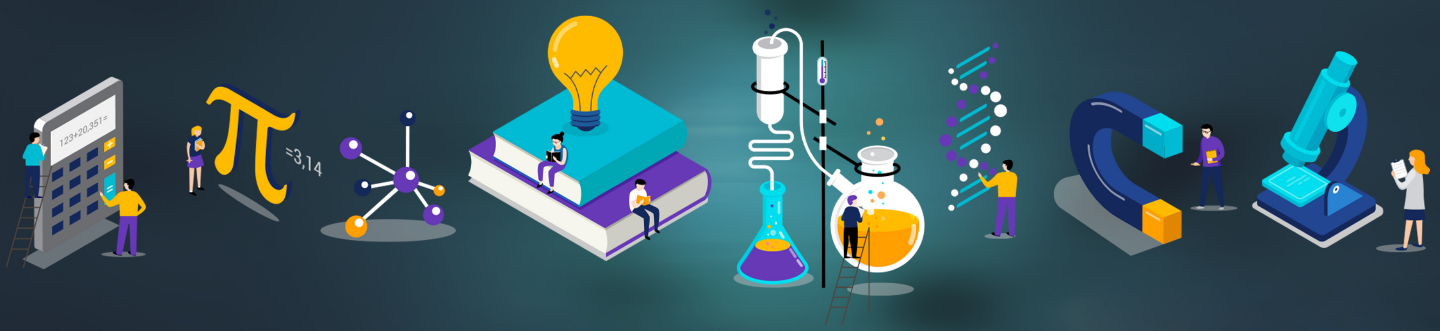

1. Angular Momentum Solutions

Practice set-1

1. The Hamiltonian of an electron in a constant magnetic field $\vec{B}$ is given by $H=\mu\vec{\sigma}\cdot\vec{B}$. where μ is a positive constant and $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ denotes the Pauli matrices. Let $\omega=\mu B/\hbar$ and I be the 2×2 unit matrix. Then the operator $e^{iHt/\hbar}$ simplifies to

[NET JUNE 2011]

A. $I\cos\frac{\omega t}{2}+\frac{i\vec{\sigma}\cdot\vec{B}}{B}\sin\frac{\omega t}{2}$

B. $I\cos\omega t+\frac{i\vec{\sigma}\cdot\vec{B}}{B}\sin\omega t$

C. $I\sin\omega t+\frac{i\vec{\sigma}\cdot\vec{B}}{B}\cos\omega t$

D. $I\sin 2\omega t+\frac{i\vec{\sigma}\cdot\vec{B}}{B}\cos 2\omega t$

Solution: $H=\mu\vec{\sigma}\vec{B}$ where $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ are pauli spin matrices and $\vec{B}$ are constant magnetic field. $\vec{\sigma}=\left(\sigma_1\hat{i},\sigma_2\hat{j},\sigma_3\hat{k}\right),\vec{B}=\left(B_x\hat{i}+B_y\hat{j}+B_z\hat{k}\right)$ and Hamiltonion $H=\mu\vec{\sigma}\cdot\vec{B}$ in matrices form is given by

$$H=\mu\begin{pmatrix} B_z & B_x-iB_y \\ B_x+iB_y & -B_z\end{pmatrix}.$$

Eigenvalue of given matrices are given by $+\mu$B and $-\mu$B.H matrices are not diagonals so $e^{iHt/\hbar}$ is equivalent to

$$S^{-1}\begin{pmatrix} e^{\frac{i\mu Bt}{\hbar}} & 0 \\ 0 & e^{\frac{-i\mu Bt}{\hbar}}\end{pmatrix}S$$

where S is unitary matrices

and $\quad S^{-1}=S=\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}.$

$$S^{-1}\begin{pmatrix} e^{\frac{i\mu Bt}{\hbar}} & 0 \\ 0 & e^{\frac{-i\mu Bt}{\hbar}}\end{pmatrix}S=\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix} e^{\frac{i\mu Bt}{\hbar}} & 0 \\ 0 & e^{\frac{-i\mu Bt}{\hbar}}\end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix},$$

where $\omega=\mu\text{B}/\hbar$.

