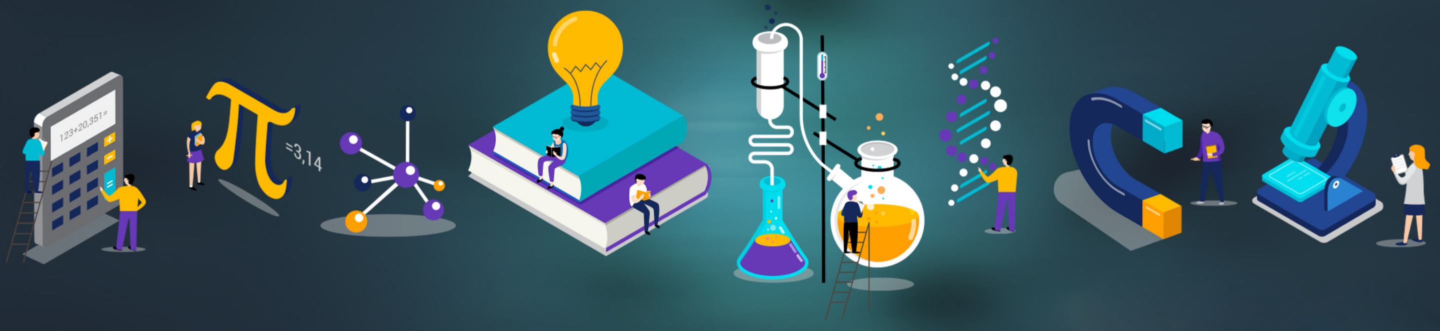


1. Hydrogen Atom Solutions

Practice Set-1

1. The energy levels of the non-relativistic electron in a hydrogen atom (i.e. in a Coulomb potential $V(r) \propto -1/r$) are given by $E_{nlm} \propto -1/n^2$, where n is the principal quantum number, and the corresponding wave functions are given by ψ_{nlm}, where l is the orbital angular momentum quantum number and m is the magnetic quantum number. The spin of the electron is not considered. Which of the following is a correct statement? **[NET JUNE 2011]**

   A. There are exactly $(2l+1)$ different wave functions ψ_{nlm}, for each E_{nlm}.

   B. There are $l(l+1)$ different wave functions ψ_{nlm}, for each E_{nlm}.

   C. E_{nlm} does not depend on l and m for the Coulomb potential.

   D. There is a unique wave function ψ_{nlm} for each E_{nlm}.

   Solution: The correct option is **(c)**

2. Let ψ_{nlm_l} denote the eigenfunctions of a Hamiltonian for a spherically symmetric potential $V(r)$. The wavefunction $\psi = \frac{1}{4}\left[\psi_{210} + \sqrt{5}\psi_{21-1} + \sqrt{10}\psi_{211}\right]$ is an eigenfunction only of **[NET JUNE 2012]**

   A. H, L^2 and L_2

   B. H and L_z

   C. H and L^2

   D. L^2 and L_z

   Solution: $H\psi = E_n\psi$
   $L^2\psi = l(l+1)\hbar^2\psi$ and $L_2\psi \neq m\hbar\psi$.
   The correct option is **c**

3. The wave function of a state of the Hydrogen atom is given by,

$$\psi = \psi_{200} + 2\psi_{211} + 3\psi_{210} + \sqrt{2}\psi_{21-1}$$

where ψ_{nlm} is the normalized eigen function of the state with quantum numbers n, l, m in the usual notation. The expectation value of L_z in the state ψ is

[NET DEC 2012]

A. $\frac{15\hbar}{6}$ **B.** $\frac{11\hbar}{6}$

C. $\frac{3\hbar}{8}$ **D.** $\frac{\hbar}{8}$

Solution: Firstly normalize ψ, $\psi = \frac{1}{\sqrt{16}}\psi_{200} + \frac{2}{\sqrt{16}}\psi_{211} + \frac{3}{\sqrt{16}}\psi_{210} + \frac{\sqrt{2}}{\sqrt{16}}\psi_{21-1}$
$P(0\hbar) = \frac{1}{16} + \frac{9}{16} = \frac{10}{16}$.
Probability of getting $(1\hbar)$ i.e. $P(\hbar) = \frac{4}{16}$ and $P(-\hbar) = \frac{2}{16}$.
Now, $\langle L_z \rangle = \frac{\langle \psi | L_z | \psi \rangle}{\langle \psi | \psi \rangle} = 0\hbar \times \frac{10}{16} + 1\hbar \times \frac{4}{16} + (-1\hbar) \times \frac{2}{16} = \frac{4}{16}\hbar - \frac{2}{16}\hbar = \frac{2}{16}\hbar = \frac{\hbar}{8}$
The correct option is **(d)**

4. Let ψ_{nlm} denote the eigenfunctions of a Hamiltonian for a spherically symmetric potential $V(r)$. The expectation value of L_z in the state

$$\psi = \frac{1}{6}\left[\psi_{200} + \sqrt{5}\psi_{210} + \sqrt{10}\psi_{21-1} + \sqrt{20}\psi_{211}\right] \text{ is}$$

