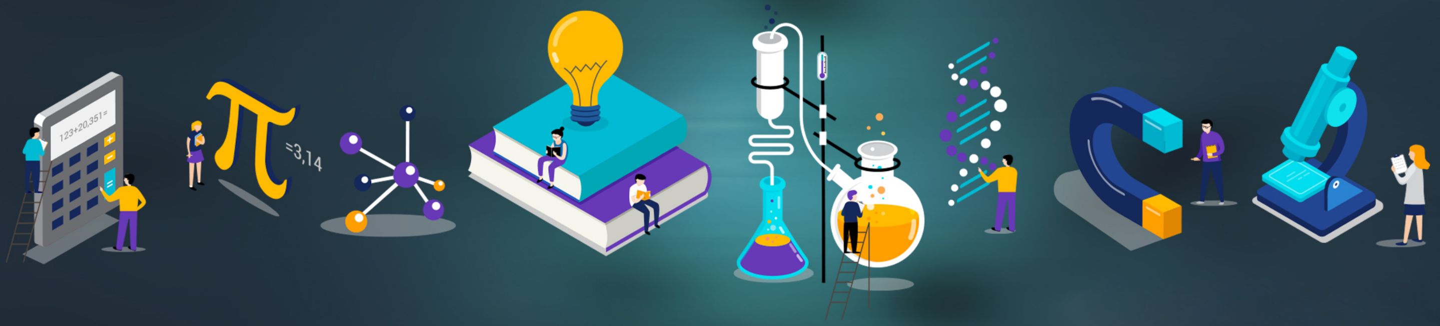


1. Differential Equations

In Physics we encounter differential equations all the time. In fact the whole programme of classical mechanics is to develop a second order differential equation using Newton's laws of motion and then solving it. Sometimes these are ordinary differential equations in one variable (abbreviated ODEs). More often the equations are partial differential equations (PDEs) in two or more variables. Simply we can say , differential equations is a relation between a function and its derivatives.

Definition 1.0.1 A differential equation is an equation which involves independent and dependent variables and their derivatives or differentials.

■ Example 1.1

- $\frac{dy}{dx} = 4x - 2$
- $\frac{d^2y}{dx^2} = 5\frac{dy}{dx} + 10$
- $(1 + \frac{dy}{dx})^3 = k\frac{dy}{dx}$
- $\frac{dy}{dx} + xy = x^3y^3$
- $\frac{\partial^2 y}{\partial^2 x} = \frac{1}{c^2}\frac{\partial^2 y}{\partial^2 x}$
- $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$

■

1.1 Types of differential equation

There are mainly two types of differential equations,

- **Ordinary differential equations.**

  A differentnial equation involving derivatives with respect to a single variable is called an ordinary differential equation.

- **Partial differential equations.**

  A differential equation involving partial derivatives with respect to more than one independent variable is called a partial differential equation.

1.2 Order and Degree of a differential equation

Order:
The order of a differential equation is the highest differential in the equation.

Degree:
The degree of a differential equation is the power of the highest differential in the equation.

Example 1.2

- $\left(\frac{\partial^2 y}{\partial x^2}\right)^2 + \left(\frac{\partial y}{\partial x}\right) - \left(\frac{\partial^3 y}{\partial x^3}\right) = xy$ Order=3 ,Degree=2
- $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E\sin\omega t$ Order=2 ,Degree=1
- $\frac{dy}{dx} + xy = x^3y^3$ Order=1 ,Degree=1
- $\left(\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2}\right)^3 = \left[1 + \left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^4\right]^5$ Order=2 ,Degree =3
- $\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^{\frac{1}{2}} = 0$ Order=3 ,Degree=2

Exercise 1.1 Find the order and degree of the given differential equations,

1. $\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^{\frac{1}{2}} = 0$
2. $\left[1 + \frac{d^2y}{dx^2}\right]^{\frac{3}{2}} = a\frac{d^2y}{dx^2}$

Solution:

1. Here we need to eliminate the radical sign. For this write the equation as

$$\frac{d^3y}{dx^3} = \left(\frac{dy}{dx}\right)^{\frac{1}{2}}$$

Squaring both sides, we get

$$\left(\frac{d^3y}{dx^3}\right)^2 = \frac{dy}{dx}$$

$$\therefore \quad \text{Order} = 3, \text{degree} = 2$$

2. Here we eliminate the radical sign. For this write the equation as

$$\left[1 + \frac{d^2y}{dx^2}\right]^{\frac{3}{2}} = a\frac{d^2y}{dx^2}$$

Squaring both sides, we get

$$\left[1+\frac{d^2y}{dx^2}\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$$

$$\therefore \quad \text{Order} = 2, \text{degree} = 3$$

Note The direction of a curve at a particular point is given by the tangent line at that point and the slope of the tangent is given by $\frac{dy}{dx}$ at that point.

1.3 First order differential equation

An equation of the general form

$$\frac{dy}{dx} = -\frac{f(x,y)}{g(x,y)}$$

Is said to be a first order differential equation.The equation contains first and no higher derivatives. The only derivative here $\frac{dy}{dx}$ is a total or ordinarry derivative not a partial one.

1.4 Geometrical meaning of First order First degree differential equations

The solution of every first order first degree differential equations represent a family of curves.

Let, $f\left(x,y,\frac{dy}{dx}\right) = 0$ represents a differential equation of first order and first degree.

Taking $A(x_0,y_0)$ as an initial point,we can find $\frac{dy}{dx}$ at $A(x_0,y_0)$. And with the help of that we can draw the tangent at the point A.

On the tangent line take a neighbouring point $B(x_1,y_1)$. Find $\frac{dy}{dx}$ at the point $B(x_1,y_1)$and draw the tangent at B. And in this way draw another tangent at the point C on the tangent line B . Similarly draw, some more tangents by taking the neighbouring points on them. Again we take another starting point $A'(x_0',y_0')$. We can draw another curve starting from A'. In this way we can draw a number of curves. They form a smooth curve.
That is the given difrential equation represents a family of curves.

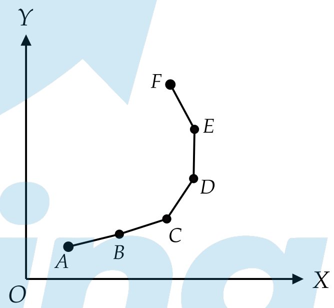


Figure 1.1: Geometrical meaning of Differential equation

1.5 Solution of a differential equation

A solution of a differential equation is any relation between variables which is free of derivatives and which satisfies the differential equation.

General solution:
A general solution is the solution in which the number of arbitrary constants and the order of the differential equation are same.

