Problem Set -1

1. Let $x_1(t)$ and $x_2(t)$ be two linearly independent solutions of the differential equation $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+f(t)x=0$ and let $w(t)=x_1(t)\frac{dx_2(t)}{dt}-x_2(t)\frac{dx_1(t)}{dt}$. If $w(0)=1$, then $w(1)$ is given by

[NET/JRF(DEC-2011)]

A. 1 **B.** e^2 **C.** $1/e$ **D.** $1/e^2$

Solution:

$W(t)$ is Wronskian of D.E.

$$W=e^{-\int \mathrm{P}dt}=e^{-2t}\Rightarrow W(1)$$
$$=e^{-2}\text{ since }P=2$$

So the correct answer is **Option (D)**

2. Let $y(x)$ be a continuous real function in the range 0 and 2π, satisfying the inhomogeneous differential equation: $\sin x\frac{d^2y}{dx^2}+\cos x\frac{dy}{dx}=\delta\left(x-\frac{\pi}{2}\right)$ The value of $dyldx$ at the point $x=\pi/2$

[NET/JRF (JUNE-2012)]

A. Is continuous **B.** Has a discontinuity of 3

C. Has a discontinuity of $1/3$ **D.** Has a discontinuity of 1

Solution:

After dividing by $\sin x$, $\frac{d^2y}{dx^2}+\cot x\frac{dy}{dx}=\operatorname{cosec} x\cdot\delta\left(x-\frac{\pi}{2}\right)$

Integrating both sides, $\frac{dy}{dx}+\int\cot x\left(\frac{dy}{dx}\right)dx=\int\operatorname{cosec} x\,\delta\left(x-\frac{\pi}{2}\right)dx$

$$\frac{dy}{dx}+\cot x\cdot y-\int\operatorname{cosec}^2 x\cdot y dx=1$$

Using Dirac delta property: $\int f(x)\delta\left(x-x_0\right)=f\left(x_0\right)$ (it lies with the limit).

$$\frac{dy}{dx}+y\cdot\frac{\cos x}{\sin x}-\int y\operatorname{cosec}^2 x dx=1,$$ at $x=\pi;\sin x=0$. So this is point of discontinuity.

So the correct answer is **Option (D)**

3. The solution of the partial differential equation

$$\frac{\partial^2}{\partial t^2}u(x,t)-\frac{\partial^2}{\partial x^2}u(x,t)=0$$

satisfying the boundary conditions $u(0,t)=0=u(L,t)$ and initial conditions $u(x,0)=\sin(\pi x/L)$ and $\left.\frac{\partial}{\partial t}u(x,t)\right|_{t=0}=\sin(2\pi x/L)$ is

[NET/JRF(JUNE-2013)]

A. $\sin(\pi x/L)\cos(\pi t/L)+\frac{L}{2\pi}\sin(2\pi x/L)\cos(2\pi t/L)$

B. $2\sin(\pi x/L)\cos(\pi t/L)-\sin(\pi x/L)\cos(2\pi t/L)$

C. $\sin(\pi x/L)\cos(2\pi t/L)+\frac{L}{\pi}\sin(2\pi x/L)\sin(\pi t/L)$

D. $\sin(\pi x/L)\cos(\pi t/L)+\frac{L}{2\pi}\sin(2\pi x/L)\sin(2\pi t/L)$

Solution:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=0, u(x,0)=\sin\frac{\pi x}{L} \text{ and } \left.\frac{\partial u}{\partial t}\right|_{t=0}=\sin\frac{2\pi x}{L}$$

This is a wave equation

$$\text{So solution is given by } u(x,t)=\sum_n\left(A_n\cos\frac{an\pi t}{L}+B_n\sin\frac{an\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$\text{with } A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx,$$

$$B_n=\frac{2}{an\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

$$\text{Comparing } a^2\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}, \text{ We have } a=1 \text{ and } f(x)$$

$$=\sin\frac{\pi x}{L}, g(x)=\sin\frac{2\pi x}{L}$$

$$A_n=\frac{2}{L}\int_0^L\sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}dx\Rightarrow\frac{2}{L}\int_0^L\sin^2\frac{\pi x}{L}dx$$

$$=\frac{2}{L}\int_0^L\left(\frac{1-\cos\frac{2\pi x}{L}}{2}\right)dx=\frac{2}{L}\cdot\frac{L}{2}=1(\text{let } n=1)$$

