Practice set-1

1. Consider the matrix $M=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

A. The eigenvalues of M are

[NET/JRF(JUNE-2011)]

A. $0,1,2$ **B.** $0,0,3$ **C.** $1,1,1$ **D.** $-1,1,3$

Solution:

$$\text{For eigen values}\begin{bmatrix}1-\lambda&1&1\\1&1-\lambda&1\\1&1&1-\lambda\end{bmatrix}=0$$
$$(1-\lambda)\left((1-\lambda)^2-1\right)-(1-\lambda-1)+1(1-(1-\lambda))=0$$
$$(1-\lambda)\left(1+\lambda^2-2\lambda-1\right)+\lambda+\lambda=0\Rightarrow\lambda^2-2\lambda-\lambda^3+2\lambda^2+2\lambda=0$$
$$\lambda^3-3\lambda^2=0\Rightarrow\lambda^2(\lambda-3)=0\Rightarrow\lambda=0,0,3$$

For any $n\times n$ matrix having all elements unity eigenvalues are $0,0,0,\ldots,n$.

So the correct answer is **Option (B)**

B. The exponential of M simplifies to (I is the 3×3 identity matrix)

A. $e^M=I+\left(\frac{e^3-1}{3}\right)M$ **B.** $e^M=I+M+\frac{M^2}{2!}$

C. $e^M=I+3^3M$ **D.** $e^M=(e-1)M$

Solution:

We know that

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots..$$
$$e^M=1+M+\frac{M^2}{2!}+\frac{M^3}{3!}+\ldots..$$
$$M=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\Rightarrow M^2=\begin{bmatrix}3&3&3\\3&3&3\\3&3&3\end{bmatrix}=3M$$
$$\text{similarly}\quad M^3=9M=3^2M$$

we can rewrite e^M as,

$$e^M=I+M+\frac{3M}{2!}+\frac{3^2M}{3!}+\frac{3^3M}{4!}+\cdots$$
$$=I+\frac{M}{3}\left[3+\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\cdots\right]$$

$$= I + \frac{M}{3}\left[e^3 - 1\right]$$

2. A 3×3 matrix M has $\mathrm{Tr}[M]=6, \mathrm{Tr}\left[M^2\right]=26$ and $\mathrm{Tr}\left[M^3\right]=90$. Which of the following can be a possible set of eigenvalues of M?

[NET/JRF(DEC-2011)]

A. $\{1,1,4\}$ **B.** $\{-1,0,7\}$ **C.** $\{-1,3,4\}$ **D.** $\{2,2,2\}$

Solution:

$$\begin{aligned} T_r[M] &= \text{sum of eigen values} \\ T_r[M^2] &= \text{sum of square of eigen values} \\ \mathrm{Tr}\left[M^2\right] &= (-1)^2+(3)^2+(4)^2 \text{ also } \mathrm{Tr}\left[M^3\right] \\ &= (-1)^3+(3)^3+(4)^3 = 90 \end{aligned}$$

So the correct answer is **Option (C)**

3. The eigen values of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$ are

[NET/JRF(JUNE-2012)]

A. $(1,4,9)$ **B.** $(0,7,7)$ **C.** $(0,1,13)$ **D.** $(0,0,14)$

Solution:

The given matrix A has identical rows and columns

So it's eigen values are,

$$\lambda = 0,0 \text{ Trace} = 0,0,14$$

Another solution

$$\text{For eigenvalues } |A-\lambda I| = 0 \Rightarrow \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)[(4-\lambda)(9-\lambda)-36]-2[2(9-\lambda)-18]+3[12-3(4-\lambda)]=0$$

$$(1-\lambda)(4-\lambda)(9-\lambda)-36(1-\lambda)-4(9-\lambda)+36+9\lambda=0$$

$$\lambda^3-14\lambda^2=0 \Rightarrow \lambda^2(\lambda-14)=0 \Rightarrow \lambda=0,0,14$$