Particular solution:
A particular solution is the solution which can be obtained by giving particular values to arbitrary constants of general solution.

1.6 Solution of First order differential equations

The solutions of first order differential equations are obtained by various methods,

1.7 Method of seperation of variables

If all functions of x and dx can be arranged on one side and y and dy on the other side, then the variables are separable. The solution of this equation is found by integrating the functions of x and y.

$$f(x)dx = g(y)dy \Longrightarrow \int f(x)dx = \int g(y)dy + C$$

Method of solving:

1. Separate the variables as $f(x)dx = g(y)dy$.
2. Integrate both sides as $\int f(x)dx = \int g(y)dy$.
3. Add an arbitrary constant C on R.H.S.

Exercise 1.2 Solve $\cos(x+y)dy = dx$ ■

Solution:

$$\begin{aligned}
\cos(x+y)dy &= dx \\
\frac{dy}{dx} &= \sec(x+y) \\
\text{Let, } x+y &= z \\
\text{Then, } 1+\frac{dy}{dx} = \frac{dz}{dx} \quad &\Rightarrow \quad \frac{dy}{dx} = \frac{dz}{dx} - 1 \\
\frac{dz}{dx} - 1 = \sec z \quad &\Rightarrow \quad \frac{dz}{dx} = 1 + \sec z \\
\text{Separating the variables, we get,} & \\
\frac{dz}{1+\sec z} &= dx \\
\text{On integrating,} & \\
\int \frac{\cos z}{\cos z + 1} dz &= \int dx \\
\int \left[1 - \frac{1}{\cos z + 1}\right] dz &= x + C \\
\int \left[1 - \frac{1}{2\cos^2 \frac{z}{2} - 1 + 1}\right] dz &= x + C \\
\int \left(1 - \frac{1}{2}\sec^2 \frac{z}{2}\right) dz &= x + C \\
\int \left(1 - \frac{1}{2}\sec^2 \frac{z}{2}\right) dz &= x + C \\
z - \tan \frac{z}{2} &= x + C \\
x + y - \tan \frac{x+y}{2} &= x + C \\
y - \tan \frac{x+y}{2} &= C
\end{aligned}$$

Exercise 1.3 Solve $e^{dy/dx} = (x+1)$; given $y = 3$ at $x = 0$ ■

Solution: Taking log of both sides we get,

$$\frac{dy}{dx} = \ln(x+1) \quad \Rightarrow \quad dy = \ln(x+1)dx$$

on integration ,

$$\int dy = \int 1 \cdot \ln(x+1)dx \quad \Rightarrow \quad y = x\ln(x+1) - \int \frac{x}{(x+1)}dx + C$$

$$y = x\ln(x+1) - \int \frac{(x+1)-1}{(x+1)}dx + C$$

$$y = x\ln(x+1) - \int \frac{(x+1)}{(x+1)}dx + \int \frac{1}{(x+1)}dx + C$$

$$= x\ln(x+1) - x + \ln(x+1) + C$$

$$y = (x+1)\ln(x+1) - x + C$$

Given: at, $x = 0 \quad y = 3 \Rightarrow \quad C = 3$

Therefore, $y = (x+1)\ln(x+1) - x + 3$

1.8 Solution of Homogeneous differential equation

Homogeneous equations are of the form,

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

Where $f(x,y)$ and $g(x,y)$ are homogeneous functions of the same degree in x and y. Homogeneous functions are those in which all the terms are of n^{th} degree.

Method of solving

1. Put, $y = vx$, then $\frac{dy}{dx} = v + x\frac{dy}{dx}$.
2. Separate v and x and then integrate.

$$\frac{dy}{dx} = f(y/x)$$

$$\Rightarrow \quad y/x = v$$

$$\Rightarrow \quad \frac{dv}{f(v)-v} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{f(v)-v} = \log x + C$$

Exercise 1.4 Solve the differential equation $\left(x^2 - y^2\right)dx + 2xy\,dy = 0$, given that $y = 1$ when $x = 1$ ■

Solution:

$$\left(x^2 - y^2\right)dx + 2xydy = 0$$

$$\left(x^2 - y^2\right)dx = -2xydy$$

$$\frac{dy}{dx} = -\frac{x^2 - y^2}{2xy} = \frac{y^2 - x^2}{2xy}$$

$$\text{Putting } y = vx \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\text{We get} \quad v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x \cdot vx}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2-1}{2v}$$

$$\Rightarrow \quad x\cdot\frac{dv}{dx} = \frac{v^2-1}{2v} - v$$

$$= \frac{v^2-1-2v^2}{2v}$$

$$= -\left[\frac{v^2+1}{2v}\right]$$

$$\Rightarrow \quad \frac{2v}{v^2+1}\cdot dv = -\frac{dx}{x}, \quad x \neq 0$$

$$\Rightarrow \quad \int \frac{2v}{v^2+1}\cdot dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \log\left(v^2+1\right) = -\log|x| + c$$

$$\Rightarrow \quad \log\left(v^2+1\right) + \log|x| = \log c$$

$$\Rightarrow \quad \left(v^2+1\right)|x| = c$$

Now, putting $v = y/x$

$$\left(y^2/x^2+1\right)|x| = c$$

$$\Rightarrow \left(x^2+y^2\right) = c|x|$$

Which is similar to $x = 1$ and $y = 1$, we get, $c = 2$

Putting value of $c = 2$, We get

$$x^2+y^2 = 2x \text{ or } x^2+y^2 = 2(-x)$$

$x = 1$ and $y = 1$

Do not satisfy $x^2+y^2 = 2(-x)$

Hence, $x^2+y^2 = 2x$ is the required solution.

1.8.1 Equations reducible to homogeneuos form

Let a differential equation be,

$$\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$$

Type-1

If, in the above equation, $\frac{a}{A} \neq \frac{b}{B}$

Then we can substitute $x = X + h$, $y = Y + k$, (h, k being constants)

The given differential equation reduces to

$$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{A(X+h)+B(Y+k)+C}$$

$$= \frac{aX+bY+ah+bk+c}{AX+BY+Ah+Bk+C}$$

Choose h, k so that $ah+bk+c = 0$

$$Ah+Kk+C = 0$$

Then the given equation becomes homogeneous

$$\frac{dY}{dX} = \frac{aX+bY}{AX+BY}$$

Type-2

$$\text{If } \frac{a}{A} = \frac{b}{B},$$

Then the value of h, k will not be finite.

$$\frac{a}{A} = \frac{b}{B} = \frac{1}{m}$$

$$A = am \quad , \quad B = b\,\mathrm{m}$$

$$\text{The given equation becomes} \quad \frac{dy}{dx} = \frac{ax+by+c}{m(ax+by)+C}$$

Now put $ax+by = z$ and apply the method of variables separable.