$$\text{Putting } n=2, B_n=\frac{2}{an\pi}\int_0^L\sin\frac{2\pi x}{L}\cdot\sin\frac{n\pi x}{L}dx$$

$$\Rightarrow\frac{2}{2\pi}\int_0^L\sin^2\frac{2\pi x}{L}dx=\frac{2}{2\pi}\int_0^L\left(\frac{1-\cos\frac{4\pi x}{L}}{2}\right)dx=\frac{2}{2\pi}\cdot\frac{L}{2}=\frac{L}{2\pi}$$

So the correct answer is **Option (D)**

4. The solution of the differential equation

$$\frac{dx}{dt}=x^2$$

with the initial condition $x(0)=1$ will blow up as t tends to

[NET/JRF(JUNE-2013)]

A. 1 **B.** 2 **C.** $\frac{1}{2}$ **D.** ∞

Solution:

$$\frac{dx}{dt}=x^2\Rightarrow\int\frac{dx}{x^2}=\int dt\Rightarrow\frac{x^{-2+1}}{-2+1}$$

$$=t+C\Rightarrow\frac{-1}{x}=t+C$$

$$\Rightarrow x(0)=1\Rightarrow\frac{-1}{1}=0+C\Rightarrow C=-1\Rightarrow\frac{-1}{x}$$

$$=t-1\Rightarrow x=\frac{1}{1-t} \text{ as } t\to 1, x \text{ blows up}$$

So the correct answer is **Option (A)**

5. Consider the differential equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0$$

with the initial conditions $x(0)=0$ and $\dot{x}(0)=1$. The solution $x(t)$ attains its maximum value when t is

[NET/JRF(JUNE-2014)]

A. $1/2$ **B.** 1 **C.** 2 **D.** ∞

Solution:

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0\Rightarrow m^2+2m+1$$
$$=0\Rightarrow(m+1)^2=0\Rightarrow m=-1,-1$$
$$\Rightarrow x=(c_1+c_2t)e^{-t},\text{ since } x(0)$$
$$=0\Rightarrow 0=c_1\Rightarrow x=c_2te^{-t}$$
$$\Rightarrow \dot{x}=c_2\left[-te^{-t}+e^{-t}\right]$$
$$\text{Since } \dot{x}(0)=1\Rightarrow 1=c_2\Rightarrow x=te^{-t}$$
$$\text{For maxima or minima } \dot{x}=0\Rightarrow \dot{x}=-te^{-t}+e^{-t}=0\Rightarrow \dot{x}=e^{-t}(1-t)$$
$$\Rightarrow e^{-t}=0,1-t=0\Rightarrow t=\infty,t=1$$
$$\ddot{x}=e^{-t}(-1)+(1-t)e^{-t}(-1)$$
$$=-e^{-t}+(t-1)e^{-t}\Rightarrow \ddot{x}(1)$$
$$=-e^{-1}+0e^{-t}<0$$

So the correct answer is **Option (B)**

6. Consider the differential equation $\frac{d^2x}{dt^2}-3\frac{dx}{dt}+2x=0$. If $x=0$ at $t=0$ and $x=1$ at $t=1$, the value of x at $t=2$ is

[NET/JRF(JUNE-2015)]

A. e^2+1 **B.** e^2+e **C.** $e+2$ **D.** $2e$

Solution:

$$D^2-3D+2=0$$
$$(D-1)(D-2)=0\Rightarrow D=1,2\Rightarrow x=c_1e^{2t}+c_2e^t$$
$$\text{using boundary condition } x=0,t=0\Rightarrow c_1=-C_2$$
$$\text{again using boundary condition } x=1,t=1$$
$$c_2=\frac{1}{e-e^2},c_1=\frac{1}{e^2-e}\Rightarrow x$$
$$=\frac{e^{2t}}{e^2-e}+\frac{1}{e-e^2}e^t$$
$$\text{again using } t=2 \text{ then } x=e^2+e$$

So the correct answer is **Option (B)**

7. If $y=\frac{1}{\tanh(x)}$, then x is

[NET/JRF(DEC-2015)]

