1. Hamiltonian Equation of Motion-Solutions

Practice set-1

1. The Hamiltonian of a system with n degrees of freedom is given by $H(q_1,\ldots.q_n;p_1,\ldots\ldots.p_n;t)$, with an explicit dependence on the time t. Which of the following is correct?

   [NET/JRF (June-2011)]

   a. Different phase trajectories cannot intersect each other.

   b. H always represents the total energy of the system and is a constant of the motion.

   c. The equations $\dot{q}_i = \partial H/\partial p_i, \dot{p}_i = -\partial H/\partial q_i$ are not valid since H has explicit time dependence.

   d. Any initial volume element in phase space remains unchanged in magnitude under time evolution.

Solution: So the correct answer is **Option (a)**

2. If the Lagrangian of a particle moving in one dimensions is given by $L = \frac{\dot{x}^2}{2x} - V(x)$ the Hamiltonian is

   [NET/JRF (June-2012)]

   a. $\frac{1}{2}xp^2 + V(x)$ **b.** $\frac{\dot{x}^2}{2x} + V(x)$

   c. $\frac{1}{2}\dot{x}^2 + V(x)$ **d.** $\frac{p^2}{2x} + V(x)$

Solution:

$$\text{Since } H = p_x\dot{x} - L \text{ and } \frac{\partial L}{\partial \dot{x}} = p_x \Rightarrow \frac{\dot{x}}{x}$$
$$= p_x \Rightarrow \dot{x} = p_x x$$
$$\mathrm{H} = \mathrm{p_x}\dot{\mathrm{x}} - \frac{\dot{\mathrm{x}}^2}{2\mathrm{x}} + \mathrm{V(x)} \Rightarrow \mathrm{H}$$
$$= \mathrm{p_x} \times \mathrm{p_x x} - \frac{(\mathrm{p_x x})^2}{2\mathrm{x}} + \mathrm{V(x)} \Rightarrow H = \frac{p_x^2 x}{2} + V(x)$$

So the correct answer is **Option (a)**

3. The Hamiltonian of a relativistic particle of rest mass m and momentum p is given by $H=\sqrt{p^2+m^2}+V(x)$, in units in which the speed of light $c=1$. The corresponding Lagrangian is

[NET/JRF (DEC-2013)]

a. $L=m\sqrt{1+\dot{x}^2}-V(x)$

b. $L=-m\sqrt{1-\dot{x}^2}-V(x)$

c. $L=\sqrt{1+m\dot{x}^2}-V(x)$

d. $L=\frac{1}{2}m\dot{x}^2-V(x)$

Solution:

$$\begin{aligned}
H &= \sqrt{p^2+m^2}+V(x) \Rightarrow \frac{\partial H}{\partial p}=\dot{x} \\
&= \frac{1}{2}\frac{2p}{(p^2+m^2)^{\frac{1}{2}}} \Rightarrow \dot{x}\left(p^2+m^2\right)^{1/2}=p \\
\Rightarrow p &= \frac{\dot{x}m}{\sqrt{1-\dot{x}^2}} \\
\text{Now } L &= \sum \dot{x}p-H=\dot{x}p-H \\
&= \dot{x}p-\sqrt{p^2+m^2}-V(x) \\
\text{Put value } p &= \frac{\dot{x}m}{\sqrt{1-\dot{x}^2}} \Rightarrow L=-m\sqrt{1-\dot{x}^2}-V(x)
\end{aligned}$$

So the correct answer is **Option (b)**

4. A particle of mass m and coordinate q has the Lagrangian $L=\frac{1}{2}m\dot{q}^2-\frac{\lambda}{2}q\dot{q}^2$, where λ is a constant. The Hamiltonian for the system is given by

[NET/JRF (June-2014)]

a. $\frac{p^2}{2m}+\frac{\lambda q p^2}{2m^2}$

b. $\frac{p^2}{2(m-\lambda q)}$

c. $\frac{p^2}{2m}+\frac{\lambda q p^2}{2(m-\lambda q)^2}$

d. $\frac{p\dot{q}}{2}$

Solution:

$$\begin{aligned}
H &= \sum \dot{q}p-L \text{ where } L=\frac{1}{2}m\dot{q}^2-\frac{\lambda}{2}q\dot{q}^2 \\
\frac{\partial L}{\partial \dot{q}} &= p=m\dot{q}-\lambda q\dot{q} \Rightarrow p \\
&= \dot{q}(m-\lambda q) \Rightarrow \dot{q}=\frac{p}{m-\lambda q} \\
\Rightarrow H &= \dot{q}p-L=\frac{p^2}{(m-\lambda q)}-\frac{1}{2}m\frac{(p^2)}{(m-\lambda q)^2}+\frac{\lambda}{2}q\cdot\frac{p^2}{(m-\lambda q)^2} \\
\Rightarrow H &= \dot{q}p-L=\frac{p^2}{(m-\lambda q)}-\frac{p^2}{2(m-\lambda q)^2}(m-\lambda q) \\
\Rightarrow H &= \dot{q}p-L=\frac{p^2}{(m-\lambda q)}-\frac{p^2}{2(m-\lambda q)} \Rightarrow H=\frac{p^2}{2(m-\lambda q)}
\end{aligned}$$

So the correct answer is **Option (b)**

5. The Hamiltonian of a one-dimensional system is $H=\frac{xp^2}{2m}+\frac{1}{2}kx$, where m and k are positive constants. The corresponding Euler-Lagrange equation for the system is

[NET/JRF (June-2018)]

a. $m\ddot{x}+k=0$

b. $m\ddot{x}+2\dot{x}+kx^2=0$

c. $2mx\ddot{x}-m\dot{x}^2+kx^2=0$

d. $mx\ddot{x}+2m\dot{x}^2+kx^2=0$

Solution:

$$
\begin{aligned}
H &= \frac{xp^2}{2m}+\frac{1}{2}kx \\
\frac{\partial H}{\partial p} &= \dot{x} \Rightarrow \frac{xp}{m}=\dot{x} \quad p=\frac{m\dot{x}}{x} \\
L &= \dot{x}p-H \Rightarrow L=\dot{x}p-\frac{xp^2}{2m}-\frac{1}{2}kx \\
&= \frac{m\dot{x}^2}{x}-\frac{m\dot{x}^2}{2x}-\frac{1}{2}kx=\frac{m\dot{x}^2}{2x}-\frac{1}{2}kx
\end{aligned}
$$