$$e^{iHt/\hbar}=\begin{pmatrix} \cos\omega t & i\sin\omega t \\ i\sin\omega t & \cos\omega t\end{pmatrix},$$

which is equivalent to $I\cos\omega t + i\sigma_x \sin\omega t$ can be written as $I\cos\omega t + \frac{i\vec{\sigma}\cdot\vec{B}}{B}\sin\omega t$, where $\sigma_x = \frac{i\vec{\sigma}\cdot B}{B}$
The correct option is **(b)**

2. In a system consisting of two spin $\frac{1}{2}$ particles labeled 1 and 2, let $\vec{S}^{(1)} = \frac{\hbar}{2}\vec{\sigma}^{(1)}$ and $\vec{S}^{(2)} = \frac{\hbar}{2}\vec{\sigma}^{(2)}$ denote the corresponding spin operators. Here $\vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$ and $\sigma_x, \sigma_y, \sigma_z$ are the three Pauli matrices.
In the standard basis the matrices for the operators $S_x^{(1)}S_y^{(2)}$ and $S_y^{(1)}S_x^{(2)}$ are respectively,

[NET JUNE 2011]

A. $\frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{\hbar^2}{4}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

B. $\frac{\hbar^2}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{\hbar^2}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

C. $\frac{\hbar^2}{4}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \frac{\hbar^2}{4}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$

D. $\frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Solution: $S_x^{(1)}S_y^{(2)} = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \Rightarrow \frac{\hbar^2}{4}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$

$$S_y^{(1)}S_x^{(2)} = \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \frac{\hbar^2}{4}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

The correct option is **c**

3. These two operators of above QUESTION satisfy the relation

[NET JUNE 2011]

A. $\left\{S_x^{(1)}S_y^{(2)}, S_y^{(1)}S_x^{(2)}\right\} = S_z^{(1)}S_z^{(2)}$

B. $\left\{S_x^{(1)}S_y^{(2)}, S_y^{(1)}S_x^{(2)}\right\} = 0$

C. $\left[S_x^{(1)}S_y^{(2)}, S_y^{(1)}S_x^{(2)}\right] = iS_z^{(1)}S_z^{(2)}$

D. $\left[S_x^{(1)}S_y^{(2)}, S_y^{(1)}S_x^{(2)}\right] = 0$

Solution: We have matrix $S_x^{(1)}S_y^{(2)}$ and $S_y^{(1)}S_x^{(2)}$ from question 6(A) so commutation is given by $\left[S_x^{(1)}S_y^{(2)}, S_y^{(1)}S_x^{(2)}\right] = 0$.
The correct option is **(d)**

4. The component along an arbitrary direction $\hat{n}$, with direction cosines (n_x, n_y, n_z), of the spin of a spin $-\frac{1}{2}$ particle is measured. The result is

[NET JUNE 2012]

A. 0

B. $\pm\frac{\hbar}{2}n_z$

C. $\pm\frac{\hbar}{2}(n_x + n_y + n_z)$

D. $\pm\frac{\hbar}{2}$

Solution: $S_x=\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}, S_y=\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}, S_z=\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$\overrightarrow{\mathrm{n}}=\mathrm{n_x}\hat{\mathrm{i}}+\mathrm{n_y}\hat{\mathrm{j}}+\mathrm{n_z}\hat{\mathrm{k}}$ and $n_x^2+n_y^2+n_z^2=1, \vec{S}=S_x\hat{i}+S_y\hat{j}+S_z\hat{k}$

$\vec{n}\cdot\vec{S}=n_x\begin{pmatrix}0&\frac{\hbar}{2}\\\frac{\hbar}{2}&\end{pmatrix}+n_y\begin{pmatrix}0&-\frac{i\hbar}{2}\\\frac{i\hbar}{2}&0\end{pmatrix}+n_z\begin{pmatrix}\frac{\hbar}{2}&0\\0&\frac{-\hbar}{2}\end{pmatrix}$