So the correct answer is **Option (D)**

4. The eigenvalues of the antisymmetric matrix,

$$A = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

where n_1, n_2 and n_3 are the components of a unit vector, are

[NET/JRF(JUNE-2012)]

A. $0, i, -i$ **B.** $0, 1, -1$ **C.** $0, 1+i, -1, -i$ **D.** $0, 0, 0$

Solution:

$$A = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \Rightarrow -A^T = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

$$(A - \lambda I) = 0, \begin{bmatrix} 0-\lambda & -n_3 & n_2 \\ n_3 & 0-\lambda & -n_1 \\ -n_2 & n_1 & 0-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \lambda_1 = 0 \Rightarrow \lambda_2 = -\sqrt{-n_1^2 - n_2^2 - n_3^2} \Rightarrow \lambda_3$$

$$= \sqrt{-n_1^2 - n_2^2 - n_3^2}$$

$$\text{but } \sqrt{n_1^2 + n_2^2 + n_3^2} = 1$$

For an antisymmetric matrix the eigen values are

$$\lambda = 0, \pm i$$

sum of sqares of non diagonal elements

$$= 0, \pm i\sqrt{n_1^2 + n_2^2 + n_3^2} = 0, \pm i$$

$\therefore n_1, n_2, n_3$are components of a unit vector

so, $\lambda_1 = 0, \lambda_2 = i, \lambda_3 = -i$

$A = -A^T$ (Antisymmetric). Eigenvalues are either zero or purely imaginary.

So the correct answer is **Option (A)**

5. Consider an $n \times n (n > 1)$ matrix A, in which A_{ij} is the product of the indices i and j (namely $A_{ij} = ij$). The matrix A

[NET/JRF(DEC-2013)]

A. Has one degenerate eigevalue with degeneracy $(n-1)$

B. Has two degenerate eigenvalues with degeneracies 2 and $(n-2)$

C. Has one degenerate eigenvalue with degeneracy n

D. Does not have any degenerate eigenvalue

Solution:

The matrix A will be like,

$$A_{ij} = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 4 & 6 & 8 & \cdots & \cdots \\ 3 & 6 & 9 & 12 & \cdots & \cdots \\ 4 & 8 & 12 & 16 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n & \vdots & \vdots & \vdots & & \end{bmatrix}$$

The matrix A is having identical rows and columns then it's eigen values will be, $(n-1)$ number of zeros and it's trace.

$$\lambda = 0, 0, \ldots\ldots, [1^2 + 2^2 + \ldots . n^2]$$

Thus the matrix has one degenerate eigen value with $n-1$ degeneracy

So the correct answer is **Option (A)**

6. Consider the matrix

$$M = \begin{pmatrix} 0 & 2i & 3i \\ -2i & 0 & 6i \\ -3i & -6i & 0 \end{pmatrix}$$

The eigenvalues of M are

[NET/JRF(JUNE-2014)]

A. $-5, -2, 7$ **B.** $-7, 0, 7$ **C.** $-4i, 2i, 2i$ **D.** $2, 3, 6$

Solution:

$$M = \begin{pmatrix} 0 & 2i & 3i \\ -2i & 0 & 6i \\ -3i & -6i & 0 \end{pmatrix}, M^+ = \begin{pmatrix} 0 & 2i & 3i \\ -2i & 0 & 6i \\ -3i & -6i & 0 \end{pmatrix}$$

$$M^+ = M$$

Matrix is Hermitian so roots are real and

$$\lambda = 0, \pm i\sqrt{(2i)^2 + (3i)^2 + (6i)^2} \quad \text{property of antisymmetric matrix}$$
$$= 0, \pm i\sqrt{-49}$$
$$= 0, \pm 7 = -7, 0, 7$$

So the correct answer is **Option (B)**

7. The column vector $\begin{pmatrix} a \\ b \\ a \end{pmatrix}$ is a simultaneous eigenvector of $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ if

[NET/JRF(DEC-2014)]