Eular Lagrangas equation is given by

$$
\begin{aligned}
\frac{d}{dr}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x} &= 0 \\
\frac{m\ddot{x}}{x}-\frac{m\dot{x}\dot{x}}{2x^2}+\frac{1}{2}k &= 0 \\
2xm\ddot{x}-m\dot{x}^2+kx^2 &= 0
\end{aligned}
$$

So the correct answer is **Option (c)**

6. The Hamiltonian of a particle of unit mass moving in the xy-plane is given to be: $H=xp_x-yp_y-\frac{1}{2}x^2+\frac{1}{2}y^2$ in suitable units. The initial values are given to be $(x(0),y(0))=(1,1)$ and $(p_x(0),p_y(0))=\left(\frac{1}{2},-\frac{1}{2}\right)$. During the motion, the curves traced out by the particles in the xy-plane and the p_xp_y- plane are **[NET/JRF (June-2011)]**

a. Both straight lines

b. A straight line and a hyperbola respectively

c. A hyperbola and an ellipse, respectively

d. Both hyperbolas

Solution:

$$
H=xp_x-yp_y-\frac{1}{2}x^2+\frac{1}{2}y^2
$$

Solving Hamiltonion equation of motion

$$
\begin{aligned}
\frac{\partial H}{\partial x} &= -\dot{p}_x \Rightarrow p_x-x=-\dot{p}_x \text{ and } \frac{\partial H}{\partial y} \\
&= -\dot{p}_y \Rightarrow -p_y+y=-\dot{p}_y \\
\frac{\partial H}{\partial p_x} &= \dot{x} \Rightarrow x=\dot{x} \text{ and } \frac{\partial H}{\partial p_y}=\dot{y} \Rightarrow -y=\dot{y}
\end{aligned}
$$

After solving these four differential equation and eliminating time t and using boundary condition one will get $\Rightarrow x=\frac{1}{y}$ and $p_x=\frac{1}{2}\frac{1}{p_y}$

So the correct answer is **Option (d)**

7. If the Lagrangian of a dynamical system in two dimensions is $L = \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{y}$, then its Hamiltonian is **[NET/JRF (June-2015)]**

a. $H = \frac{1}{m}p_x p_y + \frac{1}{2m}p_y^2$ **b.** $H = \frac{1}{m}p_x p_y + \frac{1}{2m}p_x^2$
c. $H = \frac{1}{m}p_x p_y - \frac{1}{2m}p_y^2$ **d.** $H = \frac{1}{m}p_x p_y - \frac{1}{2m}p_x^2$

Solution:

$$L = \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{y} \Rightarrow \frac{\partial L}{\partial \dot{x}} = m\dot{x} + m\dot{y} = p_x \quad \text{(i)}$$

$$\Rightarrow \quad \frac{\partial L}{\partial \dot{y}} = m\dot{x} = p_y \quad \text{or} \quad \dot{x} = \frac{p_y}{m} \quad \text{(ii)}$$

$$\text{put } \dot{x} = \frac{p_y}{m} \text{ in equation (i) } \Rightarrow p_y + m\dot{y} = p_x \Rightarrow \dot{y} = \frac{p_x - p_y}{m}$$

$$H = p_x\dot{x} + p_y\dot{y} - L = p_x\dot{x} + p_y\dot{y} - \frac{1}{2}m\dot{x}^2 - m\dot{x}\dot{y}$$

$$\text{put value of } \dot{x} \text{ and } \dot{y} \Rightarrow H = \frac{p_x p_y}{m} - \frac{p_y^2}{2m}$$

So the correct answer is **Option (c)**

8. The Hamiltonian of a system with generalized coordinate and momentum (q, p) is $H = p^2q^2$.A solution of the Hamiltonian equation of motion is (in the following A and B are constants) **[NET/JRF (June-2016)]**

a. $p = Be^{-2At}, \quad q = \frac{A}{B}e^{2At}$ **b.** $p = Ae^{-2At}, \quad q = \frac{A}{B}e^{-2At}$
c. $p = Ae^{At}, \quad q = \frac{A}{B}e^{-At}$ **d.** $p = 2Ae^{-A^2t}, \quad q = \frac{A}{B}e^{A^2t}$

Solution:

$$H = p^2q^2$$

From Hamilton's equation

$$\frac{\partial H}{\partial q} = -\dot{p} \Rightarrow \frac{dp}{dt} = -2p^2q \quad \text{(i)}$$

$$\frac{\partial H}{\partial p} = \dot{q} \Rightarrow \frac{dq}{dt} = 2pq^2 \quad \text{(ii)}$$

from equations (i) and (ii)

$$\frac{dp}{p} = -\frac{dq}{q}$$

Integrating both sides, $\ln p = -\ln q + \ln A$

$$pq = A \quad \text{(iii)}$$

from equation (i)

$$\frac{dp}{dt} = -2p^2q = -2pA$$

$$\Rightarrow \quad \int \frac{dp}{p} = -\int 2A dt + \ln B \Rightarrow \ln \frac{p}{B} = -2At \Rightarrow p = Be^{-2At}$$

