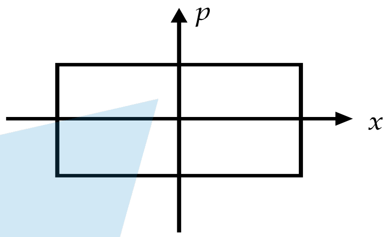
B.



C.



D.

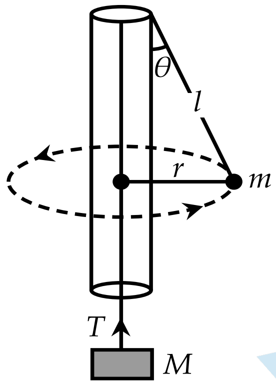


Answer key

Q.No.	Answer	Q.No.	Answer
1	**d**	2	**a**
3	**b**		

Practise set-3

1. A large mass M and a small mass m hang at the two ends of a string that passes through a smooth tube as shown in fig. The mass m moves around a circular path in a horizontal plane. The length of the string from mass m to the top of the tube is l, and θ is the angle the string makes with the vertical. What should be the frequency (ν) of rotation of mass m so that mass M remains stationary?



Solution:

$$\text{Tension in the string } \mathrm{T} = \mathrm{Mg}$$

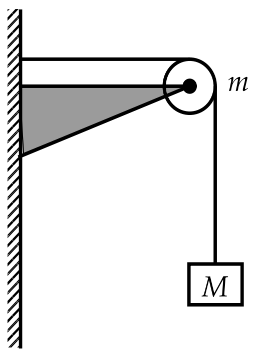
$$\text{Centripetal force on the body } = \mathrm{mr}\,\omega^2 = \mathrm{mr}(2\pi \mathrm{v})^2$$

This is provided by the component of tension acting horizontally i.e. $T\sin\theta = Mg\sin\theta$.

$$\therefore \mathrm{mr}(2\pi\nu)^2 = \sin\theta = \mathrm{Mg}\frac{r}{l}$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{\mathrm{Mg}}{\mathrm{ml}}}$$

2. A string of negligible mass going over a clamped pulley of mass m supports a block of mass M as shown in fig. The force on the pulley by the clamp is given by



A. $\sqrt{2}\mathrm{Mg}$

B. $\sqrt{2}\mathrm{mg}$

C. $\left[\sqrt{(\mathrm{M}+\mathrm{m})^2+\mathrm{m}^2}\right]\mathrm{g}$

D. $\left[\sqrt{(\mathrm{M}+\mathrm{m})^2+\mathrm{M}^2}\right]\mathrm{g}$

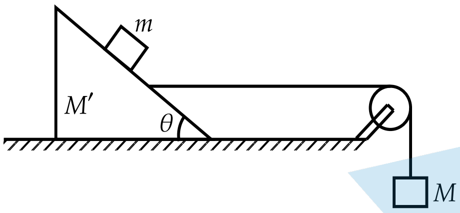
Solution:

Force on the pulley by the clamp = resultant of $T = (M+m)g$ and mg acting along horizontal and vertical respectively

$$\therefore F = \sqrt{[(M+m)g]^2 + (mg)^2}$$
$$= \left[\sqrt{(M+m)^2 + m^2}\right] g$$

So the correct answer is **Option (C)**

3. Find the mass M of the hanging block in figure which will prevent smaller block from slipping over the triangular block. All surfaces are frictionless and the string and the pulley are light.



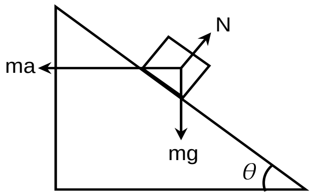
Solution:

Since m does not slip on M' (relative velocity of m w.r.t. M' is zero)

$\therefore M', m$ will move with same acceleration as that of M, Since surfaces are smooth

$$\therefore \text{frictional force is zero}$$
$$\text{Net force } = Mg = (M + M' + m)\, a$$
$$\therefore a = \frac{Mg}{M + M' + m} \tag{1.34}$$

Now let us see m, w.r.t. M'



Downward acceleration of m on slope = 0

$$\therefore N - ma\sin\theta + mg\cos\theta = 0 \tag{1.35}$$

$$(\text{net } \perp \text{ force } = 0)$$

$$\text{and } mg\sin\theta - \text{ma}\cos\theta = 0 \tag{1.36}$$

$$[\because \text{net force along slope } = 0]$$

$$\text{From eq}^{n} \cdot (1.36) g\sin\theta = a\cos\theta \text{ or } a = g\tan\theta \tag{1.37}$$

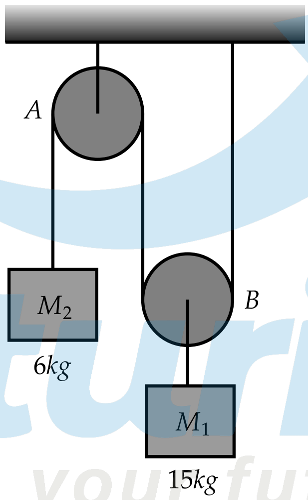
From eq $^{\text{n}}$. (1.37) and (1.34),

$$\text{we have } \tan\theta = \frac{M}{M+M'+m}$$

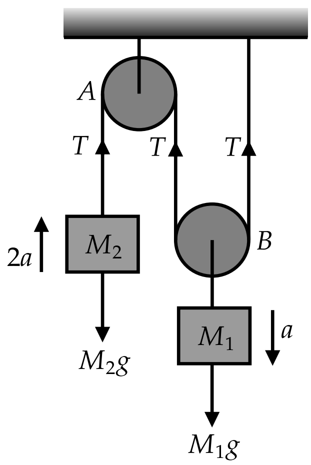
$$\Rightarrow M\cot\theta = M+M'+m$$

$$\Rightarrow \text{M} = \frac{\text{M}'+\text{m}}{\cot\theta - 1}$$

4. A mass of 15*kg* and another of mass 6*kg* are attached to a pulley system as shown in the fig. A is a fixed pulley while *B* is a movable one. Both are considered light and frictionless. Find the acceleration of 6*kg* mass.



Solution: Tension is the same throughout the string. It is clear that M_1 will descend downwards while M_2 rises up. If the acceleration of M_1 is a downwards, M_2 will have an acceleration ' 2a ' upward.



Now,

$$\begin{aligned} \mathrm{M_1\,g - 2\,T} &= \mathrm{M_1 a} \\ \mathrm{T - M_2\,g} &= \mathrm{M_2 \cdot 2a} \end{aligned}$$

or

$$\begin{aligned} \mathrm{M_1\,g - 2M_2\,g} &= \mathrm{a\,(M_1 + 4M_2)} \\ \Rightarrow a &= \frac{M_1 - 2M_2}{M_1 + 4M_2} g = \frac{15-12}{15-24} g = \frac{3}{39} g \\ \therefore \mathrm{a} &= \frac{\mathrm{g}}{13} \\ \therefore \text{ acceleration of 6 kg mass } &= 2\mathrm{a} = \frac{2\,\mathrm{g}}{13} \end{aligned}$$

5. A mass m is revolving in a vertical circle at the end of a string of length 20 cm. By how much does the tension of the string at the lowest point exceed the tension at the topmost point?

Solution:

The tension T_1 at the topmost point is given by,

$$\mathrm{T_1 = \frac{mv_1^2}{20} - mg}$$

Centrifugal force acting outward while weight acting downward

$$\text{The tension } T_2 \text{ at the lowest point, } T_2 = \frac{mv_2^2}{20} + mg$$

Centrifugal force and weight (both) acting downward

$$\begin{aligned} \mathrm{T_2 - T_1} &= \frac{\mathrm{mv_2{}^2 - mv_1^2}}{20} + 2\mathrm{mg}; \\ \mathrm{v_1^2} &= \mathrm{v_2^2 - 2\,g\,h \text{ or}} \\ \mathrm{v_2^2 - v_1^2} &= \mathrm{2\,g(40) = 80\,g} \\ \therefore \quad T_2 - T_1 &= \frac{80\mathrm{mg}}{20} + 2\,\mathrm{m\,g} = 6\mathrm{mg} \end{aligned}$$

6. A block of mass m is placed on a smooth wedge of inclination θ. The whole system is accelerated horizontally so that the block does not slip on the wedge. The force exerted by the wedge on the block (g is acceleration due to gravity) will be

A. $\mathrm{mg}/\cos\theta$ **B.** $\mathrm{mg}\sin\theta$ **C.** $mg\cos\theta$ **D.** mg

Solution:

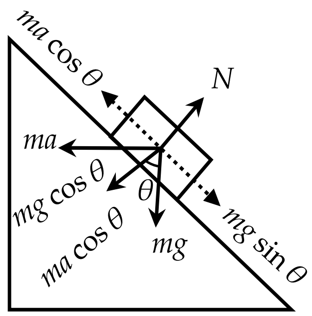
$$N = ma\sin\theta + mg\cos\theta \tag{1.38}$$

$$\text{also } \mathrm{mg}\sin\theta = \mathrm{ma}\cos\theta \tag{1.39}$$

$$\text{from } (1.39)\, a = g\tan\theta$$

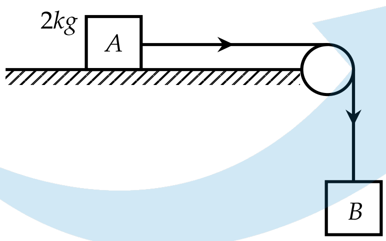
$$\therefore \mathrm{N} = \mathrm{mg}\frac{\sin^2\theta}{\cos\theta} + \mathrm{mg}\cos\theta$$

$$\text{or } N = \frac{mg}{\cos\theta}$$



So the correct answer is **Option (A)**

7. The coefficient of static friction, μ_s, between block A of mass 2 kg and the table as shown in the figure is 0.2. What would be the maximum mass value of block B so that the two blocks do not move? The string and the pulley are assumed to be smooth and massless. ($g = 10\ \mathrm{m/s^2}$)



A. 0.4 kg **B.** 4.0 kg **C.** 2.0 kg **D.** 0.2 kg

Solution:

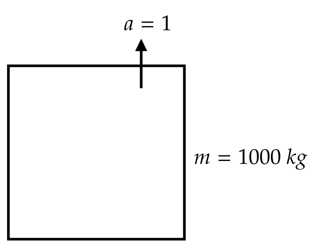
$$\mathrm{m_B g} = \mu_s \mathrm{m_A g} \quad \{\because \mathrm{m_A g} = \mu_s \mathrm{m_A g}\}$$
$$\Rightarrow \mathrm{m_B} = \mu_s \mathrm{m_A}$$
$$\text{or } m_B = 0.2 \times 2 = 0.4\ \text{kg}$$

So the correct answer is **Option (A)**

8. A person of mass 60 kg is inside a lift of mass 940 kg and presses the button on control panel. The lift starts moving upwards with an acceleration $1.0\ \mathrm{m/s^2}$. If $g = 10\ \mathrm{ms^{-2}}$, the tension in the supporting cable is

A. 8600 N **B.** 9680 N **C.** 11000 N **D.** 1200 N

Solution:



$$\text{Total mass } = (60+940)\text{kg} = 1000 \text{ kg}$$

Let T be the tension in the supporting cable, then

$$\begin{aligned} \mathrm{T} - 1000 \text{ g} &= 1000 \times 1 \\ \Rightarrow \mathrm{T} &= 1000 \times 11 = 11000 \text{ N} \end{aligned}$$

So the correct answer is **Option (C)**

9. A given object takes n times as much time to slide down a 45° rough incline as it takes to slide down a perfectly smooth 45° incline. The coefficient of kinetic friction between the object and incline is given by

A. $\left(1-\frac{1}{n^2}\right)$ **B.** $\frac{1}{1-\mathrm{n}^2}$ **C.** $\sqrt{\left(1-\frac{1}{n^2}\right)}$ **D.** $\sqrt{\left(\frac{1}{1-\mathrm{n}^2}\right)}$

Solution:

$$\text{We have } \sqrt{\frac{2s}{g(\sin\theta-\mu\cos\theta)}} = n\sqrt{\frac{2s}{g\sin\theta}}$$

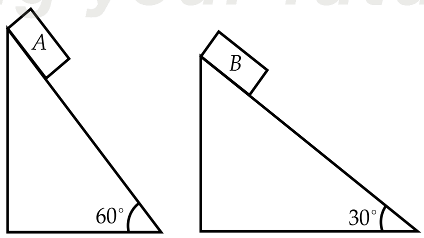
$$\frac{2s}{g(\sin\theta-\mu\cos\theta)} = \frac{2s\times n^2}{g\sin\theta}$$

$$\text{here } \theta = 45^\circ \Rightarrow \frac{1}{1-\mu} = \mathrm{n}^2$$

$$\text{or } \mu = \left(1-1/\mathrm{n}^2\right)$$

So the correct answer is **Option (A)**

10. Two fixed frictionless inclined planes making an angle 30° and 60° with the vertical are shown in the figure. Two blocks A and B are placed on the two planes. What is the relative vertical acceleration of A with respect to B ?



Solution:

$$\begin{aligned} mg\sin\theta &= ma \\ \therefore a &= g\sin\theta \end{aligned}$$

where a is along the inclined plane

$\therefore$vertical component of acceleration is $g\sin^2\theta$

Then the relative vertical acceleration of A with respect to B is

$$\begin{aligned} g\left(\sin^2 60 - \sin^2 30\right] &= \frac{g}{2} \\ &= 4.9 \text{ m/s}^2 \quad \text{(in vertical direction)} \end{aligned}$$

2. Central force motions

It was Newton's fascination with planetary motion that led him to formulate his laws of motion and the law of universal gravitation. His success in explaining Kepler's empirical laws of planetary motion was an overwhelming argument in favor of the new mechanics and marked the beginning of modern mathematical physics. Planetary motion and the more general problem of motion under a central force continue to play an important role in most branches of physics and turn up in such topics as particle scattering, atomic structure, and space navigation.

2.1 Central force

A central force acting on a particle only depends upon magnitude of a distance from a fixed center. If r is the instanteneous position vector of the particle relative to the fixed center. Then the central force is represented by the relation,

$$F = f(r)\hat{r} \tag{2.1}$$

Where, $f(r)$ is a scalar function of distance, r and $\hat{r} = \frac{\vec{r}}{r}$. Then the torque acting on the particle is,

$$\tau = r \times F \tag{2.2}$$

This type of motion is particularly relevant when studying the orbital movement of planets and satellites. The laws which govern this motion were first postulated by Kepler and deduced from observation. In this lecture, we will see that these laws are a consequence of Newton's second law. An understanding of central force motion is necessary for the design of satellites and space vehicles.

2.1.1 General Properties of Central Force Motion

1. The central force $f(r)\hat{r}$ is along r and can exert no torque on the reduced mass μ , therefore angular momentum about centre of force is conserved. This implies that central force motion is a planar motion.

$$\begin{aligned} \tau &= r \times F \\ &= r\hat{r} \times f(r)\hat{r} \\ &= 0 \quad (\text{Since, } \hat{r} \times \hat{r} = 0) \end{aligned}$$

2. Central forces are conservative, therefore total energy of a particle moving under central force is conserved.

3. The magnitude of the angular momentum $|L| \equiv l$, and the total energy E , of central force motion is constant.

4. Areal velocity is constant, that is area swept by line joining the particle to the centre of force per unit time is constant.

$$\frac{\Delta A}{\Delta t} = \text{constant} \tag{2.3}$$

2.2 The two body central force problems

2.2.1 Reduction of two-body problems to one body problem

A two-body system can be effectively reduced to one-body system by introducing the concept of reduced mass. Suppose a system is composed of two masses m_1 and m_2, then for an inertial observer the relative motion of these masses may be expressed by a fictitious particle of reduced mass μ.
Let the instantaneous position of masses m_1 and m_2 be represented by position vectors r_1 and r_2 relative to an arbitrary origin 0

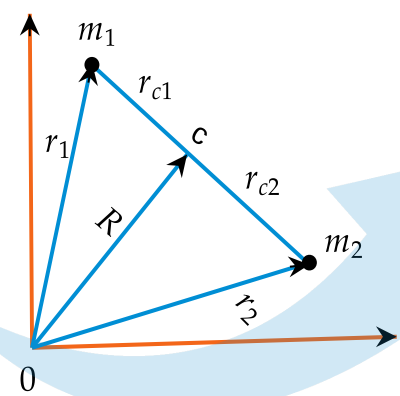


Figure 2.1

Let us assume that no external force acts on the system and the internal force is simply due to mutual interaction of the particles, so the potential energy function is dependent only on internal forces, so the potential energy function may be assumed to be the function of vector between two particles $\mathbf{r}_2 - \mathbf{r}_1$ or their relative velocity $(\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1)$. This system has six degrees of freedom and hence requires six independent generalised coordinates. We choose these to be three components of radius vector $\mathbf{R}$ of centre of mass and three components of difference vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.
The Lagrangian of the system has the form $L = T(\dot{\mathbf{R}}, \dot{\mathbf{r}}) - V(\mathbf{r}, \dot{\mathbf{r}}, \ldots)$ The kinetic energy of the system may be expressed as the sum of kinetic energy of the motion of centre of mass plus the kinetic energy of the motion about the centre of mass, i.e.,

$$T = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \left(\frac{1}{2}m_1 \cdot \mathbf{r}_{C_1}^2 + \frac{1}{2}m_2 \cdot \mathbf{r}_{C_2}^2\right)$$

where $\mathbf{r}_{C_1}$ and $\mathbf{r}_{C_2}$ are the radii vectors of the two particles relative to centre of mass. $\mathbf{r}_{C_1}$ and $\mathbf{r}_{C_2}$ are related to $\mathbf{r}$ by the equations

$$\left.\begin{array}{l} \mathbf{r}_{C_1} = -\frac{m_2}{m_1+m_2}\mathbf{r} \\ \mathbf{r}_{C_2} = \frac{m_1}{m_1+m_2}\mathbf{r} \end{array}\right\}$$

Substituting these values, we get

$$T = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\dot{\mathbf{r}}^2$$

so the Lagrangian of the system

$$L = T - V = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\dot{\mathbf{r}}^2 - V(\mathbf{r}, \dot{\mathbf{r}})$$

$\mathbf{R}$ does not occurs in Lagrangian, so three components of $\mathbf{R}$ are cyclic, so centre of mass is either at rest or in uniform motion. No equation of motion for $\mathbf{r}$, will contain terms containing $\mathbf{R}$ or $\dot{\mathbf{R}}$, so assuming centre of mass at rest, me may drop the first term from the Lagrangian, i.e.,

$$L = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\dot{\mathbf{r}}^2 - V(\mathbf{r}, \dot{\mathbf{r}}, \ldots)$$

This is the Lagrangian of a single particle of a fictitious mass $\mu = \frac{m_1 m_2}{m_1+m_2}$ placed at the at the location of m_2 with the original potential function.Thus two body problem is equivalent to one body problem.

2.2.2 The equation of motion and first integral

Let us consider only the central force, where the potential V is the function of r only, so that the force is always directed along $\mathbf{r}$. Let a single particle move about a fixed centre of force which we assume to be the origin of co-ordinate system. Using polar co-ordinates (r, θ), the kinetic energy of particle is given by
$T = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$, μ being reduced mass.
The potential energy $V = V(r)$.
The Lagrangian of the system is given by

$$\begin{aligned} L &= T - V && (2.4)\\ &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r). && (2.5) \end{aligned}$$

Lagrange's equation for

$$\theta \text{ is } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{2.6}$$

$\therefore$ From (2.5)

$$\frac{\partial L}{\partial \theta} = 0 \text{ and } \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta}$$

$\therefore$ From (2.6)

$$\frac{d}{dt}\left(\mu r^2 \dot{\theta}\right) = 0$$

Integrating, we get

$$\mu r^2 \dot{\theta} = constant = J(say), \tag{2.7}$$

where J is first integral (constant of motion) and represents the magnitude of angular momentum.
As μ is constant, equation (2.6) gives

$$\begin{aligned} &\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0\\ &\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0\\ &\frac{1}{2}r^2\dot{\theta} = \text{ constant} && (2.8) \end{aligned}$$

The term $\frac{1}{2}r^2\dot{\theta}$ represents the areal velocity, i.e., the area swept out by the radius vector per unit time.

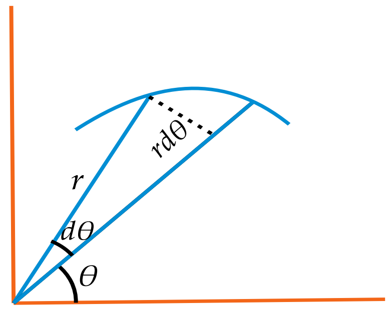


Figure 2.2

If vector **r** rotates by an angle $d\theta$ in time dt, the area swept out by r in time dt is $\frac{1}{2}r.(rd\theta) = dA$ (say), so that

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta}$$

From equation (2.8);

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \text{ constant} \tag{2.9}$$

Equation (2.4) gives

$$\frac{\partial L}{\partial \dot{r}} = \mu\dot{r}$$

$$\frac{\partial L}{\partial r} = \mu\dot{\theta}^2 - \frac{\partial V}{\partial r}$$

The Lagrangian equation in terms of r is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}(\mu\dot{r}) - \mu r\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \tag{2.10}$$

If we represent the force along **r** by $F(r)$, then we have

$$F(r) = -\frac{\partial V}{\partial r},$$

so that equation 2.10 can be written as

$$\mu\ddot{r} - \mu r\dot{\theta}^2 = F(r) \tag{2.11}$$

This is the general equation of motion.
But from equation (2.7),

$$\dot{\theta} = \frac{J}{\mu r^2}$$

$$\dot{\theta}^2 = \frac{J^2}{\mu^2 r^4},$$

so that equation (2.11) gives

$$\mu\ddot{r} - \frac{J^2}{\mu r^3} = F(r) \tag{2.12}$$

This is second order differential equation in r only
Equation (2.12) gives
$\mu\ddot{r} = \frac{J^2}{\mu r^3} + F(r)$,

$$\mu\ddot{r} = \frac{j^2}{\mu r^3} - \frac{\partial V}{\partial r} = -\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{J^2}{\mu r^2}\right) - \frac{\partial V}{\partial r}$$

$$\mu \ddot{r} = -\frac{\partial}{\partial r}\left(\frac{1}{2}\frac{J^2}{\mu r^2} + V\right) \tag{2.13}$$

Multiplying both sides of this equation by $\dot{r}$, we get

$$\begin{aligned}
&\mu \dot{r}\ddot{r} = -\frac{\partial}{\partial r}\left(\frac{1}{2}\cdot\frac{J^2}{\mu r^2} + V\right)\dot{r} \\
&\frac{d}{dt}\left(\frac{1}{2}\mu \dot{r}^2\right) = -\frac{d}{dt}\left(\frac{1}{2}\frac{J^2}{\mu r^2} + V\right) \\
&\frac{d}{dt}\left(\frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{J^2}{\mu r^2} + V\right) = 0 \\
&\frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{j^2}{\mu r^2} + V = \text{constant}
\end{aligned}$$

But

$$\begin{aligned}
&\text{K.E.} = T = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \\
&= \frac{1}{2}\mu\left(\dot{r}^2 + \frac{J^2}{\mu^2 r^2}\right) \\
&= \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{J^2}{\mu r^2} \\
&\text{potential energy } = V \\
&\text{total energy} \quad E = T + V = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{j^2}{\mu r^2} + V.
\end{aligned}$$

From these equations we get

$$\frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu^2 + V = E = \text{ constant} \tag{2.14}$$

ie total energy of the system is constant ie the total energy E , constant of motion.This is the another first integral of motion .This equation (2.12) represents equation of motion while angular momentum and total energy are constant of motion.
From equation (2.14) we have

$$\begin{aligned}
&\frac{1}{2}\mu \dot{r}^2 = E - \frac{J^2}{2\mu r^2} - V \\
&\dot{r} = \sqrt{\left[\frac{2}{\mu}\left\{E - \frac{J^2}{2\mu r^2} - V\right\}\right]} \\
&\frac{dr}{dt} = \sqrt{\left[\frac{2}{\mu}\left(E - \frac{J^2}{2\mu r^2} - V\right)\right]} \\
&\sqrt{\left[\frac{2}{\mu}\left(E - \frac{j^2}{2\mu r^2} - V\right)\right]}
\end{aligned} \tag{2.15}$$

Let the initial value of r be r_0; then integrating equation (2.15), we get

$$\int_{r_\theta}^{r} \frac{dr}{\sqrt{\left\{\frac{2}{\mu}\left(E - V - \frac{J^2}{2\mu r^2}\right)\right\}}} = \int_{\theta}^{t} dt$$

$$\left. t = \int_{r_\theta}^{r} \frac{dr}{\sqrt{\left\{\frac{2}{\mu}\left(E - V - \frac{J^2}{2\mu r^2}\right)\right.}} \right\} \tag{2.16}$$

This equation gives t as a function of r. However, from this equation we can find r as function of t and the constants. From equation (2.7), we have

$$d\theta = \frac{J}{\mu r^2} dt$$

If initially $\theta = \theta_\theta$, then integration of above equation yields

$$\int_{\theta_0}^{\theta} d\theta = \int_0^t \frac{J}{\mu r^2} dt$$
$$\theta - \theta_0 = \int_0^t \frac{J}{\mu r^2} dt$$
$$\theta = \int_0^1 \frac{J}{\mu r^2} dt + \theta_0 \tag{2.17}$$

This equation gives θ as a function of t.
Equation(2.17 and 2.16) are the only integration to be solved .Therefore the problems have been reduced to quadratures with four constants E, J, r_0, θ_0

2.2.3 The equivalent one-dimensional problem, and classification of orbits

Although we have solved the one dimensional problem formally ,practically speaking the integrals (1.16) and (1.17) are usually quite unmanageable and in specific case it is often more convenient to perform the integration in some other fashion.But before obtaining the solution for any specific force laws,let us see what can be learned about the motion in the general case using only the equation of motion and conservation theorems,without requiring explicit solutions.