Exercise 1.5 Solve $(x+2y)(dx-dy) = dx+dy$ ■

Solution:

$$(x+2y)(dx-dy) = dx+dy$$

$$\Rightarrow (x+2y-1)dx - (x+2y+1)dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+2y-1}{x+2y+1}$$

$$\text{Let } x+2y = z$$

$$1+2\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dz}{dx} = \frac{3z-1}{z+1} \quad (\text{Since, } \frac{dy}{dx} = \frac{x+2y-1}{x+2y+1})$$

$$\int \frac{z+1}{3z-1}dz = \int dx$$

$$\text{L.H.L} = \int \frac{z+1}{3z-1}dz$$

multiply numerator and denominator by 3, we get,

$$\begin{aligned}
\text{L.H.L} &= \int \frac{1}{3}\frac{(3z+3)}{(3z-1)}dz \\
&= \int \frac{1}{3}\frac{(3z-1+4)}{(3z-1)}dz \\
&= \int \frac{1}{3}\frac{(3z-1)+4}{(3z-1)}dz \\
&= \int \frac{1}{3}\left\{\frac{(3z-1)}{(3z-1)} + \frac{4}{(3z-1)}\right\}dz \\
&= \int \left\{\frac{1}{3} + \frac{1}{3}\frac{4}{(3z-1)}\right\}dz \\
&= \int \left\{\frac{1}{3} + \frac{4}{9}\frac{1}{(3z-1)}\right\}dz \\
&= \frac{1}{3}z + \frac{4}{9}\ln(3z-1) + c
\end{aligned}$$

Then,

$$\int \frac{z+1}{3z-1}dz = \int dx \Rightarrow \quad \frac{1}{3}(x+2y)+\frac{4}{9}\ln(3x+6y-1) = 2x+c$$
$$3x-3y+a = 2\ln(3x+6y-1)$$
$$4\ln(3x+6y-1) = (6x-6y)+c$$
$$2\ln(3x+6y-1) = (3x-3y)+c$$

1.9 Linear equation of first order

If a differential equation has its dependent variables and its derivatives occur in the first degree and are not multiplied together, then the equation is said to be linear. The standard equation of a linear equation of first order is given as

$$\frac{dy}{dx}+Py = Q$$

Where P and Q are functions of x.

$$\text{Integrating factor } = (\text{ I.F. }) = e^{\int P\cdot dx}$$
$$\Rightarrow y\cdot e^{\int P\cdot dx} = \int Q\cdot e^{\int P\cdot dx}dx+C$$
$$\Rightarrow \quad y(\text{I.F.}) = \int Q(\text{I.F.})dx+C$$

Exercise 1.6 Solve the differential equation $\frac{dy}{dx}-\frac{y}{x} = 2x^2, x>0$. ■

Solution: we know,

$$\frac{dy}{dx}+\left(\frac{-1}{x}\right)y = 2x^2$$
$$\frac{dy}{dx}+Py = Q, \text{ where } P = -\frac{1}{x} \text{ and } Q = 2x^2$$
$$I.F = e^{\int P\cdot dx} = e^{\int -1/x\cdot dx}$$
$$= e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

Multiplying both sides with $I.F$, we get

$$\frac{1}{x}\cdot\frac{dy}{dx}-\frac{1}{x^2}\cdot y = 2x$$

Integrating both sides w.r.t. x, we get

$$y\cdot\left(\frac{1}{x}\right) = \int 2x\cdot dx+C$$
$$\Rightarrow \quad y\cdot\frac{1}{x} = x^2+C$$
$$\Rightarrow \quad y = x^3+Cx, x>0 \quad \text{is the required solution.}$$

1.9.1 Equations reducible to linear form

The differential equation of the form,

$$\frac{dy}{dx}+p(x)y = f(x)y^n$$

is called the Bernoulli's equation or equation reducible to linear form. It can be done by dividing by y^n and substituting $\frac{1}{y^{n-1}} = z$

$$\begin{aligned}
&\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}}P = Q \\
&\text{Put } \frac{1}{y^{n-1}} = z \\
&\frac{(1-n)}{y^n}\frac{dy}{dx} = \frac{dz}{dx} \\
&\Rightarrow \frac{1}{y^n}\frac{dy}{dx} = \frac{dz}{1-n} \\
&\frac{1}{1-n}\frac{dz}{dx} + Pz = Q \quad \text{or} \\
&\frac{dz}{dx} + P(1-n)z = Q(1-n)
\end{aligned}$$

Which is a linear equation and can be solved easily.

Exercise 1.7 Solve $\frac{dy}{dx} + xy = x^3y^3$ ■

Solution: We have,

$$\begin{aligned}
&\frac{dy}{dx} + xy = x^3y^3 \\
&\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x^3 \\
&\text{putting } \frac{1}{y^2} = z \\
&\Rightarrow \frac{-2}{y^3}\frac{dy}{dx} = \frac{dz}{dx} \quad \Rightarrow \frac{1}{y^3}\frac{dy}{dx} = \frac{-1}{2}\frac{dz}{dx} \\
&\therefore \quad -\frac{1}{2}\frac{dz}{dx} + xz = x^3 \quad \Rightarrow \frac{dz}{dx} - 2xz = -2x^3 \\
&\therefore \text{I.F.} = e^{-\int 2xdx} = e^{-x^2} \\
&ze^{-x^2} = -2\int x^3 e^{-x^2} dx \\
&\text{Let } -x^2 = t \Rightarrow -2xdx = dt \\
&ze^{-x^2} = \int te^t dt = te^t - e^t + c \\
&\text{put } z = y^{-2} \text{and } t = -x^2 \\
&\therefore \frac{e^{-x^2}}{y^2} = -x^2e^{-x^2} - e^{-x^2} + c \\
&\frac{1}{y^2} = -x^2 - 1 + Ce^{x^2}
\end{aligned}$$

1.10 Exact differential equations

A differential equation of the form , $Mdx + Ndy = 0$ is said to be exact if it satisfy the following condition,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\frac{\partial M}{\partial y}$ – Differential co-efficient of M with respect to y keeping x constant

$\frac{\partial N}{\partial x}$ – Differential co-efficient of N with respect to x, keeping y constant.

Method of solving:
Step 1: Integrate M w.r.t. x keeping y constant
Step 2: Integrate w.r.t. y, only those terms of N which do not contain x.
Step 3: Result of 1 + Result of 2 = Constant.

