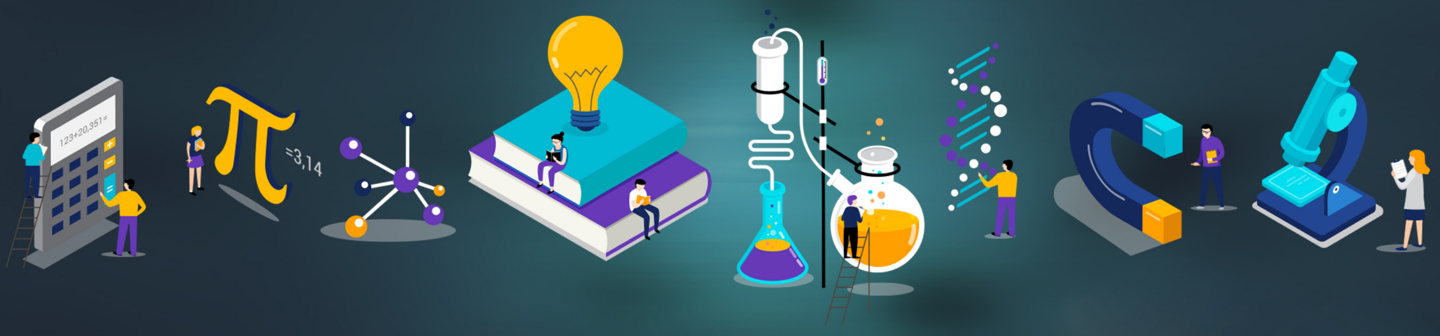


1. Eigen Value Problems

Practice Set-1

1. The energy of the first excited quantum state of a particle in the two-dimensional potential $V(x,y) = \frac{1}{2}m\omega^2\left(x^2+4y^2\right)$ is

[NET DEC 2011]

A. $2\hbar\omega$ **B.** $3\hbar\omega$

C. $\frac{3}{2}\hbar\omega$ **D.** $\frac{5}{2}\hbar\omega$

Solution: $V(x,y) = \frac{1}{2}m\omega^2\left(x^2+4y^2\right) = \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m4\omega^2y^2$,
$E = \left(n_x+\frac{1}{2}\right)\hbar\omega + \left(n_y+\frac{1}{2}\right)2\hbar\omega$
For ground state energy $n_x = 0, n_y = 0 \Rightarrow E = \frac{\hbar\omega}{2} + \frac{1}{2}2\hbar\omega = \frac{3\hbar\omega}{2}$
First exited state energy $n_x = 1, n_y = 0 \Rightarrow \frac{3\hbar\omega}{2} + \hbar\omega = \frac{5\hbar\omega}{2}$
The correct option is **(d)**

2. Let $|0\rangle$ and $|1\rangle$ denote the normalized eigenstates corresponding to the ground and first excited states of a one dimensional harmonic oscillator. The uncertainty Δp in the state $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$, is

[NET DEC 2011]

A. $\Delta p = \sqrt{\hbar m\omega}/2$ **B.** $\Delta p = \sqrt{\hbar m\omega/2}$

C. $\Delta p = \sqrt{\hbar m\omega}$ **D.** $\Delta p = \sqrt{2\hbar m\omega}$

Solution:

$$\text{Solution: } |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), p = i\sqrt{\frac{m\omega\hbar}{2}}\left(a^{\dagger} - a\right)$$

$$a^{\dagger}|\psi\rangle = \frac{1}{\sqrt{2}}(\sqrt{1}|1\rangle + \sqrt{2}|2\rangle) \text{ and } a|\psi\rangle = \frac{1}{\sqrt{2}}(0 + \sqrt{1}|0\rangle)$$

$$\langle p\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\left(\langle\psi|a^{\dagger} - a|\psi\rangle\right) = 0, p^2 = -\frac{m\omega\hbar}{2}\left(a^{\dagger 2} + a^2 - (2N+1)\right)$$

$$\langle p^2\rangle = \frac{-m\omega\hbar}{2}\left[\left\langle a^{+2}\right\rangle + \left\langle a^2\right\rangle - \langle 2N+1\rangle\right] = \frac{m\omega\hbar}{2}\langle 2N+1\rangle = \frac{m\omega\hbar}{2}\left(2\cdot\frac{1}{2}+1\right) = m\omega\hbar$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{m\omega\hbar}$$

The correct option is **(c)**

3. A particle of mass m is in a cubic box of size a. The potential inside the box $(0 \le x < a, 0 \le y < a, 0 \le z < a)$ is zero and infinite outside. If the particle is in an eigenstate of energy $E = \frac{14\pi^2\hbar^2}{2ma^2}$, its wavefunction is **[NET JUNE 2012]**

A. $\psi = \left(\frac{2}{a}\right)^{3/2} \sin\frac{3\pi x}{a} \sin\frac{5\pi y}{a} \sin\frac{6\pi z}{a}$

B. $\psi = \left(\frac{2}{a}\right)^{3/2} \sin\frac{7\pi x}{a} \sin\frac{4\pi y}{a} \sin\frac{3\pi z}{a}$

C. $\psi = \left(\frac{2}{a}\right)^{3/2} \sin\frac{4\pi x}{a} \sin\frac{8\pi y}{a} \sin\frac{2\pi z}{a}$

D. $\psi = \left(\frac{2}{a}\right)^{3/2} \sin\frac{\pi x}{a} \sin\frac{2\pi y}{a} \sin\frac{3\pi z}{a}$

Solution: $E_{n_x,n_y,n_z} = \left(n_x^2 + n_y^2 + n_z^2\right)\frac{\pi^2\hbar^2}{2ma^2} = \frac{14\pi^2\hbar^2}{2ma^2} \Rightarrow n_x^2 + n_y^2 + n_z^2 = 14 \Rightarrow n_x = 1, n_y = 2, n_z = 3.$
The correct option is **(d)**

4. A particle in one-dimension is in the potential $V(x) = \begin{cases} \infty & , \text{ if } x < 0 \\ -V_0 & , \text{ if } 0 \le x \le l \\ 0 & \text{ if } x > l \end{cases}$

If there is at least one bound state, the minimum depth of potential is **[NET JUNE 2012]**