[NET DEC 2013]

A. $-\frac{5}{18}\hbar$ **B.** $\frac{5}{6}\hbar$

C. $\hbar$ **D.** $\frac{5}{18}\hbar$

Solution:

$$\langle L_z \rangle = \langle \psi | L_z | \psi \rangle = \frac{1}{36} \times 0\hbar + \frac{5}{36} \times 0\hbar + \frac{10}{36} \times (-1\hbar) + \frac{20}{36}(1\hbar) = \frac{10}{36}\hbar = \frac{5}{18}\hbar \quad \because \langle \psi \mid \psi \rangle = 1$$

The correct option **(d)**

5. An electron is in the ground state of a hydrogen atom. The probability that it is within the Bohr radius is approximately equal to

[NET JUNE 2014]

A. 0.60 **B.** 0.90

C. 0.16 **D.** 0.32

Solution:

$$\int_0^{a_0} \left| \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} \right|^2 4\pi r^2 dr = \frac{4\pi}{\pi a_0^3} \int_0^{a_0} r^2 e^{-2r/a_0} dr$$

$$= \frac{4}{a_0^3}\left\{\left[r^2 e^{-2r/a_0}\left(-\frac{a_0}{2}\right)\right]_0^{a_0} - \left[2r\left(e^{-2r/a_0}\right)\left(-\frac{a_0}{2}\right)\left(-\frac{a_0}{2}\right)\right]_0^{a_0} + \left[2e^{-2r/a_0}\left(-\frac{a_0}{2}\right)\left(-\frac{a_0}{2}\right)\left(-\frac{a_0}{2}\right)\right]_0^{a_0}\right\}$$

$$= \frac{4}{a_0^3}\left[a_0^2 e^{\frac{2a_0}{a_0}}\left(-\frac{a_0}{2}\right) - 2a_0\left(\frac{a_0^2}{4}\right)e^{-2a_0/a_0} - \frac{a_0^3}{4}e^{-2a_0/a_0} + 2e^{-0}\left(\frac{a_0^3}{8}\right)\right]$$

$$= \frac{4}{a_0^3}\left[-\frac{a_0^3}{2}\frac{1}{e^2} - \frac{a_0^3}{2}\frac{1}{e^2} - \frac{a_0^3}{4e^2} + \frac{a_0^3}{4}\right] = 4\left[-\frac{5}{4e^2} + \frac{1}{4}\right] = \left[-5 \times \frac{1}{e^2} + 1\right]$$

$$= [-5 \times 0.137 + 1] = [-0.685 + 1] = 0.32$$

The correct option is **(d)**

6. Let ψ_{nlm} denote the eigenstates of a hydrogen atom in the usual notation. The state

$$\frac{1}{5}\left[2\psi_{200}-3\psi_{211}+\sqrt{7}\psi_{210}-\sqrt{5}\psi_{21-1}\right]$$

is an eigenstate of

[NET DEC 2015]

A. L^2, but not of the Hamiltonian or L_z

B. the Hamiltonian, but not of L^2 or L_z

C. the Hamiltonian, L^2 and L_z

D. L^2 and L_z, but not of the Hamiltonian

Solution: $|\psi\rangle=\frac{1}{5}\left[2\psi_{200}-3\psi_{211}+\sqrt{7}\psi_{210}-\sqrt{5}\psi_{21-1}\right]$
$H|\psi\rangle=-\frac{13.6}{4}|\psi\rangle$
So $|\psi\rangle$ is eigen state of H
But $L^2|\psi\rangle\neq\alpha|\psi\rangle$ and $L_z|\psi\rangle\neq\beta|\psi\rangle$
So $|\psi\rangle$ is not eigen state of L^2 and L_z
The correct option is **(b)**

7. If the position of the electron in the ground state of a Hydrogen atom is measured, the probability that it will be found at a distance $r\geq a_0$ (a_0 being Bohr radius) is nearest to

[NET DEC 2018]

A. 0.91

B. 0.66

C. 0.32

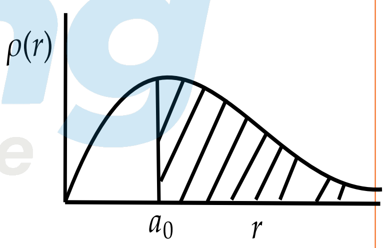
D. 0.13

Solution:

$$P\left(a_0\leq r<\infty\right)=\int_{a_0}^{\infty}r^2\left|R_{10}\right|^2 dr$$

$$R_{10}=\frac{2}{a_0^{3/2}}$$

$$P\left(a_0\leq r<\infty\right)=\frac{4}{a_0^3}\int_{a_0}^{\infty}r^2e^{-\frac{2r}{a_0}}dr=0.66$$



The correct option is **(b)**

Practice Set-2

1. The normalized ground state wavefunciton of a hydrogen atom is given by $\psi(r) = \frac{1}{\sqrt{4\pi}}\frac{2}{a^{3/2}}e^{-r/a}$, where a is the Bohr radius and r is the distance of the electron from the nucleus, located at the origin. The expectation value $\left\langle \frac{1}{r^2} \right\rangle$ is