Exercise 1.8 Solve $\left(x^2+2xy\right)dx+\left(x^2+y^2\right)dy=0$ ■

Solution:

$$\text{Here,}\quad M=\left(x^2+2xy\right)\quad\text{and}\quad N=\left(x^2+y^2\right)$$
$$\Rightarrow \frac{\partial M}{\partial y}=2x$$
$$\text{and}\quad \frac{\partial N}{\partial x}=2x$$
Hence, the given equation is exact
$$\int\left(x^2+2xy\right)dx+\int y^2dy=c$$
$$\frac{x^3}{3}+x^2y+\frac{y^3}{3}=C$$
The solution is:
$$\int\left(x^2+2xy\right)dx+\int y^2dy=c=\frac{x^3}{43}+x^2y+\frac{y^3}{3}=6$$

1.10.1 Equations reducible to Exact form

A differential equation which is not exact can be reduced to exact form by multiplying it by a constant , here the integrating factor.

Type 1: If $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$ is a function of x alone, say $f(x)$, then I.F $=e^{\int f(x)dx}$

Type 2: If $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$ is a function of y alone, say $f(y)$, then I.F $=e^{\int f(y)dy}$

Exercise 1.9 Solve $(2x\log x-xy)dy+2ydx=0$ ■

Solution:

$$M=2y,\quad N=2x\log x-xy$$
$$\frac{\partial M}{\partial y}=2,\quad \frac{\partial N}{\partial x}=2(1+\log x)-y$$
Here,
$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2-2-2\log x+y}{2x\log x-xy}$$
$$=\frac{-(2\log x-y)}{x(2\log x-y)}=-\frac{1}{x}=f(x)$$
$$\text{I.F.}=e^{\int f(x)dx}=e^{\int -\frac{1}{x}dx}$$
$$=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$$
On multiplying the given differential equation by $\frac{1}{x}$, we get

$$\frac{2y}{x}dx+(2\log x-y)dy=0$$
$$\Rightarrow \quad \int\frac{2y}{x}dx+\int -ydy=c$$
$$\Rightarrow \quad 2y\log x-\frac{1}{2}y^2=c$$

1.11 Orthogonal trajectories and Family of curves

Given a one-parameter family of plane curves, it's orthogonal trajectories are another one-parameter family of curves, each one of which is perpendicular to all the curves in the original family. For instance, if the original family consisted of all circles having center at the origin, it's orthogonal trajectories would be all rays (half-lines) starting at the origin.

Definition 1.11.1 Two families of curves are such that every curve of either family cuts each curve of the other family at right angles. They are called orthogonal trajectories of each other.

Orthogonal trajectories arise in different contexts in applications. If the original family represents the lines of force in a gravitational or electrostatic field, its orthogonal trajectories represent the equipotentials, the curves along which the gravitational or electrostatic potential is constant.

Example 1.3

1. The path of an electric field is perpendicular to equipotential curves.
2. In fluid flow, the stream lines and equipotential lines are orthogonal trajectories.
3. The lines of heat flow is perpendicular to isothermal curves.

■

1.11.1 Finding orthogonal trajectories to the curve

Let a family of curves be given by the equation

$$g(x,y)=C$$

Where C is a constant. For the given family of curves, we can draw the orthogonal trajectories, that is another family of curves $f(x,y)=C$ that cross the given curves at right angles.

Method of solving

Family of curves given ,then to find orthogonal trajectories.

1. By differentiating the equation of curves find the differential equations in the form $f\left(x,y,\frac{dy}{dx}\right)=0$
2. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$
   ($m_1m_2=-1$, where,m_1=given family,m_2=orthogonal family)
3. Solve the differential equation of the orthogonal trajectories i.e., $f\left(x,y,-\frac{dx}{dy}\right)=0$

Orthogonal trajectories given , then to find Family of curves .

1. Solve the differential equation of the orthogonal trajectory $f\left(x,y,\frac{dx}{dy}\right)=0$ using appropriate method.

Note **Self-orthogonal:**. If the family of orthogonal trajectory is the same as the given family of curves,then it's called self orthogonal.

Exercise 1.10 Find the orthogonal trajectories of the family of straight lines $y = Cx$, where C is a parameter. ■

Solution:

$$\begin{aligned}
\text{we have,}\quad y &= Cx \\
\text{differentiating the given equation we get} \\
dy &= cdx \\
dy &= \frac{y}{x}dx \quad (\because c = \frac{y}{x}) \\
\frac{dy}{dx} &= \frac{y}{x} \\
\frac{dy}{dx_{(ortho)}} &= \frac{-x}{y} \\
\text{using variable separable method} \\
(-xdx &= ydy) \\
\int -xdx &= \int ydy \\
-\frac{x^2}{2} &= \frac{y^2}{2} + C \\
\frac{x^2}{2} + \frac{y^2}{2} &= C \\
x^2 + y^2 &= 2C \Longrightarrow \text{Represents the family of circles}
\end{aligned}$$

Exercise 1.11 Find the family of curves of the given trajectory, $\frac{dy}{dx} = \frac{y}{x}$ ■

Solution:

$$\begin{aligned}
\text{given,}\ \frac{dy}{dx} &= \frac{y}{x} \\
\text{using variable seperable method, we get} \\
\int xdx &= \int ydy \\
\frac{x^2}{2} &= \frac{y^2}{2} + C \\
\frac{x^2}{2} - \frac{y^2}{2} &= C \\
x^2 - y^2 &= 2C \Longrightarrow \text{family of hyperbolas}
\end{aligned}$$

1.12 Second order differential equations

1.12.1 Linear differential equation

If the degree of the dependent variable and all derivatives is one, such differential equations are called linear differential equations.

■ **Example 1.4**

1. $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = x^2 + x + 1$

2. $\frac{d^2x}{dx^2} - \frac{dy}{dx} - 3y = x$

3. $2\frac{d^2x}{dt^2} - \frac{dx}{dt} - 3x = f(t)$

■

1.12.2 Non-Linear differential equation

If the degree of the dependent variable and / or its derivatives are of greater than 1 such differential equations are called non-linear differential equations.