A. $b = 0$ or $a = 0$ **B.** $b = a$ or $b = -2a$

C. $b = 2a$ or $b = -a$ **D.** $b = a/2$ or $b = -a/2$

Solution:

Let $b = a$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ a \\ a \end{pmatrix} = \begin{pmatrix} a \\ a \\ a \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ a \\ a \end{pmatrix} = 2\begin{pmatrix} a \\ a \\ a \end{pmatrix}$$

Let $b = -2a$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ -2a \\ a \end{pmatrix} = \begin{pmatrix} a \\ -2a \\ a \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ -2a \\ a \end{pmatrix}$$
$$= \begin{pmatrix} -a \\ 2a \\ -a \end{pmatrix} = -1\begin{pmatrix} a \\ -2a \\ a \end{pmatrix}$$

For other combination above relation is not possible.

So the correct answer is **Option (B)**

8. The matrix $M = \begin{pmatrix} 1 & 3 & 2 \\ 3 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ satisfies the equation

[NET/JRF(DEC-2016)]

A. $M^3 - M^2 - 10M + 12I = 0$

B. $M^3 + M^2 - 12M + 10I = 0$

C. $M^3 - M^2 - 10M + 10I = 0$

D. $M^3 + M^2 - 10M + 10I = 0$

Solution:

he characteristic equation is

$$\begin{vmatrix} (1-\lambda) & 3 & 2 \\ 3 & (-1-\lambda) & 0 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = 0$$
$$\Rightarrow \quad (1-\lambda)(-1-\lambda)(1-\lambda) - (3)\times 3(1-\lambda) = 0$$
$$\Rightarrow \quad -\left(\lambda^2-1\right)(\lambda-1) - 9(1-\lambda) = 0$$
$$\Rightarrow \lambda^3 - 10\lambda - \lambda^2 + 10 = 0$$

Thus the matrix M satisfies the equation

$$M^3 - M^2 - 10M + 10I = 0$$

So the correct answer is **Option (C)**

9. Which of the following can not be the eigen values of a real 3×3 matrix

[NET/JRF(JUNE-2017)]

A. $2i, 0, -2i$

B. $1, 1, 1$

C. $e^{i\theta}, e^{-i\theta}, 1$

D. $i, 1, 0$

Solution:

If the matrix is real then the complex eigen values always occurs with its complex conjugate. In option (d) if i is an eigen value then $-i$ must also be an eigen value. But $-i$ is not given in option, hence option (d) is incorrect.

So the correct answer is **Option (D)**

10. Let $\sigma_x, \sigma_y, \sigma_z$ be the Pauli matrices and $x'\sigma_x + y'\sigma_y + z'\sigma_z = \exp\left(\frac{i\theta\sigma_z}{2}\right) \times$

$$[x\sigma_x + y\sigma_y + z\sigma_z]\exp\left(-\frac{i\theta\sigma_z}{2}\right)$$

Then the coordinates are related as follows

[NET/JRF(JUNE-2017)]

A. $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

B. $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

C. $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} & 0 \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

D. $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} & 0 \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Solution:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\text{Hence, } x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$$

$$x'\sigma_x + y'\sigma_y + z'\sigma_z = \begin{pmatrix} z' & x^1-iy' \\ x'+iy' & -z' \end{pmatrix}$$

$$\exp\left(\frac{i\theta\sigma_z}{z}\right) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \text{ and } \exp\left(\frac{-i\theta\sigma_z}{2}\right) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

$$\text{Hence, } \begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}\begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix} = \begin{pmatrix} z & e^{i\theta}(x-iy) \\ e^{-i\theta}(x+iy) & -z \end{pmatrix}$$

$$\text{Hence, } z' = z \text{ and } x'-iy' = e^{i\theta}(x-iy)$$

$$\text{Thus } x'-iy' = [(\cos\theta)x + (\sin\theta)y] - i[(\cos\theta)y - (\sin\theta)x]$$

$$\text{Thus } x' = (\cos\theta)x + (\sin\theta)y$$

$$\text{And } y' = (-\sin\theta)x + (\cos\theta)y$$

$$\text{Thus, } \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

So the correct answer is **Option (B)**

11. Let A be a non-singular 3×3 matrix, the columns of which are denoted by the vectors $\vec{a},\vec{b}$ and $\vec{c}$, respectively. Similarly, $\vec{u},\vec{v}$ and $\vec{w}$ denote the vectors that form the corresponding columns of $\left(A^T\right)^{-1}$. Which of the following is true?