A. $\frac{\hbar^2\pi^2}{8ml^2}$

B. $\frac{\hbar^2\pi^2}{2ml^2}$

C. $\frac{2\hbar^2\pi^2}{ml^2}$

D. $\frac{\hbar^2\pi^2}{ml^2}$

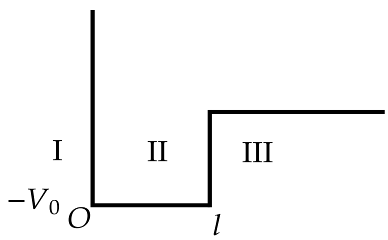


Figure 1.1

Solution: For bound state, $-V_0 < E < 0$

Wave function in region I, $\psi_I = 0, \psi_{II} = A\sin kx + B\cos kx, \psi_{III} = ce^{-\gamma x}$ where $k = \frac{\sqrt{2m(V_0+E)}}{\hbar^2}, \gamma = \frac{\sqrt{2m(-E)}}{\hbar^2}$.
Use Boundary condition at $x = 0$ and $x = l$
(wave function is continuous and differential at $x = 0$ and $x = l$), one will get $k\cot kl = -\gamma \Rightarrow kl\cot kl = -\gamma l \Rightarrow \eta = -\xi\cot\xi$
where $\gamma l = \eta, kl = \xi$.
$\Rightarrow \eta^2 + \xi^2 = \frac{2mV_0 l^2}{\hbar^2}$

For one bound state $\left(\frac{2mV_0l^2}{\hbar^2}\right)^{1/2} = \frac{\pi}{2} \Rightarrow V_0 = \frac{\pi^2\hbar^2}{8ml^2}$.

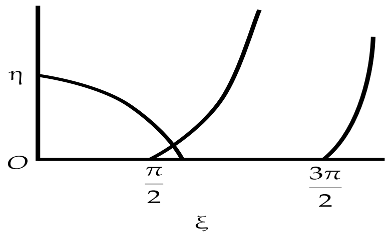


Figure 1.2

The correct option is **(a)**

5. The energy eigenvalues of a particle in the potential $V(x) = \frac{1}{2}m\omega^2x^2 - ax$ are

[NET DEC 2012]

A. $En = \left(n+\frac{1}{2}\right)\hbar\omega - \frac{a^2}{2m\omega^2}$

B. $En = \left(n+\frac{1}{2}\right)\hbar\omega + \frac{a^2}{2m\omega^2}$

C. $En = \left(n+\frac{1}{2}\right)\hbar\omega - \frac{a^2}{m\omega^2}$

D. $En = \left(n+\frac{1}{2}\right)\hbar\omega$

Solution: Hamiltonian (H) of Harmonic oscillator, $H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2x^2$
Eigenvalue of this, $E_n = \left(n+\frac{1}{2}\right)\hbar\omega$
But here, $H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2x^2 - ax \Rightarrow H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2\left[x^2 - \frac{2ax}{m\omega^2} + \frac{a^2}{m^2\omega^4}\right] - \frac{a^2}{2m\omega^2}$
$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2\left[x - \frac{a}{m\omega^2}\right]^2 - \frac{a^2}{2m\omega^2}$
Energy eigenvalue, $E_n = \left(n+\frac{1}{2}\right)\hbar\omega - \frac{a^2}{2m\omega^2}$
The correct option is **(a)**

6. A particle is in the ground state of an infinite square well potential is given by,

$$V(x) = \begin{cases} 0 & \text{for } -a \leq x \leq a \\ \infty & \text{otherwise} \end{cases}$$

The probability to find the particle in the interval between $-\frac{a}{2}$ and $\frac{a}{2}$ is

[NET DEC 2013]

A. $\frac{1}{2}$

B. $\frac{1}{2} + \frac{1}{\pi}$

C. $\frac{1}{2} - \frac{1}{\pi}$

D. $\frac{1}{\pi}$

Solution: The probability to find the particle in the interval between $-\frac{a}{2}$ and $\frac{a}{2}$ is

$$= \int_{-a/2}^{a/2} \sqrt{\frac{2}{2a}} \cdot \sqrt{\frac{2}{2a}} \cos\frac{\pi x}{2a} \cdot \cos\frac{\pi x}{2a} dx = \int_{-a/2}^{a/2} \frac{1}{a}\cos^2\frac{\pi x}{2a} dx = \frac{1}{a} \times \frac{1}{2}\left[\int_{-a/2}^{a/2}\left(1+\cos\frac{2\pi x}{2a}\right)dx\right]$$
$$= \frac{1}{2a}\left[x + \frac{a}{\pi}\sin\frac{\pi x}{a}\right]_{-a/2}^{a/2} = \frac{1}{2a}\left[\frac{a}{2}+\frac{a}{2}+\frac{a}{\pi}(1+1)\right] = \frac{1}{2a}\left[a+\frac{2a}{\pi}\right] = \left(\frac{1}{2}+\frac{1}{\pi}\right)$$

The correct option is **(b)**

7. A particle of mass m in the potential $V(x,y) = \frac{1}{2}m\omega^2\left(4x^2+y^2\right)$, is in an eigenstate of energy $E = \frac{5}{2}\hbar\omega$. The corresponding un-normalized eigen function is

[NET JUNE 2014]

A. $y\exp\left[-\frac{m\omega}{2\hbar}\left(2x^2+y^2\right)\right]$

B. $x\exp\left[-\frac{m\omega}{2\hbar}\left(2x^2+y^2\right)\right]$

C. $y\exp\left[-\frac{m\omega}{2\hbar}\left(x^2+y^2\right)\right]$

D. $xy\exp\left[-\frac{m\omega}{2\hbar}\left(x^2+y^2\right)\right]$

Solution: $V(x,y)=\frac{1}{2}m\omega^2\left(4x^2+y^2\right), E=\frac{5}{2}\hbar\omega$

$\Rightarrow V(x,y)=\dfrac{1}{2}m(2\omega)^2x^2+\dfrac{1}{2}m\omega^2y^2$

Now, $E_n=\left(n_x+\dfrac{1}{2}\right)\hbar\omega_x+\left(n_y+\dfrac{1}{2}\right)\hbar\omega_y=\left(n_x+\dfrac{1}{2}\right)2\hbar\omega+\left(n_y+\dfrac{1}{2}\right)\hbar\omega$

$\Rightarrow E_n=\left(2n_x+n_y+\dfrac{3}{2}\right)\hbar\omega$

$\because E_n=\dfrac{5}{2}\hbar\omega$ when $n_x=0$ and $n_y=1$.