$\vec{n}\cdot\vec{S}=\begin{pmatrix}n_z\frac{\hbar}{2}&\frac{\hbar}{2}(n_x-in_y)\\\frac{\hbar}{2}(n_x+in_y)&-n_z\frac{\hbar}{2}\end{pmatrix}$

Let λ is eigen value of $\vec{n}\cdot\vec{S}$

$$\begin{vmatrix}n_z\frac{\hbar}{2}-\lambda&\frac{\hbar}{2}(n_x-in_y\\\frac{\hbar}{2}(n_x+in_y)&-n_z\frac{\hbar}{2}-\lambda\end{vmatrix}=0$$

$\Rightarrow-\left(\frac{\mathrm{n_z}\hbar}{2}-\lambda\right)\left(\frac{\mathrm{n_z}\hbar}{2}+\lambda\right)-\frac{\hbar^2}{4}\left(\mathrm{n_x^2}+\mathrm{n_y^2}\right)=0\Rightarrow-\left(\frac{n_z^2\hbar^2}{4}-\lambda^2\right)-\frac{\hbar^2}{4}\left(n_x^2+n_y^2\right)=0$

$\Rightarrow-\frac{\hbar^2}{4}\left(n_x^2+n_y^2+n_z^2\right)+\lambda^2=0\Rightarrow\lambda=\pm\frac{\hbar}{2}$

The correct option is **(d)**

5. In a basis in which the z - component S_z of the spin is diagonal, an electron is in a spin state $\psi=\begin{pmatrix}(1+i)/\sqrt{6}\\\sqrt{2/3}\end{pmatrix}$. The probabilities that a measurement of S_2 will yield the values $\hbar/2$ and $-\hbar/2$ are, respectively,

[NET JUNE 2013]

A. $1/2$ and $1/2$

B. $2/3$ and $1/3$

C. $1/4$ and $3/4$

D. $1/3$ and $2/3$

Solution: Eigen state of S_z is $|\phi_1\rangle=\begin{pmatrix}1\\0\end{pmatrix}$ and $|\phi_2\rangle=\begin{pmatrix}0\\1\end{pmatrix}$ corresponds to Eigen value $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ respectively.

$$P\left(\frac{\hbar}{2}\right)=\frac{|\langle\phi_1\mid\psi\rangle|^2}{\langle\psi\mid\psi\rangle}=\left|\frac{1+i}{\sqrt{6}}\right|^2=\frac{2}{6}=\frac{1}{3},\quad P\left(-\frac{\hbar}{2}\right)=\frac{|\langle\phi_2\mid\psi\rangle|^2}{\langle\psi\mid\psi\rangle}=\frac{2}{3}$$

The correct option is **(d)**

6. A spin $-\frac{1}{2}$ particle is in the state $\chi=\frac{1}{\sqrt{11}}\begin{pmatrix}1+i\\3\end{pmatrix}$ in the eigenbasis of S^2 and S_2. If we measure S_z, the probabilities of getting $+\frac{h}{2}$ and $-\frac{h}{2}$, respectively are

[NET DEC 2013]

A. $\frac{1}{2}$ and $\frac{1}{2}$

B. $\frac{2}{11}$ and $\frac{9}{11}$

C. 0 and 1

D. $\frac{1}{11}$ and $\frac{3}{11}$

Solution: $P\left(\frac{\hbar}{2}\right)=\left|\frac{1}{\sqrt{11}}(10)\begin{pmatrix}1+i\\3\end{pmatrix}\right|^2=\frac{1}{11}\times2=\frac{2}{11}\quad\because\langle\psi\mid\psi\rangle=1$

$P\left(-\frac{\hbar}{2}\right)=\left|\frac{1}{\sqrt{11}}(01)\begin{pmatrix}1+i\\3\end{pmatrix}\right|^2=\frac{9}{11}$

i.e. probability of S_z getting $\left(\frac{\hbar}{2}\right)$ and $\left(-\frac{\hbar}{2}\right)$

The correct option is **b**

7. Let $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$, where $\sigma_1,\sigma_2,\sigma_3$ are the Pauli matrices. If $\vec{a}$ and $\vec{b}$ are two arbitrary constant vectors in three dimensions, the commutator $[\vec{a}\cdot\vec{\sigma},\vec{b}\cdot\vec{\sigma}]$ is equal to (in the following I is the identity matrix)

