Problem Set -1

1. Let $p_n(x)$ (where $n = 0, 1, 2, \ldots\ldots$) be a polynomial of degree n with real coefficients, defined in the interval $2 \le n \le 4$. If $\int_2^4 p_n(x) p_m(x) dx = \delta_{nm}$, then

[NET/JRF(JUNE-2011)]

A. $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{\frac{3}{2}}(-3-x)$

B. $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{3}(3+x)$

C. $p_0(x) = \frac{1}{2}$ and $p_1(x) = \sqrt{\frac{3}{2}}(3-x)$

D. $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{\frac{3}{2}}(3-x)$

Solution:

$$\int_2^4 p_n(x) p_m(x) dx = \delta_{nm}$$

For $n \neq m$, $\delta_{nm} = 0$.
One positive and one negative term can make integral zero. So answer may be (C) or (D). Now take $n = m = 0$ so $p_0(x) = \frac{1}{\sqrt{2}}$ and then integrate. (D) is correct option because it satisfies the equation Check by integration and by orthogonal property of Legendre polynomial also.

So the correct answer is **Option (D)**

2. The generating function $F(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n$ for the Legendre polynomials $P_n(x)$ is $F(x,t) = \left(1 - 2xt + t^2\right)^{-1/2}$. The value of $P_3(-1)$ is

[NET/JRF(DEC-2011)]

A. $5/2$ **B.** $3/2$ **C.** $+1$ **D.** -1

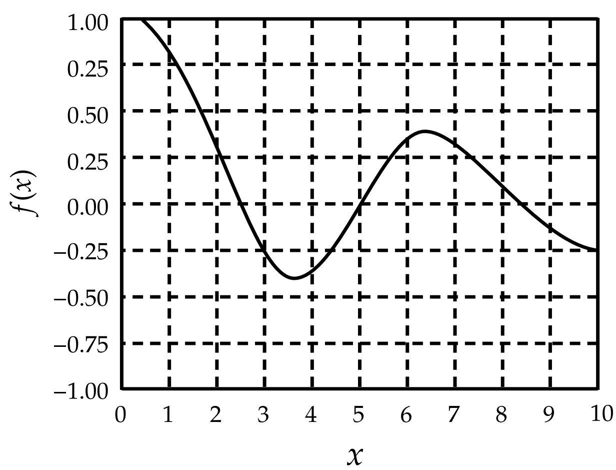
Solution:

$$P_3 = \frac{1}{2}\left(5x^3 - 3x\right) \Rightarrow P_3(-1)$$
$$= \frac{1}{2}\left(5(-1)^3 - 3(-1)\right) = \frac{1}{2}[-5+3] = -1$$

So the correct answer is **Option (D)**

3. The graph of the function $f(x)$ shown below is best described by

[NET/JRF(DEC-2012)]



A. The Bessel function $J_0(x)$
B. $\cos x$
C. $e^{-x}\cos x$
D. $\frac{1}{x}\cos x$

Solution: So the correct answer is **Option (A)**

4. Given that $\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{-t^2+2tx}$ the value of $H_4(0)$ is

[NET/JRF(JUNE-2013)]

A. 12 **B.** 6 **C.** 24 **D.** -6

Solution:

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{-t^2+2x} \Rightarrow \sum_{n=0}^{\infty} H_n(0)\frac{t^n}{n!}$$
$$= e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!}$$
$$\Rightarrow \frac{H_4(0)}{4!}t^4 = \frac{t^4}{2!} \Rightarrow H_4(0) = \frac{4!}{2!} = 12$$

So the correct answer is **Option (A)**

5. Given $\sum_{n=0}^{\infty} P_n(x)t^n = \left(1-2xt+t^2\right)^{-1/2}$, for $|t|<1$, the value of $P_5(-1)$ is

[NET/JRF(JUNE-2014)]

A. 0.26 **B.** 1 **C.** 0.5 **D.** -1

Solution:

$$P_n(-1) = -1 \text{ if } n \text{ is odd } \Rightarrow P_5(-1) = -1$$

So the correct answer is **Option (D)**

6. The function $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!}\left(\frac{x}{2}\right)^{2n+1}$, satisfies the differential equation

[NET/JRF(DEC-2014)]

A. $x^2\frac{d^2f}{dx^2} + x\frac{df}{dx} + \left(x^2+1\right)f = 0$
B. $x^2\frac{d^2f}{dx^2} + 2x\frac{df}{dx} + \left(x^2-1\right)f = 0$
C. $x^2\frac{d^2f}{dx^2} + x\frac{df}{dx} + \left(x^2-1\right)f = 0$
D. $x^2\frac{d^2f}{dx^2} - x\frac{df}{dx} + \left(x^2-1\right)f = 0$

Solution:

$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!}\left(\frac{x}{2}\right)^{2n+1}$ is generating function (Bessel Function of first kind) which satisfies the differential equation $x^2\frac{d^2f}{dx^2} + x\frac{df}{dx} + \left(x^2-n^2\right)f = 0$, put $n=1$.