Putting this value of p in equation (iii) gives $q = \frac{A}{B} e^{2At}$

So the correct answer is **Option (a)**

9. The Hamiltonian for a system described by the generalised coordinate x and generalised momentum p is

$$H = \alpha x^2 p + \frac{p^2}{2(1+2\beta x)} + \frac{1}{2}\omega^2 x^2$$

where α, β and ω are constants. The corr

[NET/JRF (JUNE-2017)]

a. esponding Lagrangian is $\frac{1}{2}\left(\dot{x} - \alpha x^2\right)^2 (1+2\beta x) - \frac{1}{2}\omega^2 x^2$

b. $\frac{1}{2(1+2\beta x)}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - \alpha x^2 \dot{x}$

c. $\frac{1}{2}\left(\dot{x}^2 - \alpha^2 x\right)^2 (1+2\beta x) - \frac{1}{2}\omega^2 x^2$

d. $\frac{1}{2(1+2\beta x)}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 + \alpha x^2 \dot{x}$

Solution:

$$\begin{aligned}
H &= ax^2 p + \frac{p^2}{2(1+2\beta x)} + \frac{1}{2}\omega^2 x^2 \\
\frac{\partial H}{\partial p} &= \dot{x} \Rightarrow ax^2 + \frac{p}{(1+2\beta x)} \Rightarrow p \\
&= \left(\dot{x} - ax^2\right)(1+2\beta x) \\
L &= \dot{x}P - H \\
&= \dot{x}P - ax^2 P - \frac{p^2}{(1+2\beta x)} - \frac{1}{2}\omega^2 x^2 \\
&= x\left(\dot{x} - \alpha x^2\right)(1+2\beta x) - \alpha x^2\left(\dot{x} - \alpha x^2\right)(1+2\beta x) - \frac{\left(\dot{x} - \alpha x^2\right)^2 (1+2\beta x)^2}{2(1+2\beta x)} \\
&= (1+2\beta x)\left(\dot{x} - \alpha x^2\right)\left[x - \alpha x^2 - \frac{\left(\dot{x} - \alpha x^2\right)}{2}\right] - \frac{1}{2}\omega^2 x^2 \\
&= (1+2\beta x)\left(\dot{x} - \alpha x^2\right)\frac{\left(\dot{x} - \alpha x^2\right)^2}{2} - \frac{1}{2}\omega^2 x^2 = (1+2\beta x)\frac{\left(\dot{x} - \alpha x^2\right)^2}{2} - \frac{1}{2}\omega^2 x^2
\end{aligned}$$

So the correct answer is **Option (a)**

10. A point mass m, is constrained to move on the inner surface of a paraboloid of revolution $x^2 + y^2 = az$ (where $a > 0$ is a constant). When it spirals down the surface, under the influence of gravity (along $-z$ direction), the angular speed about the z - axis is proportional to

[NET/JRF (JUNE-2020)]

a. 1 (independent of z)

b. z

c. z^{-1}

d. z^{-2}

Solution:

Using Lagrangian in cylindrical coordinate

$$L=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2\right)-mgz$$

$$\text{with constraint } x^2+y^2=az\Rightarrow r^2=az\Rightarrow \dot{z}=\frac{2r\dot{r}}{a}$$

$$L=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2+\left(\frac{2r\dot{r}}{a}\right)^2\right)-\frac{mgr^2}{a}$$

$$\theta \text{is cyclic coordinate so } \frac{\partial L}{\partial \theta}=0\Rightarrow \frac{\partial L}{\partial \dot{\theta}}=J\Rightarrow mr^2\dot{\theta}=J\Rightarrow \dot{\theta}\propto\frac{1}{r^2}\propto\frac{1}{z}$$

So the correct answer is **Option (c)**

11. The Poisson bracket $\{|\vec{r}|,|\vec{p}|\}$ has the value

[NET/JRF (June-2012)]

a. $|\vec{r}||\vec{p}|$ **b.** $\hat{r}\cdot\hat{p}$ **c.** 3 **d.** 1

Solution:

$$\vec{r}=x\hat{i}+\hat{y}\hat{j}+z\hat{k}, |\vec{r}|=\left(x^2+y^2+z^2\right)^{1/2}, p=p_x\hat{i}+p_y\hat{j}+p_z\hat{k}$$

$$|\vec{p}|=\left(p_x^2+p_y^2+p_z^2\right)^{1/2}$$

$$\{|\vec{r}|,|\vec{p}|\}=\left(\frac{\partial|\vec{r}|}{\partial x}\cdot\frac{\partial|\vec{p}|}{\partial p_x}-\frac{\partial|\vec{r}|}{\partial p_x}\cdot\frac{\partial|\vec{p}|}{\partial x}\right)+\left(\frac{\partial|\vec{r}|}{\partial y}\cdot\frac{\partial|\vec{p}|}{\partial p_y}-\frac{\partial|\vec{r}|}{\partial p_y}\cdot\frac{\partial|\vec{p}|}{\partial y}\right)+\left(\frac{\partial|\vec{r}|}{\partial z}\cdot\frac{\partial|\vec{p}|}{\partial p_z}-\frac{\partial|\vec{r}|}{\partial p_z}\cdot\frac{\partial|\vec{p}|}{\partial y}\right)$$

$$=\frac{x}{|\vec{r}|}\frac{p_x}{|\vec{p}|}+\frac{y}{|\vec{r}|}\frac{p_y}{|\vec{p}|}+\frac{z}{|\vec{r}|}\frac{p_z}{|\vec{p}|}=\frac{\vec{r}\cdot\vec{p}}{|\vec{r}||\vec{p}|}=(\hat{r}\cdot\hat{p})$$

So the correct answer is **Option (c)**

12. Let A, B and C be functions of phase space variables (coordinates and momenta of a mechanical system). If , represents the Poisson bracket, the value of $\{A,\{B,C\}\}-\{\{A,B\},C$ is given by

[NET/JRF (DEC-2013)]

a. 0 **b.** $\{B,\{C,A\}\}$ **c.** $\{A,\{C,B\}\}$ **d.** $\{\{C,A\},B\}$

Solution:

We know that Jacobi identity equation

$$\{A,\{B,C\}\}+\{B,\{C,A\}\}+\{C,\{A,B\}\}=0$$

$$\text{Now } \{A,\{B,C\}\}-\{\{A,B\},C\}=-\{B,\{C,A\}\}=\{\{C,A\},B\}$$

So the correct answer is **Option (d)**

13. The coordinates and momenta $x_i, p_i (i=1,2,3)$ of a particle satisfy the canonical Poisson bracket relations $\{x_i, p_j\}=\delta_{ij}$. If $C_1=x_2p_3+x_3p_2$ and $C_2=x_1p_2-x_2p_1$ are constants of motion, and if $C_3=\{C_1,C_2\}=x_1p_3+x_3p_1$, then

a. $\{C_2,C_3\}=C_1$ and $\{C_3,C_1\}=C_2$

b. $\{C_2,C_3\}=-C_1$ and $\{C_3,C_1\}=-C_2$

c. $\{C_2, C_3\} = -C_1$ and $\{C_3, C_1\} = C_2$

d. $\{C_2, C_3\} = C_1$ and $\{C_3, C_1\} = -C_2$

Solution:

$$C_1 = x_2p_3 + x_3p_2, C_2 = x_1p_2 - x_2p_1, C_3 = x_1p_3 + x_3p_1$$

$$\{C_2, C_3\} = \left(\frac{\partial C_2}{\partial x_1}\frac{\partial C_3}{\partial p_1} - \frac{\partial C_2}{\partial p_1}\frac{\partial C_3}{\partial x_1}\right) + \left(\frac{\partial C_2}{\partial x_2}\frac{\partial C_3}{\partial p_2} - \frac{\partial C_2}{\partial p_2}\frac{\partial C_3}{\partial x_2}\right) + \left(\frac{\partial C_2}{\partial x_3}\frac{\partial C_3}{\partial p_3} - \frac{\partial C_2}{\partial p_3}\frac{\partial C_3}{\partial x_3}\right)$$

$$\{C_2, C_3\} = (p_2x_3 - (-x_2)p_3) + (0 - x_1 \cdot 0) + (0 \cdot x_1 - 0 \cdot p_1)$$

$$= (p_2x_3 + x_2p_3) = C_1$$

$$\{C_3, C_1\} = \left(\frac{\partial C_3}{\partial x_1}\frac{\partial C_1}{\partial p_1} - \frac{\partial C_3}{\partial p_1}\frac{\partial C_1}{\partial x_1}\right) + \left(\frac{\partial C_3}{\partial x_2}\frac{\partial C_1}{\partial p_2} - \frac{\partial C_3}{\partial p_2}\frac{\partial C_1}{\partial x_2}\right) + \left(\frac{\partial C_3}{\partial x_3}\frac{\partial C_1}{\partial p_3} - \frac{\partial C_3}{\partial p_3}\frac{\partial C_1}{\partial x_3}\right)$$

$$\{C_3, C_1\} = (p_3 \cdot 0 - x_3 \cdot 0) + (0 \cdot x_3 - 0 \cdot p_3) + (p_1x_2 - x_1p_2)$$

$$= -(x_1p_2 - x_2p_1) = -C_2$$

So the correct answer is **Option (d)**

14. The Hamiltonian of a simple pendulum consisting of a mass m attached to a massless string of length l is $H = \frac{p_\theta^2}{2ml^2} + mgl(1 - \cos\theta)$. If L denotes the Lagrangian, the value of $\frac{dL}{dt}$ is:

[NET/JRF (DEC-2012)]

a. $-\frac{2g}{l}p_\theta \sin\theta$

b. $-\frac{g}{l}p_\theta \sin 2\theta$

c. $\frac{g}{l}p_\theta \cos\theta$

d. $lp_\theta^2 \cos\theta$

Solution:

$$\frac{dL}{dt} = [L, H] + \frac{\partial L}{\partial t} \text{ where } H$$

$$= \frac{p_\theta^2}{2ml^2} + mgl(1 - \cos\theta)$$

$$L = \sum_i p_i\dot{q}_i - H = p_\theta\dot{\theta} - H, \dot{\theta} = \frac{\partial H}{\partial P_\theta}$$

$$= \frac{p_\theta}{ml^2}, \Rightarrow L = \frac{ml^2\dot{\theta}^2}{2} - mgl(1 - \cos\theta)$$

Hence we have to calculate $[L, H]$ which is only defined into phase space i.e. p_θ and θ.

$$\text{Then } \Rightarrow L = \frac{p_\theta^2}{2ml^2} - \text{mgl}(1 - \cos\theta)$$

$$[L, H] = \frac{\partial L}{\partial \theta} \times \frac{\partial H}{\partial p_\theta} - \frac{\partial L}{\partial p_\theta} \times \frac{\partial H}{\partial \theta}$$

$$= -\frac{2g}{l}p_\theta \sin\theta \text{ and } \frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dL}{dt} = -\frac{2g}{l}p_\theta \sin\theta$$

So the correct answer is **Option (a)**

15. A particle moves in one dimension in the potential $V = \frac{1}{2}k(t)x^2$, where $k(t)$ is a time dependent parameter. Then $\frac{d}{dt}\langle V\rangle$, the rate of change of the expectation value $\langle V\rangle$ of the potential energy is

[NET/JRF (June-2015)]

a. $\frac{1}{2}\frac{dk}{dt}\langle x^2\rangle + \frac{k}{2m}\langle xp + px\rangle$

b. $\frac{1}{2}\frac{dk}{dt}\langle x^2\rangle + \frac{1}{2m}\langle p^2\rangle$

c. $\frac{k}{2m}\langle xp + px\rangle$

d. $\frac{1}{2}\frac{dk}{dt}\langle x^2\rangle$

Solution:

$$
\begin{aligned}
H &= \frac{p^2}{2m}+\frac{1}{2}k(t)x^2 \\
\frac{d}{dt}\langle V\rangle &= \langle [V,H]\rangle + \left\langle \frac{\partial V}{\partial t}\right\rangle \\
&\Rightarrow \left[\frac{1}{2}k(t)x^2, \frac{p^2}{2m}+\frac{1}{2}k(t)x^2\right]+\frac{x^2}{2}\frac{\partial k}{\partial t}=[V,H] \\
\frac{d}{dt}\langle V\rangle &= \frac{1}{2}k(t)\cdot 2\left\langle \frac{xp+px}{2m}\right\rangle + \left\langle \frac{x^2}{2}\right\rangle \frac{\partial k}{\partial t} \\
&= \left\langle \frac{x^2}{2}\right\rangle \frac{\partial k}{\partial t}+\frac{1}{2m}k(t)\langle xp+px\rangle
\end{aligned}
$$

So the correct answer is **Option (a)**

16. A particle in two dimensions is in a potential $V(x,y)=x+2y$. Which of the following (apart from the total energy of the particle) is also a constant of motion?