A. $\ln\left(\frac{y+1}{y-1}\right)$ **B.** $\ln\left(\frac{y-1}{y+1}\right)$ **C.** $\ln\sqrt{\frac{y-1}{y+1}}$ **D.** $\ln\sqrt{\frac{y+1}{y-1}}$

Solution:

$$y=\frac{1}{\tanh x}$$
$$y=\frac{e^x+e^{-x}}{e^x-e^{-x}}=\frac{e^{2x}+1}{e^{2x}-1}$$

$$ye^{2x} - y = e^{2x} + 1 \Rightarrow ye^{2x} - e^{2x}$$
$$= 1 + y \Rightarrow e^{2x}(y-1) = (1+y)$$
$$2x = \ln\left(\frac{y+1}{y-1}\right) \Rightarrow x = \frac{1}{2}\ln\left(\frac{y+1}{y-1}\right)$$
$$= \ln\left(\frac{y+1}{y-1}\right)^{\frac{1}{2}}$$

So the correct answer is **Option (D)**

8. The solution of the differential equation $\frac{dx}{dt} = 2\sqrt{1-x^2}$, with initial condition $x = 0$ at $t = 0$ is
[NET/JRF(DEC-2015)]

A. $x = \begin{cases} \sin 2t, & 0 \le t < \frac{\pi}{4} \\ \sinh 2t, & t \ge \frac{\pi}{4} \end{cases}$

B. $x = \begin{cases} \sin 2t, & 0 \le t < \frac{\pi}{2} \\ 1, & t \ge \frac{\pi}{2} \end{cases}$

C. $x = \begin{cases} \sin 2t, & 0 \le t < \frac{\pi}{4} \\ 1, & t \ge \frac{\pi}{4} \end{cases}$

D. $x = 1 - \cos 2t, \quad t \ge 0$

Solution:

$$\frac{dx}{dt} = 2\sqrt{1-x^2}, \frac{dx}{\sqrt{1-x^2}}$$
$$= 2dt, \sin^{-1} x = 2t + c, x = 0, t = 0$$
$$\text{so, } c = 0 \Rightarrow x = \sin 2t$$
$$x \text{ should not be greater than 1 at } x = 1$$
$$1 = \sin 2t, \quad \sin\frac{\pi}{2} = \sin 2t, t = \frac{\pi}{4}$$
$$\text{So,} \quad x = \begin{cases} \sin 2t, & 0 \le t < \frac{\pi}{4} \\ 1, & t \ge \frac{\pi}{4} \end{cases}$$

So the correct answer is **Option (C)**

9. The function $y(x)$ satisfies the differential equation $x\frac{dy}{dx} + 2y = \frac{\cos \pi x}{x}$. If $y(1) = 1$, the value of $y(2)$ is
[NET/JRF(JUNE-2017)]

A. π **B.** 1 **C.** 1/2 **D.** 1/4

Solution:

The given differential equation can be written as

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos \pi x}{x^2}$$

This is a linear differential equation with Integrating factor $= e^{\int \frac{2}{x} dx} = x^2$

$$\text{Hence } y.x^2 = \int x^2 \cdot \frac{\cos \pi x}{x^2} dx + c \Rightarrow y$$
$$= \frac{\sin \pi x}{\pi x^2} + \frac{c}{x^2}$$
$$\text{when } x = 1, y = 1 \text{ hence } c = 1 \Rightarrow y$$
$$= \frac{\sin \pi x}{\pi x^2} + \frac{1}{x^2}$$

$$\text{hence, when } x = 2, y = \frac{1}{4}$$

So the correct answer is **Option (D)**

10. Consider the differential equation $\frac{dy}{dt} + ay = e^{-bt}$ with the initial condition $y(0) = 0$. Then the Laplace transform $Y(s)$ of the solution $y(t)$ is

[NET/JRF(DEC-2017)]

A. $\frac{1}{(s+a)(s+b)}$ **B.** $\frac{1}{b(s+a)}$ **C.** $\frac{1}{a(s+b)}$ **D.** $\frac{e^{-a}-e^{-b}}{b-a}$

Solution:

$$\text{Given } \frac{dy}{dt} + ay = e^{-bt}$$

Taking Laplace transform of both sides

$$\text{We obtain}$$

$$L\left\{\frac{dy}{dt}\right\} + aL\{y(t)\} = L\left\{e^{-bt}\right\} \Rightarrow sY(s) - y(0) + aY(s) = \frac{1}{s+b}$$

$$\text{Since, } y(0) = 0, \text{ we obtain}$$

$$(s+a)Y(s) = \frac{1}{s+b} \Rightarrow Y(s) = \frac{1}{(s+a)(s+b)}$$

So the correct answer is **Option (A)**

11. The number of linearly independent power series solutions, around $x = 0$, of the second order linear differential equation $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$, is