■ Example 1.5

1. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y^2 = \sin x$
2. $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y^2 = e^x$
3. $\left(\frac{d^2x}{dt^2}\right)^2 + \frac{dx}{dt} + x = f(t)$

■

1.12.3 Homogeneous differential equation

A differential equation of the form $y'' + P(x)y' + Q(x)y = F(x)$, is said to be non-homogeneous if $F(x) = 0$

■ Example 1.6

1. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$
2. $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y^2 = 0$

■

1.12.4 Nonhomogeneous differential eqaution

A differential equation of the form $y'' + P(x)y' + Q(x)y = F(x)$, is said to be non-homogeneous if $F(x) \neq 0$

■ Example 1.7

1. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + y = x^2 + 2$
2. $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y^2 = e^x$

■

1.13 Linear independance and dependance of solutions

Two solutions of a differential equation, $y_1(x)$ and $y_2(x)$ are said to be linearly independant if

$$Ay_1(x) + By_2(x) \neq 0$$

given, $A \neq 0$ and $B \neq 0$

1.13.1 Wronskian

The Wronskian of two functions $y_1(x)$ and $y_2(x)$ is given by,

$$W(y_1, y_2, x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

1. If $W(y_1, y_2, x) = 0$, then $y_1(x)$ and $y_2(x)$ are linearly dependent.
2. If $W(y_1, y_2, x) \neq 0$, then $y_1(x), y_2(x)$ are linearly independent.

1.14 Linear Second Order Differential Equations With Constant Coefficients

The general form of the linear differential equation of second order is

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

Where P and Q are constants and R is a function of x or constant.

Note **Differential operator:**
A differential operator can be represented as, $D=\frac{d}{dx}$
Then a differential equation can be written in terms of differntial operators as,

$$D^2y+PDy+Qy=R$$
$$\left(D^2+PD+Q\right)y=R$$
$$\text{Where,}\quad Dy=\frac{dy}{dx},\text{ and } D^2y=\frac{d^2y}{dx^2}$$

$\frac{1}{D}$ stands for the operation of integration.

1.15 Solution of Second order homogeneous differential equation.

1.15.1 Method of solving

1. Let $y=C_1e^{mx}$ be the trial solution

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0 \tag{1.1}$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in 1.1 then,
$C_1e^{mx}\left(m^2+Pm+Q\right)=0\Rightarrow m^2+Pm+Q=0$. It is called Auxiliary equation.

2. Solve the auxiliary equation.

Case 1:

Roots are real and distinct
If m_1 and m_2 are the roots, then the C.F. is

$$y=C_1e^{m_1x}+C_2e^{m_2x}$$

Case 2:

Roots are real and equal
If both the roots are m,m then the C.F. is

$$y=\left(C_1+C_2x\right)e^{mx}$$

Case 3:

Roots are Imaginary
If the roots are $\alpha\pm i\beta$, then the solution will be

$$\begin{aligned}y&=C_1e^{(\alpha+i\beta)x}+C_2e^{(\alpha-i\beta)x}=e^{\alpha x}\cdot\left[C_1e^{i\beta x}+C_2e^{-i\beta x}\right]\\&=e^{\alpha x}\left[C_1\left(\cos\beta x+i\sin\beta x\right)+C_2\left(\cos\beta x-i\sin\beta x\right)\right]\\&=e^{\alpha x}\left[\left(C_1+C_2\right)\cos\beta x+i\left(C_1-C_2\right)\sin\beta x\right]\\&=e^{\alpha x}[A\cos\beta x+B\sin\beta x]\end{aligned}$$

Roots	Basis of solution	General solution
Real and equal(repeated root m)	e^{mx} and xe^{mx}	$y=(C_1+C_2x)\,e^{mx}$
Real and distinct(m_1 , m_2)	e^{m_1x} , e^{m_2x}	$y=C_1e^{m_1x}+C_2e^{m_2x}$
Imaginary roots ($\alpha\pm i\beta$)	$e^{(\alpha+i\beta)x}, e^{(\alpha-i\beta)x}$	$e^{\alpha x}[A\cos\beta x+B\sin\beta x]$

Table 1.1: Roots of homogeneous second order DE

Exercise 1.12 Solve $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+15y=0$. ■

Solution:

$$\begin{aligned}
\text{Given equation can be written as,}&\\
\left(D^2-8D+15\right)y&=0\\
\text{Here auxiliary equation is,}&\\
m^2-8m+15&=0\\
(m-3)(m-5)&=0 \quad \therefore m=3,5\\
\text{Hence, the required solution is,}&\\
y&=C_1e^{3x}+C_2e^{5x}
\end{aligned}$$

Exercise 1.13 Solve the differential equation: $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=0$ ■

Solution: We have

$$\begin{aligned}
\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y&=0\\
\Rightarrow \left(D^2+6D+9\right)y&=0\\
\text{Auxiliary equation is}\quad D^2+6D+9&=0\\
\Rightarrow \quad (D+3)^2&=0\Rightarrow D=-3,-3\\
\text{the solution, y}&=(c_1+c_2x)\,e^{-3x}
\end{aligned}$$

Exercise 1.14 Solve $\left(D^3-1\right)y=0$ ■

Solution:

$$\begin{aligned}
\text{we have, }\left(D^3-1\right)y&=0\\
\text{The characteristic equation is,}&\\
m^3-1=0\Rightarrow m&=1,\frac{-1\pm\sqrt{3}i}{2}\\
\text{y }&=Ae^x+e^{-x/2}\cdot\left[B\cos\frac{\sqrt{3}}{2}x+C\sin\frac{\sqrt{3}}{2}x\right]
\end{aligned}$$

1.16 Solution of Second order nonhomogeneous differential equation.

The solution of a differential equation of the form,

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

consists of two parts,a complementary function and a particular integral.
Complete Solution = Complementary Function + Particular Integral

$$\boxed{\text{y} \;=\; \text{C.F} \;+\; \text{P.I}}$$

Finding complementary function

Complementary function is the solution obtained by solving the equation replacing R.H.S by 0.Same as that explained in finding solution to homogeneous differential equations.

Finding Particular solution

Particular integral (P.I) depends on the form of $R(x)$
If the differential equation is of the form,

$$\left(D^n+k_1D^{n-1}+k_2D^{n-2}+\cdots+k_n\right)y=R(x)$$

Then the particular integral of the equation is given by,

$$\text{P.I.} = \frac{1}{D^n+k_1D^{n-1}+k_2D^{n-2}+\cdots+k_n}R(x)$$

The following cases arise for particular integrals:

1. When $X=e^{ax}$, then

$$\begin{aligned}\text{P.I.} &= \frac{1}{f(D)}e^{ax}\\ &= \frac{1}{f(a)}e^{ax}, \quad \text{If } f(a)\neq 0\end{aligned}$$

If $f(a)=0$, then

$$\text{P.I.} = \frac{x}{f'(a)}e^{ax}, \quad \text{If } f'(a)\neq 0$$

If $f'(a)=0$, then

$$\text{P.I.} = \frac{x^2}{f''(a)}e^{ax}, \quad \text{If } f''(a)\neq 0$$

2. When $X=\sin ax$, then

$$\text{P.I.} = \tfrac{1}{f(D^2)}\sin ax$$

$$= \tfrac{1}{f(-a^2)}\sin ax, \text{ if } f\left(-a^2\right)\neq 0$$

3. When $X=\cos ax$, then

$$\begin{aligned}\text{P.I.} &= \frac{1}{f(D^2)}\cos ax\\ &= \frac{1}{f(-a^2)}\cos ax, \text{ if } f\left(-a^2\right)\neq 0\end{aligned}$$

4. When $X=x^m$, then

$$\text{P.I.} = \frac{1}{f(D)}x^m=[f(D)]^{-1}x^m$$

Expansion of $[f(D)]^{-1}$ is to be carried up to the term D^m because $(m+1)^{\text{th}}$ and higher derivatives of x^m are zero.