The correct option is **(a)**

8. A particle of mass m in three dimensions is in the potential

$$V(r)=\begin{cases}0, & r<a\\ \infty, & r>a\end{cases}$$

Its ground state energy is

[NET JUNE 2014]

A. $\frac{\pi^2\hbar^2}{2ma^2}$

B. $\frac{\pi^2\hbar^2}{ma^2}$

C. $\frac{3\pi^2\hbar^2}{2ma^2}$

D. $\frac{9\pi^2\hbar^2}{2ma^2}$

Solution: $\left(-\frac{\hbar^2}{2m}\right)\frac{d^2u(r)}{dr^2}+\frac{l(l+1)}{2mr^2}+V(r)u(r)=Eu(r)\ \frac{d^2u(r)}{dr^2}=-K^2u(r)$

$\because K=\sqrt{\frac{2mE}{\hbar^2}}, l=0, V(r)=0\ u(r)=A\sin Kr+B\cos Kr$

Using boundary condition, $B=0$,

$u(r)=A\sin Kr, r=a, u(r)=0\Rightarrow \sin Ka=0\Rightarrow Ka=n\pi\Rightarrow E=\frac{\pi^2\hbar^2}{2ma^2}\quad \because n=1$

9. A particle in the infinite square well potential

$$V(x)=\begin{cases}0 & ,0<x<a\\ \infty & \text{otherwise}\end{cases}$$

is prepared in a state with the wavefunction

$$\psi(x)=\begin{cases}A\sin^3\left(\frac{\pi x}{a}\right), & 0<x<a\\ 0 & \text{otherwise}\end{cases}$$

The expectation value of the energy of the particle is

[NET JUNE 2014]

A. $\frac{5\hbar^2\pi^2}{2ma^2}$

B. $\frac{9\hbar^2\pi^2}{2ma^2}$

C. $\frac{9\hbar^2\pi^2}{10ma^2}$

D. $\frac{\hbar^2\pi^2}{2ma^2}$

Solution: $V(x)=\left\{\begin{array}{ll}0, & 0<x<a\\ \infty, & \text{otherwise}\end{array}\right.\quad \psi(x)=\left\{\begin{array}{ll}A\sin^3\left(\frac{\pi x}{a}\right), & 0<x<a\\ 0 & \text{otherwise}\end{array}\right\}$

$$\psi(x) = A\sin^3\left(\frac{\pi x}{a}\right) = A\frac{3}{4}\sin\frac{\pi x}{a} - A\frac{1}{4}\sin\frac{3\pi x}{a} \quad (\because \sin 3A = 3\sin A - 4\sin^3 A)$$

$$= \frac{A}{4}\left[\sqrt{\frac{a}{2}}\sqrt{\frac{2}{a}} \times 3\sin\frac{\pi x}{a} - \sqrt{\frac{a}{2}}\sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a}\right] \Rightarrow \psi(x) = \frac{A}{4}\left[3\sqrt{\frac{a}{2}}\phi_1(x) - \sqrt{\frac{a}{2}}\phi_3(x)\right]$$

$$\langle\psi \mid \psi\rangle = 1 \Rightarrow 9\frac{a}{32}A^2 + \frac{a}{32}A^2 = 1 \Rightarrow \frac{10a}{32}A^2 = 1 \Rightarrow A = \sqrt{\frac{32}{10a}}$$

$$\psi(x) = \frac{1}{4}\left(3\cdot\sqrt{\frac{a}{2}}\sqrt{\frac{32}{10a}}\phi_1(x) - \sqrt{\frac{a}{2}}\sqrt{\frac{32}{10a}}\phi_3(x)\right) = \frac{3}{\sqrt{10}}\phi_1(x) - \frac{1}{\sqrt{10}}\phi_3(x)$$

Now, $E_1 = \frac{\pi^2\hbar^2}{2ma^2}$, $E_3 = \frac{9\pi^2\hbar^2}{2ma^2} \Rightarrow \langle E\rangle = a_n P(a_n)$ Probability $P(E_1) = \frac{\langle\langle\varphi_1|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{9}{10}, P(E_3) = \frac{|\langle\phi_2|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{1}{10}$ $\langle E\rangle = \frac{9}{10} \times \frac{\pi^2\hbar^2}{2ma^2} + \frac{1}{10} \times \frac{9\pi^2\hbar^2}{2ma^2} \Rightarrow \langle E\rangle = \frac{9\pi^2\hbar^2}{10ma^2}$

The correct option is **(c)**

10. The ground state energy of the attractive delta function potential

$$V(x) = -b\delta(x),$$

where $b > 0$, is calculated with the variational trial function

$$\psi(x) = \begin{cases} A\cos\frac{\pi x}{2a}, & \text{for} \quad -a < x < a, \\ 0, & \text{otherwise,} \end{cases} \text{ is}$$

[NET DEC 2014]

A. $-\frac{mb^2}{\pi^2\hbar^2}$

B. $-\frac{2mb^2}{\pi^2\hbar^2}$

C. $-\frac{mb^2}{2\pi^2\hbar^2}$

D. $-\frac{mb^2}{4\pi^2\hbar^2}$

Solution: $V(x) = -b\delta(x);$ $\quad b > 0$ and $\quad \psi(x) = \{A\cos\frac{\pi x}{2a}; \quad -a < x < a$

Normalized $\psi = \sqrt{\frac{2}{2a}}\cos\frac{\pi x}{2a}$

$\langle T\rangle = \int_{-a}^{a}\psi^*\left(\frac{-\hbar^2}{2m}\right)\frac{\partial^2}{\partial x^2}\psi dx = \frac{\pi^2\hbar^2}{8ma^2}$

$\langle V\rangle = \int_{-a}^{a}\psi^* - b\delta(x)\psi dx = \frac{2}{2a}(-b) = -\frac{b}{a}$