[NET/JRF (DEC-2016)]

a. p_y-2p_x **b.** p_x-2p_y **c.** p_x+2p_y **d.** p_y+2p_x

Solution:

$$
\begin{aligned}
V(x,y) &= x+2y \\
H &= \frac{p_x^2}{2m}+\frac{p_y^2}{2m}+x+2y \\
\frac{d(p_y-2p_x)}{dt} &= [p_y-2p_x,H]+\frac{\partial}{\partial t}(p_y-2p_x) \\
&= [p_y-2p_x,H]=[p_y-2p_x,x+2y] \\
&= [p_y,2y]-[2p_x,x]=-2+2=0
\end{aligned}
$$

So the correct answer is **Option (a)**

17. The Hamiltonian of a classical one-dimensional harmonic oscillator is $H=\frac{1}{2}\left(p^2+x^2\right)$, in suitable units. The total time derivative of the dynamical variable $(p+\sqrt{2}x)$ is

[NET/JRF (JUNE-2018)]

a. $\sqrt{2}p-x$ **b.** $p-\sqrt{2}x$ **c.** $p+\sqrt{2}x$ **d.** $x+\sqrt{2}p$

Solution:

$$
\begin{aligned}
H &= \frac{p^2}{2}+\frac{x^2}{2} \quad \text{Let say dynamical variable } A=(p+\sqrt{2}x) \\
\frac{dA}{dt} &= [A,H]+\frac{\partial A}{\partial t} \\
\text{It is given } \frac{\partial A}{\partial t} &= 0 \Rightarrow \frac{dA}{dt}=[A,H] \\
\frac{dA}{dt} &= \left[p+\sqrt{2}x, \frac{p^2}{2}+\frac{x^2}{2}\right]=\left[p,\frac{x^2}{2}\right]+\left[\sqrt{2}x,\frac{p^2}{2}\right] \\
&= \frac{-2x}{2}+\frac{\sqrt{2}2p}{2}=-x+\sqrt{2}p=\sqrt{2}p-x
\end{aligned}
$$

So the correct answer is **Option (a)**

18. A system is governed by the Hamiltonian

$$H=\frac{1}{2}\left(p_x-ay\right)^2+\frac{1}{2}\left(p_x-bx\right)^2$$

where a and b are constants and p_x, p_y are momenta conjugate to x and y respectively. For what values of a and b will the quantities $\left(p_x-3y\right)$ and $\left(p_y+2x\right)$ be conserved?

[NET/JRF (June-2013)]

a. $a=-3, b=2$
b. $a=3, b=-2$
c. $a=2, b=-3$
d. $a=-2, b=3$

Solution:

$$\text{Poisson bracket } \left[p_x-3y, H\right]=0 and \left[p_y+2y, H\right]=0$$
$$p_y(b-3)+x\left(3b-b^2\right)=0 \text{ and } p_x(a+2)-y\left(2a+a^2\right)=0$$
$$\Rightarrow \quad a=-2, b=3$$

So the correct answer is **Option (d)**

19. The Lagrangian of a system moving in three dimensions is

$$L=\frac{1}{2}m\dot{x}_1^2+m\left(\dot{x}_2^2+\dot{x}_3^2\right)-\frac{1}{2}kx_1^2-\frac{1}{2}k\left(x_2+x_3\right)^2$$

The independent constants of motion is/are

[NET/JRF (June-2016)]

a. Energy alone
b. Only energy, one component of the linear momentum and one component of the angular momentum
c. Only energy, one component of the linear momentum
d. Only energy, one component of the angular momentum

Solution:

The motion is in $3D$. So don't get confine with $x_1, x_2 x_3$ they are actually x, y, z Langrangian is then

$$L=\frac{1}{2}m\dot{x}^2+m\left(\dot{y}^2+\dot{z}^2\right)-\frac{1}{2}kx^2-\frac{1}{2}k(y+z)^2,$$
$$\text{when } \frac{\partial L}{\partial x}\neq 0, \frac{\partial L}{\partial y}\neq 0, \frac{\partial L}{\partial z}\neq 0$$

So, not any component at Linear momentum is conserve.

Now transform the Lagrangian to Hamiltonian

$$H=\frac{P_x^2}{2m}+\frac{P_y^2}{4m}+\frac{P_z^2}{4m}+\frac{1}{2}kx^2+\frac{1}{2}k(y+z)^2$$
$$\frac{\partial H}{\partial t}=0 \text{ so energy is conserved}$$

Now let us assume $L_x=yP_z-zP_y$

$$\frac{dL_x}{dt}=\left[L_x, H\right]+\frac{\partial L_x}{\partial t}$$
$$\left[L_x, H\right]=\left[yP_z-zP_y, H\right]$$

$$= [y,H]P_z + y[P_z,H] - [z,H]P_y - z[P_y,H]$$

$$\Rightarrow [L_x,H] = \left[y, \frac{P_y^2}{4m}\right] P_z + y\left[P_z, \frac{1}{2}k(y+z)^2\right] - \left[z, \frac{P_z^2}{4m}\right] P_y - z\left[P_y, \frac{1}{2}k(y+z)^2\right]$$

$$= 2P_y \frac{P_z}{4m} + y\left[0 - \frac{1}{2}k \cdot 2(y+z)\right] - \left[2P_y \frac{P_z}{4m}\right] - z\left[0 - \frac{1}{2}k \cdot 2(y+z)\right]$$

$$= -k\left(y^2 + yz\right) + k\left(z^2 + yz\right) = -k\left[y^2 - z^2\right] = k\left[z^2 - y^2\right]$$

$$\Rightarrow \frac{dL_x}{dt} \neq 0. \text{ Similarly } \frac{dL_y}{dt} \neq 0 \text{and} \Rightarrow \frac{dL_z}{dt} \neq 0$$