[NET/JRF(DEC-2017)]

A. 0 (this equation does not have a power series solution)
B. 1
C. 2
D. 3

Solution: The given differential equation will have one power series solution.
So the correct answer is **Option (B)**

12. The differential equation $\frac{dy(x)}{dx} = \alpha x^2$, with the initial condition $y(0) = 0$, is solved using Euler's method. If $y_E(x)$ is the exact solution and $y_N(x)$ the numerical solution obtained using n steps of equal length, then the relative error $\left|\frac{(y_N(x)-y_E(x))}{y_E(x)}\right|$ is proportional to (Question belongs to the topic numerical methods)

[NET/JRF(DEC-2017)]

A. $\frac{1}{n^2}$ **B.** $\frac{1}{n^3}$ **C.** $\frac{1}{n^4}$ **D.** $\frac{1}{n}$

Solution:

$$\frac{dy}{dx} = \alpha x^2, y(0) = 0$$

$$y_E = \frac{\alpha x^3}{3}, \text{ but } x = n\hbar$$

$$\text{Exact solution, } y_E = \frac{\alpha n^3 h^3}{3}$$

$$\text{Numerically, } f(x,y) = \alpha x^2$$
$$\text{Euler's method, } y_i = y_{i-1} + hf(x_{i-1}, y_{i-1})$$
$$y_1 = 0, y_2 = \alpha h^3 \quad y_3 = 5\alpha h^3$$
$$y_n = \frac{(n-1)n(2n-1)}{6}\alpha h^3$$

Since, $0, 5, 14, 30, \ldots$ different from square terms

At, $x_0 = 0 \quad x_1 = x_0 + h = h \quad x_2 = x_0 + 2h = 2h \quad x_3 = x_0 + 3h = 3h$

$$x_{n-1} = x_0 + (n-1)h = (n-1)h. \text{ Now, } x_n = nh$$
$$f(x_0, y_0) = 0, f(x_1, y_1) = \alpha h^2, f(x_2, y_2) = 4\alpha h^2$$
$$f(x_{n-1}, y_{n-1}) = \alpha(n-1)^2 h^2$$
$$\left|\frac{(y_N - y_E)}{y_E}\right| = \left|\frac{\frac{(n-1)n(2n-1)\alpha h^3}{6} - \frac{\alpha n^3 h^3}{3}}{\frac{\alpha n^3 h^3}{3}}\right|$$
$$\text{By solving, } \left|\frac{y_N - y_E}{y_E}\right| \propto \frac{1}{n}$$

So the correct answer is **Option (D)**

13. Consider the following ordinary differential equation

$$\frac{d^2x}{dt^2} + \frac{1}{x}\left(\frac{dx}{dt}\right)^2 - \frac{dx}{dt} = 0$$

with the boundary conditions $x(t=0) = 0$ and $x(t=1) = 1$. The value of $x(t)$ at $t = 2$ is **[NET/JRF(JUNE-2018)]**

A. $\sqrt{e-1}$ **B.** $\sqrt{e^2+1}$ **C.** $\sqrt{e+1}$ **D.** $\sqrt{e^2-1}$

Solution:

The given equation can be written as

$$\frac{1}{x}\frac{d}{dt}\left(x\frac{dx}{dt}\right) - \frac{dx}{dt} = 0 \Rightarrow \frac{d}{dt}\left(x\frac{dx}{dt}\right) - x\frac{dx}{dt} = 0$$
$$\text{putting } y = x\frac{dx}{dt} \text{ gives}$$
$$\frac{dy}{dt} - y = 0 \Rightarrow \ln y = t + \ln c_1 \Rightarrow y = c_1 e^t$$