[NET DEC 2014]

A. $(\vec{a}\cdot\vec{b})(\sigma_1+\sigma_2+\sigma_3)$

B. $2i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$

C. $(\vec{a}\cdot\vec{b})I$

D. $|\vec{a}||\vec{b}|I$

Solution: $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}, \sigma=\sigma_x\hat{i}+\sigma_y\hat{j}+\sigma_z\hat{k}$

$$[\vec{a}\cdot\vec{\sigma},\vec{b}\cdot\vec{\sigma}]=[a_1\sigma_x+a_2\sigma_y+a_3\sigma_z,b_1\sigma_x+b_2\sigma_y+b_3\sigma_z]$$
$$[\vec{a}\cdot\vec{\sigma},\vec{b}\cdot\vec{\sigma}]=a_1b_1[\sigma_x,\sigma_x]+a_1b_2[\sigma_x,\sigma_y]+a_1b_3[\sigma_x,\sigma_z]+a_2b_1[\sigma_y,\sigma_x]+a_2b_2[\sigma_y,\sigma_y]$$
$$+a_2b_3[\sigma_y,\sigma_z]+a_3b_1[\sigma_z,\sigma_x]+a_3b_2[\sigma_z,\sigma_y]+a_3b_3[\sigma_z,\sigma_z]$$
$$=a_1b_1\cdot 0+a_1b_2\cdot 2i\sigma_z-2ia_1b_3\sigma_y-a_2b_1\cdot 2i\sigma_z+0+a_2b_3\cdot 2i\sigma_x+a_3b_1\cdot 2i\sigma_y-a_3b_2\cdot 2i\sigma_x+0$$
$$\Rightarrow[\vec{a}\cdot\vec{\sigma},\vec{b}\cdot\vec{\sigma}]==2i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

The correct option is **(b)**

8. If L_i are the components of the angular momentum operator $\vec{L}$, then the operator $\sum_{i=1,2,3}\left[\vec{L},L_i\right]$ equals **[NET JUNE 2015]**

A. $\vec{L}$

B. $2\vec{L}$

C. $3\vec{L}$

D. $-\vec{L}$

Solution: Let $\vec{L}=L_x\hat{i}+L_y\hat{j}+L_z\hat{k}$

$$x=1,y=2,z=3$$
$$\left[\vec{L},L_x\right]=[L_y,L_x]\,j+[L_z,L_x]\,\hat{k}=-i\hbar L_z\hat{j}+L_y\hat{k}i\hbar$$
$$\left[\left[\vec{L},L_x\right],L_x\right]=i\hbar[-L_z,L_x]\,\hat{j}+[L_y,L_x]\,i\hbar-i\hbar.i\hbar L_y\hat{j}-(i\hbar)L_z(i\hbar)L_z(i\hbar)\cdot\hat{k}=\hbar^2\left[L_y\hat{j}+L_z\hat{k}\right]$$
$$\text{similarly, }\left[\left[\vec{L},L_y\right]L_y\right]=\hbar^2\left[L_x\hat{i}+L_z\hat{k}\right]$$
$$\left[\left[\vec{L},L_z\right]L_z\right]=\hbar^2\left[L_x\hat{i}+L_y\hat{j}\right]$$
$$\sum_{i=1,2,3}[[L,Li]Li]=2\hbar^2\left[L_x\hat{i}+L_y\hat{j}+L_z\hat{k}\right]=2\vec{L}\quad\text{put }\hbar=1$$

The correct option is **(b)**

9. The Hamiltonian for a spin- $\frac{1}{2}$ particle at rest is given by $H=E_0(\sigma_z+\alpha\sigma_x)$, where σ_x and σ_z are Pauli spin matrices and E_0 and α are constants. The eigenvalues of this Hamiltonian are **[NET DEC 2015]**