[NET/JRF(DEC-2017)]

A. $\vec{u}\cdot\vec{a}=0, \vec{u}\cdot\vec{b}=0, \vec{u}\cdot\vec{c}=1$

B. $\vec{u}\cdot\vec{a}=0, \vec{u}\cdot\vec{b}=1, \vec{u}\cdot\vec{c}=0$

C. $\vec{u}\cdot\vec{a}=1, \vec{u}\cdot\vec{b}=0, \vec{u}\cdot\vec{c}=0$

D. $\vec{u}\cdot\vec{a}=0, \vec{u}\cdot\vec{b}=0, \vec{u}\cdot\vec{c}=0$

Solution:

We can take any 3×3 non singular matrix in order to avoid long calculation.

$$\text{Take } A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \\ \downarrow & \downarrow & \downarrow \\ \vec{a} & \vec{b} & \vec{c} \end{bmatrix} \Rightarrow \left(A^T\right)^{-1}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \\ \downarrow & \downarrow & \downarrow \\ \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$$

We see that

$$\vec{u}\cdot\vec{a}=1.1+0.0+0.0=1$$
$$\vec{u}\cdot\vec{b}=1.0+0.2+0.0=0$$
$$\vec{u}\cdot\vec{C}=1.0+0.0+0.3=0$$

So the correct answer is **Option (C)**

12. Consider the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & b & 2c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The condition for existence of a non-trivial solution and the corresponding normalised solution (upto a sign) is

[NET/JRF(DEC-2017)]

A. $b=2c$ and $(x,y,z)=\frac{1}{\sqrt{6}}(1,-2,1)$

B. $c=2b$ and $(x,y,z)=\frac{1}{\sqrt{6}}(1,1,-2)$

C. $c=b+1$ and $(x,y,z)=\frac{1}{\sqrt{6}}(2,-1,-1)$

D. $b=c+1$ and $(x,y,z)=\frac{1}{\sqrt{6}}(1,-2,1)$

Solution:

Solution: We know that the matrix equation, $AX=0$, where A is the given matrix and X is a column vector has a non-zero solution if and only if $|A|=0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & b & 2c \end{vmatrix}=0\Rightarrow 4c-3b-2c+6+b-4=0$$
$$\Rightarrow 2c-2b+2=0\Rightarrow b=c+1$$

we do not need to perform further calculation,

So the correct answer is **Option (D)**

13. Which of the following statements is true for a 3×3 real orthogonal matrix with determinant $+1$?

[NET/JRF(JUNE-2018)]

A. The modulus of each of its eigenvalues need not be 1, but their product must be 1

B. At least one of its eigenvalues is $+1$

C. All of its eigenvalues must be real

D. None of its eigenvalues must be real

Solution:

Solution: The characteristic equation of any 3×3 matrix is of thee form $\lambda^3+a\lambda^2+b\lambda+c=0$ which implies that at least one of the eigenvalues must be real. It is a proven fact that modulus of each eigenvalues of an orthogonal matrix is 1 .