Since $x\frac{dx}{dt} = c_1 e^t$ hence by integrating

$$\frac{x^2}{2} = c_1 e^t + c_2 \qquad \text{(i)}$$

Using boundary conditions we obtain

$$c_1 + c_2 = 0 \text{ and } c_1 e + c_2 = \frac{1}{2}$$

Solving these equations we obtain $c_1 = \frac{1}{2(e-1)}$ and $c_2 = -\frac{1}{2(e-1)}$

$$\text{Thus, } \frac{x^2}{2} = \frac{1}{2(e-1)}e^t - \frac{1}{2(e-1)}$$

$$\text{When } t=2, \text{ we obtain, } \quad x^2 = \frac{e^2}{(e-1)} - \frac{1}{(e-1)}$$

$$= \frac{(e^2-1)}{(e-1)} = e+1$$

$$\text{Therefore} x(2) = \sqrt{e+1}$$

So the correct answer is **Option (C)**

14. In terms of arbitrary constants A and B, the general solution to the differential equation $x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 3y = 0$ is

[NET/JRF(DEC-2018)]

A. $y = \frac{A}{x} + Bx^3$ **B.** $y = Ax + \frac{B}{x^3}$ **C.** $y = Ax + Bx^3$ **D.** $y = \frac{A}{x} + \frac{B}{x^3}$

Solution:

The given equation is Euler-Cauchy differential equation. The characteristic equation of

$$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 6y = 0$$

$$\text{is } ,m^2 + 4m + 6 = 0 \Rightarrow m = -3 or m = -1$$

$$\text{Thus, } y_1 = x^{-1} = \frac{1}{x} \text{ and } y_2 = x^2 = \frac{1}{x^3}$$

Therefore the general solution is

$$y = \frac{A}{x} + \frac{B}{x^3}$$

So the correct answer is **Option (D)**

15. The solution of the differential equation $x\frac{dy}{dx} + (1+x)y = e^{-x}$ with the boundary condition $y(x=1) = 0$, is

[NET/JRF(JUNE-2019)]

A. $\frac{(x-1)}{x}e^{-x}$ **B.** $\frac{(x-1)}{x^2}e^{-x}$ **C.** $\frac{(1-x)}{x^2}e^{-x}$ **D.** $(x-1)^2e^{-x}$

Solution:

$$x\frac{dy}{dx} + (1+x)y = e^{-x} \Rightarrow \frac{dy}{dx} + \frac{(1+x)}{x}y = \frac{e^{-x}}{x}$$

$$\text{Let } p = \frac{1+x}{x}$$

$$I.F = e^{\int p dx} = e^{\int\left(1+\frac{1}{x}\right)dx} = e^x \cdot e^{\ln x} = xe^x$$

$$y \cdot x \cdot e^x = \int \frac{e^{-x}}{x} \cdot xe^x dx + C \Rightarrow y \cdot x \cdot e^x = x + C$$

$$y = 0 \text{ at } x = 1 \quad \Rightarrow C = -1 \quad \Rightarrow y \cdot x \cdot e^x$$

$$= x-1 \Rightarrow y = \left[\frac{x-1}{x}\right] e^{-x}$$

So the correct answer is **Option (A)**

16. The solution of the differential equation $\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} = e^y$, with the boundary conditions $y(0) = 0$ and $y'(0) = -1$, is

[NET/JRF(JUNE-2020)]

A. $-\ln\left(\frac{x^2}{2} + x + 1\right)$ **B.** $-x\ln(e+x)$ **C.** $-xe^{-x^2}$ **D.** $-x(x+1)e^{-x}$

Solution:

$$\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} = e^y \text{ put } y = \ln p$$

$$\frac{dy}{dx} = \frac{1}{p}\frac{dp}{dx} \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{p}\frac{dp}{dx}\right)$$

$$= \frac{1}{p}\frac{d^2p}{dx^2} - \frac{1}{p^2}\left(\frac{dp}{dx}\right)^2$$

$$\text{Thus } \left(\frac{1}{p}\frac{dp}{dx}\right)^2 - \frac{1}{p}\frac{d^2p}{dx^2} + \frac{1}{p^2}\left(\frac{dp}{dx}\right)^2 = p$$

$$\frac{2}{p^2}\left(\frac{dp}{dx}\right)^2 - \frac{1}{p}\frac{d^2p}{dx^2} = p \Rightarrow \frac{2}{p^3}\left(\frac{dp}{dx}\right)^2 - \frac{1}{p^2}\frac{d^2p}{dx^2}$$