$\langle E\rangle = \frac{\pi^2\hbar^2}{8ma^2} - \frac{b}{a} \Rightarrow \frac{\partial\langle E\rangle}{\partial a} = \frac{-2\pi^2\hbar^2}{8ma^3} + \frac{b}{a^2} = 0 \Rightarrow \frac{-\pi^2\hbar^2}{4ma} + b = 0 \Rightarrow a = \frac{\pi^2\hbar^2}{4mb}$

Put the value of a in equation: $\langle E\rangle = \frac{\pi^2\hbar^2}{8ma^2} - \frac{b}{a} = \frac{\pi^2\hbar^2(4mb)^2}{8m(\pi^2\hbar^2)^2} - \frac{b(4mb)}{(\pi^2\hbar^2)} = -\frac{2mb^2}{\pi^2\hbar^2}$

The correct option is **(b)**

11. Let $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ (where c_0 and c_1 are constants with $c_0^2 + c_1^2 = 1$) be a linear combination of the wavefunctions of the ground and first excited states of the onedimensional harmonic oscillator. For what value of c_0 is the expectation value $\langle x\rangle$ a maximum?

[NET DEC 2014]

A. $\langle x\rangle = \sqrt{\frac{\hbar}{m\omega}}, \quad c_0 = \frac{1}{\sqrt{2}}$

B. $\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}, \quad c_0 = \frac{1}{2}$

C. $\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}, \quad c_0 = \frac{1}{\sqrt{2}}$

D. $\langle x\rangle = \sqrt{\frac{\hbar}{m\omega}}, \quad c_0 = \frac{1}{2}$

Solution:

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$
$$\langle X\rangle = \langle\psi|X|\psi\rangle$$
$$\Rightarrow \langle X\rangle = 2c_0c_1\langle 0|X|1\rangle = \left[\left(c_0^2+c_1^2\right)-(c_0-c_1)^2\right]\langle 0|X|1\rangle = \left[1-(c_0-c_1)^2\right]\langle 0|X|1\rangle$$
For max $\langle X\rangle = c_0 = c_1 \quad \because c_0^2+c_1^2 = 1 \Rightarrow c_0 = \frac{1}{\sqrt{2}}$
$$\Rightarrow \langle X\rangle = 2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle 0|X|1\rangle = \langle 0|X|1\rangle$$
$$\sqrt{\frac{\hbar}{2m\omega}}\left(\langle 0|a+a^+|1\rangle\right) \Rightarrow \langle X\rangle = \sqrt{\frac{\hbar}{2m\omega}}$$

The correct option is **(C)**

12. The ratio of the energy of the first excited state E_1, to that of the ground state E_0, to that of a particle in a three-dimensional rectangular box of side L, L and $\frac{L}{2}$, is

[NET JUNE 2015]

A. $3:2$ **B.** $2:1$

C. $4:1$ **D.** $4:3$

Solution: $E = \frac{\pi^2\hbar^2}{2mL^2}\left[n_x^2+n_y^2+4n_z^2\right]$, for ground state $n_x = 1, n_y = 1, n_z = 1 \Rightarrow E_0 = \frac{6\pi^2\hbar^2}{2mL^2}$
For first excited state $n_x = 1, n_y = 2, n_z = 1 \Rightarrow E = E_1 = \frac{\pi^2\hbar^2}{2mL^2}(1+4+4) = \frac{9\pi^2\hbar^2}{2mL^2} \therefore \frac{E_1}{E_0} = \frac{9}{6} = \frac{3}{2}$
The correct option is **(a)**

13. The ground state energy of a particle in potential $V(x) = g|x|$, estimated using the trail wavefunction
$$\psi(x) = \begin{cases}\sqrt{\frac{c}{a^5}}\left(a^2-x^2\right), & x<|a| \\ 0, & x\geq|a|\end{cases}$$
(where g and c are constants) is

[NET DEC 2015]

A. $\frac{15}{16}\left(\frac{\hbar^2g^2}{m}\right)^{1/3}$ **B.** $\frac{5}{6}\left(\frac{\hbar^2g^2}{m}\right)^{1/3}$

C. $\frac{3}{4}\left(\frac{\hbar^2g^2}{m}\right)^{1/3}$ **D.** $\frac{7}{8}\left(\frac{\hbar^2g^2}{m}\right)^{1/3}$

Solution:

on: $\int_{-a}^{a}\psi^*\psi dx = 1 \Rightarrow c = \frac{15}{16}$
$$\langle T\rangle = \frac{-\hbar^2}{2m}\left(\frac{15}{16a^2}\right)\int_{-a}^{a}\left(a^2-x^2\right)\frac{\partial^2}{\partial x^2}\left(a^2-x^2\right)dx \Rightarrow \langle T\rangle = \frac{10\hbar^2}{4ma^2}$$
$$\langle V\rangle = \frac{15\times 2g}{16a^5}\int_0^a x\left(a^2-x^2\right)dx \Rightarrow \langle V\rangle = \frac{5}{16}ga$$
$$E = \langle T\rangle + \langle V\rangle$$

$$E = \frac{10\hbar^2}{4ma^2}+\frac{5ga}{16}$$
$$\frac{dE}{da} = 0 \Rightarrow a^3 = \frac{8\hbar}{mg} \Rightarrow a = 2\left(\frac{\hbar^2}{mg}\right)^{\frac{1}{3}}$$

put the value of a in equation (i)

$$E = \frac{15}{16}\left(\frac{\hbar^2g^2}{m}\right)^{\frac{1}{3}}$$

The correct option is **(a)**

14. The state of a particle of mass m in a one dimensional rigid box in the interval 0 to L is given by the normalized wavefunction $\psi(x)=\sqrt{\frac{2}{L}}\left(\frac{3}{5}\sin\left(\frac{2\pi x}{L}\right)+\frac{4}{5}\sin\left(\frac{4\pi x}{L}\right)\right)$. If its energy is measured the possible outcomes and the average value of energy are, respectively

[NET JUNE 2016]