If all eigenvalues of 3×3 orthogonal matrix are real then only possibilities for eigenvalues are

$$\lambda_1=1,\lambda_2=1 \text{ and } \lambda_3=1 \text{ or}$$
$$\lambda_1=-1,\lambda_2=-1,\lambda_3=1 \text{ or}$$
$$\lambda_1=-1,\lambda_2=1,\lambda_3=-1$$

Thus we see that at least one eigenvalue is $+1$. Suppose one eigenvalues is real and other two eigenvalues are complex conjugates. Now

$$\lambda_1\lambda_2\lambda_3=1$$
$$\Rightarrow \lambda_1(a+ib)(a-ib)=1 \Rightarrow \lambda_1\left(a^2+b^2\right)=1$$

Since a^2+b^2 is always positive hence $\lambda_1=1$

In this case also we see that at least one eigenvalue must be $+1$

So the correct answer is **Option (B)**

14. One of the eigenvalues of the matrix e^A is e^a, where $A=\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix}$. The product of the other two eigenvalues of e^A is

[NET/JRF(DEC-2018)]

A. e^{2a} **B.** e^{-a} **C.** e^{-2a} **D.** 1

Solution:

Eigenvalues of matrix A are a,a and $-a$. The product of two other eigenvalues of A are $e^a a^{-a}=1$

Alternativety

$$e^{\text{TraceA}}=e^{\lambda_1+\lambda_2+\lambda_3}=\det e^A$$
$$\Rightarrow e^{\lambda_1}\cdot e^{\lambda_2+\lambda_3}=\det e^A \Rightarrow e^a\cdot e^{\lambda_2}\cdot e^{\lambda_3}=e^a$$
$$\Rightarrow e^{\lambda_2}\cdot e^{\lambda_3}=1$$

So the correct answer is **Option (D)**

15. A 4×4 complex matrix A satisfies the relation $A^{\dagger}A = 4I$, where I is the 4×4 identity matrix. The number of independent real parameters of A is

[NET/JRF(DEC-2018)]

A. 32 **B.** 10 **C.** 12 **D.** 16

Solution:

$$\text{Given that } A^{\dagger}A = 4I \Rightarrow \frac{1}{4}\left(A^{\dagger}A\right) = I$$
$$\text{Let } A = 2B \text{ then}$$
$$A^{\dagger} = 2B^{\dagger}$$
$$\text{Therefore, } B^{\dagger}B = I$$

This shows that B is a unitary matrix. The number of independent real parameters needed to specify an $n\times n$ unitary matrix is n^2. Thus, the number of independent parameter needed to specify matrix B is $4^2 = 16$.

Now, the number of independent parameters needed to specify matrix A is same as that of matrix B.

Thus the number of independent parameters needed to specify A is 16

So the correct answer is **Option (D)**

16. The element of a 3×3 matrix A are the products if its row and column indices $A_{ij} = ij$ (where $i, j = 1,2,3$). The eigenvalues of A are

[NET/JRF(JUNE-2019)]

A. $(7,7,0)$ **B.** $(7,4,3)$ **C.** $(14,0,0)$ **D.** $\left(\frac{14}{3},\frac{14}{3},\frac{14}{3}\right)$

Solution:

$$\text{Since } A_{ij} = ij (\text{where } i, j = 1,2,3,)$$
$$\text{We obtain the matrix } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$
$$\text{For calculating eigen values } \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{vmatrix} = 0$$

$(1-\lambda)[(4-\lambda)(9-\lambda)-36]-2[2(9-\lambda)-18]+3(12-3(4-\lambda)) = 0$

$\Rightarrow -\lambda^3+\lambda^2\cdot 14 = 0 \Rightarrow \lambda^2(-\lambda+14) = 0 \quad \Rightarrow \lambda = 0,0,14$

Also, directly for a 3x3 matrix we can write $(0, 0, \text{Trace of A})$ as Eigen values.