So the correct answer is **Option (C)**

7. The Hermite polynomial $H_n(x)$, satisfies the differential equation

$$\frac{d^2H_n}{dx^2} - 2x\frac{dH_n}{dx} + 2nH_n(x) = 0$$

The corresponding generating function $G(t,x) = \sum_{n=0}^{\infty} \frac{1}{n!}H_n(x)t^n$, satisfies the equation

[NET/JRF(DEC-2015)]

A. $\frac{\partial^2 G}{\partial x^2}-2x\frac{\partial G}{\partial x}+2t\frac{\partial G}{\partial t}=0$

B. $\frac{\partial^2 G}{\partial x^2}-2x\frac{\partial G}{\partial x}-2t^2\frac{\partial G}{\partial t}=0$

C. $\frac{\partial^2 G}{\partial x^2}-2x\frac{\partial G}{\partial x}+2\frac{\partial G}{\partial t}=0$

D. $\frac{\partial^2 G}{\partial x^2}-2x\frac{\partial G}{\partial x}+2\frac{\partial^2 G}{\partial x\partial t}=0$

Solution:

$$G=\frac{1}{n!}H_n(x)t^n, G'=\frac{1}{n!}H_n'(x)t^n, G''=\frac{1}{n!}H_n''(x)t^n$$
$$\frac{\partial G}{\partial t}=\frac{1}{n!}H_n(x)nt^{n-1}$$

Let's check the options one by one

$$\frac{\partial G}{\partial x^2}-2x\frac{\partial G}{\partial x}+2t\frac{\partial G}{\partial t}=0$$
$$\frac{1}{n!}H_n''(x)t^n-2x\frac{1}{n!}H_n'(x)t^n+2t\frac{1}{n!}H_n(x)nt^{n-1}$$
$$H_n''(x)-2xH_n'(x)+2xH_n(x)=0,\ \text{which is Hermite Differential Equation.}$$

So the correct answer is **Option (A)**

8. A stable asymptotic solution of the equation $x_{n+1}=1+\frac{3}{1+x_n}$ is $x=2$. If we take $x_n=2+\varepsilon_n$ and $x_{n+1}=2+\varepsilon_{n+1}$, where ε_n and ε_{n+1} are both small, the ratio $\frac{\varepsilon_{n+1}}{\varepsilon_n}$ is approximately
[NET/JRF(DEC-2016)]

A. $-\frac{1}{2}$ **B.** $-\frac{1}{4}$ **C.** $-\frac{1}{3}$ **D.** $-\frac{2}{3}$

Solution: So the correct answer is **Option (C)**

9. The generating function $G(t,x)$ for the Legendre polynomials $P_n(t)$ is

$$G(t,x)=\frac{1}{\sqrt{1-2xt+x^2}}=\sum_{n=0}^{\infty}x^nP_n(t),\ \text{for } |x|<1$$

If the function $f(x)$ is defined by the integral equation $\int_0^x f(x')\,dx'=xG(1,x)$, it can be expressed as
[NET/JRF(DEC-2017)]

A. $\sum_{n,m=0}^{\infty}x^{n+m}P_n(1)P_m\left(\frac{1}{2}\right)$

B. $\sum_{n,m=0}^{\infty}x^{n+m}P_n(1)P_m(1)$

C. $\sum_{n,m=0}^{\infty}x^{n-m}P_n(1)P_m(1)$

D. $\sum_{n,m=0}^{\infty}x^{n-m}P_n(0)P_m(1)$

Solution:

$$G(t,x)=\frac{1}{\sqrt{1-2xt+x^2}}=\sum_{n=0}^{\infty}x^nP_n(t)\ \text{for } |x|<1$$
$$G(1,x)=\frac{1}{\sqrt{1-2x+x^2}}=\sum_{n=0}^{\infty}x^nP_n(1)$$
$$\sum_{n=0}^{\infty}x^nP_n(1)=\frac{1}{\sqrt{(1-x)^2}}$$
$$=\frac{1}{1-x}\ \text{Since } |x|<1$$
$$\text{Now, } x\cdot\frac{1}{1-x}=\int_0^x f(x')\,dx'$$

Differentiating both sides,

$$f(x) = \frac{d}{dx}\frac{x}{1-x} = \frac{1}{(1-x)^2}$$

Which can be represented as,

$$\sum_{n,m=0}^{\infty} x^{n+m} P_n(1) P_m(1) = \frac{1}{(1-x)^2}$$

So the correct answer is **Option (B)**

10. In the function $P_n(x)e^{-x^2}$ of a real variable $x, P_n(x)$ is polynomial of degree n. The maximum number of extrema that this function can have is