The equation of motion in r,with $\dot{\theta}$ expressed in terms of l,quation $m\ddot{r} - \frac{l^2}{mr^3} = f(r)$ involves only r and its derivatives.It is the same equation as would be obtained for a fictitious one-dimensional problem in which a particle of mass m is subject to a force

$$f' = f + \frac{l^2}{mr^3}$$

The significance of the additional term is clear if it is written as $mr\dot{\theta}^2 = mv_\theta^2/r$, which is the familiar centrifugal force. An equivalent statement can be obtained from the conservation theorem for energy. By Eq. $\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2} + V = constant$ the motion of the particle in r is that of a one-dimensional problem with a fictitious potential energy:

$$V' = V + \frac{1}{2}\frac{l^2}{mr^2}$$

As a check, note that

$$f' = -\frac{\partial V'}{\partial r} = f(r) + \frac{l^2}{mr^3}$$

The energy conservation theorem can thus also be written as

$$E = V' + \frac{1}{2}m\dot{r}^2$$

As an illustration of this method of examining the motion, consider a plot of V' against r for the specific case of an attractive inverse-square law of force:

$$f = -\frac{k}{r^2}.$$

(For positive k, the minus sign ensures that the force is toward the center of force.) The potential energy for this force is

$$V = -\frac{k}{r}$$

and the corresponding fictitious potential is

$$V' = -\frac{k}{r} + \frac{l^2}{2mr^2}$$

Such a plot is shown in Figure the two dashed lines represent the separate components

$$-\frac{k}{r} \quad \text{and} \quad \frac{l^2}{2mr^2}$$

and the solid line is the sum V'.

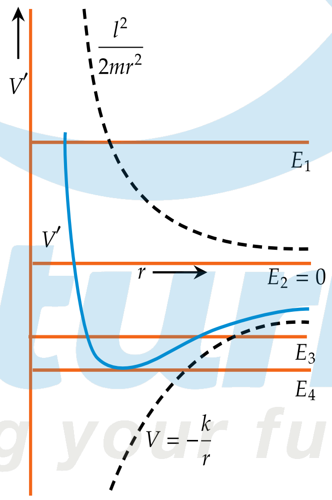


Figure 2.3: The equivalent one dimensional potential for attractive inverse square law of force

Let us consider now the motion of a particle having the energy E_1, as shown in Figures. Clearly this particle can never come closer than r_1 Otherwise with $r < r_1, V'$ exceeds E_1 and by conservation of energy the kinetic energy would have to be negative, corresponding to an imaginary velocity! on the other hand, there is no upper limit to the possible value of r, so the orbit is not bounded. A particle will come in from infinity, strike the "repulsive centrifugal barrier," be repelled, and travel back out to infinity. The distance between E and V' is $\frac{1}{2}m\dot{r}^2$, i.e., proportional to the square of the radial velocity, and becomes zero, naturally, at the turning point r_1. At the same time. the distance between E and V on the plot is the kinetic energy $\frac{1}{2}mv^2$ at the given value of r. Hence, the distance between the V and V' curves is $\frac{1}{2}mr^2\theta^2$. These curves therefore supply the magnitude of the particle velocity and its components for any distance r,at the given energy and angular momentum .This information is sufficient to produce an approximate picture of the form of the orbit.

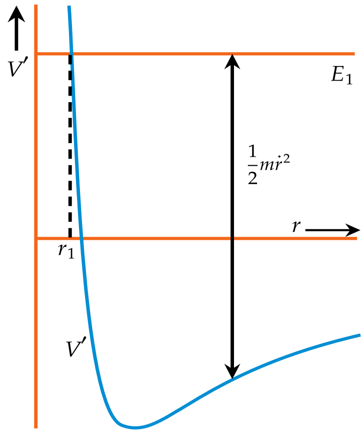


Figure 2.4: Unbounded motion at positive energies for inverse square law of force

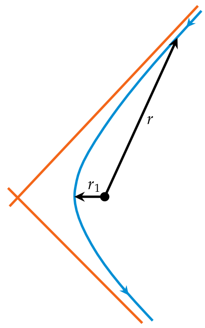


Figure 2.5: The orbit for E_1 corresponding to unbounded motion

For the energy $E_2 = 0$, a roughly similar picture of the orbit behavior is obtained. But for any lower energy, such as E_3 indicated in fig (1.6) we have a different story. In addition to a lower bound r_1, there is also a max. imum value r_2 that cannot be exceeded by r with positive kinetic energy. The motion is then "bounded," and there are two turning points, r_1 and r_2, also known as apsidal distances. This does not necessarily mean that the orbits are closed. All that can be said is that they are bounded, contained between two circles of radius r_1 and r_2 with turning points always lying on the circles

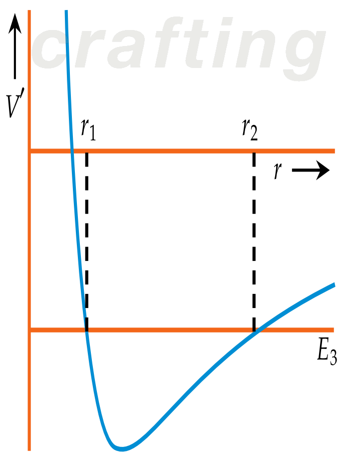


Figure 2.6: The equivalent one dimensional potential for inverse square law of force illustrating bounded motion at negative energies.

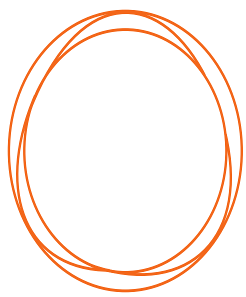


Figure 2.7: The nature of the orbit for bounded motion

If the energy is E_4 at the minimum of the fictitious potential as shown in Fig. (1.8), then the two bounds coincide. In such case, motion is possible at only one radius; $\dot{r} = 0$, and the orbit is a circle.

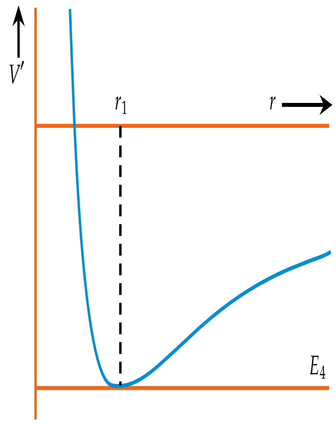


Figure 2.8: The equivalent one dimensional potential of inverse square law of force,illustrating the condition for circular orbits

2.2.4 Condition for circular orbit

Remembering that the effective "force" is the negative of the slope of the V' curve, the requirement for circular orbits is simply that f' be zero, or

$$f(r) = -\frac{l^2}{mr^3} = -mr\dot{\theta}^2$$

which can be also derived by $\frac{\partial V_{\text{effective}}}{\partial r}\Big|_{r=r_0} = 0$ and $\dot{\theta} = \omega_0$ is identified as angular frequency circular orbit. where

$$V_{eff} = \frac{J^2}{2mr^2} - \frac{k}{r^n}$$

Radius $r = r_0$ of circular orbit is also identified as stable equilibrium point so $\frac{\partial^2 V_{effective}}{\partial r}|_{r=r_0} \geq 0$ Somehow particle of mass m changes its orbit without changing its angular momentum and orbit is bounded then new orbit is identified as elliptical orbit .The angular frequency in new elliptical orbit is

$$\omega = \sqrt{\frac{\frac{\partial^2 V_{effective}}{\partial r}|_{r-r_0}}{m}}$$

2.2.5 The differential equation for the orbit

Under central force

$$\text{We have } J = \mu r^2 \dot{\theta} = \text{constant} \tag{2.18}$$

$$and E = \frac{1}{2}\mu \dot{r}^2 + \frac{J^2}{2\mu r^2} + V = \text{constant} \tag{2.19}$$

Differential equation in r is

$$\mu \ddot{r} - \frac{J^2}{\mu r^3} = F(r) \tag{2.20}$$

From equation (2.18)

$$J = \mu r^2 \dot{\theta}$$
$$J = \mu r^2 \frac{d\theta}{dt}$$
$$Jdt = \mu r^2 d\theta \tag{2.21}$$

The corresponding relation between the derivative relative to t and θ can be written as

$$\frac{d}{dt} = \frac{J}{\mu r^2}\frac{d}{d\theta} \tag{2.22}$$

second derivative w.r.t. t can be written as

$$\frac{d^2}{dt^2} = \frac{J}{\mu r^2}\frac{d}{d\theta}\left[\frac{J}{\mu r^2}\frac{d}{d\theta}\right] \tag{2.23}$$

from equation (2.20)

$$\mu\frac{J}{\mu r^2}\frac{d}{d\theta}\left\{\frac{J}{\mu r^2}\frac{dr}{d\theta}\right\} - \frac{J^2}{\mu r^3} = F(r).$$
$$\frac{J}{r^2}\frac{d}{d\theta}\left\{\frac{J}{\mu r^2}\frac{dr}{d\theta}\right\} - \frac{J^2}{\mu r^3} = F(r) \tag{2.24}$$

To simplify above equation we must remember that

$$\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{d(1/r)}{d\theta} \tag{2.25}$$

Using (2.25), equation (2.24) gives

$$\frac{J^2}{\mu r^2}\frac{d}{d\theta}\left[-\frac{d(1/r)}{d\theta}\right] - \frac{J^2}{\mu r^3} = F(r) \tag{2.26}$$

Substituting

$$u = \frac{1}{r}$$

equation (2.26) gives

$$-\frac{J^2u^2d^2u}{\mu}\frac{J^2}{\mu^2} - \frac{J^2}{\mu} = F\left(\frac{1}{u}\right)$$
$$\frac{J^2u^2}{\mu}\left[\frac{d^2u}{d\theta^2} + u\right] = -F\left(\frac{1}{u}\right) \tag{2.27}$$

This is the differential equation for the orbit if the force F is known.

2.2.6 The kepler problem:Inverse square law of force

The inverse square law is most important of all central force laws.It results in the deduction of Kepler's laws of planetary motion.
The kepler's laws of planetary motion are:
(i) All planets move in elliptical orbits having the sun as one focus.
(ii)The area swepts out by the radius vector of planet relative to the sun in equal times are equal.
(iii)The square of the period of revolution of any planet about the sun is proportional to the cube of the semi major axis
Deduction of Kepler's laws
Kepler's first law
Under central force the constant of motion are angular momentum and energy

$$J = \mu r^2\dot{\theta} \tag{2.28}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{J^2}{2\mu r^2} + V \tag{2.29}$$

From equation (2.28 and 2.29)

$$\frac{d\theta}{dt} = \frac{J}{\mu r^2} \tag{2.30}$$

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}\left(E - \frac{J^2}{2\mu r^2} - V\right)} \tag{2.31}$$

Dividing equation (2.31) by (2.30)

$$\frac{dr}{d\theta} = \frac{\mu r^2}{J} \times \sqrt{\left\{\frac{2}{\mu}\left(E - V - \frac{j^2}{2\mu r^2}\right)\right\}} \tag{2.32}$$

$$d\theta = \frac{Jdr}{\mu r^2 \sqrt{\left\{\frac{2}{\mu}\left(E - V - \frac{J^2}{2\mu r^2}\right)\right\}}} \tag{2.33}$$

Under inverse square law of force, we have

$$F(r) = -\frac{k}{r^2}.$$

$$F(r) = -\frac{\partial V}{\partial r}$$

$$-\frac{\partial V}{\partial r} = -\frac{k}{r^2} \Rightarrow dV = \frac{k}{r^2}dr$$

Integrating $\quad V = \int^r \frac{k}{r^2}dr$

or the potential energy $V = -\frac{k}{r}$.

Substituting this value of V in equation (2.33), we get

$$d\theta = \frac{Jdr}{\mu r^2 \sqrt{\left\{\frac{2}{\mu}\left(E + \frac{k}{r} - \frac{J^2}{2\mu r^2}\right)\right\}}}$$

Integrating, we get

$$\theta = \int \frac{Jdr}{\mu r^2 \sqrt{\left\{\frac{2}{\mu}\left(E + \frac{k}{r} - \frac{J^2}{2\mu r^2}\right)\right\}}} + \theta'$$

where θ' is constant of integration.

Substituting $r = 1/u$, we get

$$\theta = -\int \frac{Jdu}{\mu \sqrt{\left\{\frac{2}{\mu}\left(E + ku - \frac{J^2u^2}{2\mu}\right)\right\}}} + \theta'$$

$$= \theta' - \int \frac{du}{\sqrt{\left(\frac{2\mu E}{J^2} + \frac{2\mu k u}{J^2} - u^2\right)}}$$

$$= \theta' - \int \frac{du}{\sqrt{\left[\left(\frac{2\mu E}{J^2} + \frac{\mu^2 k^2}{J^4}\right) - \left(u - \frac{\mu k}{J^2}\right)^2\right]}}$$

$$= \theta' - \cos^{-1}\frac{u - \frac{\mu k}{J^2}}{\sqrt{\left(\frac{2\mu E}{J^2} + \frac{\mu^2 k^2}{J^4}\right)}} = \theta' - \cos^{-1}\frac{\frac{uJ^2}{\mu k} - 1}{\sqrt{\left(\frac{2EJ^2}{\mu k^2} + 1\right)}}$$

$$\frac{\frac{uJ^2}{\mu k}-1}{\sqrt{\left(\frac{2EJ^2}{\mu k^2}+1\right)}}=\cos\left(\theta-\theta'\right)$$

$$\frac{uJ^2}{\mu k}-1=\sqrt{\left(\frac{2EJ^2}{\mu k^2}+1\right)}\cos\left(\theta-\theta'\right)$$

$$u=\frac{\mu k}{J^2}\left[1+\sqrt{\left(\frac{2EJ^2}{\mu k^2}+1\right)}\cos\left(\theta-\theta'\right)\right] \tag{2.34}$$

Substituting

$$c=\frac{\mu k}{J^2} \tag{2.35}$$

$$\varepsilon=\sqrt{\left(1+\frac{2EJ^2}{\mu k^2}\right)} \tag{2.36}$$

Equation (2.34) gives

$$u=c\left[1+\varepsilon\cos\left(\theta-\theta'\right)\right]$$

$$\frac{1}{r}=c\left[1+\varepsilon\cos\left(\theta-\theta'\right)\right] \tag{2.37}$$

which is the equation of the conic with ε as eccentricity and one focus as the origin. Thus the equation of the path of the two-body problem of reduced mass μ is always a conic section, which is the generalisation of Kepler's first law.

The nature of the conic dependes on the value of eccentricity given by eqn. (2.36).

If $\in>1$, i.e., if $\sqrt{\left(\frac{2EJ^2}{\mu k^2}+1\right)}>1$ or $E>0$, the conic is hyperbola.

If $\in=1$, i.e., if $\sqrt{\left(\frac{2EJ^2}{\mu k^2}+1\right)}=1$ or $E=0$, the conic is a parabola.

If $\in<1$, i.e., if $\sqrt{\left(\frac{2E^2}{\mu k^2}+1\right)}<1$ or $E<0$, the conic is an ellipse.

If $\in=0$, i.e., if $\sqrt{\left(\frac{2E^2}{\mu k^2}+1\right)}=0$ or $E=-\frac{\mu k^2}{2J^2}$, the conic is a circle.

In the case of elliptical orbits, when $\theta-\theta'=0, r=r_1=$ perihelion,

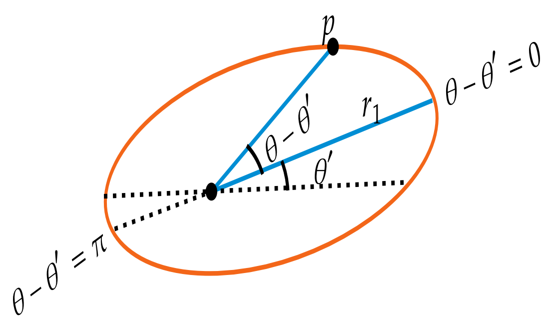


Figure 2.9

Then from equation (2.37) we have

$$r_1=\frac{1}{c(1+\varepsilon)}$$

When $\theta-\theta'=\pi, r=r_2=$ aphelion, then eqn. (2.37) gives

$$r_2=\frac{1}{c(1-\varepsilon)}$$

The semi-major axis, which is one-half the sum of perihelion r_1 and aphelion r_2 is given by

$$a = \frac{r_1 + r_2}{2} = \frac{1}{2}\left[\frac{1}{c(1+\varepsilon)} + \frac{1}{c(1-\varepsilon)}\right]$$

Substituting values of c and ε , we get

$$\begin{aligned} a &= \frac{1}{\frac{\mu k}{J^2}\left\{1 - \left(1 + \frac{2EJ^2}{\mu k^2}\right)\right\}} = -\frac{k}{2E} \\ E &= -\frac{k}{2a}. \end{aligned} \tag{2.38}$$

This shows that in the case of elliptical orbits the total energy depends solely on the major axis.

Deduction of II law

$$J = \mu r^2 \dot{\theta} = \text{ constant}$$

This implies

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \text{ constant}$$

which represents the areal velocity, i.e., the area swept out by the radius vector per unit time is constant. This means that the areas swept out by the radius vector in equal times are equal which is Kepler's II law.

Deduction of kepler's III Law

If T is the periodic time of describing the complete orbit, the area of the orbit is given by

$$\begin{aligned} A &= \int_0^T \frac{dA}{dt} dt = \int_0^T \frac{1}{2} r^2 \ddot{\theta} dt \\ &= \int_0^T \frac{J}{2\mu} dt \quad \left(\text{ since } J = \mu r^2 \dot{\theta}\right) \\ &= \frac{JT}{2\mu} \end{aligned} \tag{2.39}$$

But area of the ellipse

$$\text{A} = \pi ab \tag{2.40}$$

where a and b are the semi-major and semi-minor axes of the ellipse respectively.

$$\begin{aligned} &\text{Also} \quad b = a\sqrt{(1-\varepsilon^2)} = a\sqrt{\left(1 - 1 - \frac{2EJ^2}{\mu k^2}\right)} = a\sqrt{\left(-\frac{2EJ^2}{\mu k^2}\right)} \\ &\text{But} \quad E = -\frac{k}{2a} \end{aligned}$$

Therefore

$$b = a\sqrt{\left(\frac{kJ^2}{a\mu k^2}\right)} = a^{1/2}\sqrt{\left(\frac{J^2}{\mu k}\right)} \tag{2.41}$$

Substituting value of b in equation (2.40) we get

$$A = \pi a^{3/2}\sqrt{\left(\frac{J^2}{\mu k}\right)} \tag{2.42}$$

Comparing equation (2.39 and 2.42)

$$\begin{aligned} \frac{JT}{2\mu} &= \pi a^{3/2}\sqrt{\left(\frac{J^2}{\mu k}\right)} \\ \frac{J^2 T^2}{4\mu^2} &= \pi^2 a^3 \frac{J^2}{\mu k} \end{aligned}$$

$$T^2 = 4\pi^2 a^3 \frac{\mu}{k}$$
$$T^2 \propto a^3 \tag{2.43}$$

ie the square of the period of revolution of the planet around the sun is proportional to the cube of the semimajor axis, which is kepler's III law.

2.3 Two body collisions

When discussing conservation of momentum, we considered examples in which two objects collide and stick together, and either there are no external forces acting in some direction (or the collision was nearly instantaneous) so the component of the momentum of the system along that direction is constant. We shall now study collisions between objects in more detail. In particular we shall consider cases in which the objects do not stick together. The momentum along a certain direction may still be constant but the mechanical energy of the system may change. We will begin our analysis by considering two-particle collision. We introduce the concept of the relative velocity between two particles and show that it is independent of the choice of reference frame. We then show that the change in kinetic energy only depends on the change of the square of the relative velocity and therefore is also independent of the choice of reference frame. We will then study one- and two-dimensional collisions with zero change in potential energy. In particular we will characterize the types of collisions by the change in kinetic energy and analyze the possible outcomes of the collisions.

2.3.1 Laboratory frame of reference

Let $\overrightarrow{\mathbf{R}}$ be the vector from the origin of frame S to the origin of reference frame S'. Denote the position vector of the j^{th} particle with respect to the origin of reference frame S by $\overrightarrow{\mathbf{r}}_j$ and similarly, denote the position vector of the j^{th} particle with respect to the origin of reference frame S' by $\overrightarrow{\mathbf{r}}'_j$.

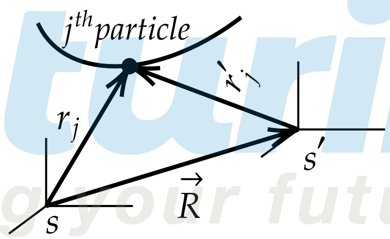


Figure 2.10

The position vectors are related by

$$\overrightarrow{\mathbf{r}}_j = \overrightarrow{\mathbf{r}}'_j + \overrightarrow{\mathbf{R}}$$

The relative velocity (call this the boost velocity) between the two reference frames is given by

$$\overrightarrow{\mathbf{V}} = \frac{d\overrightarrow{\mathbf{R}}}{dt}$$

Assume the boost velocity between the two reference frames is constant. Then, the relative acceleration between the two reference frames is zero,

$$\overrightarrow{\mathbf{A}} = \frac{d\overrightarrow{\mathbf{V}}}{dt} = \overrightarrow{\mathbf{0}}$$

When the equation is satisfied, the reference frames S and S' are called relatively inertial reference frames. Suppose the j^{th} particle in Figure is moving; then observers in different reference frames will measure different velocities. Denote the velocity of j^{th} particle in frame S by $\overrightarrow{\mathbf{v}}_j = d\overrightarrow{\mathbf{r}}_j/dt$, and the velocity of the same particle in frame S' by $\overrightarrow{\mathbf{v}}'_j = d\overrightarrow{\mathbf{r}}'_j/dt$. Taking derivative, the velocities of the particles in two different reference frames are related according to

$$\overrightarrow{\mathbf{v}}_j = \overrightarrow{\mathbf{v}}'_j + \overrightarrow{\mathbf{V}}$$

2.3.2 Center-of-mass Reference Frame

Let $\vec{\mathbf{r}}_{cm}$ be the vector from the origin of frame S to the center-of-mass of the system of particles, a point that we will choose as the origin of reference frame S_{cm}, called the center-of-mass reference frame. Denote the position vector of the j^{th} particle with respect to origin of reference frame S by $\vec{\mathbf{r}}_j$ and similarly, denote the position vector of the j^{th} particle with respect to origin of reference frame S_{cm} by $\vec{\mathbf{r}}'_j$

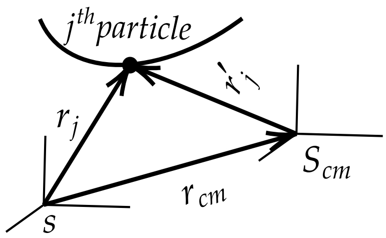


Figure 2.11

The position vector of the j^{th} particle in the center-of-mass frame is then given by

$$\vec{\mathbf{r}}'_j = \vec{\mathbf{r}}_j - \vec{\mathbf{r}}_{cm}.$$

The velocity of the j^{th} particle in the center-of-mass reference frame is then given by

$$\vec{\mathbf{v}}'_j = \vec{\mathbf{v}}_j - \vec{\mathbf{v}}_{cm}$$

There are many collision problems in which the center-of-mass reference frame is the most convenient reference frame to analyze the collision.