So the correct answer is **Option (a)**

20. The Hamiltonian of a system with two degrees of freedom is $H = q_1 p_1 - q_2 p_2 + a q_1^2$, where $a > 0$ is a constant. The function $q_1 q_2 + \lambda p_1 p_2$ is a constant of motion only if λ is

[NET/JRF (DEC-2019)]

a. 0 **b.** 1 **c.** $-a$ **d.** a

Solution: So the correct answer is **Option (a)**

Answer key			
Q.No.	Answer	Q.No.	Answer
1	**a**	2	**a**
3	**b**	4	**b**
5	**c**	6	**d**
7	**c**	8	**a**
9	**a**	10	**c**
11	**b**	12	**d**
13	**d**	14	**a**
15	**a**	16	**a**
17	**a**	18	**d**
19	**a**	20	**a**

Practice set-2

1. Consider the Lagrangian $L = a\left(\frac{dx}{dt}\right)^2 + b\left(\frac{dy}{dt}\right)^2 + cxy$, where a, b and c are constants. If p_x and p_y are the momenta conjugate to the coordinates x and y respectively, then the Hamiltonian is

[GATE- 2020]

a. $\frac{p_x^2}{4a} + \frac{p_y^2}{4b} - cxy$

b. $\frac{p_x^2}{2a} + \frac{p_y^2}{2b} - cxy$

c. $\frac{p_x^2}{2a} + \frac{p_y^2}{2b} + cxy$

d. $\frac{p_x^2}{a} + \frac{p_y^2}{b} + cxy$

Solution:

$$L = a\dot{x}^2 + b\dot{y}^2 + cxy$$

$$\frac{\partial L}{\partial \dot{x}} = p_x = 2a\dot{x} \Rightarrow \dot{x} = \frac{p_x}{2a} \text{ and } \frac{\partial L}{\partial \dot{y}} = p_y = 2a\dot{y} \Rightarrow \dot{y} = \frac{p_y}{2a}$$

$$H = p_x\dot{x} + p_y\dot{y} - L \Rightarrow H = 2a\dot{x}^2 + 2b\dot{y}^2 - \left(a\dot{x}^2 + b\dot{y}^2 + cxy\right)$$

$$\Rightarrow H = a\dot{x}^2 + b\dot{y}^2 - cxy = \frac{p_x^2}{4a} + \frac{p_y^2}{4b} - cxy$$

So the correct answer is **Option (a)**

2. If Hamiltonion is given by $H = \frac{P_\theta^2}{2ml^2} + mgl(1-\cos\theta)$ Hamilton's equations are then given by

[GATE- 2010]

a. $\dot{p}_\theta = -mgl\sin\theta; \quad \dot{\theta} = \frac{p_\theta}{ml^2}$

b. $\dot{p}_\theta = mgl\sin\theta; \quad \dot{\theta} = \frac{p_\theta}{ml^2}$

c. $\dot{p}_\theta = -m\ddot{\theta}; \quad \dot{\theta} = \frac{p_\theta}{m}$

d. $\dot{p}_\theta = -\left(\frac{g}{l}\right)\theta; \quad \dot{\theta} = \frac{p_\theta}{ml}$

Solution:

$$H = \frac{P_\theta^2}{2ml^2} + mgl(1-\cos\theta) \Rightarrow \frac{\partial H}{\partial \theta} = -\dot{P}_\theta \Rightarrow \dot{P}_\theta = -mgl\sin\theta; \frac{\partial H}{\partial P_\theta} = \dot{\theta} \Rightarrow \dot{\theta} = \frac{P_\theta}{ml^2}.$$

So the correct answer is **Option (a)**

3. A particle of mass m is attached to a fixed point O by a weightless inextensible string of length a. It is rotating under the gravity as shown in the figure. The Lagrangian of the particle is $L(\theta, \phi) = \frac{1}{2}ma^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - mga\cos\theta$ where θ and ϕ are the polar angles. The Hamiltonian of the particles is

[GATE- 2012]

a. $H = \frac{1}{2ma^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - mga\cos\theta$

b. $H = \frac{1}{2ma^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) + mga\cos\theta$

c. $H = \frac{1}{2ma^2}\left(p_\theta^2 + p_\phi^2\right) - mga\cos\theta$

d. $H = \frac{1}{2ma^2}\left(p_\theta^2 + p_\phi^2\right) + mga\cos\theta$

Solution:

$$H = P_\theta \dot{\theta} + P_\phi \dot{\phi} - L = P_\theta \dot{\theta} + P_\phi \dot{\phi} - \frac{1}{2}ma^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mga\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = P_\theta \Rightarrow ma^2\dot{\theta} = P_\theta \Rightarrow \dot{\theta} = \frac{P_\theta}{ma^2} \text{ and } P_\phi = \frac{\partial L}{\partial \dot{\phi}} = ma^2\sin^2\theta\dot{\phi} \Rightarrow \dot{\phi} = \frac{P_\phi}{ma^2\sin^2\theta}$$

Put the value of $\dot{\theta}$ and $\dot{\phi}$

$$H = P_\theta \times \frac{P_\theta}{ma^2} + P_\phi \times \frac{P_\phi}{ma^2\sin^2\theta} - \frac{1}{2}ma^2\left(\left(\frac{P_\theta}{ma^2}\right)^2 + \sin^2\theta\left(\frac{P_\phi}{ma^2\sin^2\theta}\right)^2\right) + mga\cos\theta$$

$$H = \frac{P_\theta^2}{ma^2} - \frac{P_\theta^2}{2ma^2} + \frac{P_\phi^2}{ma^2\sin^2\theta} - \frac{P_\phi^2}{2ma^2\sin^2\theta} + mga\cos\theta$$

$$H = \frac{1}{2ma^2}\left(P_\theta^2 + \frac{P_\phi^2}{\sin^2\theta}\right) + mga\cos\theta$$