A. $\pm E_0\sqrt{1+\alpha^2}$

B. $\pm E_0\sqrt{1-\alpha^2}$

C. E_0 (doubly degenerate)

D. $E_0\left(1\pm\frac{1}{2}\alpha^2\right)$

Solution: $H=E_0(\dot{\sigma}_z+\alpha\sigma_x)=E_0\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\alpha\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]\Rightarrow H=E_0\begin{pmatrix}1&\alpha\\\alpha&-1\end{pmatrix}$ if λ is eigen value, then

$$H-\lambda I=0\Rightarrow E_0\begin{pmatrix}(1-\lambda)&\alpha\\\alpha&-(1+\lambda)\end{pmatrix}=0,\quad\lambda=\pm E_0\sqrt{1+\alpha^2}$$

The correct option is **(a)**

10. If $\hat{L}_x, \hat{L}_y, \hat{L}_z$ are the components of the angular momentum operator in three dimensions the commutator $\left[\hat{L}_x, \hat{L}_x\hat{L}_y\hat{L}_z\right]$ may be simplified to

[NET JUNE 2016]

A. $i\hbar L_x\left(\hat{L}_z^2-\hat{L}_y^2\right)$

B. $i\hbar\hat{L}_z\hat{L}_y\hat{L}_x$

C. $i\hbar L_x\left(2\hat{L}_z^2-\hat{L}_y^2\right)$

D. 0

Solution:

$$\begin{aligned}
&: \left[L_x, L_xL_yL_z\right] = L_x\left[L_x, L_yL_z\right] + \left[L_x, L_x\right]L_yL_z \\
&= L_x\left[L_x, L_y\right]L_z + L_xL_y\left[L_x, L_z\right] + 0 = L_x\left[i\hbar L_z\right]L_z + L_xL_y\left(-i\hbar L_y\right) \\
&= i\hbar L_xL_z^2 - i\hbar L_xL_y^2 = i\hbar L_x\left(L_z^2 - L_y^2\right)
\end{aligned}$$

The correct option is **(a)**

11. The Hamiltonian of a spin $\frac{1}{2}$ particle in a magnetic field $\vec{B}$ is given by $H=-\mu\cdot\vec{B}\cdot\vec{\sigma}$, where μ is a real constant and $\vec{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ are the Pauli spin matrices. If $\vec{B}=(B_0,B_0,0)$ and the spin state at time $t=0$ is an eigenstate of σ_x, then of the expectation values $\langle\sigma_x\rangle, \langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$

[NET JUNE 2018]

A. only $\langle\sigma_x\rangle$ changes with time

B. only $\langle\sigma_y\rangle$ changes with time

C. only $\langle\sigma_z\rangle$ changes with time

D. all three change with time

Solution: $\langle\sigma_x\rangle, \langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$ will changes with time because Eigen state of σ_x ie $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ and can be written in basis of eigen state of $H=-\mu\cdot\vec{B}\cdot\vec{\sigma}=-B_0\begin{pmatrix}0 & 1-i\\1+i & 0\end{pmatrix}$
THe correct option is **(d)**

Practice Set-2

1. For a spin-s particle, in the eigen basis of $\vec{S}^2, S_x$ the expectation value $\langle sm|S_x^2|sm\rangle$ is

[GATE 2010]

A. $\frac{\hbar^2\{s(s+1)-m^2\}}{2}$ **B.** $\hbar^2\{s(s+1)-2m^2\}$

C. $\hbar^2\{s(s+1)-m^2\}$ **D.** $\hbar^2 m^2$

Solution:

$$\langle sm|S_x^2|sm\rangle = \frac{1}{4}\left\langle sm\left|(S_+ + S_-)^2\right|sm\right\rangle = \frac{1}{4}\langle sm|S_+^2 + S_-^2 + S_+S_- + S_-S_+|sm\rangle$$
$$= \frac{1}{4}\langle sm|S_+S_- + S_-S_+|sm\rangle = \frac{\hbar^2}{2}\left[s(s+1)-m^2\right] \quad \left[\because S_+S_- + S_-S_+ = 2\left(S^2 - S_z^2\right)\right]$$

The correct option is **(a)**

2. If L_x, L_y and L_z are respectively the x, y and z components of angular momentum operator L. The commutator $[L_xL_y, L_z]$ is equal to