Consider a system consisting of two particles, which we shall refer to as particle 1 and particle 2. We can determine the velocities of particles 1 and 2 in the center-of-mass, as

$$\vec{\mathbf{v}}'_1 = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_{cm} = \vec{\mathbf{v}}_1 - \frac{m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2}{m_1+m_2} = \frac{m_2}{m_1+m_2}\left(\vec{\mathbf{v}}_1, -\vec{\mathbf{v}}_2\right) = \frac{\mu}{m_1}\vec{\mathbf{v}}_{1,2}$$

where $\vec{\mathbf{v}}_{12} = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2$ is the relative velocity of particle 1 with respect to particle 2 . A similar result holds for particle 2 :

$$\vec{\mathbf{v}}'_2 = \vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_{cm} = \vec{\mathbf{v}}_2 - \frac{m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2}{m_1+m_2} = -\frac{m_1}{m_1+m_2}\left(\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2\right) = -\frac{\mu}{m_2}\vec{\mathbf{v}}_{1,2}$$

The momentum of the system the center-of-mass reference frame is zero as we expect,

$$m_1\vec{\mathbf{v}}'_1 + m_2\vec{\mathbf{v}}'_2 = \mu\vec{\mathbf{v}}_{12} - \mu\vec{\mathbf{v}}_{12} = \vec{\mathbf{0}}$$

2.3.3 Characterizing Collisions

In a collision, the ratio of the magnitudes of the initial and final relative velocities is called the coefficient of restitution and denoted by the symbol e,

$$e = \frac{v_B}{v_A}$$

If the magnitude of the relative velocity does not change during a collision, $e = 1$, then the change in kinetic energy is zero. Collisions in which there is no change in kinetic energy are called elastic collisions,

$$\Delta K = 0, \text{elastic collision}$$

If the magnitude of the final relative velocity is less than the magnitude of the initial relative velocity, $e < 1$, then the change in kinetic energy is negative. Collisions in which the kinetic energy decreases are called inelastic collisions,

$$\Delta K < 0, \text{inelastic collision}$$

If the two objects stick together after the collision, then the relative final velocity is zero, $e = 0$. Such collisions are called totally inelastic. The change in kinetic energy can be written as

$$\Delta K = -\frac{1}{2}\mu v_A^2 = -\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} v_A^2, \text{ totally inelastic collision .}$$

If the magnitude of the final relative velocity is greater than the magnitude of the initial relative velocity, $e > 1$, then the change in kinetic energy is positive. Collisions in which the kinetic energy increases are called superelastic collisions,

$$\Delta K > 0, \textbf{superelastic collision}$$

2.3.4 Two-dimensional Elastic Collision in Laboratory Reference Frame

Consider the elastic collision between two particles in which we neglect any external forces on the system consisting of the two particles. Particle 1 of mass m_1 is initially moving with velocity $\overrightarrow{\mathbf{v}}_{1,i}$ and collides elastically with a particle 2 of mass m_2 that is initially at rest. We shall refer to the reference frame in which one particle is at rest, 'the target', as the laboratory reference frame. After the collision particle 1 moves with velocity $\overrightarrow{\mathbf{v}}_{1,f}$ and particle 2 moves with velocity $\overrightarrow{\mathbf{v}}_{2,f}$, (Figure). The angles $\theta_{1,f}$ and $\theta_{2,f}$ that the particles make with the positive forward direction of particle 1 are called the laboratory scattering angles.

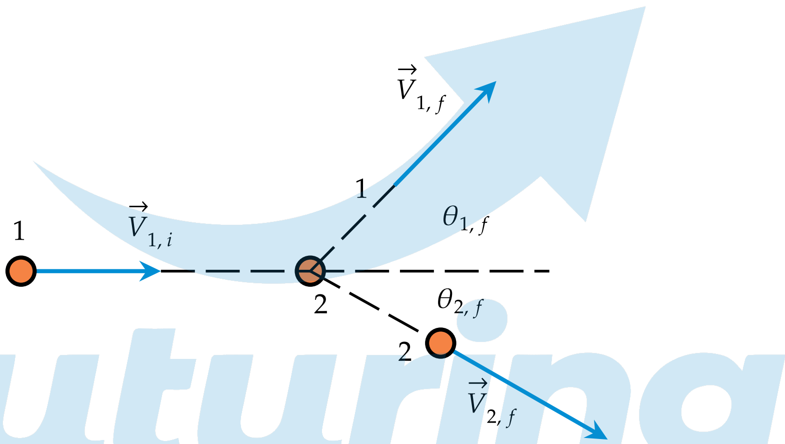


Figure 2.12

Generally the initial velocity $\overrightarrow{\mathbf{v}}_{1,i}$ of particle 1 is known and we would like to determine the final velocities $\overrightarrow{\mathbf{v}}_{1,f}$ and $\overrightarrow{\mathbf{v}}_{2,f}$, which requires finding the magnitudes and directions of each of these vectors, $v_{1,f}, v_{2,f}, \theta_{1,f}$, and $\theta_{2,f}$. These quantities are related by the two equations describing the constancy of momentum, and the one equation describing constancy of the kinetic energy. Therefore there is one degree of freedom that we must specify in order to determine the outcome of the collision. In what follows we shall express our results for $v_{1,f}, v_{2,f}$, and $\theta_{2,f}$ in terms of $v_{1,i}$ and $\theta_{1,f}$.
The components of the total momentum $\overrightarrow{\mathbf{p}}_i^{\text{sys}} = m_1 \overrightarrow{\mathbf{v}}_{1,i} + m_2 \overrightarrow{\mathbf{v}}_{2,i}$ in the initial state are given by

$$\begin{aligned} p_{x,i}^{\text{sys}} &= m_1 v_{1,i} \\ p_{y,i}^{\text{sys}} &= 0. \end{aligned}$$

The components of the momentum $\overrightarrow{\mathbf{p}}_f^{\text{sys}} = m_1 \overrightarrow{\mathbf{v}}_{1,f} + m_2 \overrightarrow{\mathbf{v}}_{2,f}$ in the final state are given by

$$\begin{aligned} p_{x,f}^{\text{sys}} &= m_1 v_{1,f} \cos\theta_{1,f} + m_2 v_{2,f} \cos\theta_{2,f} \\ p_{y,f}^{\text{sys}} &= m_1 v_{1,f} \sin\theta_{1,f} - m_2 v_{2,f} \sin\theta_{2,f}. \end{aligned}$$

There are no any external forces acting on the system, so each component of the total momentum remains constant during the collision,

$$\begin{aligned} p_{x,i}^{\text{sys}} &= p_{x,f}^{\text{sys}} \\ p_{y,i}^{\text{sys}} &= p_{y,f}^{\text{sy}} \end{aligned}$$

substituting the values

$$m_1 v_{1,i} = m_1 v_{1,f} \cos\theta_{1,f} + m_2 v_{2,f} \cos\theta_{2,f}$$
$$0 = m_1 v_{1,f} \sin\theta_{1,f} - m_2 v_{2,f} \sin\theta_{2,f}$$

rewriting the expressions we will get

$$m_2 v_{2,f} \cos\theta_{2,f} = m_1 \left(v_{1,i} - v_{1,f} \cos\theta_{1,f}\right) \tag{2.44}$$
$$m_2 v_{2,f} \sin\theta_{2,f} = m_1 v_{1,f} \sin\theta_{1,f} \tag{2.45}$$

Squaring and adding and using the identity the identity $\cos^2\theta + \sin^2\theta = 1$ yielding

$$v_{2,f}^2 = \frac{m_1^2}{m_2^2}\left(v_{1,i}^2 - 2v_{1,i}v_{1,f}\cos\theta_{1,f} + v_{1,f}^2\right)$$

The collision is elastic and therefore the system kinetic energy of is constant

$$K_i^{\text{sys}} = K_f^{\text{sys}}$$
$$\frac{1}{2}m_1 v_{1,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2$$

substituting the value of $v_{2,f}^2$ in this equation

$$\frac{1}{2}m_1 v_{1,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}\frac{m_1^2}{m_2}\left(v_{1,i}^2 - 2v_{1,i}v_{1,f}\cos\theta_{1,f} + v_{1,f}^2\right)$$
$$0 = \left(1 + \frac{m_1}{m_2}\right)v_{1,f}^2 - \frac{m_1}{m_2}2v_{1,i}v_{1,f}\cos\theta_{1,f} - \left(1 - \frac{m_1}{m_2}\right)v_{1,i}^2$$

Let $\alpha = m_1/m_2$ then Equation can be written as

$$0 = (1+\alpha)v_{1,f}^2 - 2\alpha v_{1,i}v_{1,f}\cos\theta_{1,f} - (1-\alpha)v_{1,i}^2$$

The solution to this quadratic equation is given by

$$v_{1,f} = \frac{\alpha v_{1,i}\cos\theta_{1,f} \pm \left(\alpha^2 v_{1,i}^2\cos^2\theta_{1,f} + (1-\alpha)v_{1,i}^2\right)^{1/2}}{(1+\alpha)}$$

Divide equation (2.44 and 2.25) yields

$$\frac{v_{2,f}\sin\theta_{2,f}}{v_{2,f}\cos\theta_{2,f}} = \frac{v_{1,f}\sin\theta_{1,f}}{v_{1,i} - v_{1,f}\cos\theta_{1,f}}$$
$$\tan\theta_{2,f} = \frac{v_{1,f}\sin\theta_{1,f}}{v_{1,i} - v_{1,f}\cos\theta_{1,f}}.$$

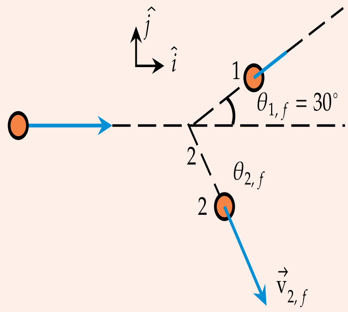
The relationship between the scattering angles is independent of the masses of the colliding particles. Thus the scattering angle for particle 2 is

$$\theta_{2,f} = \tan^{-1}\left(\frac{v_{1,f}\sin\theta_{1,f}}{v_{1,i} - v_{1,f}\cos\theta_{1,f}}\right)$$

From (2.45) final velocity of the particle 1

$$v_{2,f} = \frac{v_{1,f}\sin\theta_{1,f}}{\alpha\sin\theta_{2,f}}$$

Exercise 2.1 Object 1 with mass m_1 is initially moving with a speed $v_{1,i} = 3.0\ \mathrm{m \cdot s^{-1}}$ and collides elastically with object 2 that has the same mass, $m_2 = m_1$, and is initially at rest. After the collision, object 1 moves with an unknown speed $v_{1,f}$ at an angle $\theta_{1,f}$ with respect to its initial direction of motion and object 2 moves with an unknown speed $v_{2,f}$, at an unknown angle $\theta_{2,f}$ (as shown in the Figure 15.10). Find the final speeds of each of the objects and the angle $\theta_{2,f}$.



Solution: Because the masses are equal, $\alpha = 1$. We are given that $v_{1,i} = 3.0\ \mathrm{m \cdot s^{-1}}$ and $\theta_{1,f} = 30°$. Hence $0 = (1+\alpha)v_{1,f}^2 - 2\alpha v_{1,i}v_{1,f}\cos\theta_{1,f} - (1-\alpha)v_{1,i}^2$ reduces to

$$v_{1,f} = v_{1,i}\cos\theta_{1,f} = \left(3.0\ \mathrm{m \cdot s^{-1}}\right)\cos 30° = 2.6\ \mathrm{m \cdot s^{-1}}$$

substitute this value in

$$\tan\theta_{2,f} = \frac{v_{1,f}\sin\theta_{1,f}}{v_{1,i} - v_{1,f}\cos\theta_{1,f}}$$

$$\theta_{2,f} = \tan^{-1}\left(\frac{v_{1,f}\sin\theta_{1,f}}{v_{1,i} - v_{1,f}\cos\theta_{1,f}}\right)$$

$$\theta_{2,f} = \tan^{-1}\left(\frac{\left(2.6\ \mathrm{m \cdot s^{-1}}\right)\sin(30°)}{3.0\ \mathrm{m \cdot s^{-1}} - \left(2.6\ \mathrm{m \cdot s^{-1}}\right)\cos(30°)}\right)$$

$$= 60°.$$

The above results for $v_{1,f}$ and $\theta_{2,f}$ may be substituted into either of the expressions in $m_2 v_{2,f}\cos\theta_{2,f} = m_1\left(v_{1,i} - v_{1,f}\cos\theta_{1,f}\right)$ to find $v_{2,f} = 1.5\ \mathrm{m \cdot s^{-1}}$.

2.3.5 Two-Dimensional Collision in Center-of-Mass Reference Frame

Consider a collision between particle 1 of mass m_1 and velocity $\vec{\mathbf{v}}_{1,i}$ and particle 2 of mass m_2 at rest in the laboratory frame. Particle 1 is scattered elastically through a scattering angle Θ in the center-of-mass frame. The center-of-mass velocity is given by

$$\vec{\mathbf{v}}_{cm} = \frac{m_1\vec{\mathbf{v}}_{1,i}}{m_1 + m_2}$$

In the center-of-mass frame, the momentum of the system of two particles is zero

$$\vec{\mathbf{0}} = m_1\vec{\mathbf{v}}'_{1,i} + m_2\vec{\mathbf{v}}'_{2,i} = m_1\vec{\mathbf{v}}'_{1,f} + m_2\vec{\mathbf{v}}'_{2,f}$$

Therefore

$$\vec{\mathbf{v}}'_{1,i} = -\frac{m_2}{m_1}\vec{\mathbf{v}}'_{2,i}$$

$$\vec{\mathbf{v}}'_{1,f} = -\frac{m_2}{m_1}\vec{\mathbf{v}}'_{2,f}$$

The energy condition in the center-of-mass frame is

$$\frac{1}{2}m_1 v'^2_{1,i} + \frac{1}{2}m_2 v'^2_{2,i} = \frac{1}{2}m_1 v'^2_{1,f} + \frac{1}{2}m_2 v'^2_{2,f}.$$

substitute the value of velocities in this equation yields

$$v'_{1,i} = v'_{1,f}$$

(we are only considering magnitudes). Therefore

$$v'_{2,i} = v'_{2,f}.$$

Because the magnitude of the velocity of a particle in the center-of-mass reference frame is proportional to the relative velocity of the two particles, imply that the magnitude of the relative velocity also does not change

$$\left|\vec{\mathbf{V}}'_{1,2,i}\right| = \left|\vec{\mathbf{V}}'_{1,2,f}\right|$$

Recall that the relative velocity is independent of the reference frame

$$\vec{\mathbf{V}}_{1,i} - \vec{\mathbf{V}}_{2,i} = \vec{\mathbf{V}}'_{1,i} - \vec{\mathbf{V}}'_{2,i}$$

In the laboratory reference frame $\vec{\mathbf{v}}_{2,i} = \vec{\mathbf{0}}$, hence the initial relative velocity is $\vec{\mathbf{v}}'_{1,2,i} = \vec{\mathbf{v}}_{1,2,i} = \vec{\mathbf{v}}_{1,i}$, and the velocities in the center-of-mass frame of the particles are then

$$\vec{\mathbf{v}}'_{1,i} = \frac{\mu}{m_1}\vec{\mathbf{v}}_{1,i}$$
$$\vec{\mathbf{V}}'_{2,i} = -\frac{\mu}{m_2}\vec{\mathbf{v}}_{1,i}$$

Therefore the magnitudes of the final velocities in the center-of-mass frame are

$$v'_{1,f} = v'_{1,i} = \frac{\mu}{m_1}v'_{1,2,i} = \frac{\mu}{m_1}v_{1,2,i} = \frac{\mu}{m_1}v_{1,i}.$$
$$v'_{2,f} = v'_{2,i} = \frac{\mu}{m_2}v'_{1,2,i} = \frac{\mu}{m_2}v_{1,2,i} = \frac{\mu}{m_2}v_{1,i}.$$

2.3.6 Relation between scattering angles in laboratory and centre of mass frame for particle undergoing elastic collision

Consider a particle of mass m_1 moving with velocity $\vec{u_1}$ in the laboratory frame and let it collide with particles of mass m_2 at rest ,the collision being perfectly elastic .After collision the incident particle moves with a velocity $\vec{v_1}$ making scattering angle θ_1 with the initial direction and the target particle of mass m_2 moves with a velocity $\vec{v_2}$ making recoil angle θ_2 with the initial direction of motion of m_1.The initial path of m_1 is along the X- axis and the plane containing u_1 and v_1 is the X-Y plane as ahown in figure.

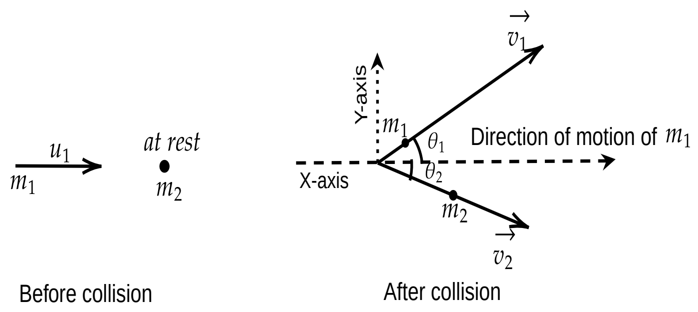


Figure 2.13

Let $\vec{v'_1}$ and $\vec{v'_2}$ be the final velocities of the particle m_1 and m_2 after collision in the center of mass frame making an angle θ with the X-axis as shown in the figure.

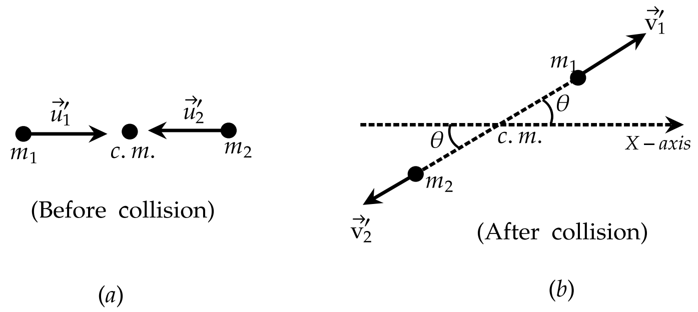


Figure 2.14

Then

$$\vec{v}_1' = \vec{v}_1 - \vec{V}_{cm} \qquad \vec{v}_2' = \vec{v}_2 - \vec{V}_{cm}$$

As

$$\vec{u}_2 = 0, \quad \vec{V}_{cm} = \frac{m_1 u_1}{m_1 + m_2}$$

ie $\vec{V}_{cm}$ and u_1 have the same direction along X-axis.Therefore $\vec{V}_{cm}$ has no component along the Y-axis .The y component of the velocity of the particle of mass m_1 is the same in both frames.

$$v_1 \sin\theta_1 = v_1' \sin\theta \tag{2.46}$$

As the center of mass has a velocity $\vec{V}_{cm}$ along X-axis with respect to laboratory frame.

$$v_1 \cos\theta_1 = v_1' \cos\theta + V_{cm} \tag{2.47}$$

Divide (1.46) by (1.47),we have

$$\tan\theta_1 = \frac{v_1' \sin\theta}{v_1' \cos\theta + V_{cm}} = \frac{\sin\theta}{\cos\theta + \frac{V_{cm}}{v_1'}}$$

But

$$V_{,} = \frac{m_1}{m_1 + m_2} u_1$$

and

$$v_1' = \frac{m_2}{m_1 + m_2} u_1$$

Dividing we get

$$\frac{V_{cm}}{v_1'} = \frac{m_1}{m_2}$$

$$\tan\theta_1 = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}}$$

Special cases (i)**when** $m_1 <<<< m_2$
in this case m_1/m_2 can be neglected and we have

$$\tan\theta_1 = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

Thus if the incident particle is very light as compared to the target particle ,the angle of scattering for the incident particle in the laboratory and center of mass system are very nearly equal.
Case (ii)**when** $m_1 = m_2$
In this case m_1/m_2=1
Hence

$$\tan\theta_1 = \frac{\sin\theta}{1 + \cos\theta} = \tan(\theta/2)$$

$$\theta_1 = \theta/2$$

Thus if the incident and target particle are of equal masses ,the angle of scattering in the laboratory system is the half the angle of scattering in the CM system.