A. $\frac{h^2}{2mL^2}, \frac{2h^2}{mL^2}$ and $\frac{73}{50}\frac{h^2}{mL^2}$

B. $\frac{h^2}{8mL^2}, \frac{h^2}{2mL^2}$ and $\frac{19}{40}\frac{h^2}{mL^2}$

C. $\frac{h^2}{2mL^2}, \frac{2h^2}{mL^2}$ and $\frac{19}{10}\frac{h^2}{mL^2}$

D. $\frac{h^2}{8mL^2}, \frac{2h^2}{mL^2}$ and $\frac{73}{200}\frac{h^2}{mL^2}$

Solution: $\psi(x)=\sqrt{\frac{2}{L}}\left(\frac{3}{5}\sin\left(\frac{2\pi x}{L}\right)+\frac{4}{5}\sin\left(\frac{4\pi x}{L}\right)\right)$
Measurement $E=\frac{n^2\pi^2\hbar^2}{2mL^2}$ $\because n=2\Rightarrow E_2=\frac{h^2}{2mL^2}$ and $n=4\Rightarrow E_4=\frac{2h^2}{mL^2}$
Probability $p(E_2)=\frac{9}{25}$ and $p(E_4)=\frac{16}{25}$
Now, average value of energy is $\langle E\rangle=\sum a_n p(a_n)=\frac{9}{25}\times\frac{h^2}{2mL^2}+\frac{16}{25}\times\frac{2h^2}{mL^2}=\frac{73h^2}{50mL^2}$
The correct option is **(a)**

15. A particle of charge q in one dimension is in a simple harmonic potential with angular frequency ω. It is subjected to a time- dependent electric field $E(t)=Ae^{-\left(\frac{t}{\tau}\right)^2}$, where A and τ are positive constants and $\omega\tau\gg 1$. If in the distant past $t\to-\infty$ the particle was in its ground state, the probability that it will be in the first excited state as $t\to+\infty$ is proportional to

[NET DEC 2016]

A. $e^{-\frac{1}{2}(\omega\tau)^2}$

B. $e^{\frac{1}{2}(\omega\tau)^2}$

C. 0

D. $\frac{1}{(\omega\tau)^2}$

Solution: Transition probability is proportional to $P_{if}\propto\left|\int_{-\infty}^{\infty}e^{-\frac{t^2}{\tau^2}}e^{i\omega_{f}t}\right|^2$ where

$$\omega_{fi}=\frac{\frac{3}{2}\hbar\omega-\frac{1}{2}\hbar\omega}{\hbar}=\omega$$

$$P_{if}=\left|\int_{-\infty}^{\infty}\exp-\frac{t^2}{\tau^2}+i\omega t dt\right|^2$$

Now calculate $\int_{-\infty}^{\infty}\exp\left(-\frac{t^2}{\tau^2}+i\omega t\right)dt=\int_{-\infty}^{\infty}\exp-\frac{1}{\tau^2}\left(t^2-i\omega t\tau^2+\left(\frac{i\omega\tau^2}{2}\right)^2-\left(\frac{i\omega\tau^2}{2}\right)^2\right)$

$$=\exp\left(-\frac{\omega^2\tau^2}{4}\right)\int_{-\infty}^{\infty}\exp\frac{1}{\tau^2}\left(t-\frac{i\omega t}{2}\right)^2 dt$$

$$P_{if}=\left|\int_{-\infty}^{\infty}\exp-\frac{t^2}{\tau^2}+i\omega t dt\right|^2\Bigg|^2$$

$$P_{if}=\left|\exp\left(-\frac{\omega^2\tau^2}{4}\right)\int_{-\infty}^{\infty}\exp\frac{1}{\tau^2}\left(t-\frac{i\omega t}{2}\right)^2 dt\right|^2$$

$$P_{if}\propto\exp-\frac{\omega^2\tau^2}{2}$$
The correct option is **(a)**

16. Consider a potential barrier A of height V_0 and width b, and another potential barrier B of height $2V_0$ and the same width b. The ratio T_A/T_B of tunnelling probabilities T_A and T_B, through barriers A and B respectively, for a particle of energy $V_0/100$ is best approximated by

[NET JUNE 2017]

A. (a) $\exp\left[(\sqrt{1.99}-\sqrt{0.99})\sqrt{8mV_0b^2/\hbar^2}\right]$

B. $\exp\left[(\sqrt{1.98}-\sqrt{0.98})\sqrt{8mV_0b^2/\hbar^2}\right]$

C. $\exp\left[(\sqrt{2.99}-\sqrt{0.99})\sqrt{8mV_0b^2/\hbar^2}\right]$ **D.** $\exp\left[(\sqrt{2.98}-\sqrt{0.98})\sqrt{8mV_0b^2/\hbar^2}\right]$

Solution: $T\alpha e^{-\sqrt{2m(V-E)}}$, where $E=\frac{V_0}{100}$ For potential A, $V=V_0$

$$T_A\alpha e^{-\sqrt{\frac{2m}{\hbar^2}\left(V_0-\frac{V_0}{100}\right)}}\Rightarrow T_A\alpha e^{-\sqrt{\frac{2m}{\hbar^2}\left(\frac{99}{100}V_0\right)}}\alpha e^{-\sqrt{2m(0.99V_0)}}$$

For Potential B, $V=2V_0$ and $E=\frac{V_0}{100}$ $T_B\alpha e^{-\sqrt{\frac{2m}{\hbar^2}\left(2V_0-\frac{V_0}{100}\right)}}\Rightarrow T_B\alpha e^{-\sqrt{\frac{2m}{\hbar^2}\left(\frac{199V_0}{100}\right)}}\alpha e^{-\sqrt{2m(1.99V_0)}}$

$$\frac{T_A}{T_B}=\frac{e^{-\sqrt{0.99V_0}}}{e^{-\sqrt{1.99V_0}}}$$
$$\frac{T_A}{T_B}=\left(e^{\sqrt{1.99V_0}}-e^{-\sqrt{0.99V_0}}\right)$$

THe correct option is **(a)**

17. Using the trial function $\psi(x)=\begin{cases}A\left(a^2-x^2\right), & -a<x<a\\ 0 & \text{otherwise}\end{cases}$
the ground state energy of a one-dimensional harmonic oscillator is

[NET JUNE 2017]