[GATE 2011]

A. $i\hbar\left(L_x^2 + L_y^2\right)$ **B.** $2i\hbar L_z$

C. $i\hbar\left(L_x^2 - L_y^2\right)$ **D.** 0

Solution: $[L_xL_y, L_z] = L_x[L_yL_z] + [L_x, L_z]L_y = i\hbar\left(L_x^2 - L_y^2\right)$
The correct option is **(c)**

3. Which one of the following commutation relations is NOT CORRECT? Here, symbols have their usual meanings.

[GATE 2013]

A. $[L^2, L_z] = 0$ **B.** $[L_x, L_y] = i\hbar L_z$

C. $[L_z, L_+] = \hbar L_+$ **D.** $[L_z, L_-] = \hbar L_-$

Solution: The correct option is **(d)**

4. A spin-half particle is in a linear superposition $0.8|\uparrow\rangle + 0.6|\downarrow\rangle$ of its spin-up and spindown states. If $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of σ_z, then what is the expectation value up to one decimal place, of the operator $10\sigma_z + 5\sigma_x$? Here, symbols have their usual meanings.

[GATE 2013]

Solution:

$$\psi\rangle = .8|\uparrow\rangle + .6|\downarrow\rangle = 0.8\begin{pmatrix}1\\0\end{pmatrix} + 0.6\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0.8\\0.6\end{pmatrix}$$
$$\text{Operator } A = 10\sigma_z + 5\sigma_x = 10\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} + 5\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \Rightarrow A = \begin{pmatrix}10 & 5\\5 & -10\end{pmatrix}$$
$$\langle A\rangle = \langle\psi|A|\psi\rangle = \begin{pmatrix}0.8 & 0.6\end{pmatrix}\begin{pmatrix}10 & 5\\5 & -10\end{pmatrix}\begin{pmatrix}0.8\\0.6\end{pmatrix} = (8.8-1.2) = 7.6$$

5. If $\vec{L}$ is the orbital angular momentum and $\bar{S}$ is the spin angular momentum, then $\vec{L}\cdot\vec{S}$ does not commute with

[GATE 2014]

A. S_z

B. L^2

C. S^2

D. $(\vec{L}+\vec{S})^2$

Solution: The correct option is **(d)**

6. If $L_+$and $L_-$are the angular momentum ladder operators then the expectation value of $(L_+L_- + L_-L_+)$in the state $|l=1, m=1\rangle$ of an atom is $\hbar^2$

[GATE 2014]

Solution: $(L_+L_- + L_-L_+) = 2\left(L^2 - L_z^2\right) = 2\left(l.(l+1) - m^2\right)\hbar^2 = 2\hbar^2$

7. The Pauli matrices for three spin $-\frac{1}{2}$ particles are $\vec{\sigma}_1, \vec{\sigma}_2$ and $\vec{\sigma}_3$, respectively. The dimension of the Hilbert space required to define an operator $\hat{O} = \vec{\sigma}_1 \cdot \vec{\sigma}_2 \times \vec{\sigma}_3$ is

[GATE 2015]

Solution: $\sigma_2 \times \sigma_3$ has dimension of 4 and $\sigma_1 . \sigma_2 \times \sigma_3$ has dimension of $2 \times 4 = 8$

8. Let the Hamiltonian for two spin-1/2 particles of equal masses m, momenta $\vec{p}_1$ and $\vec{p}_2$ and positions $\vec{r}_1$ and $\vec{r}_2$ be $H = \frac{1}{2m}p_1^2 + \frac{1}{2m}p_2^2 + \frac{1}{2}m\omega^2\left(r_1^2 + r_2^2\right) + k\vec{\sigma}_1 \cdot \vec{\sigma}_2$, where $\vec{\sigma}_1$ and $\vec{\sigma}_2$ denote the corresponding Pauli matrices, $\hbar\omega = 0.1\text{eV}$ and $k = 0.2\text{eV}$. If the ground state has net spin zero, then the energy (in eV) is

[GATE 2015]