Practice set 1

1. The acceleration due to gravity (g) on the surface of Earth is approximately 2.6 times that on the surface of Mars. Given that the radius of Mars is about one half the radius of Earth, the ratio of the escape velocity on Earth to that on Mars is approximately

[NET JUNE 2011]

A. 1.1 **B.** 1.3

C. 2.3 **D.** 5.2

2. Two particles of identical mass move in circular orbits under a central potential $V(r) = \frac{1}{2}kr^2$. Let l_1 and l_2 be the angular momenta and r_1, r_2 be the radii of the orbits respectively. If $\frac{l_1}{l_2} = 2$, the value of $\frac{r_1}{r_2}$ is:

[NET DEC 2011]

A. $\sqrt{2}$ **B.** $1/\sqrt{2}$

C. 2 **D.** $1/2$

3. A planet of mass m moves in the inverse square central force field of the Sun of mass M. If the semi-major and semi-minor axes of the orbit are a and b, respectively, the total energy of the planet is:

[NET DEC 2011]

A. $-\frac{GMm}{a+b}$ **B.** $-GMm\left(\frac{1}{a}+\frac{1}{b}\right)$

C. $-\frac{GMm}{a}\left(\frac{1}{b}-\frac{1}{a}\right)$ **D.** $-GMm\left(\frac{a-b}{(a+b)^2}\right)$

4. A planet of mass m moves in the gravitational field of the Sun (mass M). If the semimajor and semi-minor axes of the orbit are a and b respectively, the angular momentum of the planet is

[NET DEC 2012]

A. $\sqrt{2GMm^2(a+b)}$ **B.** $\sqrt{2GMm^2(a-b)}$

C. $\sqrt{\frac{2GMm^2ab}{a-b}}$ **D.** $\sqrt{\frac{2GMm^2ab}{a+b}}$

5. A planet of mass m and an angular momentum L moves in a circular orbit in a potential, $V(r) = -k/r$, where k is a constant. If it is slightly perturbed radially, the angular frequency of radial oscillations is

[NET JUNE 2013]

A. $mk^2/\sqrt{2}L^3$ **B.** mk^2/L^3

C. $\sqrt{2}mk^2/L^3$ **D.** $\sqrt{3}mk^2/L^3$

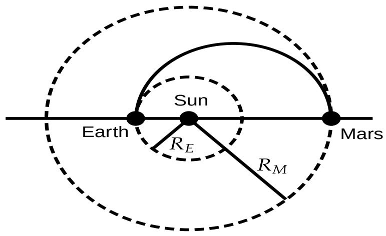
6. The radius of Earth is approximately 6400 km. The height h at which the acceleration due to Earth's gravity differs from g at the Earth's surface by approximately 1% is

[NET DEC 2014]

A. 64 km **B.** 48 km

C. 32 km **D.** 16 km

7. The probe Mangalyaan was sent recently to explore the planet Mars. The inter-planetary part of the trajectory is approximately a half-ellipse with the Earth (at the time of launch),Sun and Mars (at the time the probe reaches the destination) forming the major axis. Assuming that the orbits of Earth and Mars are approximately circular with radii R_E and R_M, respectively, the velocity (with respect to the Sun) of the probe during its voyage when it is at a distance



$r\,(R_E << r << R_M)$ from the Sun, neglecting the effect of Earth and Mars, is

[NET DEC 2014]

A. $\sqrt{2GM\frac{(R_E+R_M)}{r(R_E+R_M-r)}}$
B. $\sqrt{2GM\frac{(R_E+R_M-r)}{r(R_E+R_M)}}$
C. $\sqrt{2GM\frac{R_E}{rR_M}}$
D. $\sqrt{\frac{2GM}{r}}$

8. After a perfectly elastic collision of two identical balls, one of which was initially at rest, the velocities of both the balls are non zero. The angle θ between the final, velocities (in the lab frame) is

[NET DEC 2016]

A. $\theta = \frac{\pi}{2}$
B. $\theta = \pi$
C. $0 < \theta \le \frac{\pi}{2}$
D. $\frac{\pi}{2} < \theta \le \pi$

9. Consider circular orbits in a central force potential $V(r) = -\frac{k}{r^n}$, where $k > 0$ and $0 < n < 2$. If the time period of a circular orbit of radius R is T_1 and that of radius $2R$ is T_2, then $\frac{T_2}{T_1}$

[NET DEC 2016]

A. $2^{\frac{n}{2}}$
B. $2^{\frac{2}{3}n}$
C. $2^{\frac{n}{2}+1}$
D. 2^n

10. A ball weighing 100gm, released from a height of 5 m, bounces perfectly elastically off a plate. The collision time between the ball and the plate is 0.5 s. The average force on the plate is approximately

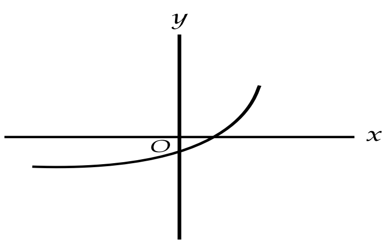
[NET JUNE 2017]

A. $3N$
B. $2N$
C. $5N$
D. $4N$

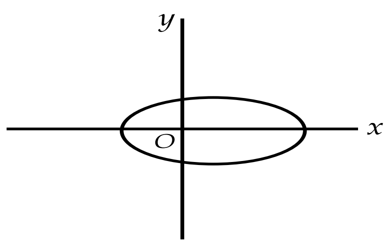
11. Which of the following figures best describes the trajectory of a particle moving in a repulsive central potential $V(r) = \frac{a}{r}(a > 0$ is a constant)?

[NET JUNE 2018]

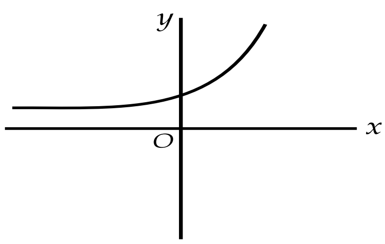
A.



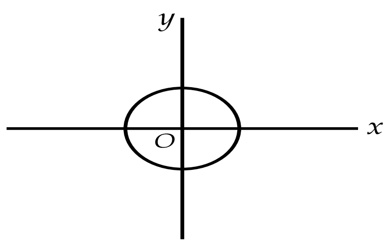
B.



C.



D.



12. A particle of mass m moves in a central potential $V(r) = -\frac{k}{r}$ in an elliptic orbit $r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}$, where $0 \le \theta < 2\pi$ and a and e denote the semi-major axis and eccentricity, respectively. If its total energy is $E = -\frac{k}{2a}$, the maximum kinetic energy is

[NET JUNE 2018]

A. $E\left(1-e^2\right)$
B. $E\frac{(e+1)}{(e-1)}$
C. $E/\left(1-e^2\right)$
D. $E\frac{(e-1)}{(e+1)}$

13. In the attractive Kepler problem described by the central potential $V(r) = \frac{-k}{r}$(where k is a positive constant), a particle of mass m with a non-zero angular momentum can never reach the centre due to the centrifugal barrier. If we modify the potential to

$$V(r) = -\frac{k}{r} - \frac{\beta}{r^3}$$

one finds that there is a critical value of the angular momentum ℓ_c below which there is no centrifugal barrier. This value of ℓ_c is

[NET DEC 2018]

A. $\left[12 \text{ km}^2\beta\right]^{1/2}$
B. $\left[12 \text{ km}^2\beta\right]^{-1/2}$
C. $\left[12 \text{ km}^2\beta\right]^{1/4}$
D. $\left[12 \text{ km}^2\beta\right]^{-1/4}$

Answer key

Q.No.	Answer	Q.No.	Answer
1	**c**	2	**a**
3	**a**	4	**d**
5	**b**	6	**c**
7	**b**	8	**a**
9	**c**	10	**d**
11	**c**	12	**b**
13	**c**		

Practice set 2

1. In a central force field, the trajectory of a particle of mass m and angular momentum L in plane polar coordinates is given by,

$$\frac{1}{r} = \frac{m}{l^2}(1+\varepsilon\cos\theta)$$

where, ε is the eccentricity of the particle's motion. Which one of the following choice for ε gives rise to a parabolic trajectory?

[GATE 2012]

A. $\varepsilon = 0$ **B.** $\varepsilon = 1$
C. $0 < \varepsilon < 1$ **D.** $\varepsilon > 1$

2. A particle of unit mass moves along the x-axis under the influence of a potential, $V(x) = x(x-2)^2$. The particle is found to be in stable equilibrium at the point $x = 2$. The time period of oscillation of the particle is

[GATE 2012]

A. $\frac{\pi}{2}$ **B.** π
C. $\frac{3\pi}{2}$ **D.** 2π

3. A particle of mass m is in a potential given by

$$V(r) = -\frac{a}{r} + \frac{ar_0^2}{3r^3}$$

where a and r_0 are positive constants. When disturbed slightly from its stable equilibrium position it undergoes a simple harmonic oscillation. The time period of oscillation is

[GATE 2014]

A. $2\pi\sqrt{\frac{mr_0^3}{2a}}$ **B.** $2\pi\sqrt{\frac{m{r_0}^3}{a}}$
C. $2\pi\sqrt{\frac{2mr_0^3}{a}}$ **D.** $4\pi\sqrt{\frac{mr_0^3}{a}}$

4. A planet of mass m moves in a circular orbit of radius r_0 in the gravitational potential $V(r) = -\frac{k}{r}$, where k is a positive constant. The orbit angular momentum of the planet is

[GATE 2014]

A. $2r_0$ km **B.** $\sqrt{2r_0 \text{ km}}$
C. $r_0 km$ **D.** $\sqrt{r_0 km}$

5. An interstellar object has speed v at the point of its shortest distance R from a star of much larger mass M. Given $v^2 = 2GM/R$, the trajectory of the object is

[GATE 2018]

A. circle **B.** ellipse
C. parabola **D.** hyperbola

Answer key

Q.No.	Answer	Q.No.	Answer
1	**b**	2	**b**
3	**a**	4	**d**
5	**c**		

Practice set 3

1. A particle moves under a central potential $V(r) = \frac{-k}{r^m}$. What should be value of m for its orbit to be stable.

Solution:

$$\text{Corresponding is } f(r) = -\frac{\partial V}{\partial r} = \frac{-mk}{r^{m+1}} = -mkr^{-(m+1)}$$
$$\text{We know that for } f = -kr^n \text{ condition for stability is } n > -3.$$

Therefore for orbit to be stable under given potential we must have.

$$-(m+1) > -3 \text{ or } m+1 < 3 \quad \therefore m < 2$$

2. Equation of the orbit of a particle moving under central force is $r\theta = \beta$, where β is a constant. Find the force acting on the particle.

Solution:

$$r\theta = \beta, \text{ therefore, } u = \frac{1}{r} = \frac{\theta}{\beta} \quad \therefore \frac{\partial^2 u}{\partial \theta^2} = 0$$
$$\text{Differential equation of orbit, } \frac{\partial^2 u}{\partial \theta^2} + u = \frac{-mf(r)}{L^2 u^2}$$
$$\therefore 0 + u = \frac{-mf(r)}{L^2 u^2} \quad \therefore \quad f(r) = \frac{-L^2 u^3}{m} = \frac{-L^2}{mr^3}$$

3. Equation of orbit of a particle moving under central force is $r^n = a\cos n\theta$. Find the force on the particle.

Solution:

$$\text{Given, } r^n = a\cos n\theta, \text{ therefore, } u^n = \frac{1}{a\cos n\theta} = \frac{1}{a}\sec n\theta \tag{2.48}$$
$$\text{Taking ln both sides we get, } n\ln u = \ln\left(\frac{1}{a}\right) + \ln\sec n\theta$$
$$\text{Differentiating w.r.t. } \theta \text{ we get, } \frac{n}{u}\frac{\partial u}{\partial \theta} = n\tan n\theta \quad \therefore \frac{\partial u}{\partial \theta} = u\tan n\theta$$
$$\begin{aligned}\text{Differentiating again w.r.t. } \theta \text{ we get } \frac{\partial^2 u}{\partial \theta^2} &= \frac{\partial u}{\partial \theta}\tan n\theta + un\sec^2 n\theta \\ &= u\tan^2 n\theta + un\sec^2 n\theta\end{aligned}$$
$$\text{Differential equation of orbit is } \frac{\partial^2 u}{\partial \theta^2} + u = \frac{-mf(r)}{L^2u^2}$$
$$u\tan^2 n\theta + un\sec^2 n\theta + u = \frac{-mf(r)}{L^2u^2}, u\sec^2 n\theta + un\sec^2 n\theta = \frac{-mf(r)}{L^2u^2}$$
$$\therefore f(r) = -\frac{L^2u^3(1+n)\sec^2 n\theta}{m}$$
$$\text{from (2.48) } \sec^2 n\theta = a^2u^{2n}$$
$$\therefore f(r) = \frac{-L^2(n+1)a^2u^{2n+3}}{m}$$

$$\therefore \quad f(r) = \frac{-L^2(n+1)a^2}{m} \cdot \frac{1}{r^{2n+3}}$$

Or $f(r) \propto \frac{1}{r^{2n+3}}$ and the force is attractive in nature.

4. For a particle moving under gravitational force pericentre distance in parabolic orbit is r_p while the radius of the circular orbit with same angular momentum is r_c. What is relation between r_p and r_c ?

Solution: Pericentre distance is the minimum distance $(\cos\theta = \max = 1)$ and for parabolic orbit $e = 1$.

$$\text{Therefore, } r_{\min} = \frac{l}{1+e\cos\theta} = \frac{l}{2} = r_p$$

$$\text{for circular orbit } e = 0\text{, therefore } r_c = \frac{l}{1+e\cos\theta} = l \quad \therefore r_p = \frac{r_c}{2}$$

5. Ratio of maximum to minimum speed of a planet revolving around the sun in an elliptical orbit is 2 : 1, What is eccentricity of the orbit?

Solution:

$$\text{Given } \frac{v_{\max}}{v_{\min}} = \frac{2}{1} \therefore \frac{\sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}}{\sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}} = \frac{2}{1} \quad \text{or} \quad \frac{1+e}{1-e} = \frac{2}{1} \quad \therefore e = \frac{1}{3}$$

6. A planet is revolving around the sun in a circular orbit. Due to some reason the speed of the planet suddenly becomes double. What is new orbit of the planet.

Solution:

$$\text{Orbital speed of the planet is } \sqrt{\frac{GM}{r}}$$

$$\text{New speed of the planet } = 2\sqrt{\frac{GM}{r}}$$

$$\text{Therefore, new energy of the planet } = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}m\cdot\frac{4GM}{r} - \frac{GMm}{r} = \frac{2GMm}{r} > 0$$

Total energy of the planet becomes positive on doubling its speed therefore new orbit of the planet will be hyperbolic.

7. Two masses constrained to move in a horizontal plane collide. Given initially $m_1 = 85\text{gms}, m_2 = 200\text{gms}; u_1 = 6.48\text{cms/sec}$ and $u_2 = -6.78\text{cms/sec}$, find the velocity of centre of mass.

Solution:

The velocity of centre of mass is given by

$$\vec{V}_{\text{cm}} = \frac{m_1\vec{u_1} + m_2\vec{u_2}}{m_1+m_2}$$

$$\vec{V}_{\text{cm}} = \frac{85\times 6.48 + 200\times(-6.78)}{85+200} = 2.82 \text{ cm/sec} \text{in the direction of motion of } m_2$$

8. Two particles each of mass 2kg are moving with velocities $2\vec{i}+4\vec{j}$m/s and $5\vec{i}+6\vec{j}$m/s respectively.Find the kinetic energy of the system relative to the center of mass.

Solution:

$$\text{Given } m_1 = m_2 = 2 \text{ kg.}; u_1 = 3\hat{i} + 4\hat{j}; \quad u_2 = 5\hat{i} + 6\hat{j}$$

$$\text{velocity of centre of mass} \quad \vec{V}_{cm} = \frac{m_1 \vec{u_1} + m_2 \vec{u_2}}{m_1 + m_2} = \frac{2(3\hat{i} + 4\hat{j}) + 2(5\hat{i} + 6\hat{j})}{2+2} = 4\hat{i} + 5\hat{j}$$

Velocity of m_1 in centre of mass frame

$$\vec{u}_1' = \vec{u}_1 - \vec{V}_{cm}$$
$$\vec{u}_1' = 3\hat{i} + 4\hat{j} - 4\hat{i} - 5\hat{j} = -\hat{i} - \hat{j}$$

Velocity of m_2 in centre of mass frame

$$\vec{u}_2' = \vec{u}_2 - \vec{V}_{cm}$$
$$\vec{u}_2' = 5\hat{i} + 6\hat{j} - 4\hat{i} - 5\hat{j}$$

Kinetic energy relative to centre of mass before collision

$$= \frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2$$
$$= \frac{1}{2} m_1 |\sqrt{(-1)^2 + (-1)^2}|^2 + \frac{1}{2} m_2 |\sqrt{(1)^2 + (1)^2}|^2$$
$$= 2 + 2 = 4 Joule$$

9. A particle of mass m_1 moving with a velocity $\vec{u_1}$ is elastically scattered from another particle of mass m_2. After collision the two particles move in opposite directions with the same speed. Find the relation between the two masses.

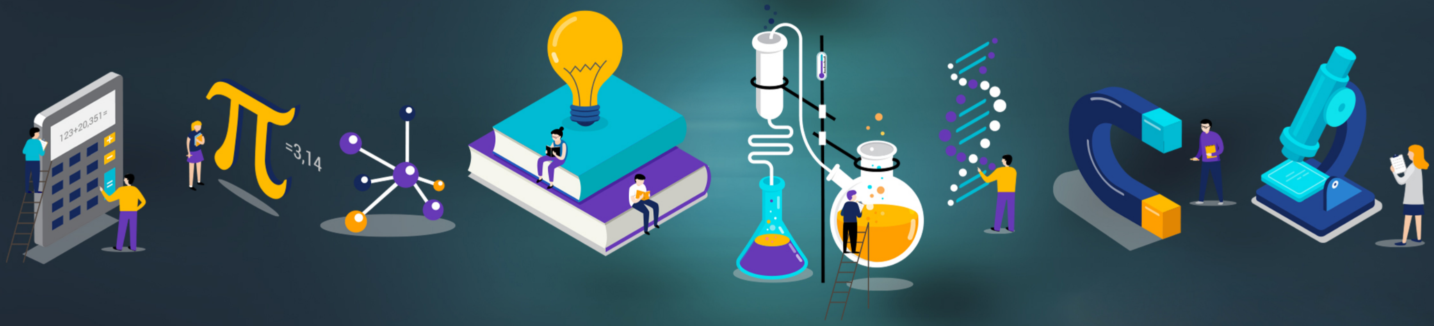
Solution: Ans. Let $\vec{v_1}$ and $\vec{v_2}$ be the velocities of the two particles after collision, then
$\vec{v_1} = -\vec{v_2}$
The particle of mass m_2 is at rest
$\therefore$ Linear momentum before collision $= m_1 \vec{u_1}$
+ Linear momentum after collision $= m_1 \vec{v_1} + m_2 verrightarrowv_2 = m_1 \vec{v_1} - m_2 \vec{v_1} = (m_1 - m_2) \vec{v_1}$
According to the law of conservation of linear momentum

$$m_1 \vec{u_1} = (m_1 - m_2) \vec{v_1}$$
$$\vec{v_1} = \frac{m_1}{m_1 - m_2} \vec{u_1}$$

Considering magnitudes only $|\vec{v_1}| = \frac{m_1}{m_1 - m_2} |\vec{u_1}|$
According to the law of conservation of energy

$$\frac{1}{2} m_1 |\vec{u_1}|^2 = \frac{1}{2} m_1 |\vec{v_1}|^2 + \frac{1}{2} m_2 |\vec{v_1}|^2$$
$$= \frac{1}{2}(m_1 + m_2) |\vec{v_1}|^2 = \frac{1}{2} \frac{m_1 + m_2}{(m_1 - m_2)^2} m_1^2 |\vec{u_1}|^2$$
$$(m_1 - m_2)^2 = (m_1 + m_2) m_1$$
$$m_1^2 + m_2^2 - 2m_1 m_2 = m_1^2 + m_1 m_2$$
$$m_2^2 = 3m_1 m_2 \quad \text{or} \quad m_2 = 3m_1$$

Futuring
crafting your future



3. Non Inertial frames and Fictitious force

3.1 Non- inertial frame

By now we were discussing inertial frames , in which Newton's second law of motion '$F = ma$' holds true . There are other frames of references in which Newton's law of inertia does not hold and are called non- inertial frames. All the accelerated and rotating frames are the non-inertial frames of reference.
In an accelerated frame , a force-free particle will seem to have an acceleration. If we do not consider the acceleration of the frame but apply Newton's laws to the motion of the force free-particle, then it will appear that a force is acting on it.

3.2 Fictitious or Pseudo force (Translational.)

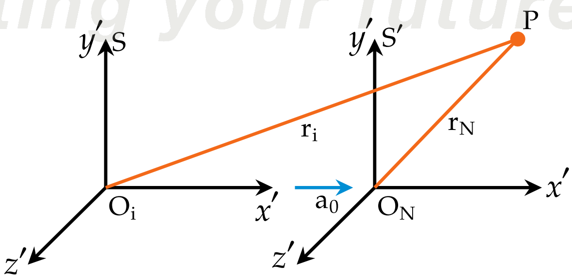


Figure 3.1

Let us consdier two frames S and S' where, S is an inertial frame and frame S' is moving with an acceleration a_0 relative to S. The acceleration of a particle P, on which no external force is acting, will be zero in the frame S. But in frame S' the observer will find that an acceleration - a_0 is acting on it. Thus, in frame S' the observed force on the particle is $-ma_0$, where m is the mass of the particle. **Such a force, which does not really act on the particle but appears due to the acceleration of the frame, is called a Fictitious or pseudo force**. The fictitious force on the particle P is

$$F_0 = -ma_0 \tag{3.1}$$

Hence the accelerated frame is non inertial. If a force F_i is applied on the particle and a_i is the observed acceleration in the inertial frame S, then according to Newton's law

$$F_i = ma_i \tag{3.2}$$

Since the non inertial frame S' is moving with an acceleration a_0, We can connect the position vectors of the two frames as,

$$\mathrm{r}_i = \mathrm{r}_n + \frac{1}{2}\mathrm{a}_0 t^2 \tag{3.3}$$

Differentiating twice we get,

$$\frac{d^2\mathrm{r}_i}{dt^2} = \frac{d^2\mathrm{r}_n}{dt^2} + \mathrm{a}_0 \tag{3.4}$$

$$\frac{d^2\mathrm{r}_i}{dt^2} = \mathrm{a}_i \quad (\text{ The acceleration in the inertial frame.})$$

$$\frac{d^2\mathrm{r}_n}{dt^2} = \mathrm{a}_n \quad (\text{ The acceleration observed in the non-inertial frame.})$$

$$a_i = a_n + a_0$$

$$a_i - a_0 = a_n \tag{3.5}$$

$$ma_i - ma_0 = ma_n \tag{3.6}$$

And using equations 3.1 and 3.2 we get

$$\mathrm{F}_n = \mathrm{F}_i + \mathrm{F}_0 \tag{3.7}$$

Thus the observer in the accelerated frame will measure the resultant (total) force which is the sum of real and fictitious forces on the particle i.e.,

$$\text{Total force} = \text{True force} + \text{Fictitious force} \tag{3.8}$$

3.2.1 Free fall of a body inside a box

Suppose that a box is falling in the gravitational field of the earth with an acceleration $\mathrm{a}_0 = -g\hat{\mathrm{n}}$, where g is the acceleration due to gravity and $\hat{n}$ is a unit vector in the upward direction. Now, if we consider a particle, falling freely inside the box, the fictitious force on the particle is $\mathrm{F}_0 = -m\mathrm{a}_0 = mg\hat{n}$. As the real force on the particle due to the attraction of the earth is $-mg\hat{\mathrm{n}}$, the force observed by the observer inside the box is

$$\mathrm{F}_n = \mathrm{F}_i + \mathrm{F}_0 = -mg\hat{\mathrm{n}} + mg\hat{\mathrm{n}} = 0 \tag{3.9}$$

Case-1: If the particle has no initial velocity relative to the box, it will seem to remain suspended in mid-air at the same place inside the box.

Case-2: Suppose that the box is moved with an acceleration $a_0 = g\hat{\mathrm{n}}$ in the upward direction relative to the ground. In such a case, the real force (F_i) and fictitious force (F_0) on the particle are given by,

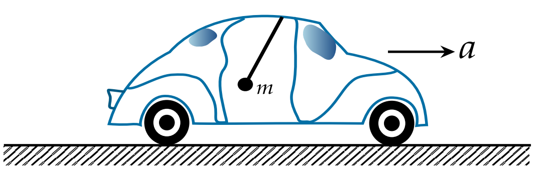
$$\mathrm{F}_i = -mg\hat{\mathrm{n}} \quad ; \quad \mathrm{F}_0 = -m\mathrm{a}_0 = -mg\hat{\mathrm{n}} \tag{3.10}$$

Then, the total force in the accelerated frame (box) is

$$\mathrm{F}_n = \mathrm{F}_i + \mathrm{F}_0 = -mg\hat{\mathrm{n}} - mg\hat{\mathrm{n}} = -2mg\hat{\mathrm{n}} \tag{3.11}$$

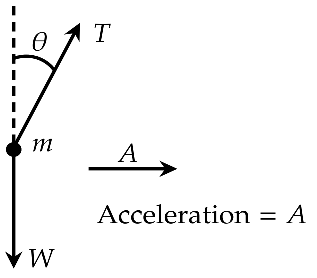
This means that the observer, stationed in the box having an acceleration g upward, will measure a force $2mg$ downward on the particle.

Exercise 3.1 The Apparent Force of Gravity:
A small weight of mass m hangs from a string in an automobile which accelerates at rate A. What is the static angle of the string from the vertical, and what is its tension? ■

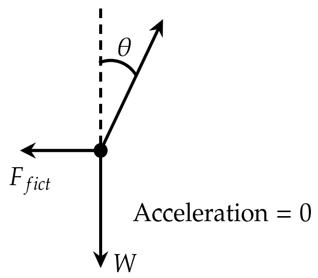


Solution: We shall analyze the problem both in an inertial frame and in a frame accelerating with the car.