A. $\hbar\omega$ **B.** $\sqrt{\frac{5}{14}}\hbar\omega$

C. $\frac{1}{2}\hbar\omega$ **D.** $\sqrt{\frac{5}{7}}\hbar\omega$

Solution: $\psi(x)=\begin{cases}A\left(a^2-x^2\right), & -a<x<a\\ 0 & \text{otherwise}\end{cases}$ For normalization

$$\int\psi^*\psi dx=1$$
$$A^2=\frac{15}{16a^5}\Rightarrow A=\sqrt{\frac{15}{16a^5}}$$

$$\langle T\rangle=\frac{-\hbar^2}{2m}\int_{-a}^{a}\psi^*\frac{\partial^2}{\partial x^2}\psi dx=\frac{-\hbar^2}{2m}\frac{15}{16a^5}\cdot(-2)(2)\int_0^a\left(a^2-x^2\right)dx$$
$$\langle T\rangle=\frac{5\hbar^2}{4ma^2}$$
$$\langle V\rangle=\int_{-a}^{a}\psi^*V\psi dx,\text{ where }V(x)=\frac{1}{2}m\omega^2x^2=\frac{1}{2}m\omega^2\frac{15}{16a^5}2\int_0^a x^2\left(a^2-x^2\right)^2dx.$$
$$\langle V\rangle=\frac{m\omega^2a^2}{14}$$
$$E=T+V=\frac{5\hbar^2}{4ma^2}+\frac{m\omega^2a^2}{14}$$
$$\frac{dE}{da}=0\Rightarrow\frac{5\times(-2)\hbar^2}{4ma^3}+\frac{m\omega^2a}{7}=0\Rightarrow a^4=\frac{35}{2}\left(\frac{\hbar^2}{m^2\omega^2}\right).$$
$$a^2=\left(\frac{35}{2}\right)^{1/2}\left(\frac{\hbar}{m\omega}\right).$$
$$E=\frac{5}{4}\times\frac{\hbar^2}{m}\cdot\frac{m\omega}{\hbar}\sqrt{\frac{2}{35}}+\frac{m\omega^2}{14}\sqrt{\frac{35}{2}}\frac{\hbar}{m\omega}.$$
$$=\frac{\hbar\omega}{2}\left(\frac{5}{2}\sqrt{\frac{2}{35}}+\frac{1}{7}\sqrt{\frac{35}{2}}\right)=\frac{\hbar\omega}{2}\left(\sqrt{\frac{5}{14}}+\sqrt{\frac{5}{14}}\right)=\hbar\omega\sqrt{\frac{5}{14}}$$

The correct option is **(b)**

18. A particle of mass m is confined in a three-dimensional box by the potential

$$V(x,y,z) = \begin{cases} 0, & 0 \le x,y,z \le a \\ \infty & \text{otherwise} \end{cases}$$

The number of eigenstates of Hamiltonian with energy $\frac{9\hbar^2\pi^2}{2ma^2}$ is

[NET JUNE 2018]

A. 1

B. 6

C. 3

D. 4

Solution:

$$E_{n_x,n_y,n_z} = \frac{9\pi^2\hbar^2}{2ma^2}$$

$$\left.\begin{array}{ccc} n_x & n_y & n_z \\ 1 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{array}\right\}$$

where $E_{xx,xy,xz} = \left(n_x^2 + n_y^2 + n_z^2\right)\dfrac{\pi^2\hbar^2}{2ma^2}$

The correct option is **(c)**

19. At $t = 0$, the wavefunction of an otherwise free particle confined between two infinite walls at $x = 0$ and $x = L$ is $\psi(x,t=0) = \sqrt{\frac{2}{L}}\left(\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}\right)$. Its wave function at a later time $t = \frac{mL^2}{4\pi\hbar}$ is

[NET JUNE 2018]

A. $\sqrt{\frac{2}{L}}\left(\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}\right)e^{i\pi/6}$

B. $\sqrt{\frac{2}{L}}\left(\sin\frac{\pi x}{L} + \sin\frac{3\pi x}{L}\right)e^{-i\pi/6}$

C. $\sqrt{\frac{2}{L}}\left(\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}\right)e^{-i\pi/8}$

D. $\sqrt{\frac{2}{L}}\left(\sin\frac{\pi x}{L} + \sin\frac{3\pi x}{L}\right)e^{-i\pi/6}$

Solution:

$$\text{solution } \psi(x,t=0) = \left(\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L} - \sqrt{\frac{2}{L}}\sin\frac{3\pi x}{L}\right)$$

$$\psi(x,t=0) = |\varphi_1\rangle - |\varphi_3\rangle$$

$$\psi(x,t) = |\varphi_1\rangle e^{\frac{-iE_1t}{\hbar}} - |\varphi_3\rangle e^{\frac{-iE_3t}{\hbar}}$$

$$E_1 = \frac{\pi^2\hbar^2}{2mL^2} E_3 = \frac{9\pi^2\hbar^2}{2mL^2} t = \frac{mL^2}{4\pi\hbar}$$

$$\psi(x,t) = |\varphi_1\rangle e^{\frac{-i\pi}{8}} - |\varphi_3\rangle e^{\frac{-9i\pi}{8}} = e^{\frac{-i\pi}{8}}\left(|\varphi_1\rangle - |\varphi_3\rangle e^{-i\pi}\right)$$

$$= e^{\frac{-i\pi}{8}}\left(|\varphi_1\rangle + |\varphi_3\rangle\right) = e^{\frac{-i\pi}{8}}\left(\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L} + \sqrt{\frac{2}{L}}\sin\frac{3\pi x}{L}\right)$$

The correct option is **(d)**

20. The ground state energy of an anisotropic harmonic oscillator described by the potential $V(x,y,z) = \frac{1}{2}m\omega^2x^2 + 2m\omega^2y^2 + 8m\omega^2z^2$ (in units of $\hbar\omega$) is

[NET DEC 2018]