Solution: $H = \frac{1}{2m}p_1^2 + \frac{1}{2m}p_2^2 + \frac{1}{2}m\omega^2\left(r_1^2 + r_2^2\right) + k\vec{\sigma}_1 \cdot \vec{\sigma}_2$

$$\vec{\sigma} = \vec{\sigma_1} + \vec{\sigma}_2 \Rightarrow \vec{\sigma}^2 = \sigma_1^2 + \sigma_2^2 + 2\vec{\sigma}_1 \cdot \vec{\sigma}_2 \Rightarrow 2\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \vec{\sigma}^2 - \sigma_1^2 - \sigma_2^2$$
$$\Rightarrow 2\vec{\sigma}_1 \cdot \vec{\sigma}_2 = 0 - 3I - 3I = -6I \Rightarrow \vec{\sigma}_1 \cdot \vec{\sigma}_2 = -3$$

Now energy $E = 2 \times \frac{3}{2}\hbar\omega + k(-3) = 3 \times (0.1) + (0.2)(-3) = -0.3\text{eV}$

9. If $\vec{s}_1$ and $\vec{s}_2$ are the spin operators of the two electrons of a He atom, the value o $\langle \vec{s}_1 \cdot \vec{s}_2 \rangle$ for the ground state is

[GATE 2016]

A. $-\frac{3}{2}\hbar^2$

B. $-\frac{3}{4}\hbar^2$

C. 0

D. $\frac{1}{4}\hbar^2$

Solution: $\vec{s} = \vec{s}_1 + \vec{s}_2, s_1 = \frac{1}{2}, s_1 = \frac{1}{2}, s = 0, 1$

$$\langle \vec{s}_1 \cdot \vec{s}_2 \rangle = \frac{s(s+1)\hbar^2 - s_1(s_1+1)\hbar^2 - s_2(s_2+1)\hbar^2}{2}$$

For

$$s = 1, \langle \vec{s}_1 \cdot \vec{s}_2 \rangle = \frac{2\hbar^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2}{2} = \frac{3}{4}\hbar^2$$
$$s = 0, \langle \vec{s}_1 \cdot \vec{s}_2 \rangle = \frac{0\hbar^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2}{2} = -\frac{3}{4}\hbar^2$$

The correct option is **(b)**

10. σ_x, σ_y and σ_z are the Pauli matrices. The expression $2\sigma_x\sigma_y + \sigma_y\sigma_x$ is equal to

[GATE 2016]

A. $-3i\sigma_z$

B. $-i\sigma_z$

C. $i\sigma_z$

D. $3i\sigma_z$

Solution: $2\sigma_x\sigma_y + \sigma_y\sigma_x \Rightarrow \sigma_x\sigma_y + \sigma_x\sigma_y + \sigma_y\sigma_x \Rightarrow \sigma_x\sigma_y = i\sigma_z$
The correct option is **(c)**

11. For the Hamiltonian $H = a_0 I + \vec{b}\cdot\vec{\sigma}$ where $a_0 \in R, \vec{b}$ is a real vector, I is the 2×2 identity matrix, and $\vec{\sigma}$ are the Pauli matrices, the ground state energy is

[GATE 2017]

A. $|b|$

B. $2a_0 - |b|$

C. $a_0 - |b|$

D. a_0

Solution: $a_0 I + \vec{b}\cdot\vec{\sigma} = a_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b_x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + b_y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + b_z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$= \begin{pmatrix} a_0 + b_z & b_x - ib_y \\ b_x + ib_y & a_0 - b_z \end{pmatrix}$
$H = a_0 I + \vec{b}\cdot\vec{\sigma} = \begin{pmatrix} a_0 + b_z & b_x - ib_y \\ b_x + ib_y & a_0 - b_z \end{pmatrix}$
For eigen value $\begin{pmatrix} a_0 + b_z - \lambda & b_x - ib_y \\ b_x + ib_y & a_0 - b_z - \lambda \end{pmatrix} = 0$

$$(a_0 + b_z - \lambda)(a_0 - b_z - \lambda) - (b_x^2 + b_y^2) = 0$$
$$\lambda_1 = a_0 - |b|, \lambda_1 = a_0 + |b|$$

The correct option is **(c)**