Inertial System



System accelerating with auto.



$$\begin{aligned}
T\cos\theta - W &= 0\\
T\sin\theta &= mA\\
\tan\theta &= \frac{mA}{W}\\
&= \frac{A}{g}\\
T &= m\left(g^2+A^2\right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
T\cos\theta - W &= 0\\
T\sin\theta - F_{\text{fict}} &= 0\\
F_{\text{fict}} &= -mA\\
\tan\theta &= \frac{A}{g}\\
T &= m\left(g^2+A^2\right)^{1/2}
\end{aligned}$$

From the point of view of a passenger in the accelerating car, the fictitious force acts like a horizontal gravitational force. The effective gravitational force is the vector sum of the real and fictitious forces.

3.3 Centrifugal force

Let us consider a mass m rest in a non -inertial frame of reference, so that in this frame, the observed acceleration or the acceleration of the particle is zero. Now suppose that a frame is rotating with an angular velocity $\vec{\omega}$ relative to an inertial frame . In this noninertial (rotating) frame, the observed acceleration (a_n) of the mass m is zero.

$$\begin{aligned}
i.e.,\ a_n &= 0\\
\text{Then the total force, } \mathrm{F}_n = m\mathrm{a}_n &= 0\\
\mathrm{F}_i + \mathrm{F}_0 &= \mathrm{F}_n\\
-m\vec{\omega}^2\mathrm{r} + \mathrm{F}_0 &= 0\\
\text{Then, } \mathrm{F}_0 &= m\vec{\omega}^2\mathrm{r}
\end{aligned}$$

This fictitious force F_0 is directed away from the centre , along r, and is called the centrifugal force.

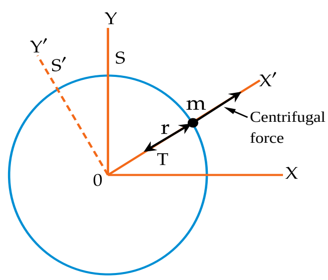


Figure 3.2

We know that the centrifugal force is a pseudo force and appears in the rotating frame due to its rotation . Here in the noninertial frame the centrifugal force is balanced by the inward tension in the string. In general, in the rotating frame, the centrifugal force is equal and opposite to the actual force and both are acting on the same particle.

3.4 Uniformely Rotating Frames

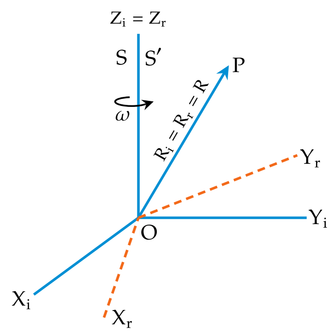


Figure 3.3

Suppose that a frame $S'(X_r, Y_r, Z_r)$ is rotating with an angular velocity $\vec{\omega}$ relative to an inertial frame $S(X_i, Y_i, Z_i)$. For simplicity, we assume that both of the frames have common origin O and common Z-axis.
The position vector of a particle P in both frames will be the same $^+$, i.e., $\mathrm{R}_i = \mathrm{R}_r = \mathrm{R}$, because the origins are coincident. Now, if the particle P is stationary in the frame S, the observer in the rotating frame S' will see that the particle is moving oppositely with linear velocity $-\vec{\omega} \times \mathrm{R}$.

If the velocity of the particle in the frame S,

$$v_i = \left(\frac{d\mathrm{R}}{dt}\right)_i \tag{3.12}$$

Then it's velocity in the rotating frame, S' ,

$$v_r = \left(\frac{d\mathrm{R}}{dt}\right)_r \tag{3.13}$$

It can be written as, $$\left(\frac{d\mathrm{R}}{dt}\right)_r = \left(\frac{d\mathrm{R}}{dt}\right)_i - \vec{\omega} \times \mathrm{R} \tag{3.14}$$

$$\left(\frac{d\mathrm{R}}{dt}\right)_i = \left(\frac{d\mathrm{R}}{dt}\right)_r + \vec{\omega} \times \mathrm{R} \tag{3.15}$$

In fact this equation holds for all vectors and relates the time derivatives of a vector in the frames Therefore, equation.3.15 may be written in the form of operator equation,

$$\left(\frac{d}{dt}\right)_i = \left(\frac{d}{dt}\right)_r + \vec{\omega} \times \mathrm{R} \tag{3.16}$$

Equation.3.15 can be written in terms of velocity as,

$$\mathrm{v}_i = \mathrm{v}_r + \vec{\omega} \times \mathrm{R} \qquad \text{Since, } \frac{d\mathrm{R}}{dt} = \mathrm{v} \tag{3.17}$$

Now, if we operate equation.3.16 on velocity vector v_i, we have

$$\left(\frac{d\mathrm{v}_i}{dt}\right)_i = \left(\frac{d\mathrm{v}_i}{dt}\right)_r + \vec{\omega} \times \mathrm{v}_i \tag{3.18}$$

Substituting the value of v_i in the right hand side of the equation.3.18 . We obtain,

$$\left(\frac{d\mathrm{v}_i}{dt}\right)_i = \left[\frac{d}{dt}(\mathrm{v}_r + \vec{\omega} \times \mathrm{R})\right]_r + \vec{\omega} \times (\mathrm{v}_r + \vec{\omega} \times \mathrm{R}) \tag{3.19}$$

$$= \left(\frac{d\mathrm{v}_r}{dt}\right)_r + \frac{d\vec{\omega}}{dt} \times \mathrm{R} + \vec{\omega} \times \left(\frac{d\mathrm{R}}{dt}\right)_r + \vec{\omega} \times \mathrm{v}_r + \vec{\omega} \times (\vec{\omega} \times \mathrm{R}) \tag{3.20}$$

If we write the acceleration ,

$$\frac{d\mathrm{v_r}}{dt} = \mathrm{a}_r \quad ; \quad \frac{d\mathrm{v_i}}{dt} = \mathrm{a}_i \quad \text{And} \quad \left(\frac{d\mathrm{R}}{dt}\right)_r = \mathrm{v}_r \tag{3.21}$$

$$\text{Then, } \mathrm{a}_i = \mathrm{a}_r + 2\vec{\omega} \times \mathrm{v}_r + \vec{\omega} \times (\vec{\omega} \times \mathrm{R}) + \frac{d\vec{\omega}}{dt} \times \mathrm{R} \tag{3.22}$$

$$m\mathrm{a_i} = m\mathrm{a_r} + 2m(\vec{\omega} \times \mathrm{v}_r) + m\vec{\omega} \times (\vec{\omega} \times \mathrm{R}) + m\frac{d\vec{\omega}}{dt} \times \mathrm{R} \tag{3.23}$$

$$\mathrm{F_i} = \mathrm{F_r} + m\frac{d^2\mathrm{R}}{dt^2} + m\vec{\omega} \times (\vec{\omega} \times \mathrm{r}) + 2m\vec{\omega} \times \mathrm{v} + m\frac{d\vec{\omega}}{dt} \times \mathrm{r} \tag{3.24}$$

$$\mathrm{F_r} = \mathrm{F_i} - m\frac{d^2\mathrm{R}}{dt^2} - m\vec{\omega} \times (\vec{\omega} \times \mathrm{r}) - 2m\vec{\omega} \times \mathrm{v} - m\frac{d\vec{\omega}}{dt} \times \mathrm{r} \tag{3.25}$$

$$\mathrm{F_r} = \mathrm{F_i} + \mathrm{F_{translation}} + \mathrm{F_{centrifugal}} + \mathrm{F_{Coriolis}} + \mathrm{F_{azimuthal}} \tag{3.26}$$

Rotation of Earth

In the case of earth, the common origin O may be considered as the centre of the earth, Z-axes as coinciding with its rotational axis and the frame S' as rotating with earth relative to the non-rotating frame S.

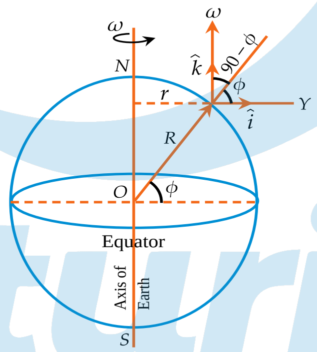


Figure 3.4: Rotation of earth

For earth, $\vec{\omega}$ is constant, so,

$$\frac{d\vec{\omega}}{dt} = 0$$

$$\text{Then, } \mathrm{a}_i = \mathrm{a}_r + 2\vec{\omega} \times \mathrm{v}_r + \vec{\omega} \times (\vec{\omega} \times \mathrm{R})$$

If m is the mass of the particle, then force in the rotating frame is,

$$m\mathrm{a}_r = m\mathrm{a}_i - 2m\vec{\omega} \times \mathrm{v}_r - m\vec{\omega} \times (\vec{\omega} \times \mathrm{R})$$

$$\text{But, } m\mathrm{a}_r = \mathrm{F}_i + \mathrm{F}_0$$

Therefore fictitious force F_0 is given by,

$$\mathrm{F}_0 = -2m\vec{\omega} \times \mathrm{v}_r - \vec{\omega} \times (\vec{\omega} \times \mathrm{R})$$

Here $-2m\vec{\omega} \times \mathrm{v}_r$ is the Coriolis force and $-m\vec{\omega} \times (\vec{\omega} \times \mathrm{R})$, the centrifugal force.

3.4.1 Azimuthal Force

Azimuthal Force appears to act on the particles which are being observed from a rotating frame which has non-uniform angular velocity i.e. $\frac{d\vec{\omega}}{dt} \neq 0$. It's direction is tangential to the rotation of frame.

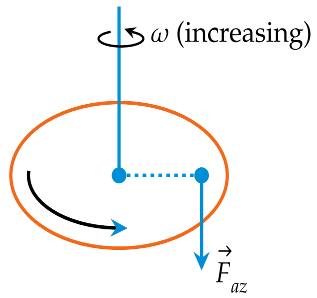


Figure 3.5: Azimuthal force.

$$\text{Azimuthal Force} = -m\frac{d\vec{\omega}}{dt} \times r = \vec{F}_{az} \tag{3.27}$$

The direction of azimuthal force on a particle lying on a non-uniformly rotating disc is shown in the figure .3.5

3.4.2 Centrifugal force

The centrifugal force is the only fictitious force, acting on a particle which is at rest ($\mathrm{v}_r = 0$) in the rotating frame. This force goes hand in hand with the ,$\frac{mv^2}{r} = mr\vec{\omega}^2$, centripetal acceleration as viewed by someone in an inertial frame. The centrifugal force may be written as,

$$-m\vec{\omega} \times (\vec{\omega} \times \mathrm{R}) = m\vec{\omega}^2 \mathrm{r}$$

Where, r is the vector from the axis of the earth to the particle and normal to it, because,

$$\begin{aligned}
\vec{\omega} \times (\vec{\omega} \times \mathrm{R}) &= (\vec{\omega} \cdot \mathrm{R})\vec{\omega} - (\vec{\omega} \cdot \vec{\omega})\mathrm{R} \\
&= \vec{\omega}^2 R \sin\phi \hat{\mathrm{k}} - \vec{\omega}^2 R(\hat{\mathrm{i}}\cos\phi + \hat{\mathrm{k}}\sin\phi) \qquad [\because \vec{\omega} = \vec{\omega}\hat{\mathrm{k}} \text{ and } \mathrm{R} = R(\hat{\mathrm{i}}\cos\phi + \hat{\mathrm{k}}\sin\phi)] \\
&= -\vec{\omega}^2 R\cos\phi \hat{\mathrm{i}} \\
&= -\vec{\omega}^2 \mathrm{r}
\end{aligned}$$

Effective gravity force ($\mathrm{mg_{eff}}$).

Consider a person standing motionless on the earth, at a polar angle θ. In the rotating frame of the earth, the person feels a centrifugal force (directed away from the axis) in addition to the gravitational force, mg. Note that we're using g to denote the acceleration due solely to the gravitational force. The sum of the gravitational and centrifugal forces doesn't point radially, unless the person is at the equator or at a pole. Let us denote the sum by $m\ \mathrm{g_{eff}}$. To calculate $m\ \mathrm{g_{eff}}$, we must calculate $\mathrm{F_{cent}} = -m\vec{\omega} \times (\vec{\omega} \times \mathrm{r})$. The $\vec{\omega} \times \mathrm{r}$ part has magnitude $R\vec{\omega}\sin\theta$, where R is the radius of the earth, and it is directed tangentially along the latitude circle of radius $R\sin\theta$. So $-m\vec{\omega} \times (\vec{\omega} \times \mathrm{r})$ points outward from the axis, with magnitude $mR\vec{\omega}^2\sin\theta$, which is just what we expect for something traveling at frequency $\vec{\omega}$ in a circle of radius $R\sin\theta$. Therefore, the effective gravitational force,

$$mg_{\text{eff}} \equiv m(g - \vec{\omega} \times (\vec{\omega} \times \mathrm{r}))$$

Figure 3.6: Effective Gravity force

3.4.3 Coriolis force: $-2m\vec{\omega} \times v_r$

Coriolis force is a fictitious force which acts on a particle only if it is in motion with respect to the rotating frame. In the rotating frame, if a particle moves with velocity v_r , then it's always experience a force, $(-2m\vec{\omega} \times v_r)$, perpendicular to it's path opposite to the direction of vector product, $\vec{\omega} \times v_r$.

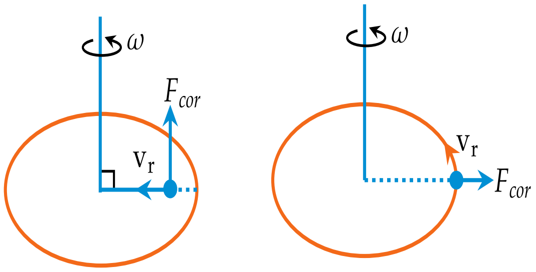


Figure 3.7: Coriolis force

$$\begin{aligned} \text{Coriolis force, } \vec{F}_{\text{cor}} &= -2m\vec{\omega} \times \vec{v}_r \\ \text{Or} \quad \vec{F}_{cor} &= 2m\vec{v}_r \times \vec{\omega} \end{aligned} \tag{3.28}$$

The classic Coriolis example is to imagine you are sitting on the north pole and just happen to have a Howizter handy. You fire the thing off and because the earth is turning under the projectile to you the shell is turning to the right. From the point of view of an observer in an inertial frame, the projectile follows a straight path. The angular deflection of the projectile must be ωt, the earth's rotation in time t. The effect of coriolis force is appreciable in the condition if it acts horizontal or has a horizontal componet because in the vertical direction it's effect is masked by the large gravitational force.

Practise set-3

1. **(a)** Given that earth rotates once every 23 h56 min around the axis from the North to South Pole, calculate the angular velocity, ω, of the earth. When viewed from above the North Pole, the earth rotates counter-clockwise (west to east). Which way does ω point?

   (b) Foucault's pendulum is a simple pendulum suspended by a long string from a high ceiling. The effect of Coriolis force on the motion of the pendulum is to produce a precession or rotation of the plane of oscillation with time. Find the time for one rotation for the plane of oscillation of the Foucault pendulum at 30° latitude.

   Solution:

   (a) ω points in the south to north direction along the rotational axis of the earth.

   $$\omega = \frac{2\pi}{T} = \frac{2\pi}{86,160} = 7.292 \times 10^{-5} \text{rad/s}$$

   (b) The period of rotation of the plane of oscillation is given by

   $$T' = \frac{2\pi}{\omega'} = \frac{2\pi}{\omega \sin\lambda} = \frac{T_0}{\sin\lambda} = \frac{24}{\sin 30^\circ} = 48 \text{ h}$$

2. An iceberg of mass 5×10^5 tons near the North Pole moves west at the rate of 8 km/ day. Neglecting the curvature of the earth, find the magnitude and direction of the Coriolis force.

Solution:

$$
\begin{aligned}
F_{\text{coriolis}} &= -2m\omega \times v_{\text{R}} \\
F_{\text{cor}} &= 2m\omega v_{\text{R}} \sin\theta \\
&= 2\times 5\times 10^{8}\times 7.27\times 10^{-5}\times \frac{8000}{86,400} \quad (\because \theta = 90^\circ) \\
&= 6730 \text{ N due north}
\end{aligned}
$$

3. A train of mass 1000 tons moves in the latitude 60° north. Find the magnitude and direction of the lateral force that the train exerts on the rails if it moves with a velocity of 15 m/s.

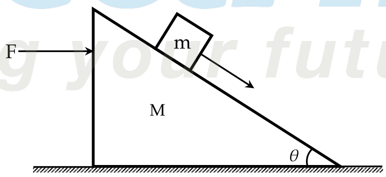
Solution:

$$
\begin{aligned}
F_{\text{cor}} &= 2mv\omega \sin\theta \\
&= 2\times 10^{6}\times 15\times 7.27\times 10^{-5} \sin 60^\circ \\
&= 1889 \text{ N on the right rail.}
\end{aligned}
$$

4. A train of mass m is travelling with a uniform velocity v along a parallel latitude. Show that the difference between the lateral force on the rails when it travels towards east and when it travels towards west is $4mv\omega\cos\lambda$, where λ is latitude and ω is the angular velocity of the earth.

Solution: The difference between the lateral forces on the rails arises because when the train reverses its direction of motion Coriolis force also changes its sign, the magnitude remaining the same. Therefore, the difference between the lateral force on the rails will be equal to $2mv\omega\cos\lambda - (-2mv\omega\cos\lambda)$ or $4mv\omega\cos\lambda$.

5. A small block of mass ' m ' lies on a wedge of mass M as shown in figure. All the contact surfaces are smooth. When a horizontal force F is applied to the wedge, the block does not slide on the wedge. What must be value of F.



Solution:

$$\text{Horizontal acceleration of the system is } a_0 = \frac{F}{M+m}$$

According to question, the block does not slide on the wedge, therefore if block is seen from the reference frame of the wedge it will appear stationary. The wedge has a linear acceleration a_0 therefore if observation is made from the wedge frame a pseudo force $-ma_0$ must be applied on the block as shown in figure. Block is stationary on the wedge so component of forces parallel to the inclined plane must cancel.

$$
\begin{aligned}
\text{Therefore, } mg\sin\theta &= ma_0\cos\theta \\
\text{or } \tan\theta &= \frac{a_0}{g} \\
&= \frac{F}{(M+m)g} \\
\text{or } F &= (M+m)g\tan\theta
\end{aligned}
$$

6. Assuming earth to be a sphere, calculate the linear speed of an object lying on the earth's surface at an altitude of $\lambda = 60^\circ$ and also calculate the centrifugal force experienced by an object of mass 50 kg. Compare this force with gravitational force. fictitious pset-3

Solution:

Angular velocity of earth about its axis due to its spinning motion is ,

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{(24 \times 60 \times 60)} \text{rad/sec}$$

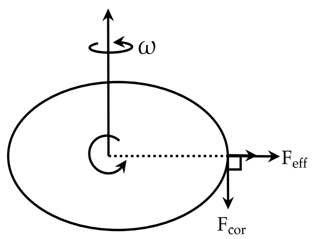
At latitude $\lambda = 60^\circ$ objects move in circle of radius $r = R\cos\lambda$

$$\begin{aligned}
\therefore \text{Linear speed } &= \omega r \\
&= \frac{2\pi}{24 \times 60 \times 60} \times 6400 \times 10^3 \cos 60^\circ \text{m/s} \\
&= \frac{2\pi \times 10^3}{27} \text{ m/s} \quad 233 \text{ m/s} \\
\text{Centrifugal force } &= m\omega^2 r \\
&= 50 \times \left(\frac{2\pi}{24 \times 60 \times 60}\right)^2 \times 6400 \times 10^3 \\
&\simeq 1.70 \text{ N} \\
\text{Gravitational force } &= mg = 50 \times 9.8 \\
&= 490 \text{ N}
\end{aligned}$$

Since centrifugal force due to spinning of earth is very small compared to the gravitational force due to this reason we do not feel the rotation of earth.

7. A person is standing at the edge of a disc of radius R. The dise is rotating about its axis with uniform angular velocity ω. The person throws a stone in radially outward direction with speed $\frac{\omega R}{2}$ relative for the disc. Calculate acceleration of stone as seen by the person soon after throwing. (neglect gravity).

Solution:



As seen by the person the stone is acted upon by centrifugal and coriolis forces. The direction of the two forces soon after throwing is shown in the figure. Net force on the stone is

$$F' = \sqrt{F_{cor}^2 + F_{cf}^2}$$

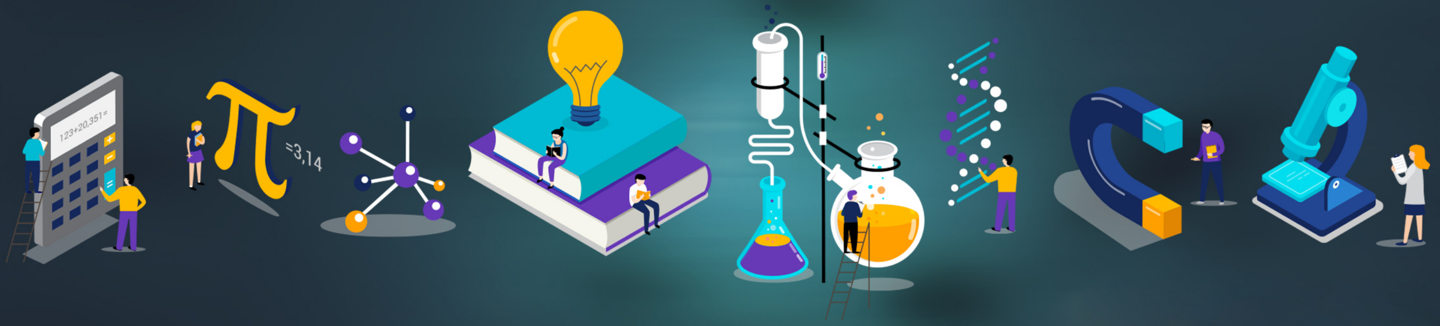
$$= \sqrt{|2m\vec{v}' \times \vec{\omega}|^2 + (m\omega^2 r_\perp)^2}$$

$$= \sqrt{\left(2m\frac{\omega R}{2}\omega \sin 90^\circ\right)^2 + (m\omega^2 R)^2}$$

$$F' = \sqrt{2}m\omega^2$$

Then the acceleration,

$$a' = \frac{F'}{m} = \sqrt{2}\omega^2 R$$



4. Rigid body Dynamics

4.1 Rigid body

A body is called a rigid body if the distance between any two points in the body does not change in time . i.e.,a rigid body do not stretch, compress, or shear. Rigid bodies, unlike point masses, can have forces applied at different points in the body.

4.2 Fixed axis rotations

The simplest motion of a rigid body is the rotation of a rigid body about an axis fixed in space. So the axis is neither translating nor rotating.