$$= 1 \Rightarrow \frac{1}{p^2}\frac{d^2p}{dx^2} - \frac{2}{p^3}\left(\frac{dp}{dx}\right)^2 = -1$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{p^2}\frac{dp}{dx}\right) = -1$$

$$\text{Let } \frac{1}{p^2}\frac{dp}{dx} = z \Rightarrow \frac{dz}{dx} = -1 \Rightarrow z = -x + c$$

$$\text{Thus } \frac{1}{p^2}\frac{dp}{dx} = -x + c \Rightarrow \int \frac{dp}{p^2} = \int(-x+c)dx$$

$$-\frac{1}{p} = -\frac{x^2}{2} + cx + d \Rightarrow p = \frac{1}{\frac{x^2}{2} - cx - d}$$

$$y = \ln p = \ln\left(\frac{1}{\frac{x^2}{2} - cx - d}\right) = \ln\left(\frac{x^2}{2} - cx - d\right)$$

$$y(0) = 0 \Rightarrow y(0) = -\ln(-d) \Rightarrow d = -1$$

$$y = -\ln\left(\frac{x^2}{2} - cx + 1\right)$$

$$y'(x) = -\frac{1}{\left(\frac{x^2}{2} - cx + 1\right)}(x - c), \quad y'(0)$$

$$= -1 \Rightarrow -\frac{(-c)}{1} = c = -1, \quad y = -\ln\left(\frac{x^2}{2} + x + 1\right)$$

So the correct answer is **Option (A)**

Answer key			
Q.No.	Answer	Q.No.	Answer
1	**D**	2	**D**
3	**D**	4	**A**
5	**B**	6	**B**
7	**D**	8	**C**
9	**D**	10	**A**
11	**B**	12	**D**
13	**C**	14	**D**
15	**A**	16	**A**

Problem Set -2

1. The solution of the differential equation for $y(t)$: $\frac{d^2y}{dt^2} - y = 2\cosh(t)$, subject to the initial conditions $y(0) = 0$ and $\left.\frac{dy}{dt}\right|_{t=0} = 0$, is

[GATE 2010]

A. $\frac{1}{2}\cosh(t) + t\sinh(t)$
B. $-\sinh(t) + t\cosh(t)$
C. $t\cosh(t)$
D. $t\sinh(t)$

Solution:

$$\text{For C.F } (D^2-1)\,y = 0 \Rightarrow m = \pm 1 \Rightarrow C.F.$$
$$= C_1e^t + C_2e^{-t}$$
$$P.I. = \frac{1}{D^2-1}(2\cosh t) = \frac{1}{D^2-1}2\left(\frac{e^t+e^{-t}}{2}\right)$$
$$= \frac{1}{D^2-1}\left(e^t\right) + \frac{1}{D^2-1}\left(e^{-t}\right)$$
$$= \frac{t}{2}e^t + \frac{t}{2}\left(-e^{-t}\right)$$
$$\Rightarrow y = C_1e^t + C_2e^{-t} + \frac{t}{2}e^t - \frac{t}{2}e^{-t}$$
$$\text{As, } y(0) = 0 \Rightarrow C_1 + C_2 = 0 \tag{1}$$
$$\frac{dy}{dt} = C_1e^t - C_2e^{-t} + \frac{t}{2}e^t + \frac{1}{2}e^t + \frac{t}{2}e^{-t} - \frac{1}{2}e^{-t}$$
$$\text{Also, } \left.\frac{dy}{dt}\right|_{t=0} = 0 \Rightarrow C_1 - C_2 + 0 + \frac{1}{2} + 0 - \frac{1}{2} = 0 \Rightarrow C_1 - C_2 = 0 \tag{2}$$
From equation (1) and (2),
$$C_1 = 0, C_2 = 0$$
$$\text{Thus } y = \frac{t}{2}e^t - \frac{t}{2}e^{-t} \Rightarrow y = t\sinh t$$

So the correct answer is **Option (D)**

2. The solutions to the differential equation $\frac{dy}{dx} = -\frac{x}{y+1}$ are a family of

[GATE 2011]

A. Circles with different radii
B. Circles with different centres
C. Straight lines with different slopes
D. Straight lines with different intercepts on the y-axis

Solution:

$$\frac{dy}{dx} = -\frac{x}{y+1} \Rightarrow xdx + ydy + dy$$
$$= 0 \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + y$$
$$= C_1 \Rightarrow x^2 + y^2 + 2y$$
$$= 2C_1 \Rightarrow (x-0)^2 + (y+1)^2$$

$$= 2C_1 + 1 = C$$

which is a family of circles with different radii.