So the correct answer is **Option (b)**

4. The Hamiltonian for a particle of mass m is $H = \frac{p^2}{2m} + kqt$ where q and p are the generalized coordinate and momentum, respectively, t is time and k is a constant. For the initial condition, $q = 0$ and $p = 0$ at $t = 0, q(t) \propto t^\alpha$. The value of α is ———

[GATE - 2019]

Solution:

$$\frac{\partial H}{\partial p} = \dot{q} = \frac{p}{m}$$

$$\frac{\partial H}{\partial q} = -\dot{p} = kt \Rightarrow p = -\frac{kt^2}{2}$$

$$\frac{dq}{dt} = -\frac{kt^2}{2} \Rightarrow q = -\frac{kt^3}{6} q \propto t^3 \quad \text{so } \alpha = 3$$

So the correct answer is **Option (3)**

5. Consider the Hamiltonian $H(q,p) = \frac{ap^2q^4}{2} + \frac{\beta}{q^2}$, where α and β are parameters with appropriate dimensions, and q and p are the generalized coordinate and momentum, respectively. The corresponding Lagrangian $L(q,\dot{q})$ is

[GATE - 2019]

a. $\frac{1}{2\alpha}\frac{\dot{q}^2}{q^4} - \frac{\beta}{q^2}$

b. $\frac{1}{2\alpha}\frac{\dot{q}^2}{q^4} + \frac{\beta}{q^2}$

c. $\frac{1}{\alpha}\frac{\dot{q}^2}{q^4} + \frac{\beta}{q^2}$

d. $-\frac{1}{2\alpha}\frac{\dot{q}^2}{q^4} + \frac{\beta}{q^2}$

Solution:

$$L = p\dot{q} - H \Rightarrow p\dot{q} - \frac{ap^2q^4}{2} - \frac{\beta}{q^2} \quad \text{from Hamiltonian equation of motion}$$

$$\frac{\partial H}{\partial p} = \dot{q} \Rightarrow p = \frac{\dot{q}}{aq^4}$$

$$L = \frac{1}{2\alpha}\frac{\dot{q}^2}{q^4} - \frac{\beta}{q^2}$$

So the correct answer is **Option (a)**

6. The Lagrangian for a particle of mass m at a position $\vec{r}$ moving with a velocity $\vec{v}$ is given by $L =$

$\frac{m}{2}\vec{v}^2 + C\vec{r}\cdot\vec{v} - V(r)$, where $V(r)$ is a potential and C is a constant. If $\vec{p}_c$ is the canonical momentum, then its Hamiltonian is given by

[GATE- 2015]

a. $\frac{1}{2m}(\vec{p}_c + C\vec{r})^2 + V(r)$
b. $\frac{1}{2m}(\vec{p}_c - C\vec{r})^2 + V(r)$
c. $\frac{p_c^2}{2m} + V(r)$
d. $\frac{1}{2m}p_c^2 + C^2r^2 + V(r)$

Solution:

$$L = \frac{m}{2}\vec{v}^2 + C\vec{r}\cdot\vec{v} - V(r) \quad \text{where } v = \dot{r}$$
$$H = \sum \dot{r}p_c - L = \dot{r}p_c - L$$
$$\Rightarrow \frac{\partial L}{\partial \dot{r}} = p_c = (m\dot{r} + Cr) \Rightarrow \dot{r} = \frac{p_c - Cr}{m}$$
$$\Rightarrow H = \left(\frac{p_c - Cr}{m}\right)p_c - \frac{m}{2}\left(\frac{p_c - Cr}{m}\right)^2 - cr\left(\frac{p_c - Cr}{m}\right) + V(r)$$
$$\Rightarrow H = \left(\frac{p_c - Cr}{m}\right)(p_c - Cr) - \frac{m}{2}\left(\frac{p_c - Cr}{m}\right)^2 + V(r)$$
$$\Rightarrow H = \frac{(p_c - Cr)^2}{m} - \frac{(p_c - Cr)^2}{2m} + V(r) \quad \Rightarrow H$$
$$= \frac{1}{2m}(p_c - Cr)^2 + V(r)$$

So the correct answer is **Option (b)**

7. The Hamiltonian for a system of two particles of masses m_1 and m_2 at $\vec{r}_1$ and $\vec{r}_2$ having velocities $\vec{v}_1$ and $\vec{v}_2$ is given by $H = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{C}{(\vec{r}_1-\vec{r}_2)^2}\hat{z}\cdot(\vec{r}_1 \times \vec{r}_2)$, where C is constant. Which one of the following statements is correct?

[GATE- 2015]

a. The total energy and total momentum are conserved
b. Only the total energy is conserved
c. The total energy and the z - component of the total angular momentum are conserved
d. The total energy and total angular momentum are conserved

Solution: Solution: Lagrangian is not a function of time, so energy is conserved and component of $(\vec{r}_1 \times \vec{r}_2)$ is only in $\hat{z}$ direction means potential is symmetric under ϕ, so L_z is conserved.
So the correct answer is **Option (c)**

8. The Hamilton's canonical equation of motion in terms of Poisson Brackets are

[GATE- 2014]

a. $\dot{q} = \{q,H\}; \dot{p} = \{p,H\}$
b. $\dot{q} = \{H,q\}; \dot{p} = \{H,p\}$
c. $\dot{q} = \{H,p\}; \dot{p} = \{H,p\}$
d. $\dot{q} = \{p,H\}; \dot{p} = \{q,H\}$

Solution:

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial t} + \frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial t} + \frac{\partial f}{\partial t}$$
$$\frac{df}{dt} = \frac{\partial f}{\partial q}\cdot\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\cdot\frac{\partial H}{\partial q} + \frac{\partial f}{\partial t} \Rightarrow \frac{df}{dt}$$
$$= \{f,H\} + \frac{\partial f}{\partial t}$$

$$\frac{dq}{dt} = \{q, H\} \text{ and } \frac{dp}{dt} = \{p, H\}$$

So the correct answer is **Option (a)**

9. The Poisson bracket $[x, xp_y + yp_x]$ is equal to

[GATE- 2017]

a. $-x$ **b.** y **c.** $2p_x$ **d.** p_y

Solution:

$$\begin{aligned}[x, xp_y + yp_x] &= [x, xp_y] + [x, yp_x] \\ &= 0 + y[x, p_x] = y\end{aligned}$$

So the correct answer is **Option (b)**

10. If H is the Hamiltonian for a free particle with mass m, the commutator $[x, [x, H]]$ is

[GATE - 2018]

a. $\hbar^2/m$ **b.** $-\hbar^2/m$ **c.** $-\hbar^2/(2m)$ **d.** $\hbar^2/(2m)$

Solution:

For free particle, potential is zero.

$$\begin{aligned}\Rightarrow H &= \frac{P_x^2}{2m} \\ \text{Now, } [x, H] &= \left[x, \frac{P_x^2}{2m}\right] = \frac{2i\hbar}{2m} P_x \\ [x, [x, H]] &= \frac{2i\hbar}{2m}[x, P_x] = \frac{i\hbar}{m}(i\hbar) = -\frac{\hbar^2}{m}\end{aligned}$$

So the correct answer is **Option (b)**

11. The Poisson bracket between θ and $\dot{\theta}$ is

[GATE- 2010]

a. $\{\theta, \dot{\theta}\} = 1$ **b.** $\{\theta, \dot{\theta}\} = \frac{1}{ml^2}$

c. $\{\theta, \dot{\theta}\} = \frac{1}{m}$ **d.** $\{\theta, \dot{\theta}\} = \frac{g}{l}$

Solution:

$$\begin{aligned}\{\theta, \dot{\theta}\} &= \left\{\theta, \frac{P_\theta}{ml^2}\right\} \text{ where } \dot{\theta} \\ &= \frac{P_\theta}{ml^2} \Rightarrow \frac{1}{ml^2}\left(\frac{\partial\theta}{\partial\theta}\frac{\partial\theta}{\partial P_\theta} - \frac{\partial\theta}{\partial P_\theta}\frac{\partial P_\theta}{\partial\theta}\right) \\ &= 1 \cdot \frac{1}{ml^2} - 0 = \frac{1}{ml^2}\end{aligned}$$

So the correct answer is **Option (b)**

12. A dynamical system with two generalized coordinates q_1 and q_2 has Lagrangian $L = \dot{q}_1^2 + \dot{q}_2^2$. If p_1 and p_2 are the corresponding generalized momenta, the Hamiltonian is given by

[JEST-2014]

a. $\left(p_1^2+p_2^2\right)/4$

b. $\left(\dot{q}_1^2+\dot{q}_2^2\right)/4$

c. $\left(p_1^2+p_2^2\right)/2$

d. $\left(p_1\dot{q}_1+p_2\dot{q}_2\right)/4$

Solution:

$$
\begin{aligned}
H &= \sum \dot{q}_i p_i - L \\
&= \dot{q}_1 p_1 + \dot{q}_2 p_2 - L \\
\frac{\partial L}{\partial \dot{q}_1} &= p_1 = 2\dot{q}_1 \\
\Rightarrow \dot{q}_1 &= \frac{p_1}{2} and \frac{\partial L}{\partial \dot{q}_2} = p_2 = 2\dot{q}_2 \\
\Rightarrow \dot{q}_2 &= \frac{p_2}{2} \\
H &= \frac{p_1}{2}\cdot p_1 + \frac{p_2}{2}\cdot p_2 - \frac{p_1^2}{4} - \frac{p_2^2}{4} \\
\Rightarrow H &= \frac{\left(p_1^2+p_2^2\right)}{4}
\end{aligned}
$$

So the correct answer is **Option (a)**

13. The Hamiltonian for a particle of mass m is given by $H=\frac{(p-\alpha q)^2}{(2m)}$, where α is a nonzero constant. Which one of the following equations is correct?

[JEST-2020]

a. $p=m\dot{q}$

b. $\alpha\dot{p}=\dot{q}$

c. $\ddot{q}=0$

d. $L=\frac{1}{2}m\dot{q}^2-\alpha q\dot{q}$

Solution:

$$
\begin{aligned}
H &= \frac{(P-\alpha q)^2}{2m} \\
\frac{\partial H}{\partial q} &= -\dot{p} \\
\frac{-\alpha(P-\alpha q)}{m} &= \dot{q} \\
p &= m\dot{q}+\alpha q \\
\dot{p} &= m\ddot{q}+\alpha\dot{q} = m\ddot{q}+\alpha\left(\frac{P-dq}{m}\right) \\
\alpha\frac{(P-\alpha q)}{m} &= m\ddot{q}+\frac{\alpha(P-\alpha q)}{m} \\
m\ddot{q} &= 0 = \ddot{q}
\end{aligned}
$$

So the correct answer is **Option (c)**

14. If the Poisson bracket $\{x,p\}=-1$, then the Poisson bracket $\left\{x^2+p,p\right\}$ is ?

[JEST-2013]

a. $-2x$

b. $2x$

c. 1

d. -1

Solution:

$$\left\{x^2+p,p\right\}=\left\{x^2,p\right\}+\{p,p\} \Rightarrow x\{x,p\}+\{x,p\}x+0 \Rightarrow x(-1)+(-1)x \Rightarrow -2x$$

So the correct answer is **Option (a)**

Answer key			
Q.No.	Answer	Q.No.	Answer
1	**a**	2	**a**
3	**b**	4	**3**
5	**a**	6	**b**
7	**c**	8	**a**
9	**b**	10	**b**
11	**b**	12	**a**
13	**c**	14	**a**