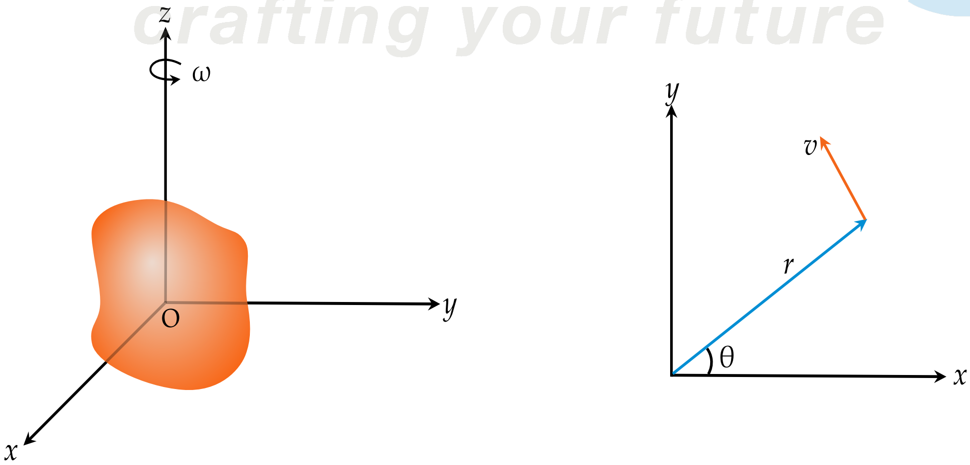


Figure 4.1: Fixed axis rotation of rigid body

The rigid body pivoted at the origin and rotates with angular speed ω around the z axis, in the counterclockwise direction (as viewed from above). Consider a little piece of the body, with mass dm and position (x,y). This little piece travels in a circle around the origin with speed

$$v = \omega r$$

$$\text{Where, } r = \sqrt{x^2+y^2} \qquad \because x = r\cos\theta \; ; \; y = r\sin\theta$$

Then, the angular momentum of this piece (relative to the origin) equals

$$\begin{aligned} \mathrm{J} &= \mathrm{r} \times \mathrm{p} \\ &= r(vdm)\hat{\mathrm{z}} \\ &= dmr^2\omega\hat{\mathrm{z}} \end{aligned}$$

Then the angular momentum of the entire body is therefore

$$\begin{aligned} \mathrm{J} &= \int r^2\omega\hat{\mathrm{z}}dm \\ &= \int \left(x^2+y^2\right)\omega\hat{\mathrm{z}}dm \end{aligned}$$

Where the integration runs over the area of the body. If the density of the object is constant, as is usually the case, then we have,

$$dm = \rho dxdy$$

If we define the moment of inertia around the z axis to be

$$\begin{aligned} I_z &= \int r^2dm \\ &= \int \left(x^2+y^2\right)dm \end{aligned}$$

Then the z component of the angular momentum

$$J_z = I_z\omega$$

And in general, the moment of inertia, $I_z = \sum_i m_i r_i^2$

$$\boxed{\begin{aligned} I &= \sum_i m_i r_i^2 \\ J &= I\omega \end{aligned}}$$

4.2.1 Torque

Torque about a point is defined as the rate of change of angular momentum about the same point.

$$\tau = \frac{dJ}{dt}$$

For a rigid body rotating about it's axis of symmetry

$$\begin{aligned} \tau &= \frac{d(I\omega)}{dt} \\ &= I\frac{d\omega}{dt} \end{aligned}$$

$\frac{d\omega}{dt}$ is the angular acceleration.

Case-1: If the axis of rotation and axis of symmetry of the bbody are not one. J and ω may not along the same direction. Then,

$$\tau_0 = \frac{d(I\omega)}{dt}$$

Case-2: If the axis of rotation fixed relative to the body . I will be constant. Then,

$$\tau_0 = I\frac{d\omega}{dt}$$

In the absence of external torque $\tau = 0$. Then the angular momentum about the axis of rotation is conserved.

$$I\omega = \text{Constant.}$$

4.3 Moment of Inertia tensor

particle P of the body, having the position vector r_i with respect to O, has an instantaneous velocity v_i relative to O, given by

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i \tag{4.1}$$

The angular velocity ω has components $\omega_x, \omega_y, \omega_z$ along x, y and z axes .

$$\omega = \omega_x \hat{\mathrm{i}} + \omega_y \hat{\mathrm{j}} + \omega_z \hat{\mathrm{k}} \tag{4.2}$$

Then the angular momentum is given by,

$$\mathrm{J}_p = \mathrm{r}_i \times m_i \mathrm{v}_i \quad \text{m}_i \text{ is the mass of the particle.} \tag{4.3}$$

Then the angular momentum ,

$$\mathrm{J} = \sum_{i=1}^{N} \mathrm{r}_i \times m_i \mathrm{v}_i \tag{4.4}$$
$$= \sum_i m_i \mathrm{r}_i \times (\omega \times \mathrm{r}_i) \tag{4.5}$$
$$= \sum_i m_i \left[(\mathrm{r}_i \cdot \mathrm{r}_i)\,\omega - (\mathrm{r}_i \cdot \omega)\,\mathrm{r}_i\right] \tag{4.6}$$
$$= \sum_i m_i \left[\mathrm{r}_i^2 \omega - (\mathrm{r}_i \cdot \omega)\,\mathrm{r}_i\right] \tag{4.7}$$

Whose direction is not along that of angular velocity If J_x, J_y, J_z are the components of angular momentum along X, Y, Z axes respectively, then

$$J_x = \sum_i m_i \left[r_i^2 \omega_x - (x_i\omega_x + y_i\omega_y + z_i\omega_z)\,x_i\right] \tag{4.8}$$

It can be written as,

$$\begin{aligned}
J_x &= \omega_x \sum_i m_i \left(r_i^2 - x_i^2\right) - \omega_y \sum_i m_i x_i y_i - \omega_z \sum_i m_i x_i z_i \\
J_y &= -\omega_x \sum_i m_i x_i y_i + \omega_y \sum_i m_i \left(r_i^2 - y_i^2\right) - \omega_z \sum_i m_i y_i z_i \\
J_z &= -\omega_x \sum_i m_i x_i z_i - \omega_y \sum_i m_i y_i z_i + \omega_z \sum_i m_i \left(r_i^2 - z_i^2\right)
\end{aligned} \tag{4.9}$$

(4.10)

$$\begin{aligned}
J_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\
J_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\
J_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z
\end{aligned} \tag{4.11}$$

Where,

$$
\begin{aligned}
I_{xx} &= \sum_i m_i\left(r_i^2 - x_i^2\right) = \sum_i m_i\left(y_i^2 + z_i^2\right)\\
I_{yy} &= \sum_i m_i\left(r_i^2 - y_i^2\right) = \sum_i m_i\left(\dot{x}_i^2 + z_i^2\right)\\
I_{zz} &= \sum_i m_i\left(r_i^2 - z_i^2\right) = \sum_i m_i\left(x_i^2 + y_i^2\right)
\end{aligned}
\tag{4.12}
$$

And,

$$
\begin{aligned}
I_{xy} &= -\sum_i m_i x_i y_i = I_{yx}\\
I_{xz} &= -\sum_i m_i x_i z_i = I_{zx}\\
I_{yz} &= -\sum_i m_i y_i z_i = I_{zy}
\end{aligned}
\tag{4.13}
$$

Then any component of the angular momentum vector can be written as,

$$J_\alpha = \sum_\beta I_{\alpha\beta}\,\omega_\beta \quad \text{Where } \alpha, \beta = x, y, z \tag{4.14}$$

$$\text{Or } \; J = I\omega \tag{4.15}$$

In matrix notation,

$$
\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{xz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{4.16}
$$

The nine elements $I_{xx}, I_{xy}, \ldots, I_{xz}$ of the (3×3) matrix may be regarded as components of a single entity I. This entity I is called inertia tensor. Since $I_{xy} = I_{yx}$ etc., I is a symmetric tensor.

$$
I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{xz} \end{bmatrix} \tag{4.17}
$$

4.4 Principle axes and Principle moment of inertia

If a body has symmetries with respect to some of the axis, then some of the products of inertia become zero and we can identify the principal axes. If a body is symmetric with respect to the plane $x' = 0$ then, we will have $I_{x'y'} = I_{y'x'} = I_{x'z'} = I_{z'x'} = 0$ and x' will be a principal axis.

If we choose the axes of the coordinate system fixed in the body with respect to which **off-diagonal elements disappear and only the diagonal elements remain in the inertia tensor, then such axes are called the principal axes** of the body and the corresponding moments of inertia as the principal moments of inertia. In general the directions of the principal axes are different to those of arbitrary axes fixed in the body. If x' , y' and z' are the principle axes then the principle moment of inertia can be written as,

$$
I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{4.18}
$$

Where, $I_{x'x'} = I_1, I_{y'y'} = I_2$ and $I_{zz'} = I_3$

If $\omega_1, \omega_2, \omega_3$ be the components of angular velocity and J_1, J_2, J_3 those of angular momentum about the principal axes, then for the principal axes we obtain the angular momenta as,

$$
\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{4.19}
$$

4.5 Parallel and perpendicular axes theorem

4.5.1 Theorem of Parallel Axis

Consider a rigid body of mass M undergoing fixed-axis rotation.

Theorem 4.5.1 According to the theorem of parallel axis, the moment of inertia of a body about any axis is equal to the sum of moment of inertia about a parallel axis through it's center of mass and the product of the mass of the body and the square of the distance between two axes.

$$I = I_{cm} + Md^2 \tag{4.20}$$

Where, $I =$ Moment of inertia about any axis.

$I_{cm} =$ Moment of inertia of the body about a parallel axis passing through it's center of mass.

$m =$ Mass of the body.

$d =$ Distance between the two axes.

4.5.2 Theorem of Perpendicular Axis

Theorem 4.5.2 According to the theorem of perpendicular axis, the moment of inertia of a plane lamina about an axis perpendicular to it's plane is equal to the sum of moments of inertia of the lamina about two axes at right angles to each other, in it's own plane, and intersecting each other at the point where the perpendicular axis passes through it.

If I_x and I_y be the moment of inertia of a plane lamina about x and y axes . Which lie in the palne of the lamina and mutually peroendicular to each other intersecting at the origin. The n the moment of inertia I about an axis which is passing through the origin and perpendicular to the oplane of the lamina is ,

$$I = I_x + I_y \tag{4.21}$$

4.6 Moment of Inertia of some simple objects

4.6.1 Moment of Inertia of a thin uniform rod

1. About an axis passing through the center of mass and perpendicular to it's length.

Consider a thin uniform rod of length L and mass m and uniform mass density λ. Choose Cartesian coordinates, with the origin at the center of mass of the rod, which is midway between the endpoints since the rod is uniform. Choose the x-axis to lie along the length of the rod, with the positive x-direction to the right, as in the figure.4.3

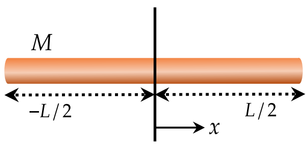


Figure 4.2: Uniform rod with an axis passing through the center.

Consider an infinitesimal mass element dm ,located at a displacement x from the center of the rod,,

$$dm = \lambda dx \qquad \text{Where, } \lambda = \frac{m}{L}$$

The moment of inertia of a continuos mass distribution is given by,

$$I = \int r^2 dm$$

$$
\begin{aligned}
\text{Then, } I &= \int_{-L/2}^{L/2} x^2 \frac{m}{L} dx \\
&= \frac{m}{L} \int_{-L/2}^{L/2} x^2 dx \\
&= \frac{m}{L} \left[\frac{x^3}{3} \right]_{-L/2}^{L/2} = \frac{m}{3L} \frac{2L^3}{8} \\
&= \frac{1}{12} mL^2
\end{aligned}
$$

2. About an axis passing through the endpoint.

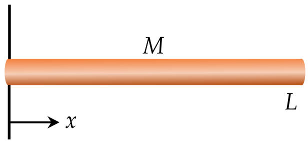


Figure 4.3: Uniform rod with an axis passing through the end point.

According to the theorem of parallel axis, the moment of inertia about any axis is given by,

$$I = I_{cm} + Md^2 \quad \text{Where } d \text{ is the distance from the axis to center of mass .}$$

$$
\begin{aligned}
\text{Here,} \quad d &= \frac{L}{2} \\
I &= \frac{1}{12} mL^2 + \frac{1}{4} mL^2 \\
&= \frac{1}{3} mL^2
\end{aligned}
$$

4.6.2 Moment of Inertia of a Uniform Disc

1. About an axis passing through the center of mass and perpendicular to the plane of the disc.

Figure 4.4: Uniform Disc with an axis passing through the center of mass

Choose cylindrical coordinates with the coordinates (r, θ) in the plane and the z-axis perpendicular to the plane. The area element,

$$
\begin{aligned}
da &= \rho d\rho d\theta \\
\sigma &= \frac{M}{A} = \frac{M}{\pi R^2}
\end{aligned}
$$

The infinitesimal mass

$$
\begin{aligned}
dm &= \frac{M}{A} da \\
&= \frac{M}{\pi R^2} \rho d\rho d\theta
\end{aligned}
$$

Then the moment of inertia ,

$$
\begin{aligned}
I &= \int \rho^2 \sigma da \\
&= \left(\frac{M}{\pi R^2}\right) \int \rho^2 da \\
&= \left(\frac{M}{\pi R^2}\right) \int_0^R \int_0^{2\pi} \rho^2 \rho d\rho d\theta \\
&= \left(\frac{2M}{R^2}\right) \int_0^R \rho^3 d\rho \\
&= \frac{1}{2} MR^2
\end{aligned}
$$

2. About an axis passing through the diameter.

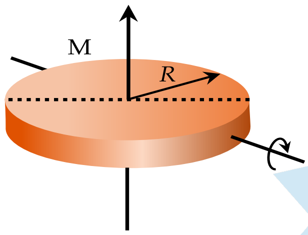


Figure 4.5: Uniform disc with an axis passing through the diameter.

By the theorem of perpendicular axis,

$$I_z = I_x + I_y$$

The moment of inertia of the disc is the same about any diameter. Let I be the moment of inertia about any diameter xx'. Then it will be the moment of inertia about a perpendicular diameter yy'

$$
\begin{aligned}
\text{Here, } I_z &= \frac{MR^2}{2} \\
\text{And , } I_x &= I_y = I \\
\frac{MR^2}{2} &= 2I \\
I &= \frac{MR^2}{4}
\end{aligned}
$$

4.6.3 Moment of inertia of a solid sphere .

1. About an axis passing through the diameter.

$$
\begin{aligned}
\text{The volume element, } dv &= r^2 dr \sin\theta d\theta d\phi \\
\text{The infinitesimal mass element, } dm &= \frac{M}{V} dv \\
&= \frac{M}{\frac{4}{3}\pi R^3} r^2 dr \sin\theta d\theta d\phi \\
\text{Here, } \; r_\perp &= r\sin\theta \\
I &= \int r_\perp^2 + dm \\
&= \iiint r^2 \sin^2\theta \cdot \frac{M}{\frac{4}{3}\pi R^3} r^2 dr \sin\theta d\theta d\phi \\
&= \frac{3M}{4\pi R^3} \int_0^R r^4 dr \int_0^\pi \sin^3\theta d\theta \int_0^{2\pi} d\phi
\end{aligned}
$$

$$= \frac{3M}{4\pi R^3} \cdot \frac{R^5}{5} \cdot \frac{4}{3} \cdot 2\pi$$
$$= \frac{2}{5} M R^2$$

2. About an axis passing through it's tangent.

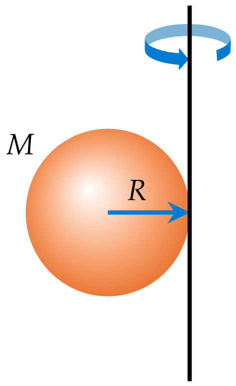


Figure 4.6: About an axis passing through it's tangent.

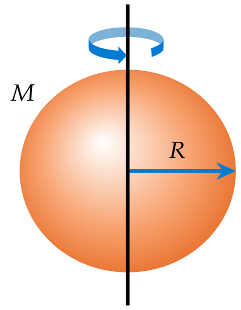


Figure 4.7: About an axis passing through it's diameter.

Any tangent to the sphere at any point is parallel to one of it's diameter and is at a distance equal to R from the centre. Now by the application of the theorem of parallel axis the moment of inertia becomes,

According to the theorem of parallel axis, the moment of inertia about any axis is given by,

$$I = I_{cm} + Md^2 \quad \text{Where } d \text{ is the distance from the axis to center of mass .}$$
$$\text{Here,} \quad d = R$$
$$I = \frac{2}{5} mR^2 + mR^2$$
$$= \frac{7}{5} mR^2$$

4.6.4 Moment of inertia of a spherical shell.

1. About an axis passing through it's diameter.

On the spherical shell the mass element is ,

$$da = R^2 \sin\theta d\theta d\phi$$
$$\sigma = \frac{M}{a}$$
$$dm = \sigma R \sin\theta d\theta R d\phi$$

where $\sigma = M/4\pi R^2$ is the surface mass density, and the distance from the rotational axis is $r = R\sin\theta$. Hence the moment of inertia to be calculated is

$$I = \int r_{\perp}^2 + dm$$
$$= \iint R^2 \sin^2\theta \cdot \frac{M}{4\pi R^2} R^2 \sin\theta d\theta d\phi$$
$$= \frac{MR^2}{4\pi} \int_0^{\pi} \sin^3\theta d\theta \int_0^{2\pi} d\phi$$
$$= \frac{MR^2}{4\pi} \cdot \frac{4}{3} \cdot 2\pi$$
$$= \frac{2}{3} MR^2$$

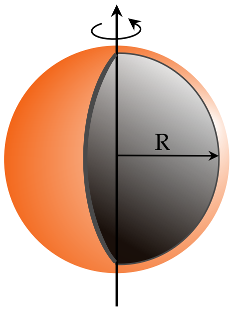


Figure 4.8: About an axis passing through it's diameter.

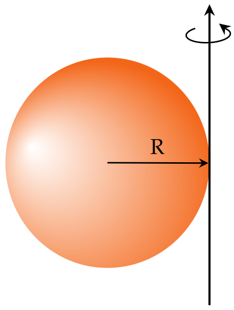


Figure 4.9: About an axis passing through it's tangent.

2. About an axis passing through it's tangent.
Any tangent to the sphere at any point is parallel to one of it's diameter and is at a distance equal to R from the centre. Now by the application of the theorem of parallel axis the moment of inertia becomes,

According to the theorem of parallel axis, the moment of inertia about any axis is given by,

$$I = I_{cm} + Md^2 \quad \text{Where } d \text{ is the distance from the axis to center of mass .}$$

$$\text{Here,} \quad d = R$$

$$I = \frac{2}{3}mR^2 + mR^2$$

$$= \frac{5}{3}mR^2$$

4.7 Kinetic energy of a rotating body

The kinetic energy of a rotating body about an axis depends not only upon it's mass and angular velocity but also depends upon the position of the axis and the distribution of mass about that axis. Suppose a body of mass M is rotating about an axis . Assume that the body is a system of small particles .

Let us consider a particle P of elementary mass m at a distance r from the axis.

$$\text{The angular velocity} = \omega$$

$$\text{The linear velocity} = r\omega$$

Then the kinetic energy of rotation,

$$\text{K.E} = \frac{1}{2}mr^2\omega^2$$

Hence the kinetic energy (K.E.) of rotation of the whole body is given by,

$$E = \Sigma\frac{1}{2}mr^2\omega^2$$

$$= \frac{1}{2}\omega^2\Sigma mr^2$$

$$\text{Now, } \Sigma mr^2 = I \quad \text{The moment of inertia about the given axis}$$

Then the kinetic energy of rotation,

$$E = \frac{1}{2}I\omega^2$$

$$\text{If } \omega = 1, \text{Obviously } I = 2E$$

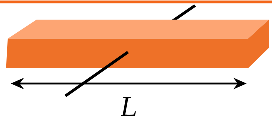
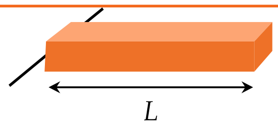
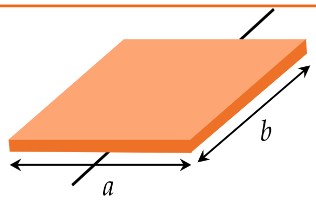
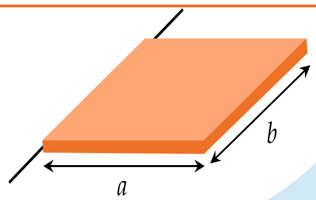
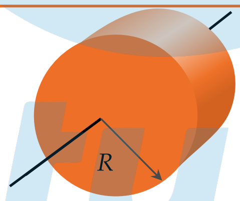
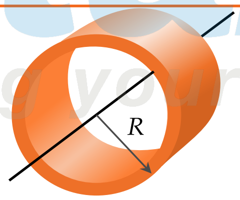
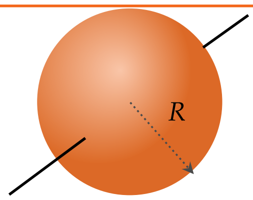
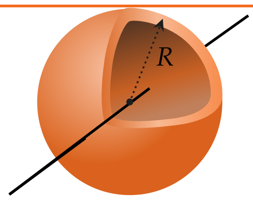
Hence the moment of inertia of a body may also be defined as twice its kinetic energy of rotation,

$$\text{But angular momentum} J = I\omega \quad \text{where } J \text{ is along the axis of rotation}$$

$$\therefore \quad J = I\sqrt{2E/I} = \sqrt{2EI}$$

$$\text{or } E = J^2/2I$$

This is the relation between rotational kinetic energy and angular momentum on the same aaxis

Center of mass of uniform systems		
Uniform body		Moment of Inertia
Thin Rod (About center)		$\frac{1}{12}ML^2$
Thin Rod (About End)		$\frac{1}{2}ML^3$
Rectangular plane (About center)		$\frac{1}{12}Ma^2$
Rectangular plane (About Edge)		$\frac{1Ma^2}{2}$
Cylinder		$\frac{1}{2}MR^2$
Cylindrical hoop		MR^2
Solid sphere(About diameter)		$\frac{2}{3}MR^2$
Spherical shell (About diameter)		$\frac{2}{5}MR^2$

Practice set 1

1. An annulus of mass M made of a material of uniform density has inner and outer radii a and b respectively. Its principle moment of inertia along the axis of symmetry perpendicular to the plane of the annulus is:
[NET DEC 2011]

A. $\frac{1}{2}M\frac{\left(b^4+a^4\right)}{\left(b^2-a^2\right)}$ **B.** $\frac{1}{2}M\pi\left(b^2-a^2\right)$

C. $\frac{1}{2}M\left(b^2-a^2\right)$ **D.** $\frac{1}{2}M\left(b^2+a^2\right)$

2. Two bodies of equal mass m are connected by a massless rigid rod of length l lying in the xy-plane with the centre of the rod at the origin. If this system is rotating about the z-axis with a frequency ω, its angular momentum is
[NET DEC 2012]

A. $ml^2\omega/4$ **B.** $ml^2\omega/2$

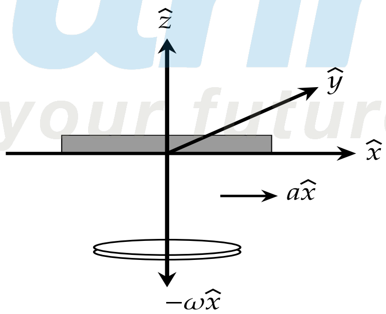
C. $ml^2\omega$ **D.** $2ml^2\omega$

3. Two masses m each, are placed at the points $(x,y)=(a,a)$ and $(-a,-a)$ and two masses, $2m$ each, are placed at the points $(a,-a)$ and $(-a,a)$. The principal moments of inertia of the system are
[NET DEC 2015]

A. $2m^2, 4ma^2$ **B.** $4ma^2, 8ma^2$

C. $4ma^2, 4ma^2$ **D.** $8ma^2, 8ma^2$

4. A disc of mass m is free to rotate in a plane parallel to the xy plane with an angular velocity $-\omega\hat{z}$ about a massless rigid rod suspended from the roof of a stationary car (as shown in the figure below). The rod is free to orient itself along any direction.



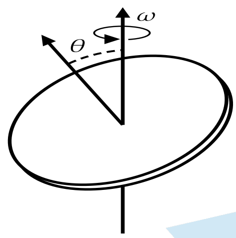
The car accelerates in the positive x-direction with an acceleration $a>0$. Which of the following statements is true for the coordinates of the centre of mass of the disc in the reference frame of the car?
[NET DEC 2017]

A. only the x and the z coordinates change **B.** only the y and the z coordinates change

C. only the x and the y coordinates change **D.** all the three coordinates change

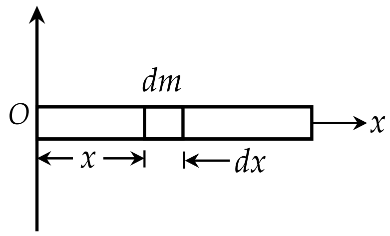
Practice set 2

1. A uniform solid cylinder is released on a horizontal surface with speed 5 m/s without any rotation (slipping without rolling). The cylinder eventually starts rolling without slipping. If the mass and radius of the cylinder are 10gm and 1 cm respectively, the final linear velocity of the cylinder is............. m/s. (up to two decimal places).
[GATE 2017]
2. A uniform circular disc of mass m and radius R is rotating with angular speed ω about an axis passing through its centre and making an angle $\theta = 30^\circ$ with the axis of the disc. If the kinetic energy of the disc is $\alpha m\omega^2R^2$, the value of α is (up to two decimal places).
[GATE 2018]



Practise set-3

1. Show that the centre of mass of a rod of mass M and length L lies midway between its ends, assuming the rod has a uniform mass per unit length.