So the correct answer is **Option (C)**

17. The operator A has a matrix representation $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ in the basis spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In another basis spanned by $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, the matrix representation of A is

[NET/JRF(JUNE-2019)]

A. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ **B.** $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ **C.** $\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$ **D.** $\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$

Solution:

The given vector $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigen vectors of operator A,

Hence in this basis matrix A is represented by diagonal matrix D consisting of eigenvalues of matrix A on the main diagonal. Therefore,

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

So the correct answer is **Option (B)**

18. If the rank of an $n \times n$ matrix A is m, where m and n are positive integers with $1 \le m \le n$, then the rank of the matrix A^2 is

[NET/JRF(DEC-2019)]

A. m **B.** $m-1$ **C.** 2 m **D.** $m-2$

Solution:

$$\text{Let } \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{\substack{2\times 2 \\ n=2}} = A \quad m = 2$$

$$1 \le 2 \le 2 \quad 1 \le m \le n$$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} \quad m = 2$$

So the correct answer is **Option (A)**

19. The eigenvalues of the 3×3 matrix $M = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}$ are

[NET/JRF(JUNE-2020)]

A. $a^2+b^2+c^2, 0, 0$ **B.** $b^2+c^2, a^2, 0$

C. $a^2+b^2, c^2, 0$ **D.** $a^2+c^2, b^2, 0$

Solution:

$$M = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}$$

To make it simple, Let $a = 1, b = 1, c = 1$ so $\quad M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times 3}$

$$\Rightarrow \lambda = 3, 0, 0$$

So the correct answer is **Option (A)**

Practice set-2

1. The eigenvalues of the matrix $\begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are

[GATE 2010]

A. $5, 2, -2$ **B.** $-5, -1, -1$ **C.** $5, 1, -1$ **D.** $-5, 1, 1$

Solution:

The characteristic equation of the matrix $A, |A - \lambda I| = 0$

$$\Rightarrow |A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$
$$\Rightarrow (1-\lambda)\left[(2-\lambda)^2 - 9\right] = 0$$
$$\Rightarrow \lambda = 1, 2-\lambda = \pm 3$$
$$\Rightarrow \lambda = 5, 1, -1$$

So the correct answer is **Option (C)**

2. Two matrices A and B are said to be similar if $B = P^{-1}AP$ for some invertible matrix P.Which of the following statements is NOT TRUE?

[GATE 2011]

A. Det A = Det B **B.** Trace of A = Trace of B

C. A and B have the same eigenvectors **D.** A and B have the same eigenvalues

Solution:

If A and B be square matrices of the same type and if P be invertible matrix, then matrices A and $B = P^{-1}AP$ have the same characteristic roots.

Then, $B - \lambda I = P^{-1}AP - P^{-1}\lambda IP = P^{-1}(A - \lambda I)P$ where I is identity matrix.

$$\begin{aligned} |B - \lambda I| &= \left|P^{-1}(A-\lambda I)P\right| = \left|P^{-1}\right||A - \lambda I||P| \\ &= \left|A - \lambda I\right|\left|P^{-1}\right|\left|P\right| = |A - \lambda I|\left|PP^{-1}\right| \\ &= |A - \lambda I| \end{aligned}$$

Thus, the matrices A and $B \left(= P^{-1}AP\right)$ have the same characteristic equation and hence same characteristic roots or eigen values. Since, the sum of the eigen values of a matrix and product of eigen values of a matrix is equal to the determinant of matrix, hence third alternative is incorrect.

So the correct answer is **Option (C)**

3. A 3×3 matrix has elements such that its trace is 11 and its determinant is 36 . The eigenvalues of the matrix are all known to be positive integers. The largest eigenvalues of the matrix is

[GATE 2011]

A. 18 **B.** 12 **C.** 9 **D.** 6

Solution: We know that for any matrix
1. The product of eigenvalues is equals to the determinant of that matrix.
2. $\lambda_1+\lambda_2+\lambda_3+\ldots\ldots=$ Trace of matrix
$\lambda_1+\lambda_2+\lambda_3=11$ and $\lambda_1\lambda_2\lambda_3=36$. Hence, the largest eigen value of the matrix is 6.

So the correct answer is **Option (D)**

4. The number of independent components of the symmetric tensor A_{ij} with indices $i, j=1,2,3$ is

[GATE 2012]

A. 1 **B.** 3 **C.** 6 **D.** 9

Solution:

$$\text{For symmetric tensor, } A_{ij}=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$\because A_{12}=A_{21}, \quad A_{23}=A_{32}, A_{13}=A_{31},$ hence there are six independent components.