5. When $X = e^{ax}v(x)$, then

$$\text{P.I.} = \tfrac{1}{f(D)} e^{ax} v(x)$$
$$\text{P.I.} = e^{ax} \tfrac{1}{f(D+a)} v(x)$$

6. When $X = xv(x)$, then

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D)} x v(x) \\ &= \left[x - \frac{f'(D)}{f(D)}\right] \cdot \frac{1}{f(D)} v(x) \end{aligned}$$

Exercise 1.15 Solve the differential equation:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$$

Solution: We have

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$$
$$\left(D^2 + 6D + 9\right) y = 5e^{3x}$$

Auxiliary equation is $D^2 + 6D + 9 = 0$

$$(D+3)^2 = 0 \Rightarrow D = -3, -3$$

The solution, $y = (c_1 + c_2 x)\, e^{-3x}$

$$\begin{aligned} \text{Particular integral } &= \frac{1}{D^2 + 6D + 9} \cdot 5e^{3x} \\ &= \frac{5e^{3x}}{(3)^2 + 6(3) + 9} = \frac{5e^{3x}}{36} \end{aligned}$$

The complete solution is given by$y = \text{C.F.} + \text{P.I}$

$$y = (c_1 + c_2 x)\, e^{-3x} + \frac{5e^{3x}}{36}$$

1.17 Euler - Cauchy Differential equation

A homogeneous differential equation of the form,

$$a_n x^n \frac{d^n y}{dx^n} + a_{(n-1)} x^{(n-1)} \frac{d^{(n-1)} y}{dx^{(n-1)}} + \cdots\cdots + a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = \text{Q(x)} \tag{1.2}$$

Where, $a_0, a_1, a_2 \cdots a_n$ are constants is called a Euler - Cauchy differential equation.

$$\text{Put, x} = \text{e}^z \Rightarrow \log \text{x} = \text{z} \Rightarrow \frac{1}{x} = \frac{dz}{dx}$$
$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dx} \cdot \frac{dz}{dx} \\ &= \frac{1}{x} \frac{dy}{dz} \\ \text{x}\frac{dy}{dx} &= \frac{dy}{dz} \end{aligned}$$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right)\\
&= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)\\
&= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}\\
x^2\frac{dy}{dx^2} &= -\frac{dy}{dz} + \frac{d^2y}{dz^2}
\end{aligned}$$

Then neglecting the higher orders , the given equation becomes,

$$a_2\frac{d^2y}{dz^2} + (a_1 - a_2)\frac{dy}{dz} + a_0y = \mathrm{Q}$$

Linear second order non-homogeneous differential equation

Exercise 1.16 $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 20y = (x+1)^2$ ■

Solution:

$$\begin{aligned}
x &= e^z\\
\log x = z \Rightarrow \frac{1}{x} &= \frac{dz}{dx}\\
\text{Let, } x\frac{dy}{dx} &= \frac{dy}{dz}\\
\text{and,} x^2\frac{d^2y}{dx^2} &= \frac{d^2y}{dz^2} - \frac{dy}{dz}\\
\text{Then we get,} \frac{d^2y}{dz^2} + \frac{dy}{dz} - 20y &= (e^z+1)^2\\
\text{C.F.} &= c_1e^{4z} + c_2e^{-5z}\\
&= c_1x^4 + c_2x^{-5}\\
\text{P . I .} &= \frac{1}{D^2+D-20}(e^z+1)^2\\
&= \frac{1}{D^2+D-20}\left(e^{2z}+2e^z+1\right)\\
&= -\frac{1}{14}e^{2z} - \frac{1}{9}e^z - \frac{1}{20}\\
\text{Total solution } y &= c_1x^4 + c_2x^{-5} - \frac{1}{14}x^2 - \frac{1}{9}x - \frac{1}{20}
\end{aligned}$$

1.18 Singular Points

The concept of singular point or singularity (as applied to a differential equation) stems from the ts usefulness in classifying Ordinary Differential Equations and investigating the feasibility of a series solution (Series Solution method will be explained in the next section) If we write our second-order homogeneous differential equation (in y) as,

$$y'' + P(x)y' + Q(x)y = 0 \tag{1.3}$$

We are ready to define ordinary and singular points. If the functions $P(x)$ and $Q(x)$ remain finite at $x = x_0$, point $x = x_0$ is an ordinary point. However, if either $P(x)$ or $Q(x)$ (or both) diverges as $x \to x_0$, point x_0 is a singular point. Using equation 1.3 we may distinguish between two kinds of singular points.

1. If either $P(x)$ or $Q(x)$ diverges as $x \to x_0$ but $(x-x_0)\,P(x)$ and $(x-x_0)^2\,Q(x)$ remain finite as $x \to x_0$, then $x = x_0$ is called a **regular**, or **nonessential, singular point.**

2. If $P(x)$ diverges faster than $\frac{1}{(x-x_0)}$ so that $(x-x_0)\,P(x)$ goes to infinity as $x \to x_0$, or $Q(x)$ diverges faster than $\frac{1}{(x-x_0)}^2$ so that $(x-x_0)^2\,Q(x)$ goes to infinity as $x \to x_0$, then point $x = x_0$ is labeled an **irregular**, or **essential, singularity**.

1.19 Series Solution Method

Series expansion is a method of obtaining one solution of the linear, second-order, homogeneous ODE. The method, will always work, provided the point of expansion is no worse than a regular singular point.In physics this very gentle condition is almost always satisfied. A linear, second-order, homogeneous ODE can be written in the form

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0 \tag{1.4}$$

The most general solution of the equation 1.4 may be written as,

$$y(x)=c_1y_1(x)+c_2y_2(x) \tag{1.5}$$

But a physical problem may lead to a nonhomogeneous, linear, second-order ODE

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=F(x) \tag{1.6}$$

Hence the most general solution to the equation will be of the form,

$$y(x)=c_1y_1(x)+c_2y_2(x)+y_p(x) \tag{1.7}$$

The constants c_1 and c_2 will eventually be fixed by boundary conditions.