A. $\frac{5}{2}$

B. $\frac{7}{2}$

C. $\frac{3}{2}$

D. $\frac{1}{2}$

Solution: $V(x,y,z)=\frac{1}{2}m\omega^2x^2+\frac{1}{2}m(2\omega)^2y^2+\frac{1}{2}m(4\omega)^2z^2$ $\omega_x=\omega\quad\omega_y=2\omega\quad\omega_z=4\omega$
$E_{n_x,n_y,n_z}=\left(n_x+\frac{1}{2}\right)\hbar\omega_x+\left(n_y+\frac{1}{2}\right)\hbar\omega_y+\left(n_z+\frac{1}{2}\right)\hbar\omega_z$
For ground state
$n_x=0,n_y=0,n_z=0$
$=\frac{1}{2}\hbar\omega+\frac{1}{2}\hbar 2\omega+\frac{1}{2}\hbar 4\omega=\frac{1}{2}\hbar\omega(1+2+4)=\frac{7}{2}\hbar\omega$
The correct option is **(b)**

Practice Set-2

1. Which of the following is an allowed wavefunction for a particle in a bound state? N is a constant and $\alpha, \beta > 0$. **[GATE 2010]**

A. $\psi = N\frac{e^{-\alpha r}}{r^3}$

B. $\psi = N\left(1 - e^{-\alpha r}\right)$

C. $\psi = Ne^{-\alpha x}e^{-\beta\left(x^2+y^2+z^2\right)}$

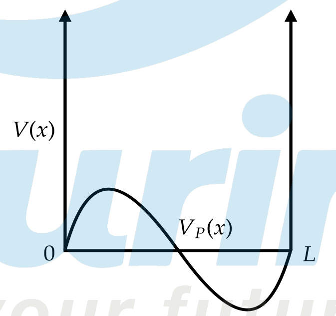
D. $\psi = \begin{cases} \text{non - zero constant} & \text{if } r < R \\ 0 & \text{if } r > R \end{cases}$

Solution: The correct option is **(c)**

2. A particle of mass m is confined in an infinite potential well:

$$V(x) = \begin{cases} 0, & \text{if } 0 < x < L, \\ \infty, & \text{otherwise.} \end{cases}$$

It is subjected to a perturbing potential $V_p(x) = V_o \sin\left(\frac{2\pi x}{L}\right)$ within the well. Let $E^{(1)}$ and $E^{(2)}$ be corrections to the ground state energy in the first and second order in V_0, respectively. Which of the following are true? **[GATE 2010]**



A. $E^{(1)} = 0; E^{(2)} < 0$

B. $E^{(1)} > 0; E^{(2)} = 0$

C. $E^{(1)} = 0; E^{(2)}$ depends on the sign of V_0

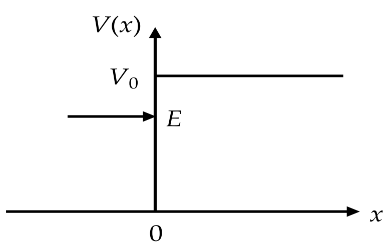
D. $E^{(1)} < 0; E^{(2)} < 0$

Solution: $E_1^1 = \frac{2}{L}\int_0^L V_0 \sin\frac{2\pi x}{L}dx = 0; E_1^2 = \sum_{m\neq 1}\frac{|\langle\psi_m|V_P|\psi_1\rangle|^2}{E_1 - E_m}$ $\quad \because E_1 < E_m$ so $E_1^2 = -ve$
The correct option is **(a)**

3. An electron with energy E is incident from left on a potential barrier, given by

$$V(x) = \begin{cases} 0, & \text{for } x < 0 \\ V_0, & \text{for } x > 0 \end{cases}$$

as shown in the figure. For $E < V_0$, the space part of the wavefunction for $x > 0$ is of the form **[GATE 2011]**



A. e^{ax} **B.** e^{-ax}

C. e^{iax} **D.** e^{-iax}

Solution: $E < V_0$, so there is decaying wave function.
The correct option is **(b)**

4. A particle of mass m is confined in a two dimensional square well potential of dimension a. This potential $V(x,y)$ is given by

$$V(x,y) = 0 \text{ for } -a < x < a \text{ and } -a < y < a$$
$$= \infty \text{ elsewhere}$$

The energy of the first excited state for this particle is given by,

[GATE 2012]

A. $\frac{\pi^2\hbar^2}{ma^2}$ **B.** $\frac{2\pi^2\hbar^2}{ma^2}$

C. $\frac{5\pi^2\hbar^2}{8ma^2}$ **D.** $\frac{4\pi^2\hbar^2}{ma^2}$

Solution:

$$E = \left(n_x^2 + n_y^2\right)\frac{\pi^2\hbar^2}{2m(2a)^2} = \left(n_x^2 + n_y^2\right)\frac{\pi^2\hbar^2}{8ma^2} = \frac{5\pi^2\hbar^2}{8ma^2} \quad \because n_x = 1, n_y = 2$$

The correct option is **(c)**

5. A proton is confined to a cubic box, whose sides have length 10^{-12} m. What is the minimum kinetic energy of the proton? The mass of proton is 1.67×10^{-27} kg and Planck's constant is $6.63\times10^{-34} J_S$.

[GATE 2013]

A. 1.1×10^{-17} J **B.** 3.3×10^{-17} J

C. 9.9×10^{-17} J **D.** 6.6×10^{-17} J

Solution: $\frac{3\pi^2\hbar^2}{2ma^2} = 9.9\times10^{-17}$
The correct option is **(c)**

6. Consider a system of eight non-interacting, identical quantum particles of spin $-\frac{3}{2}$ in a one dimensional box of length L. The minimum excitation energy of the system, in units of $\frac{\pi^2\hbar^2}{2mL^2}$ is

[GATE 2015]

Solution: spin $\frac{3}{2} \Rightarrow$ degeneracy $= (2S+1) = \left(2\times\frac{3}{2}+1\right) = 4$

$$E_{\text{ground}} = 4\times\frac{\pi^2\hbar^2}{2mL^2} + 4\times\frac{4\pi^2\hbar^2}{2mL^2} = \frac{20\pi^2\hbar^2}{2mL^2}$$
$$E_{\text{excited}}^{I'} = 4\times\frac{\pi^2\hbar^2}{2mL^2} + 3\times4\frac{\pi^2\hbar^2}{2mL^2} + 1\times9\frac{\pi^2\hbar^2}{2mL^2} = 25\frac{\pi^2\hbar^2}{2mL^2}$$