[NET/JRF(JUNE-2018)]

A. $n+2$ **B.** $n-1$ **C.** $n+1$ **D.** n

Solution:

$$\begin{aligned} y = P_n(x)e^{-x^2} &\Rightarrow P_n'(x)e^{-x^2} + P_n(x)e^{-x^2}(-2x) \\ &= 0 \Rightarrow P_n'(x) - 2xP_n(x) = 0 \\ P_0(x) = 1, P_1(x) = 2 &\Rightarrow P_0'(x) - 2xP_0(x) \\ &= 0 \Rightarrow 0 - 2x.1 = 0 \\ x &= 0, 1 \text{ extrema} \\ P_1'(x) - 2xP_1(x) &= 0 \\ 1 - 2x \cdot x = 0 &\Rightarrow x = \pm\frac{1}{\sqrt{2}} \text{ i.e., 2 extrema.} \end{aligned}$$

Thus in general there are $(n+1)$ extrema.

So the correct answer is **Option (C)**

11. The polynomial $f(x) = 1 + 5x + 3x^2$ is written as linear combination of the Legendre polynomials $\left(P_0(x) = 1, P_1(x), P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)\right)$ as $f(x) = \sum_n c_n P_n(x)$. The value of c_0 is

[NET/JRF(DEC-2018)]

A. $\frac{1}{4}$ **B.** $\frac{1}{2}$ **C.** 2 **D.** 4

Solution:

$$\begin{aligned} f(x) &= 1 + 5x + 3x^2 \\ 1 &= P_0(x) \quad x = P_1(x) \\ x^2 &= \frac{1}{3}(2P_2(x) + 1) \\ f(x) &= P_0(x) + 5P_1(x) + 2P_2(x) + P_0(x) \\ &= 2P_0(x) + 5P_1(x) + 2P_2(x) \\ &= c_0P_0(x) + c_1P_1(x) + c_2P_2(x) c_0 = 2 \end{aligned}$$

So the correct answer is **Option (C)**

Answer key			
Q.No.	Answer	Q.No.	Answer
1	**D**	2	**D**
3	**A**	4	**A**
5	**D**	6	**C**
7	**A**	8	**C**
9	**B**	10	**C**
11	**C**		

Problem Set -3

1. What is the maximum number of extrema of the function $f(x) = P_k(x)e^{-\left(\frac{x^4}{4}+\frac{x^2}{2}\right)}$, where $x \in (-\infty, \infty)$ and $P_k(x)$ is an arbitrary polynomial of degree k ?

[JEST 2015]

A. $k+2$ **B.** $k+6$ **C.** $k+3$ **D.** k

Solution:

$$f(x) = P_x(x)e^{-\left(\frac{x^4}{4}+\frac{x^2}{2}\right)}$$
$$f'(x) = \left[P_x'(x) + P_x(x)(-1)\left(x^3+x\right)\right]e^{-\left(\frac{x^4}{4}+\frac{x^2}{2}\right)}$$

For maximum number of extrema,

$$\Rightarrow f'(x) = 0 \Rightarrow \left[P_x(x)\left(x^3+x\right) - P'(x)\right] = 0$$

Then it is a polynomial of order $k+3$

From the sign scheme maximum number of extrema $= k+3$

Correct option is (C)

2. The Bernoulli polynominals $B_n(s)$ are defined by, $\frac{xe^{xs}}{e^x-1} = \sum B_n(s)\frac{x^n}{n!}$. Which one of the following relations is true?

[JEST 2015]

A. $\frac{xe^{x(1-s)}}{e^x-1} = \sum B_n(s)\frac{x^n}{(n+1)!}$

B. $\frac{xe^{x(1-s)}}{e^x-1} = \sum B_n(s)(-1)^n\frac{x^n}{(n+1)!}$

C. $\frac{xe^{x(1-s)}}{e^x-1} = \sum B_n(-s)(-1)^n\frac{x^n}{n!}$

D. $\frac{xe^{x(1-s)}}{e^x-1} = \sum B_n(s)(-1)^n\frac{x^n}{n!}$

Solution:

$$\frac{xe^{xs}}{e^x-1} = \sum B_n(s)\frac{x^n}{n!}$$

Put $s=(s-1)$, Then,

$$\frac{xe^{x(s-1)}}{e^x-1}=\sum B_n(s-1)\frac{x^n}{n!}$$
$$\text{Since, } B_n(s-1)=(-1)^n B(s)$$
$$\Rightarrow \frac{xe^{x(s-1)}}{e^x-1}=\sum B_n(s)(-1)^n\frac{x^n}{n!}$$

Correct option is (D)

3. For which of the following conditions does the integral $\int_0^1 P_m(x)P_n(x)dx$ vanish for $m \neq n$, where $P_m(x)$ and $P_n(x)$ are the Legendre polynomials of order m and n respectively?