So the correct answer is **Option (A)**

3. The solution of the differential equation $\frac{d^2y}{dt^2} - y = 0$, subject to the boundary conditions $y(0) = 1$ and $y(\infty) = 0$ is

[GATE 2014]

A. $\cos t + \sin t$ **B.** $\cosh t + \sinh t$ **C.** $\cos t - \sin t$ **D.** $\cosh t - \sinh t$

Solution:

$$D^2 - 1 = 0 \Rightarrow D = \pm 1 \Rightarrow y(t)$$
$$= c_1 e^t + c_2 e^{-t}$$

Applying boundary condition,

$$y(0) = 1 \Rightarrow 1 = c_1 + c_2 \text{ and } y(\infty)$$
$$= 0 \Rightarrow 0 = c_1 e^{\infty} + c_2 e^{-\infty} \Rightarrow c_1$$
$$= 0, c_2 = 1$$
$$\Rightarrow y(t) = e^{-t} \Rightarrow y(t) = \cosh t - \sinh\ t$$

So the correct answer is **Option (D)**

4. A function $y(z)$ satisfies the ordinary differential equation $y'' + \frac{1}{z}y' - \frac{m^2}{z^2}y = 0$, where $m = 0, 1, 2, 3, \ldots$. Consider the four statements P, Q, R, S as given below.
P : z^m and z^{-m} are linearly independent solutions for all values of m
Q: z^m and z^{-m} are linearly independent solutions for all values of $m > 0$
R : $\ln z$ and 1 are linearly independent solutions for $m = 0$
S: z^m and $\ln z$ are linearly independent solutions for all values of m
The correct option for the combination of valid statements is

[GATE 2015]

A. P, R and S only **B.** P and R only **C.** Q and R only **D.** R and S only

Solution:

$$y'' + \frac{1}{z}y' - \frac{m^2}{z^2}y = 0 \Rightarrow z^2y'' + zy' - m^2y$$
$$= 0, m = 0, 1, 2, 3, \ldots, \quad z = e^x, D = \frac{d}{dx}$$
$$\text{If } m = 0; \quad z^2y'' + zy' = 0, [D(D-1) + D]y$$
$$= 0 \Rightarrow \left[D^2 - D + D\right] y = 0$$
$$D^2 y = 0 \Rightarrow y = c_1 + c_2 x \Rightarrow y$$
$$= c_1 + c_2 \ln z \quad (R \text{ is correct})$$
$$\text{And if } m \neq 0, m > 0, \text{ then } m \neq 0, \text{ then } \left(D^2 - m^2\right) y$$
$$= 0 \Rightarrow D = \pm m$$

$$y = c_1 e^{mx} + c_2 e^{-mx} = c_1 e^{m\log z} + c_2 e^{-m\log z}$$
$$= c_1 z^m + c_2 z^{-m}$$
$$\text{or if } m \neq 0, m > 0, \text{ then}$$
$$y = c_1 \cosh(m\log(z)) + ic_2 \sinh(m\log(x)), \quad m > 0$$

So the correct answer is **Option (C)**

5. Consider the linear differential equation $\frac{dy}{dx} = xy$. If $y = 2$ at $x = 0$, then the value of y at $x = 2$ is given by **[GATE 2016]**

A. e^{-2} **B.** $2e^{-2}$ **C.** e^2 **D.** $2e^2$

Solution:

$$\frac{dy}{dx} = xy \Rightarrow \frac{1}{y}dy = xdx \Rightarrow \ln y$$
$$= \frac{x^2}{2} + \ln c \Rightarrow y = ce^{x^2/2}$$
$$\text{If } y = 2 \text{ at } x = 0 \Rightarrow c = 2 \Rightarrow y = 2e^{x^2/2}$$
$$\text{The value of } y \text{ at } x = 2 \text{ is given by } y = 2e^2$$