Solution: By symmetry, we see that $y_{CM} = z_{CM} = 0$ if the rod is placed along the x axis. Furthermore, if we call the mass per unit length λ (the linear mass density), then $\lambda = M/L$ for a uniforr rod. If we divide the rod into elements of length dx, then the mass of each element is dm =?dx. Since an arbitrary element of each element is at a distance x from the origin, equation gives,

$$\begin{aligned} x_{CM} &= \frac{1}{M}\int_0^L x\,dm \\ &= \frac{1}{M}\int_0^L x\lambda\,dx \\ &= \frac{\lambda L^2}{2M} \end{aligned}$$

$$\text{Because } \lambda = M/L \text{ this reduces to, } x_{CM} = \frac{L^2}{2M}\left(\frac{M}{L}\right) = \frac{L}{2}$$

One can also argue that by symmetry,

$$x_{CM} = \frac{L}{2}$$

2. *A* body of radius R and mass M is rolling horizontally without slipping with speed v, it then rolls *up* a hill to a maximum height h. If $h = 3v^2/4$ g,
(a) what is the moment of inertia of the body?
(b) what might be the shape of the body?

Solution:

$$\begin{aligned} \text{(a)} \quad K_{\text{total}} &= K_{\text{trans.}} + K_{\text{rot.}} = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}Mv^2 + \frac{1}{2}I\left(\frac{v^2}{R^2}\right) = \frac{v^2}{2}\left[M + \frac{I}{R^2}\right] \end{aligned}$$

When it rolls up a hill to height h, the entire kinetic energy is converted into potential energy Mgh

$$\text{Thus} \frac{v^2}{2}\left[M + \frac{I}{R^2}\right] = Mgh = Mg\left[3\frac{v^2}{4\,g}\right]$$

$$\text{or} \quad \left[M + \frac{I}{R^2}\right] = \frac{3}{2}M$$

$$\therefore I = \frac{MR^2}{2}$$

(b) The body may be a circular disc or a solid cylinder.

3. Let g be the acceleration due to gravity at earth's surface and K be the rotational kinetic energy of the earth. Suppose the earth's radius decrease by 2%, keeping all other quantities same, then

A. g decreases by 2% and K decreases by 4%

B. g decreases by 4% and k increases by 2%

C. g increases by 4% and K decreases by 4%

D. g decreases by 4% and K increases by 4%

Solution:

$$\text{We know that, } g = \frac{GM}{R^2}$$

$$\text{Differentiating, } \frac{dg}{g} = -\left(\frac{2dR}{R}\right)$$

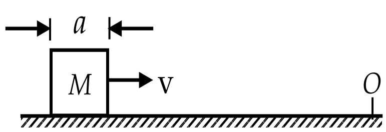
$$\text{Further, } K = \frac{1}{2}I\omega^2 = \frac{1}{2}\left[\frac{3}{5}MR^2\right]\omega^2$$

$$\text{or } \frac{dK}{K} = \frac{3}{10}M\omega^2 \times \left(\frac{2dR}{R}\right)$$

When radius decreases by 2%, then g increases by 4% and K decreases by 4%.

So the correct answer is **Option (C)**

4. A cubical block of side a is moving with velocity v on a horizontal smooth plane as shown in fig. It hits a ridge at point O. The angular speed of the block after it hits O is

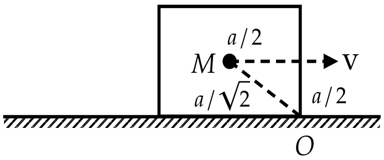


A. $3v/(4a)$ **B.** $3v/(2a)$ **C.** $\sqrt{3}v/(\sqrt{2}a)$ **D.** Zero

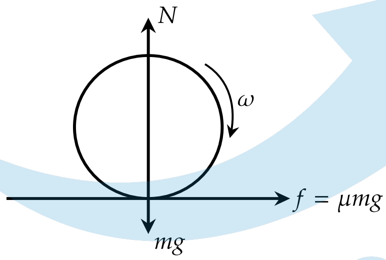
Solution:

By conservation of angular momentum, we have

$$I\omega = M \vee (a/2)$$
$$\text{Here, } I = \frac{Ma^2}{6} + M\left(\frac{a}{\sqrt{2}}\right)^2$$
$$= \frac{2Ma^2}{3}$$
$$\therefore \frac{2\text{Ma}^2}{3}\omega = \frac{\text{Mva}}{2}$$
$$\omega = \frac{3\text{v}}{4\text{a}}$$



5. A sphere is spinned in a clockwise direction by angular velocity ω and then it is released to a rough surface. Find the time elapsed by the sphere, till it starts pure rolling (coefficient of friction is μ) and radius of sphere is R.



Solution:

$$\mathbf{N} = \text{mg}$$

$\text{f} = \mu\text{mg}$ (friction opposes relative motion, so it is in forward direction)

$$a_{cm} = \frac{f}{m} = \mu g \tag{4.22}$$
$$\text{Also f} \times \text{R} = \text{I}\alpha$$
$$\Rightarrow \mu\text{mgR} = \text{I}\alpha \Rightarrow \alpha = \frac{\mu\text{mgR}}{\text{I}} = \frac{\mu\text{mgR}}{2/5\text{mR}^2} = \frac{5}{2}\frac{\mu\text{g}}{\text{R}}$$
$$\omega = \omega_o - \alpha\text{t} = \omega_o - \frac{5}{2}\frac{\mu\text{gt}}{\text{R}} \tag{4.23}$$

At the instant, m starts pure rolling

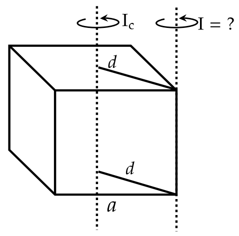
$$\omega R = v \text{ (lowest point is at rest)}$$
$$v_{cm} = 0 + \mu gt \tag{4.24}$$

[from eqns. (4.23) and (4.24) and $\omega R = v$]

$$\Rightarrow \omega\text{R} = \frac{7}{2}\mu\text{gt} \Rightarrow \text{t} = \frac{2}{7}\frac{\omega\text{R}}{\mu\text{g}}$$

6. A uniform solid cube has mass M and side ' a '. Calculate moment of inertia of the cube about an axis

which coincides with one of it's sides



Solution:

Moment of inertia of a cube about an axis passing through it's centre

$$I_c = \frac{Ma^2}{6}$$

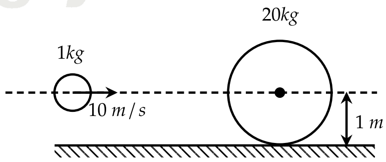
Seperation between two axes

$$d = \frac{a}{\sqrt{2}}$$

Then according to the parellel axis theorem

$$\begin{aligned} I &= I_c + Md^2 \\ &= \frac{Ma^2}{6} + M\left(\frac{a}{\sqrt{2}}\right)^2 \\ &= \frac{2Ma^2}{3} \end{aligned}$$

7. A 1 kg mass of clay, moving with a velocity of 10 m/s, strikes a stationary wheel and sticks to it. The solid wheel has a mass of 20 kg and a radius of 1 m.



Assuming that the wheel and the ground are both rigid and that the wheel is set into pure rolling motion, the angular velocity of the wheel immediately after the impact is, approximately,

a. Zero. **b.** $1/3$ rad/s

c. $\sqrt{10/3}$ rad/s **d.** $10/3$ rad/s

Solution: The moment of inertia of wheel of mass m and radius r w.r.t. a normal axis passing through circumference is

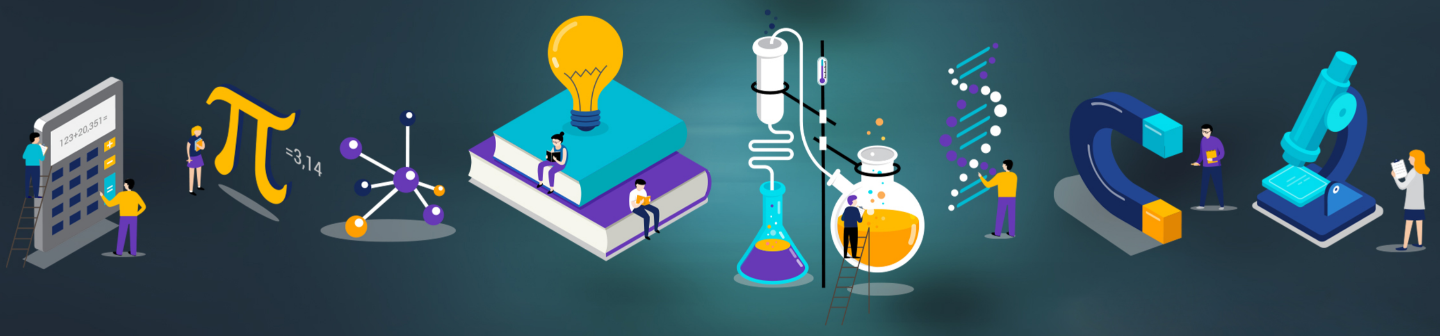
$$\begin{aligned} I &= \frac{mr^2}{2} + mr^2 \\ &= \frac{3}{2}mr^2 \end{aligned}$$

If ω is the velocity of pure rolling of wheel after the impact, total kinetic energy of both the masses before and after the impact will be equal, therefore,

$$\frac{1}{2}\times 1\times 10^2 = \frac{1}{2}\left(\frac{3}{2}\times 20\times 1^2\right)\omega^2$$

$$\omega = \sqrt{\frac{10}{3}}$$

So the correct answer is **option(c)**



5. Small oscillations

5.1 Potential energy and equilibrium

In order to understand the general theory of oscillations, it is essential to know about the potential energl at the equilibrium configuration. Let us consider a conservative system in which the potential energy is: function of position only. Let the system be specified by n generalized coordinates $q_1, q_2, \ldots, q_n$, not involve, time explicitly. For such a system, the potential energy is given by

$$V = V(q_1, q_2, \ldots, q_n)$$

and the generalized forces are given by $G_k = -\frac{\partial V}{\partial q_k}$ where $k = 1, 2, \ldots, n$
The system is said to be in equilibrium, if the generalized forces acting on the system are equal to zero: i.e.,

$$G_k = -\left[\frac{\partial V}{\partial q_k}\right]_0 = 0$$

For small oscillations
Force constant $k = \left.\frac{\partial^2 V}{\partial q_k^2}\right|_{q=q_0}$ where q_0 is stable equilibrium point.
Angular frequency $\omega = \sqrt{\frac{k}{m}}$

5.1.1 Stable,Unstable and Neutral equilibrium

Stable equilibrium

A system is said to be in stable equilibrium, if a small displacement of the system from the rest position (by giving a little energy to it) results in a small bounded motion about the equilibrium position.

Unstable equilibrium

Small displacement of the system from the equilibrium position results in an unbounded motion, it is in an unstable equilibrium.

Neutral equilibrium

Further, if the system on displacement has no tendency to move about or away the equilibrium position, it is said to be in neutral equilibrium.

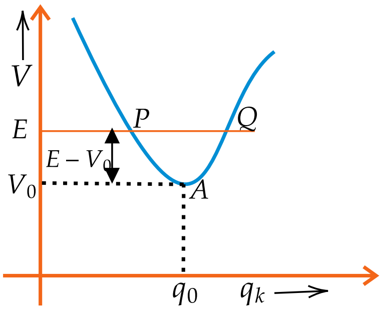


Figure 5.1

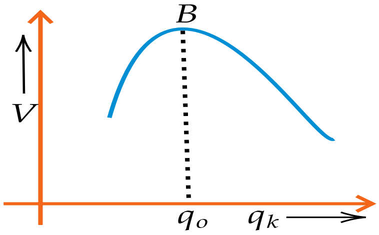


Figure 5.2

A graph drawn between the potential energy of the system and a particular coordinate q_k is called potential energy curve and The positions A and B, where the generalized force $F = -\partial V/\partial q$ vanishes, are the positions of equilibrium; potential energy V is minimum (say V_0) at A [fig.5.1] and maximum at B [fig.5.2]. Position A corresponds to the stable equilibrium, because if the system is displaced from A to Q by giving energy $(E - V_0)$ and left to itself, the system tries to come in the position of minimum potential energy. Consequently the potential energy will change to kinetic energy and at A the energy $(E - V_0)$ will be purely in the kinetic form because of the conservation law. This will change again to potential form, when the system moves towards the position P and hence a bounded motion ensues about the equilibrium position A. Obviously the position B of the maximum potential energy represents the unstable equilibrium because any energy given to the system at this position will result more and more kinetic energy when the system moves either left or right to it. In this case, the system moves away from the equilibrium position. In case of neutral equilibrium, the potential energy is independent of the coordinate and equilibrium. occurs at any arbitrary value of that coordinate.

5.2 Small oscillations

In small oscillation we generalize the harmonic oscillator problem of one degree of freedom in the lagrangian formulation to the case of small amplitude oscillations of a system of seversl degrees of freedom near the position of equilibrium.When we go from a single oscillator to the problem of two coupled oscillators the analysis results in some interesting and surprising new features.We shall see that the motion of the two coupled oscillators in general is much complicated and none of the oscillators in general executes simple harmonic motion.However for small amplitude oscillations ,we may express the general motion as a superposition of two independant simple harmonic motions ,both going on simultaneously.We call thsese two simple harmonic motions as normal modes or simply modes.Further we shall see that a system of N coupled oscillators with N degrees of freedom ,has exactly N independant modes of vibration and general motion can be expressed as the superposition of N normal modes.Each mode has its own frquency and wavelength.

5.2.1 Matrix method to solve small oscillations

We shall interested in the motion of the system with in the immediate neighbourhood of configuration of stable equilibrium .Since the departure fro the equilibrium is very small ,all functions may be expanded in a Taylor series about the equlibrium ,retaining only the lowest order terms .The deviation of the generalized coordinates from equilibrium will be denoted by η_i:

$$q_i = q_{oi} + \eta_i$$

and these may be taken as the new generalized coordinates of the motion

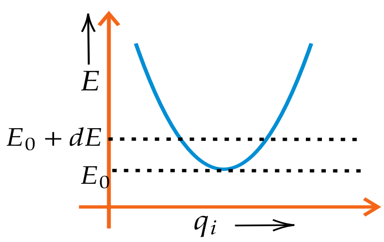


Figure 5.3

One can expand potential energy by Taylor series expansion

$$V(q_i \ldots\ldots q_n) = V(q_{00} \ldots\ldots q_{0n}) + \left.\frac{\partial V}{\partial q_i}\right|_{q_{i0}} (q_i - q_{0i}) + \frac{1}{2}\frac{\partial^2 V}{\partial q_i \partial q_j}(q_i - q_{0i})(q_j - q_{0j}) + \ldots.$$

The 0 in subscript means $q_i = q_{0i}$
So,

$$V(q) = V_0 + 0 + \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i \eta_j + \ldots\ldots \tag{5.1}$$

because at equilibrium,

$$\left(\frac{\partial V}{\partial q_i}\right)_{q_i = q_{0i}} \equiv \left(\frac{\partial V}{\partial q_i}\right)_0 = -Q_i = 0$$

where Q_i is the generalized force. Now,

$$V_0 \equiv V(q_{01}, q_{02}, \ldots\ldots, q_{0n}) \tag{5.2}$$

is the potential energy of eqilibrium configuration whis is constant.So we set V_0 is equal to zero because in equations of motions only derivative of V occur.
Neglecting higher order terms (because $\eta_i's$ are very small)

$$V = \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i \eta_j = \frac{1}{2} V_{ij} \eta_i \eta_j \tag{5.3}$$

We see that $V_{ij} = V_{ji}$ We can write V in matrix form as

$$V = \frac{1}{2}\begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_n \end{pmatrix} \begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1n} \\ V_{21} & V_{22} & \ldots & V_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ V_{n1} & V_{n2} & \ldots & V_{nn} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix} \tag{5.4}$$

Since the constraints are time independent, hence kinetic energy can be written as

$$T = \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2} m_{ij} \dot{\eta}_i \dot{\eta}_j \tag{5.5}$$

Since, $q_i = q_{0i} + \eta_i$, so $\dot{q}_i = \dot{\eta}_i$
m_{ij} can also be expanded similar to V

$$m_{ij} = m_{ij}(q_{01}, q_{02}, \ldots\ldots, q_{0n}) + \left(\frac{\partial m_{ij}}{\partial q_k}\right)_0 \dot{\eta}_k + \ldots\ldots\ldots$$

here we take first term only, because in equation 5.5 we already has $\dot{\eta}\dot{\eta}_j$.
so

$$T = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j \tag{5.6}$$

where, $\quad T_{ij} = m_{ij}(q_{01}, q_{02}, \ldots\ldots q_{0n})$
In matrix form,

$$T = \frac{1}{2}\begin{pmatrix} \dot{\eta}_1 & \dot{\eta}_2 & \ldots & \dot{\eta}_n \end{pmatrix} \begin{bmatrix} T_{11} & T_{12} & \ldots & T_{1n} \\ T_{21} & T_{22} & \ldots & T_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ T_{n1} & T_{n2} & \ldots & T_{nn} \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \vdots \\ \dot{\eta}_n \end{bmatrix} \tag{5.7}$$

Like $\mathbf{V}, \mathbf{T}$ is also symmetric, i.e. $T_{ij} = T_{ji}$ The Lagrangian is,

$$L = T - V = \frac{1}{2}(T_{ij}\dot{\eta}_i \dot{\eta}_j - V_{ij}\eta_i \eta_j) \tag{5.8}$$

The equation of motion for η_i

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right)-\frac{\partial L}{\partial \eta_i}=0$$

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\eta}_i}(T_{ij}\dot{\eta}_i\dot{\eta}_j-V_{ij}\eta_i\eta_j)\right)-\frac{\partial}{\partial \eta_i}(T_{ij}\dot{\eta}_i\dot{\eta}_j-V_{ij}\eta_i\eta_j)=0\\
&\frac{d}{dt}(T_{ij}\dot{\eta}_j)-V_{ij}\eta_j=0\\
&T_{ij}\ddot{\eta}_j+V_{ij}\eta_j=0 \qquad (5.9)
\end{aligned}$$

Note that T_{ij} and V_{ij} are constants and $\dot{\eta}'s$ and $\eta's$ appear only as multiplication with each other. For example the equation for η_1 is

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_1}\right)-\frac{\partial L}{\partial \eta_1}=0\\
&\frac{d}{dt}\left[\frac{\partial}{\partial \dot{\eta}_1}(T_{1j}\dot{\eta}_1\dot{\eta}_j)+\frac{\partial}{\partial \eta_1}(V_{1j}\eta_1\eta_j)\right]=0
\end{aligned}$$

[Since V is independent of $\dot{\eta}'s$ and T is independent of $\eta's$]. So,

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\eta}_1}(T_{11}\dot{\eta}_1\dot{\eta}_1+T_{12}\dot{\eta}_1\dot{\eta}_2+\ldots\ldots\ldots+T_{1n}\dot{\eta}_1\dot{\eta}_n+\ldots\ldots.)\right)\\
&+\frac{\partial}{\partial \eta_1}(V_{11}\eta_1\eta_1+V_{12}\eta_1\eta_2+\ldots\ldots.V_{1n}\eta_1\eta_n+\ldots\ldots)=0\\
&\frac{d}{dt}(T_{11}\dot{\eta}_1+T_{12}\dot{\eta}_2+\ldots\ldots T_{1n}\dot{\eta}_n)+(V_{11}\eta_1+V_{12}\eta_2+\ldots\ldots+V_{1n}\eta_n)=0\\
&T_{11}\ddot{\eta}_1+T_{12}\ddot{\eta}_2+\ldots..+T_{1n}\ddot{\eta}_n+(V_{11}\eta_1+V_{12}\eta_2+\ldots..+V_{1n}\eta_n)=0
\end{aligned}$$

In summation convention the above equation is

$$T_{1j}\ddot{\eta}_j+V_{1j}\eta_j=0 \qquad (5.10)$$

The point is that you should know what any expression means. So, we get η equations in total. To sovle the equation of motion η, we put a trial solution.

$$\eta_j=Ca_je^{-i\omega t} \qquad (5.11)$$

Note that in the exponential $i=\sqrt{-1}$ We get,

$$\begin{aligned}
&\dot{\eta}_j=-Ca_j(i\omega)e^{-i\omega t}\\
&\ddot{\eta}_j=+Ca_j(i\omega)^2e^{-i\omega t}=-\omega^2\eta_j \qquad (5.12)
\end{aligned}$$

putting $\ddot{\eta}_j$ in equation (5.9)

$$\begin{aligned}
&T_{ij}\ddot{\eta}_j+V_{ij}\eta_j=0\\
&T_{ij}\left(-\omega^2\eta_j\right)+V_{ij}\eta_j=0\\
&\left(V_{ij}-\omega^2T_{ij}\right)\eta_j=0 \qquad (5.13)\\
&V_{ij}a_j=\omega^2T_{ij}a_j \qquad (5.14)
\end{aligned}$$

in the matrix form equation (5.13), is (equation 5.14 also can be written in similar form)

$$\begin{bmatrix} V_{11} & V_{12} & \ldots & V_{1n}\\ V_{21} & V_{22} & \ldots & V_{2n}\\ \vdots & \vdots & & \vdots\\ V_{n1} & V_{n2} & \ldots & V_{nn}\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\\ \vdots\\ \eta_n\end{bmatrix}-\omega^2\begin{bmatrix} T_{11} & T_{12} & \ldots & T_{1n}\\ T_{21} & T_{22} & \ldots & T_{2n}\\ \vdots & \vdots & & \vdots\\ T_{n1} & T_{n2} & \ldots & T_{nn}\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\\ \vdots\\ \eta_n\end{bmatrix}=0$$

$$\begin{bmatrix} V_{11}-\omega^2T_{11} & V_{12}-\omega^2T_{12} & \ldots & V_{1n}-\omega^2T_{1n}\\ V_{21}-\omega^2T_{21} & V_{22}-\omega_2T_{22} & \ldots & V_{2n}-\omega^2T_{2n}\\ \vdots & \vdots & & \vdots\\ V_{n1}-\omega^2T_{n1} & V_{n2}-\omega^2T_{n2} & & V_{nn}-\omega^2T_{nn}\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\\ \vdots\\ \eta_n\end{bmatrix}=0 \qquad (5.15)$$

From properties of matrices we know that for equation (5.15) to be satisfied for all $\eta' s$. The determinant of matrix $\mathbf{V}-\omega^2\mathbf{T}$ must be zero i.e.

$$\left|\mathbf{V}-\omega^2\mathbf{T}\right|=0$$

$$\begin{vmatrix} V_{11}-\omega^2 T_{11} & V_{12}-\omega^2 T_{12} & \cdots & V_{1n}-\omega^2 T_{1n} \\ V_{21}-\omega^2 T_{21} & V_{22}-\omega_2 T_{22} & \cdots & V_{2n}-\omega^2 T_{2n} \\ \vdots & \vdots & & \vdots \\ V_{n1}-\omega^2 T_{n1} & V_{n2}-\omega^2 T_{n2} & & V_{nn}-\omega^2 T_{nn} \end{vmatrix}=0 \tag{5.16}$$

This equation (5.16) is called the secular equation. By solving the determinant (5.16) we get n values of ω^2. Each value of ω represents the frequency of normal mode. The equation (5.14) is a type of eigenvalues. $\mathbf{V}$ acting on eigenvector a gives $\omega^2\mathbf{Ta}$. There are n such eigenvectors. Hence, there are n normal modes. It can be shown that the eigen vector matrix of which is denoted by $\mathbf{A}$ diagonalizes both $\mathbf{T}$ and $\mathbf{V}$ (for detail see section 6.2 of Goldstein, 3$^{\text{rd}}$ eddition, but it is sufficient for us to rememher the result). A diagonalizes $\mathbf{V}$ to a matrix whose diagonal elements are the eigen- values i.e. $\omega^{2\prime} s$ (say $\lambda' s$)
(Note that in ordinary eigen alue problems, matrix acting on eigenvector produces eigenvalue times the eigenvector i.e. $\mathbf{Ma}=\lambda\mathbf{a}$ or $(\mathbf{M}-\lambda\mathbf{1})\mathbf{a}=0$, but here it is $(\mathbf{V}-\lambda\mathbf{T})\mathbf{a}=0$.)
ie

$$\tilde{\mathbf{A}}\mathbf{VA}=\lambda=\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 1, & \cdots & \lambda_n \end{bmatrix} \tag{5.17}$$

where $\lambda_1, \lambda_2 \ldots\ldots$ are eigen values of (i.e. $\omega^{21} s$)
And $\mathbf{T}$ is diagonalized to identity i.e.

$$\tilde{\mathbf{A}}\mathbf{TA}=\mathbf{1} \tag{5.18}$$

To solve a problem, the main thing we need to do is
(1) Solve the determinant (9.16) to find the frequencies of normal modes
(2) Find the eigenvectors a correspoinding to each ω^2 (the eigenvalue), in the equation.

$$\left(\mathbf{V}-\omega^2\mathbf{T}\right)\mathbf{a}=0$$

where the k^{th} eigenvector (corrersponding to k^{th} ω^2)

$$\mathbf{a}_k=\begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{bmatrix} \tag{5.19}$$

(3) The normal mode coordinates are

$$Q_k=f_k\cos\left(\omega_k t+\delta_k\right) \tag{5.20}$$

where, f_k and δ_k are amplitude and phase factor respectively. Note that in a normal mode, the hole system oscillates with same frequency. (4) The general solution for

$$\eta_i=a_{ik}Q_k \tag{5.21}$$

The matrix $\mathbf{A}$ is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{5.22}$$

So, the solution of our problem (the equation (5.21) is:

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{bmatrix} \tag{5.23}$$

One complication may arise when eigenvalues are degenerate. In this case we first choose an eigenvector (corresponding to degenerate eigenvalue) satisfying the eigenvalue equation and remaining eigenvectors (if there is k-fold degeneracy, then there are k-eigenvectors corresponding to single eigenvalue) are determined such that they satisfy the eigenvalue equation and are orthogonal to each other.