[JEST 2018]

A. all $m, m \neq n$

B. $m-n$ is an odd integer

C. $m-n$ is a nonzero even integer

D. $n=m\pm 1$

Solution:

$$\int_{-1}^{1} P_m(x)P_n(x)dx=\frac{2}{2x+1}\delta_{nm}$$
$$2\int_0^1 P_m(x)P_n(x)dx=\frac{2}{2x+1}\delta_{nm}$$

Only $P_m(x)P_n(x)=$ even

m	
0	even
1	odd
2	even
3	odd
4	even

$m-n=$ non zero even integer then only

$P_m(x)P_n(x)=$ even

$m \neq n \ \Delta\delta_{nm}=0$

Correct option is (A)

4. The Euler polynomials are defined by $\frac{2e^{xs}}{e^x+1}=\sum_{n=0}^{\infty}E_n(s)\frac{x^n}{n!}$What is the value of $E_5(2)+E_5(3)$?

[JEST 2019]

Solution:

$$\frac{2e^{xs}}{e^x+1}=\sum_{n=0}^{\infty}E_n(s)\frac{x^n}{n!}$$
$$E_n(x+1)+E_n(x)=2x^n$$
$$E_5(x+1)+E_5(x)=2x^5$$
$$c=2=2\times 2^5$$
$$=64$$

5. If $F(x,y)=x^2+y^2+xy$, its Legendre transformed function $G(u,v)$, upto a multiplicative constant, is

[JEST 2018]

A. u^2+v^2+uv **B.** u^2+v^2-uv **C.** u^2+v^2 **D.** $(u+v)^2$

Solution:

$$G = F - xu - yv$$
$$dG = dF - xdu - udv - ydv - vdy$$
$$F = x^2+y^2+xy$$
$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$
$$dF = udx + vdy$$
$$dG = udx + vdy - xdu - udx - ydv - vdy$$
$$dG = -xdu - ydv$$
$$u = \frac{\partial F}{\partial x}, v = \frac{\partial F}{\partial x}$$
$$u = 2x+y, 2u = 4x+2y$$
$$2v = 4y+2x, v = 2y+x$$
$$y = +\frac{1}{3}[2v-u]$$
$$2u - v = 3x$$
$$x = \frac{1}{3}[2u-v]$$
$$dG = -\frac{1}{3}[2u-v]du - \frac{1}{3}[2v-u]dx$$
$$dG(u,v) = -\frac{1}{3}(2u-v)du - \frac{1}{3}(2v-u)dv = \frac{\partial G}{\partial u}du + \frac{\partial G}{\partial v}dv$$
$$\frac{\partial G}{\partial u} = -\frac{1}{3}(2u-v), \frac{\partial G}{\partial v} = -\frac{1}{3}(2v-u)$$
$$G(u,v) = -\frac{1}{3}\left(u^2-uv\right) + h(v)$$
$$G(u,v) = -\frac{1}{3}(-u) + \frac{dh(v)}{d(v)} = \frac{u}{3} + \frac{dh(v)}{dv}$$
$$\frac{-2v}{3} + \frac{u}{3} = \frac{u}{3} + \frac{dh(v)}{dx}$$
$$h(v) = \frac{-v^2}{3}$$
$$G(u,v) = -\frac{1}{3}\left(u^2-uv\right) - \frac{v^2}{3} = -\frac{1}{3}\left(u^2+v^2-uv\right)$$

Correct option is (B)

6. Consider a function $f(x) = P_k(x)e^{-\left(x^4+2x^2\right)}$ in the domain $x \in (-\infty, \infty)$, where P_k is any polynomial of degree k. What is the maximum possible number of extrema of the function?

[JEST 2019]

A. $k+3$ **B.** $k-3$ **C.** $k+2$ **D.** $k+1$

Solution:

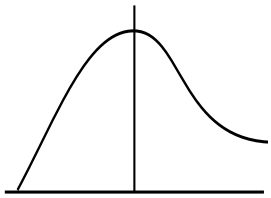
$$f(x) = p_k(x)e^{-\left(x^4+2x^2\right)}$$
$$\text{Let } k=0, f(x) = p_0(x)e^{-\left(x^4+2x^2\right)}$$

Number of extrema

$$P_0(x) = 1, k = 0$$

Number of extrema $= 1$

$$k + 1 = 0 + 1 = 1$$



Answer key			
Q.No.	Answer	Q.No.	Answer
1	**C**	2	**D**
3	**A**	4	**64**
5	**B**	6	**D**