There are two series solution method for differential equation,

1. **Simple series expansion method**
2. **Frobenious Method**

1.19.1 Simple Power Series Expansion Method

The simple series expansion method works for differential equations whose solutions are well-behaved at the expansion point $x=0$. This method can be illustrated by Linear classical oscillator problem

1.19.2 Classical Linear Oscillator

$$\frac{d^2y}{dx^2}+\omega^2y=0 \tag{1.8}$$

$$\text{with known solutions } y=\sin\omega x,\cos\omega x \tag{1.9}$$

$$\text{We try } y(x)=x^k\left(a_0+a_1x+a_2x^2+a_3x^3+\cdots\right) \tag{1.10}$$

$$=\sum_{\lambda=0}^{\infty}a_\lambda x^{k+\lambda},\quad a_0\neq 0 \tag{1.11}$$

with the exponent k and all the coefficients a_λ still undetermined. Note that k need not be an integer. By differentiating twice, we obtain

$$\frac{dy}{dx}=\sum_{\lambda=0}^{\infty}a_\lambda(k+\lambda)x^{k+\lambda-1} \tag{1.12}$$

$$\frac{d^2y}{dx^2}=\sum_{\lambda=0}^{\infty}a_\lambda(k+\lambda)(k+\lambda-1)x^{k+\lambda-2} \tag{1.13}$$

By substituting into equation.1.8, we have

$$\sum_{\lambda=0}^{\infty}a_\lambda(k+\lambda)(k+\lambda-1)x^{k+\lambda-2}+\omega^2\sum_{\lambda=0}^{\infty}a_\lambda x^{k+\lambda}=0 \tag{1.14}$$

The coefficients of each power of x on the left-hand side of equation.1.14 must vanish individually.The lowest power of x appearing in equation.1.14 is x^{k-2}, for $\lambda = 0$ in the first summation. The requirement that the coefficient vanish yields,

$$a_0 k(k-1) = 0 \tag{1.15}$$

We had chosen a_0 as the coefficient of the lowest nonvanishing terms of the series 1.11, hence, by definition, $a_0 \neq 0$. Therefore we have,

$$k(k-1) = 0 \tag{1.16}$$

This equation, coming from the coefficient of the lowest power of x, we call the indicial equation. The indicial equation and its roots are of critical importance to our analysis.
From equation.1.16, $k = 0$ or $k = 1$
The only way a power series can be zero is, it's coefficients must be equal to zero. But here the power of x in the equation do not match up. The Coefficent of x in the first term is,$k+\lambda-2$ and for the second term it is,$k+\lambda$, to make them equal, we can replace λ by $\lambda+2$ in the first term. Then we get,

$$\sum_{\lambda=2}^{\infty} a_{\lambda+2}(k+\lambda+2)(k+\lambda+1)x^{k+\lambda} + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0 \tag{1.17}$$

$$\sum_{\lambda=2}^{\infty} a_{\lambda+2}(k+\lambda+2)(k+\lambda+1) + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda = 0 \tag{1.18}$$

Here the coefficients are independent summations and λ is a dummy index. Then we get,

$$a_{\lambda+2}(k+\lambda+2)(k+\lambda+1) + \omega^2 a_\lambda = 0 \tag{1.19}$$

$$a_{\lambda+2} = -a_\lambda \frac{\omega^2}{(k+\lambda+2)(k+\lambda+1)} \tag{1.20}$$

For this example, if we start with a_0, Equation.1.20 leads to the even coefficients a_2, a_4, and so on, and ignores a_1, a_3, a_5, and so on. Since a_1 is arbitrary if $k = 0$ and necessarily zero if $k = 1$,

$$a_3 = a_5 = a_7 = \cdots = 0$$

and all the odd-numbered coefficients vanish. The odd powers of x will actually reappear when the second root of the indicial equation is used.

$$a_{\lambda+2} = -a_\lambda \frac{\omega^2}{(\lambda+2)(\lambda+1)} \tag{1.21}$$

which leads to

$$a_2 = -a_0 \frac{\omega^2}{1\cdot 2} = -\frac{\omega^2}{2!} a_0 \tag{1.22}$$

$$a_4 = -a_2 \frac{\omega^2}{3\cdot 4} = +\frac{\omega^4}{4!} a_0 \tag{1.23}$$

$$a_6 = -a_4 \frac{\omega^2}{5\cdot 6} = -\frac{\omega^6}{6!} a_0, \quad \text{and so on.} \tag{1.24}$$

By inspection (and mathematical induction),

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!} a_0 \tag{1.25}$$

and our solution is

$$y(x)_{k=0} = a_0 \left[1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} - \frac{(\omega x)^6}{6!} + \cdots \right] \tag{1.26}$$

$$= a_0 \cos \omega x \tag{1.27}$$

If we choose the indicial equation root $k = 1$ Equation.1.20, the recurrence relation becomes

$$a_{j+2} = -a_j \frac{\omega^2}{(j+3)(j+2)} \tag{1.28}$$

Substituting in $j = 0, 2, 4$, successively, we obtain

$$a_2 = -a_0 \frac{\omega^2}{2 \cdot 3} = -\frac{\omega^2}{3!} a_0 \tag{1.29}$$

$$a_4 = -a_2 \frac{\omega^2}{4 \cdot 5} = +\frac{\omega^4}{5!} a_0 \tag{1.30}$$

$$a_6 = -a_4 \frac{\omega^2}{6 \cdot 7} = -\frac{\omega^6}{7!} a_0, \quad \text{and so on.} \tag{1.31}$$

Again, by inspection and mathematical induction,

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n+1)!} a_0 \tag{1.32}$$

For this choice, $k = 1$, we obtain

$$y(x)_{k=1} = a_0 x \left[1 - \frac{(\omega x)^2}{3!} + \frac{(\omega x)^4}{5!} - \frac{(\omega x)^6}{7!} + \cdots \right] \tag{1.33}$$

$$= \frac{a_0}{\omega} \left[(\omega x) - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} - \frac{(\omega x)^7}{7!} + \cdots \right] \tag{1.34}$$

$$= \frac{a_0}{\omega} \sin \omega x \tag{1.35}$$

Power Series Solution (About an Ordinary Point)

Find the power series solution of $\left(1 - x^2\right) y'' - 2xy' + 2y = 0$ about $x = 0$