So the correct answer is **Option (D)**

6. Consider the differential equation $\frac{dy}{dx} + y\tan(x) = \cos(x)$. If $y(0) = 0, y\left(\frac{\pi}{3}\right)$ is .... (up to two decimal places)

[GATE 2017]

Solution:

The given differential equation is a linear differential equation of the form

$$\frac{dy}{dx} + p(x)y = \cos x$$
$$\text{Integrating factor } = e^{\int p(x)dx}$$
$$\text{Thus integrating factor } = e^{\int \tan x dx}$$
$$\Rightarrow I \cdot F = e^{\ln \sec x} = \sec x$$

Thus the general solution of the given differential equation is

$$y \cdot \sec x = \int \sec x \cdot \cos x dx + c$$
$$\Rightarrow y \sec x = x + c$$
$$\text{It is given that } y(0) = 0 \Rightarrow 0 \cdot \sec 0 = 0 + c \Rightarrow c = 0$$

Thus the solution satisfying the given condition is

$$y \sec x = x \Rightarrow y = \frac{x}{\sec x}$$
$$\text{Thus the value of } y\left(\frac{\pi}{3}\right) \text{ is}$$
$$y = \frac{\pi/3}{\sec \pi/3} = \frac{\pi/3}{2} = \frac{\pi}{6} = 0 \cdot 52$$

7. Given

$$\frac{d^2f(x)}{dx^2}-2\frac{df(x)}{dx}+f(x)=0$$

and boundary conditions $f(0)=1$ and $f(1)=0$, the value of $f(0.5)$ is ———(up to two decimal places). **[GATE 2018]**

Solution:

$$\frac{d^2f(x)}{dx^2}-2\frac{df(x)}{dx}+f(x)=0$$

Auxiliary equation is,

$$(m^2-2m+1)=0\Rightarrow(m-1)^2$$
$$=0\Rightarrow m=1,1$$

Hence, the solution is

$$f(x)=(c_1+c_2x)e^x$$

using boundary condition,

$$f(0)=c_1e^0\Rightarrow c_1=1 \quad (3)$$
$$f(1)=(c_1+c_2)e=0 \quad (4)$$

From (3) and (4), $c_2=-1$

Hence, $f(x)=(1-x)e^x\Rightarrow f(0.5)$

$$=(1-0.5)e^{0.5}=0.81$$

8. For the differential equation $\frac{d^2y}{dx^2}-n(n+1)\frac{y}{x^2}=0$, where n is a constant, the product of its two independent solutions is

[GATE 2019]

A. $\frac{1}{x}$ **B.** x **C.** x^n **D.** $\frac{1}{x^{n+1}}$

Solution:

$$\frac{d^2y}{dx^2}-n(n+1)\frac{y}{x^2}=0$$

$$x^2\frac{d^2y}{dx^2}-n(n+1)y=0 \quad \text{This is an Euler -Cauchy equation.}$$

$$\text{Put } x=e^z\Rightarrow \log x=z\Rightarrow\frac{1}{x}=\frac{dz}{dx}$$

$$\frac{dy}{dx}=\frac{dy}{dx}\cdot\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz}$$

$$x\frac{dy}{dx}=\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)$$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$x^2\frac{d^2y}{dx^2}=-\frac{dy}{dz}+\frac{d^2y}{dz^2}$$

Then the given equation becomes,

$$\begin{aligned}
\frac{d^2y}{dz^2} - \frac{dy}{dz} - n(n+1)y &= 0 \\
D^2 - D - n(n+1) &= 0 \\
D &= \frac{1 \pm \sqrt{1+4n(n+1)}}{2} \\
&= \frac{1 \pm \sqrt{(2n+1)^2}}{2} = \frac{1 \pm (2n+1)}{2} \\
&= (n+1), -n \\
\text{Solution, } y_1 &= c_1 e^{(n+1)}, \quad y_2 = c_2 e^{(-n)} \\
\text{Their product, } y_1 y_2 &= c_1 c_2 e^{(n+1)} e^{(-n)} = c_1 c_2 e^{z}
\end{aligned}$$

But , $e^z = x$, then,

$$\begin{aligned}
y_1 y_2 &= c_1 c_2 x \qquad \text{Let, } c_1 = c_2 = 1 \\
&= x
\end{aligned}$$

So the correct answer is **Option (B)**