So the correct answer is **Option (C)**

5. The eigenvalues of the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ are

[GATE 2012]

A. $0,1,1$ **B.** $0,-\sqrt{2},\sqrt{2}$ **C.** $\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0$ **D.** $\sqrt{2},\sqrt{2},0$

Solution:

$$|A-\lambda I|=0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$=0 \Rightarrow -\lambda\left(\lambda^2-1\right)+\lambda=0 \Rightarrow \lambda$$

$$=0,+\sqrt{2},-\sqrt{2}$$

So the correct answer is **Option (B)**

6. The degenerate eigenvalue of the matrix $\begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$ is (your answer should be an integer)—

[GATE 2013]

Solution:

$$\begin{bmatrix} 4-\lambda & -1 & -1 \\ -1 & 4-\lambda & -1 \\ -1 & -1 & 4-\lambda \end{bmatrix}=0 \Rightarrow (2-\lambda)\begin{bmatrix} 1 & -1 & -1 \\ 0 & 5-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

$$= (2-\lambda)(5-\lambda)^2 = 0 \Rightarrow \lambda$$
$$= 2,5,5$$

7. The matrix

$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \text{ is}$$

[GATE 2014]

A. Orthogonal **B.** Symmetric **C.** Anti-symmetric **D.** Unitary

Solution:

$$\text{Unitary } A^{\dagger}A = I$$

So the correct answer is **Option (D)**

8. Let X be a column vector of dimension $n > 1$ with at least one non-zero entry. The number of non-zero eigenvalues of the matrix $M = XX^T$ is

[GATE 2017]

A. 0 **B.** n **C.** 1 **D.** $n-1$

Solution:

$$\text{Let } X = \begin{bmatrix} 0 \\ 0 \\ a \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ then} X^T = \begin{bmatrix} 0 & 0 & a \ldots & 0 \end{bmatrix}$$

Here, X is an $n \times 1$ column vector with the entry in the i th row equal to a. X^T is a row vector having entry in the i th column equal to a. Then, XX^T is an $n \times 1$ matrix having the entry in the i th row and i th column equal to a^2.

Hence

$$XX^T = \begin{array}{c} \begin{bmatrix} 0\;0\;0 \ldots 0 \ldots 0\;0 \\ 0\;0\;0 \ldots 0 \ldots 0\;0 \\ 0\;0\;0 \ldots 0 \ldots 0\;0 \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ 0\;0\;0 \ldots 0 \ldots 0\;0 \end{bmatrix} \begin{array}{l} \\ \\ \rightarrow i^{th}\ row \\ \\ \\ \\ \end{array} \\ \downarrow \\ i^{th}\ row \end{array}$$

Since this matrix is diagonal, its eigenvalues are $a^2, 0, 0 \ldots\ldots 0$. Hence, the number of non zero eigenvalues of the matrix XX^T is 1 .

So the correct answer is **Option (C)**

9. The eigenvalues of a Hermitian matrix are all

[GATE 2018]

A. Real **B.** Imaginary **C.** Of modulus one **D.** Real and positive

Solution: Eigenvalue of Hermitian matrix must be real.

So the correct answer is **Option (A)**

10. During a rotation, vectors along the axis of rotation remain unchanged. For the rotation matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$, the vector along the axis of rotation is

[GATE 2019]

A. $\frac{1}{3}(2\hat{i} - \hat{j} + 2\hat{k})$

B. $\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} - \hat{k})$

C. $\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$

D. $\frac{1}{3}(2\hat{i} + 2\hat{j} - \hat{k})$

Solution: So the correct answer is **Option (B)**

Answer key

Q.No.	Answer	Q.No.	Answer
1	**C**	2	**C**
3	**D**	4	**C**
5	**B**	6	**5**
7	**D**	8	**C**
9	**A**	10	**B**

Futuring

crafting your future