Practice set 1

1. A particle of unit mass moves in a potential $V(x) = ax^2 + \frac{b}{x^2}$, where a and b are positive constants. The angular frequency of small oscillations about the minimum of the potential is

[NET JUNE 2011]

A. $\sqrt{8b}$ **B.** $\sqrt{8a}$

C. $\sqrt{8a/b}$ **D.** $\sqrt{8b/a}$

2. Consider the motion of a classical particle in a one dimensional double-well potential $V(x) = \frac{1}{4}\left(x^2 - 2\right)^2$. If the particle is displaced infinitesimally from the minimum on the x-axis (and friction is neglected), then

[NET JUNE 2012]

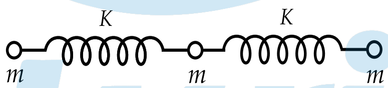
A. the particle will execute simple harmonic motion in the right well with an angular frequency $\omega = \sqrt{2}$

B. the particle will execute simple harmonic motion in the right well with an angular frequency $\omega = 2$

C. the particle will switch between the right and left wells

D. the particle will approach the bottom of the right well and settle there

3. Three particles of equal mass (m) are connected by two identical massless springs of stiffness constant (K) as shown in the figure



If x_1, x_2 and x_3 denote the horizontal displacement of the masses from their respective equilibrium positions the potential energy of the system is

[NET DEC 2012]

A. $\frac{1}{2}K\left[x_1^2 + x_2^2 + x_3^2\right]$ **B.** $\frac{1}{2}K\left[x_1^2 + x_2^2 + x_3^2 - x_2(x_1 + x_3)\right]$

C. $\frac{1}{2}K\left[x_1^2 + 2x_2^2 + x_3^2 - 2x_2(x_1 + x_3)\right]$ **D.** $\frac{1}{2}K\left[x_1^2 + 2x_2^2 - 2x_2(x_1 + x_3)\right]$

4. The time period of a simple pendulum under the influence of the acceleration due to gravity g is T. The bob is subjected to an additional acceleration of magnitude $\sqrt{3}g$ in the horizontal direction. Assuming small oscillations, the mean position and time period of oscillation, respectively, of the bob will be

[NET JUNE 2014]

A. 0° to the vertical and $\sqrt{3}T$ **B.** 30° to the vertical and $T/2$

C. 60° to the vertical and $T/\sqrt{2}$ **D.** 0° to the vertical and $T/\sqrt{3}$

5. A particle of mass m is moving in the potential $V(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$ where a, b are positive constants. The frequency of small oscillations about a point of stable equilibrium is

[NET DEC 2014]

A. $\sqrt{a/m}$ **B.** $\sqrt{2a/m}$

C. $\sqrt{3a/m}$ **D.** $\sqrt{6a/m}$

6. A particle of mass m, kept in potential $V(x) = -\frac{1}{2}kx^2 + \frac{1}{4}\lambda x^4$ (where k and λ are positive constants), undergoes small oscillations about an equilibrium point. The frequency of oscillations is

[NET JUNE 2018]

A. $\frac{1}{2\pi}\sqrt{\frac{2\lambda}{m}}$ **B.** $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$

C. $\frac{1}{2\pi}\sqrt{\frac{2k}{m}}$ **D.** $\frac{1}{2\pi}\sqrt{\frac{\lambda}{m}}$

Answer key

Q.No.	Answer	Q.No.	Answer
1	**b**	2	**b**
3	**c**	4	**c**
5	**b**	6	**c**

Practice set 2

1. A particle is placed in a region with the potential $V(x) = \frac{1}{2}kx^2 - \frac{\lambda}{3}x^3$, where $k, \lambda > 0$. Then, **[GATE 2010]**

   A. $x = 0$ and $x = \frac{k}{\lambda}$ are points of stable equilibrium

   B. $x = 0$ is a point of stable equilibrium and $x = \frac{k}{\lambda}$ is a point of unstable equilibrium

   C. $x = 0$ and $x = \frac{k}{\lambda}$ are points of unstable equilibrium

   D. There are no points of stable or unstable equilibrium

2. Two bodies of mass m and $2m$ are connected by a spring constant k. The frequency of the normal mode is **[GATE 2011]**

   A. $\sqrt{3k/2m}$ **B.** $\sqrt{k/m}$

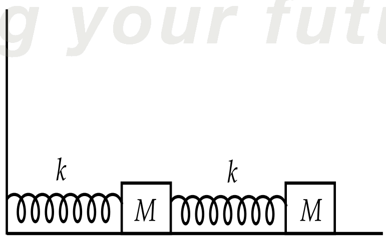
   C. $\sqrt{2k/3m}$ **D.** $\sqrt{k/2m}$

3. A particle of unit mass moves along the x-axis under the influence of a potential, $V(x) = x(x-2)^2$. The particle is found to be in stable equilibrium at the point $x = 2$. The time period of oscillation of the particle is **[GATE 2012]**

   A. $\frac{\pi}{2}$ **B.** π

   C. $\frac{3\pi}{2}$ **D.** 2π

4. Consider two small blocks, each of mass M, attached to two identical springs. One of the springs is attached to the wall, as shown in the figure. The spring constant of each spring is k. The masses slide along the surface and the friction is negligible. The frequency of one of the normal modes of the system is, **[GATE 2013]**



   A. $\sqrt{\frac{3+\sqrt{2}}{2}}\sqrt{\frac{k}{M}}$ **B.** $\sqrt{\frac{3+\sqrt{3}}{2}}\sqrt{\frac{k}{M}}$

   C. $\sqrt{\frac{3+\sqrt{5}}{2}}\sqrt{\frac{k}{M}}$ **D.** $\sqrt{\frac{3+\sqrt{6}}{2}}\sqrt{\frac{k}{M}}$

5. Two masses m and $3m$ are attached to the two ends of a massless spring with force constant K. If $m = 100$ g and $K = 0.3$ N/m, then the natural angular frequency of oscillation is Hz. **[GATE 2014]**

6. A particle of mass m is in a potential given by

$$V(r) = -\frac{a}{r} + \frac{ar_0^2}{3r^3}$$

   where a and r_0 are positive constants. When disturbed slightly from its stable equilibrium position it undergoes a simple harmonic oscillation. The time period of oscillation is **[GATE 2014]**

A. $2\pi\sqrt{\frac{mr_0^3}{2a}}$ **B.** $2\pi\sqrt{\frac{mr_0{}^3}{a}}$

C. $2\pi\sqrt{\frac{2mr_0^3}{a}}$ **D.** $4\pi\sqrt{\frac{mr_0^3}{a}}$

7. Two identical masses of 10gm each are connected by a massless spring of spring constant 1 N/m. The non-zero angular eigenfrequency of the system is. .rad/s. (up to two decimal places)
[GATE 2017]

8. In the context of small oscillations, which one of the following does NOT apply to the normal coordinates?
[GATE 2018]

A. Each normal coordinate has an eigen-frequency associated with it

B. The normal coordinates are orthogonal to one another

C. The normal coordinates are all independent

D. The potential energy of the system is a sum of squares of the normal coordinates with constant coefficients

Answer key

Q.No.	Answer	Q.No.	Answer
1	**b**	2	**a**
3	**b**	4	**c**
5	**0.318**	6	**a**
7	**14.14**	8	**b**

Practice set 3

1. A particle of mass m moves in one dimension under the influence of a potential energy

$$V(x) = -a\left(\frac{x}{\ell}\right)^2 + b\left(\frac{x}{\ell}\right)^4$$

where a and b are positive constants and ℓ is a characteristic length. The frequency of small oscillations about a point of stable equilibrium is:

a. $\frac{1}{2\pi\ell}\sqrt{\frac{b}{m}}$ **b.** $\frac{2b}{\pi\ell}\sqrt{\frac{1}{ma}}$

c. $\frac{1}{\pi\ell}\sqrt{\frac{a^2}{mb}}$ **d.** $\frac{1}{\pi\ell}\sqrt{\frac{a}{m}}$

Solution:

$$\text{We have,} V(x) = -a\left(\frac{x}{\ell}\right)^2 + b\left(\frac{x}{\ell}\right)^4$$

Here first we need to find the stable equilibrium. At equilibrium

$$\frac{\partial V(x)}{\partial x} = 0 \quad \Rightarrow \frac{-2ax}{\ell^2} + \frac{4bx^3}{\ell^4} = 0 \quad \Rightarrow x\left(\frac{2bx^2}{\ell^2} - a\right) = 0$$

$$\Rightarrow \quad x = 0, x = \pm\ell\sqrt{\frac{a}{2b}}$$

$$\text{Now,} \quad \frac{\partial^2 V}{\partial x^2} = \frac{-2a}{\ell^2} + \frac{12bx^2}{\ell^4}$$

$$\text{We see that for } x = 0, \frac{\partial^2 V}{\partial x^2} < 0 \text{hence, here there is unstable equilibrium}$$

$$\text{For} \quad x = \pm\ell\sqrt{\frac{a}{2b}}, \left(\frac{\partial^2 V}{\partial x^2}\right)_{x=\pm\ell\sqrt{\frac{a}{2b}}} = -\frac{2a}{\ell^2} + \frac{12b}{\ell^4}\ell^2 \times \frac{a}{2b} = \frac{4a}{\ell^2} > 0$$

hence, $x = \pm\ell\sqrt{\frac{a}{2b}}$ corresponds to stable equilibrium.. Now to find the frequeny, we see that the problemis one dimensional, so matrices $\mathbf{V}$ and $\mathbf{T}$ both have one element each, i.e. V_{11} and T_{11} respectively.

$$V_{11} = \left(\frac{\partial^2 V}{\partial x^2}\right)_{x=\pm f\sqrt{\frac{a}{2b}}} = \frac{4a}{\ell^2}$$

$$\text{Now, the kinetic energy } = \frac{1}{2}m\dot{x}^2, \text{ so } T_{11} = m$$

$$\text{Now, we have, } V_{11} - \omega^2 T_{11} = 0 \qquad \Rightarrow \omega^2 = \frac{V_{11}}{T_{11}} = \frac{4a}{m\ell^2} \qquad \Rightarrow \omega = \frac{2}{\ell}\sqrt{\frac{a}{m}}$$

$$\text{So, the frequency } \nu = \frac{\omega}{2\pi} = \frac{1}{\pi\ell}\sqrt{\frac{a}{m}}$$

Note: As we have seen in this problem, first we should find the equilibrium configuration of the system then expand potential energy about this configuration in Taylor series.

2. A particle of mass m is moving in a potential of the form $V(x,y,z) = \frac{1}{2}m\omega^2\left(3x^2 + 3y^2 + 2z^2 + 2x\right)$. The oscillation frequencies of the three normal modes of the particles are given by

Solution:

$$\text{The potential is, } V(x,y,z) = \frac{1}{2}m\omega^2\left(3x^2+3y^2+2z^2+2xy\right)$$

$$\text{Now, } \frac{\partial V}{\partial x} = 0 \quad \Rightarrow 6x+2y=0 \quad \Rightarrow y=-3x$$

$$\frac{\partial V}{\partial y} = 0 \quad \Rightarrow 6y+2x=0 \quad \Rightarrow y=\frac{-x}{3}$$

$$\frac{\partial V}{\partial z} = 0 \quad \Rightarrow z=0$$

$$\text{Now, } \frac{\partial^2 V}{\partial x^2} = 3m\omega^2 > 0 \text{for all } x$$

$$\frac{\partial^2 V}{\partial y^2} = 3m\omega^2 > 0 \text{ for all } y$$

$$\text{and } \frac{\partial^2 V}{\partial z^2} = 4 > 0 \text{ for all } z$$

All these conditions tell us that equilibrium point is $(0,0,0)$, because $y=-3x=-\frac{x}{3}$ satisfies only $x=y=0$ So, the given potential is in the form of expansion about $(0,0,0)$ and the matrix V_{ij} can be written just by inspection.

$$V = \frac{1}{2}m\omega^2\left(3x^2+3y^2+2z^2+2xy\right)$$

$$= \frac{1}{2}m\omega^2\begin{bmatrix} x & y & z\end{bmatrix}\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix} x \\ y \\ z\end{bmatrix}$$

$$\text{because } \quad V = \frac{1}{2}\left(3m\omega^2x^2+3m\omega^2y^2+2m\omega^2z^2+m\omega^2xy+m\omega^2yx\right)$$

$$\text{or, } \quad V = \frac{1}{2}\left(V_{11}x^2+V_{22}y^2+V_{33}z^2+V_{12}xy+V_{21}yx\right)$$

$$\text{Kinetic energy } T = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) = \frac{1}{2}\begin{bmatrix} \dot{x} & \dot{y} & \dot{z}\end{bmatrix}\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m\end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z}\end{bmatrix}$$

$$\text{Now, } \left|\mathbf{V}-\Omega^2\mathbf{T}\right| = 0 \quad \text{where } \Omega \text{ is normal mode freuency.}$$

$$\Rightarrow \begin{vmatrix} 3m\omega^2-\Omega^2 m & m\omega^2 & 0 \\ m\omega^2 & 3m\omega^2-\Omega^2 m & 0 \\ 0 & 0 & 2m\omega^2-\Omega^2 m\end{vmatrix} = 0$$

$$\Rightarrow \quad \left(2\omega^2-\Omega^2\right)\left[\left(3\omega^2-\Omega^2\right)^2-\omega^4\right] = 0 \quad \Rightarrow \Omega_1^2 = 2\omega^2$$

$$\text{and} \quad \Omega^2 = 3\omega^2 \pm \omega^2 \quad \Rightarrow \Omega_2^2 = 2\omega^2 \text{ and } \Omega_3^2 = 4\omega^2$$

So, the frequencies are $\omega\sqrt{2}, \omega\sqrt{2}$ and 2ω

Note that frequencies are always positive, hence we shouldn't write $\Omega_1 = \pm\omega\sqrt{2}$ etc.

3. The Lagrangian of a system is given by $L = \frac{1}{2}m\dot{q}_1^2 + 2m\dot{q}_2^2 - k\left(\frac{5}{4}q_1^2 + 2q_2^2 - 2q_1q_2\right)$ where m and k are positive constants. The frequencies of its normal modes are

a. $\sqrt{\frac{k}{2m}}, \sqrt{\frac{3k}{m}}$

b. $\sqrt{\frac{k}{2m}\left(13 \pm \sqrt{73}\right)}$

c. $\sqrt{\frac{5k}{2m}}, \sqrt{\frac{k}{m}}$ **d.** $\sqrt{\frac{k}{2m}}, \sqrt{\frac{6k}{m}}$

Solution:

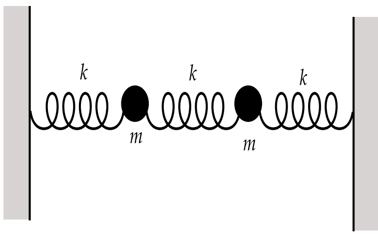
$$L = \frac{1}{2}m\dot{q}_1^2 + 2m\dot{q}_2^2 - k\left(\frac{5}{4}q_1^2 + 2q_2^2 - 2q_1q_2\right)$$
$$= \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}4m\dot{q}_2^2 - \frac{1}{2}k\left(\frac{5}{2}q_1^2 + 4q_2^2 - 4q_1q_2\right)$$
$$\hat{T} = \begin{pmatrix} m & 0 \\ 0 & 4m \end{pmatrix}, \hat{V} = \begin{pmatrix} \frac{5}{2}k & -2k \\ -2k & 4k \end{pmatrix}$$

For frequencies of normal modes:

$$\det\left|\omega^2\hat{T} - \hat{V}\right| = 0$$
$$\begin{vmatrix} \left(m\omega^2 - \frac{5}{2}k\right) & 2k \\ 2k & \left(4m\omega^2 - 4k\right) \end{vmatrix} = 0 \Rightarrow 4\left(m\omega^2 - k\right)\frac{\left(2m\omega^2 - 5k\right)}{2} - 4k^2 = 0$$
$$\Rightarrow 2m^2\omega^4 + 5k^2 - 7km\omega^2 - 2k^2 = 0$$
$$\Rightarrow 2\left(m\omega^2\right)^2 - 7k\left(m\omega^2\right) + 3k^2 = 0$$
$$\Rightarrow m\omega^2 = \frac{7k \pm \sqrt{49k^2 - 24k^2}}{4} = \frac{7k \pm \sqrt{49k^2 - 24k^2}}{4}$$
$$= \frac{7k \pm 5k}{4} = 3k, \frac{k}{2}$$
$$\therefore \omega = \sqrt{\frac{3k}{m}}, \sqrt{\frac{k}{2m}}$$

Correct answer is option **(a)**

4. Consider two masses m connected to each other and two walls by two springs as shown in figure.The three springs have the same sping constant k.If x_1 ,x_2 are generalized coordinates and displacement from mean position.



(a)Write down lagrangian of the system
(b)Write down equation of motion
(c)Write down secular equation and solve it for normal frequency.

Solution:

$$\text{(a) } L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}k\left(x_2 - x_1\right)^2 - \frac{1}{2}kx_2^2$$
$$\text{(b) } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \left(\frac{\partial L}{\partial x_1}\right) = 0 \Rightarrow m\ddot{x}_1 + kx_1 - k\left(x_2 - x_1\right) = 0$$
$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{x}_2}\right) - \left(\frac{\partial L}{\partial x_2}\right) = 0 \Rightarrow m\ddot{x}_2 + kx_2 + k\left(x_2 - x_1\right) = 0 \Rightarrow m\ddot{x}_2 + kx_2 + k\left(x_2 - x_1\right)$$

(c) $L=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2-\frac{1}{2}kx_1^2-\frac{1}{2}k\left(x_2-x_1\right)^2-\frac{1}{2}kx_2^2$

$T=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2$

$T=\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$

$V=\frac{1}{2}kx_1^2+\frac{1}{2}k\left(x_2-x_1\right)^2+\frac{1}{2}kx_2^2$

$V=\frac{1}{2}kx_1^2+\frac{1}{2}k\left(x_2^2+x_1^2-2x_1x_2\right)+\frac{1}{2}kx_2^2$

$V=\frac{1}{2}kx_1^2+\frac{1}{2}k\left(x_2^2+x_1^2-x_1x_2-x_2x_1\right)+\frac{1}{2}kx_2^2$

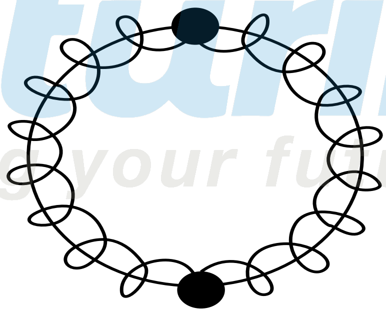
$=kx_1^2+kx_2^2+\frac{1}{2}k\left(-x_1x_2-x_2x_1\right)=\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}$

The secular equation is given by

$\left[V-\omega^2T\right]=0,\quad \begin{pmatrix} 2k-\omega^2m & -k \\ -k & 2k-\omega^2m \end{pmatrix}=0$

$\left(2k-\omega^2m\right)^2-k^2=0, \omega_1=\sqrt{\frac{k}{m}}$, which is normal frequency for oscillation and another value of $\omega_2=\sqrt{\frac{3k}{m}}$which is first over-tone.

5. Two identical particle of masses m are constrained to move on a horizontal loop .Two identical spring with constant k connected the mass and wrap around the loop.If x_1 and x_2 are displacements of first mass and second mass from equilibrium point.



(a)Write down lagrangian of the system
(b)Write down equation of motion
(c)Write down secular equation and solve it for normal frquency

Solution:

(a) $l=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2-\left(\frac{1}{2}k\left(x_2-x_1\right)^2+\frac{1}{2}k\left(x_1-x_2\right)^2\right)$

(b) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\left(\frac{\partial L}{\partial x_1}\right)=0\Rightarrow m\ddot{x}_1+2\dot{k}\left(x_1-x_2\right)=0$

$\frac{d}{dt}\left(\frac{\partial L}{\partial x_2}\right)-\left(\frac{\partial L}{\partial x_2}\right)=0\Rightarrow m\ddot{x}_2+2k\left(x_2-x_1\right)=0$

(c) $L=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2-\left(\frac{1}{2}k\left(x_2-x_1\right)^2+\frac{1}{2}k\left(x_1-x_2\right)^2\right)$

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 \Rightarrow T = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$I(x) = \left(\frac{1}{2}k\,(x_2 - x_1)^2 + \frac{1}{2}k\,(x_1 - x_2)^2\right) = \frac{1}{2}k\left(2x_1^2 + 2x_2^2 - 4x_1x_2\right)$$

$$V = \begin{pmatrix} 2k & -2k \\ -2k & 2k \end{pmatrix}$$

$$\left[V - \omega^2 T\right] = 0$$

$$\begin{pmatrix} 2k - \omega^2 m & -2k \\ -2k & 2k - \omega^2 m \end{pmatrix} = 0$$

$$\left(2k - \omega^2 m\right)^2 - (2k)^2 = 0$$

$$\left(\left(2k - \omega^2 m\right) + 2k\right)\left(\left(2k - \omega^2 m\right) - 2k\right) = 0$$

$\omega = \sqrt{\frac{4k}{ml}}$is normal frequency corresponding to oscillation and$\omega = 0$corresponding to translation motion.

Futuring

crafting your future