[GATE 2011]

A. $\frac{8\pi}{a^2}$ **B.** $\frac{4\pi}{a^2}$

C. $\frac{4}{a^2}$ **D.** $\frac{2}{a^2}$

Solution:

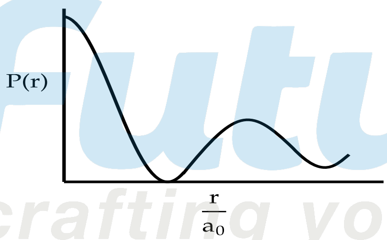
$$\left\langle \frac{1}{r^2} \right\rangle = \frac{4}{4\pi a^3}\int_0^\infty \frac{1}{r^2} r^2 e^{-\frac{2r}{a}} dr \int_0^\pi \int_0^{2\pi} \sin\theta d\theta d\phi = \frac{2}{a^2}$$

THe correct option is **(d)**

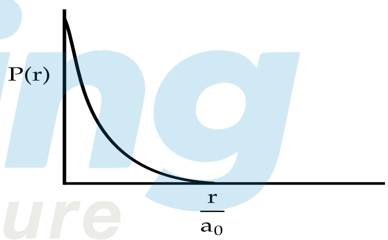
2. The ground state wavefunction for the hydrogen atom is given by $\psi_{100} = \frac{1}{\sqrt{4\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$, where a_0 is the Bohr radius. The plot of the radial probability density, $P(r)$ for the hydrogen atom in the ground state is

[GATE 2012]

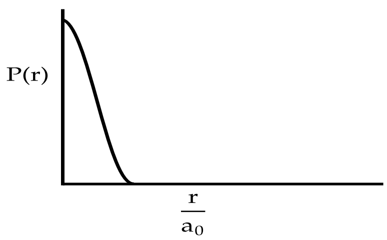
A.



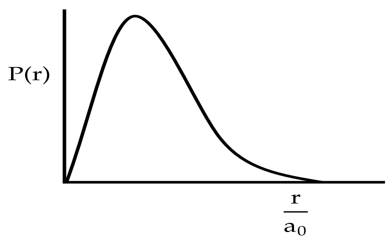
B.



C.



D.



Solution: n : The ground state is given by $\psi_{100} = \frac{1}{\sqrt{4\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$ Radial probability function $P(r) = |\psi|^2 r^2 = \frac{1}{4\pi}\frac{1}{a_0^3} r^2 e^{-2r/a_0}$

The correct option is **(d)**

3. An electron in the ground state of the hydrogen atom has the wave function $\psi(\vec{r}) = \frac{1}{\sqrt{\pi a_0^3}} e^{-\left(\frac{r}{a_0}\right)}$, where a_0 is constant. The expectation value of the operator $\hat{Q} = z^2 - r^2$, where $z = r\cos\theta$ is (Hint: $\int_0^\infty e^{-ar} r^n dr = \frac{\sqrt{n}}{a^{n+1}} = \frac{(n-1)!}{a^{n+1}}$)

[GATE 2014]

A. $\frac{-a_0^2}{2}$

B. $-a_0^2$

C. $\frac{-3a_0^2}{2}$

D. $-2a_0^2$

Solution:

$$\langle \hat{Q} \rangle = \langle z^2 \rangle - \langle r^2 \rangle \Rightarrow a_0^2 - 3a_0^2 = -2a_0^2$$

4. A hydrogen atom is in the state

$$\psi = \sqrt{\frac{8}{21}}\psi_{200} - \sqrt{\frac{3}{7}}\psi_{310} + \sqrt{\frac{4}{21}}\psi_{321},$$

where n, l, m in ψ_{nlm} denote the principal, orbital and magnetic quantum numbers, respectively. If $\vec{L}$ is the angular momentum operator, then the average value of L^2 is $\cdots \quad \hbar^2$

[GATE 2014]

Solution: If L^2 will measure on state ψ the measurement is $0\hbar^2, 2\hbar^2$ and $6\hbar^2$ with probability

$$\frac{8}{21}, \frac{3}{7}, \frac{4}{21} \text{ so }, \langle L^2 \rangle = 2\hbar^2 \times \frac{3}{7} + 6\hbar^2 \times \frac{4}{21} = 2\hbar^2$$

5. An electric field $\vec{E} = E_0\hat{z}$ is applied to a Hydrogen atom in $n = 2$ excited state. Ignoring spin the $n = 2$ state is fourfold degenerate, which in the $|l, m\rangle$ basis are given by $|0,0\rangle, |1,1\rangle, |1,0\rangle$ and $|1,-1\rangle$. If H' is the interaction Hamiltonian corresponding to the applied electric field, which of the following matrix elements is nonzero?

[GATE 2019]

A. $\langle 0,0|H'|0,0\rangle$

B. $\langle 0,0|H'|1,1\rangle$

C. $\langle 0,0|H'|1,0\rangle$

D. $\langle 0,0|H'|1,-1\rangle$

Solution: The correct option is **(c)**

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