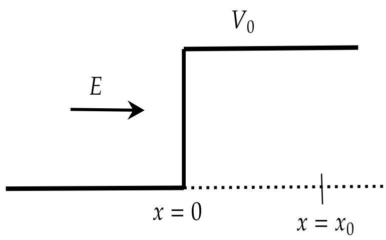
Now minimum excitation energy $\Delta E = E_{\text{excited}}^{I_s'} - E_{\text{ground}} = 25\frac{\pi^2\hbar^2}{2mL^2} - 20\frac{\pi^2\hbar^2}{2mL^2} = 5\frac{\pi^2\hbar^2}{2mL^2}$

7. A two-dimensional square rigid box of side L contains six non-interacting electrons at $T = 0K$. The mass of the electron is m. The ground state energy of the system of electrons, in units of $\frac{\pi^2\hbar^2}{2mL^2}$ is

[GATE 2016]

Solution: $2\times\frac{\left(1^2+1^2\right)\pi^2\hbar^2}{2mL^2} + 4\times\frac{\left(2^2+1^2\right)\pi^2\hbar^2}{2mL^2} = \frac{24\pi^2\hbar^2}{2mL^2}$

8. A particle of mass m and energy E, moving in the positive x direction, is incident on a step potential at $x=0$, as indicated in the figure. The height of the potential is V_0, where $V_0 > E$. At $x = x_0$, where $x_0 > 0$, the probability of finding the electron is $\frac{1}{e}$ times the probability of finding it at $x=0$. If $\alpha = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$, the value of x_0 is **[GATE 2016]**



A. $\frac{2}{\alpha}$

B. $\frac{1}{\alpha}$

C. $\frac{1}{2\alpha}$

D. $\frac{1}{4\alpha}$

Solution: $\frac{1}{e} = e^{-2\alpha x_0} = e^{-1} = e^{-2\alpha x_0} \Rightarrow x_0 = \frac{1}{2\alpha}$
The correct option is **(c)**

9. A free electron of energy 1eV is incident upon a one-dimensional finite potential step of height 0.75eV. The probability of its reflection from the barrier is.......... (up to two decimal places). **[GATE 2017]**

Solution: $R = \left(\frac{\sqrt{E}-\sqrt{E-V_0}}{\sqrt{E}+\sqrt{E-V_0}}\right)^2 = \left(\frac{1-\sqrt{0.25}}{1+\sqrt{0.25}}\right)^2 = \left(\frac{1-0.5}{1+0.5}\right)^2 = 0.11$

10. The ground state energy of a particle of mass m in an infinite potential well is E_0. It changes to $E_0\left(1+\alpha\times 10^{-3}\right)$, when there is a small potential pump of height $V_0 = \frac{\pi^2\hbar^2}{50mL^2}$ and width $a = L/100$, as shown in the figure. The value of α is (up to two decimal places). **[GATE 2018]**

Solution:

Solution: $\alpha_1 = \left(\frac{L}{2}-\frac{a}{2}\right), \alpha_2 = \left(\frac{L}{2}+\frac{a}{2}\right), \quad a = \frac{L}{100}$

$$E_1 = V_0\int_{\alpha_1}^{\alpha_2}\left(\sqrt{\frac{2}{L}}\right)^2 \sin^2\left(\frac{\pi x}{L}\right)dx$$

$$= \frac{V_0}{L}\int_{\alpha_1}^{\alpha_2}\left[1-\cos\frac{2\pi x}{L}\right]dx = \frac{V_0}{L}\left[x-\frac{L}{2\pi}\sin\frac{2\pi x}{L}\right]_{\alpha_1}^{\alpha_2}$$

$$= \frac{V_0}{L}\left[a-\frac{L}{2\pi}\left(\sin\frac{2\pi(L+a)}{2L}-\sin\frac{2\pi(L-a)}{2L}\right)\right]$$

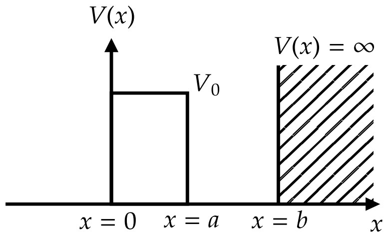
$$= \frac{V_0}{L}\left[\frac{L}{100}-\frac{L}{2\pi}\left(\sin\left(\pi+\frac{\pi a}{L}\right)-\sin\left(\pi-\frac{\pi a}{L}\right)\right)\right]$$

$$= V_0\left[\frac{1}{100}+\frac{1}{2\pi}(0.0314+0.0314)\right]$$

$$= V_0\times 10^{-3}(10+10) = E_0\times 10^{-3}\times\left(\frac{20}{25}\right) \Rightarrow \alpha E_0\times 10^{-3} = 0.81\times E_0\times 10^{-3}$$

Hence, $\alpha = 0.81$

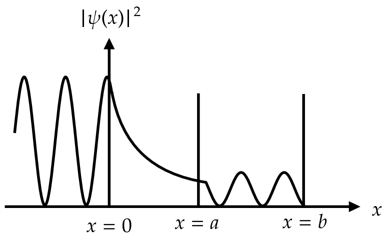
11. Consider a potential barrier $V(x)$ of the form:



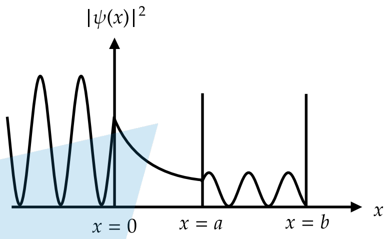
where V_0 is a constant. For particles of energy $E < V_0$ incident on this barrier from the left which of the following schematic diagrams best represents the probability density $|\psi(x)|^2$ as a function of x ?

[GATE 2019]

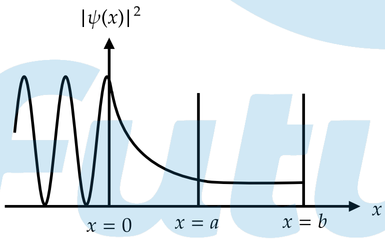
A.



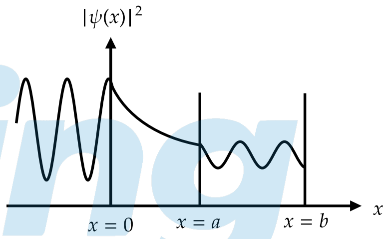
B.



C.



D.



Solution: The correct option is **(a)**