Since $x = 0$ is an ordinary point of the given differential equation, the solution can be written as

$$y = \sum_{k=0}^{\infty} a_k x^k$$

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} k a_k x^{k-1}$$

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k k(k-1) x^{k-2}$$

Substituting these values in the given equation we get,

$$\left(1 - x^2\right) \sum_k a_k k(k-1) x^{k-2} - 2x \sum_k a_k (k) x^{k-1} + 2 \sum_k a_k x^k = 0$$

$$\sum_{k=2} a_k k(k-1) x^{k-2} - \sum \left(k^2 + k - 2\right) a_k x^k = 0$$

now equating the coefficient of x^k then

$$(k+2)(k+1) a_{k+2} - \left(k^2 + k - 2\right) a_k = 0$$

$$a_{k+2} = \frac{k-1}{(k+1)} a_k$$

$$\begin{aligned}
\text{For } k = 0 &\Rightarrow a_2 = -a_0 \\
k = 1 &\Rightarrow a_3 = 0 \\
k = 2 &\Rightarrow a_4 = \frac{a_2}{3} = \frac{-a_0}{3} \\
k = 3 &\Rightarrow a_5 = \frac{2}{4}a_3 = 0
\end{aligned}$$

$$\begin{aligned}
\text{Therefore, solution } y &= a_0 + a_1x + a_2x^2 + \ldots\ldots \\
&= a_0\left[1 - x^2 - \frac{x^4}{3}\ldots\ldots\right] + a_1x
\end{aligned}$$

1.19.3 Frobenious Method

Even though the simple power series expansion method works for many functions there are some whose behaviour precludes the simple series method like the Bessel's function. The need of Frobenious method lies under the fact that, **any functions involving negative or fractional powers would not be amenable to power series solution method**. The Frobenious method extends the simple power series solution method to include negative and fractional powers, and it also allows a natural extension involving logarithm terms.
The basic idea of the Frobenius method is to look for solutions of the form

$$\begin{aligned}
y(x) &= a_0x^\lambda + a_1x^{\lambda+1} + a_2x^{\lambda+2} + a_3x^{\lambda+3} + \ldots \\
&= x^\lambda\left(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\right) \\
&= x^\lambda \sum_{k=0}^{\infty} a_kx^k \\
&= \sum_{k=0}^{\infty} a_kx^{k+\lambda}
\end{aligned}$$

The extension of the simple power series method is all in the factor x^λ. The power c must now be determined, as well as the coefficients a_k. Since λ may be negative, positive, and possibly non-integral, this extends considerably the range of functions which may be treated. Note that a_0 is the lowest non-zero coefficient, so by definition it cannot be zero.

1.19.4 Bessel Function

Problem Set -1

1. The solution of the differential equation $dz(x,y)+xz(x,y)dx+yz(x,y)dy=0$ is.........

[IIT JAM 2013]

$Ce^{-\left(x^2+y^2\right)/2}$

2. Consider a function $f(x,y)=x^3+y^3$, where y represents a parabolic curve x^2+1. The total derivative of f with respect to x, at $x=1$ is.....................

[IIT JAM 2016]

3. The tangent line to the curve $x^2+xy+5=0$ at $(1,1)$ is represented by

a. $y=3x-2$ **b.** $y=-3x+4$

c. $x=3y-2$ **d.** $x=-3y+4$

4. Consider two particles moving along the x - axis. In terms of their coordinates x_1 and x_2, their velocities are given as $\frac{dx_1}{dt}=x_2-x_1$ and $\frac{dx_2}{dt}=x_1-x_2$, respectively. When they start moving from their initial locations of $x_1(0)=1$ and $x_2(0)=-1$, the time dependence of both x_1 and x_2 contains a term of the form e^{at}, where a is a constant. The value of a (an integer) is..........

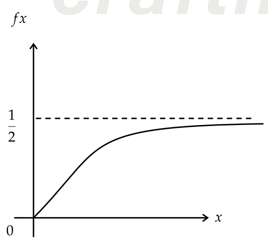
[IIT JAM 2017]

5. Consider the differential equation $y''+2y'+y=0$. If $y(0)=0$ and $y'(0)=1$, then the value of $y(2)$ is...................(Specify your answer to two digits after the decimal point)

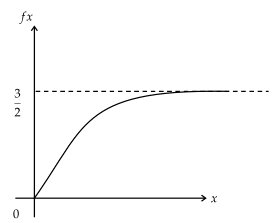
[IIT JAM 2017]

6. Which one of the following curves correctly represents (schematically) the solution for the equation $\frac{df}{dx}+2f=3:f(0)=0$?

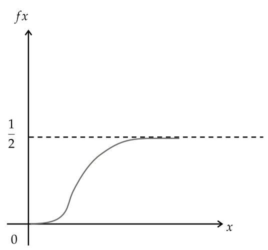
[IIT JAM 2018]



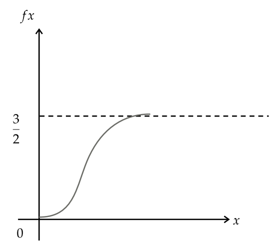
a.



b.



c.



d.

7. Let $f(x)=3x^6-2x^2-8$. Which of the following statements is (are) true?

[IIT JAM 2018]

a. The sum of all its roots is zero **b.** The product of its roots is $-\frac{8}{3}$

c. The sum of all its roots is $\frac{2}{3}$ **d.** Complex roots are conjugates of each other.

Answer key			
Q.No.	Answer	Q.No.	Answer
1	$Ce^{-\frac{\left(x^2+y^2\right)}{2}}$	2	**27**
3	**b**	4	**2**
5	**0.27**	6	**b**
7	**a,b,d**	8	
9		10	
11		12	
13		14	
15			

Problem Set -2

1. The solutions to the differential equation $\frac{dy}{dx} = -\frac{x}{y+1}$ are a family of

   a. Circles with different radii

   b. Circles with different centres

   c. Straight lines with different slopes

   d. Straight lines with different intercepts on the y -axis

2. The general solution of the differential equation $\frac{dy}{dx} = \frac{1+\cos 2y}{1-\cos 2x}$ is,

   a. $\tan y - \cot x = c$ (c is a constant)

   b. $\tan x - \cot y = c$ (c is a constant)

   c. $\tan y + \cot x = c$ (c is a constant)

   d. $\tan x + \cot y = c$ (c is a constant)

3. If $y(x)$ is solution for first order differential equation $xy' + (x+1)y = 6$, then value of $y(x)$ as $x \to \infty$ is............

4. The solution of the differential equation $xy\frac{dy}{dx} = 3y^2 + x^2$ with the initial condition $y = 2$ when $x = 1$ is

   a. $2y^2 + x^2 = 9x^6$

   b. $y^2 + 2x^2 = 9x^6$

   c. $2y^2 + x^2 = 8x^6$

   d. $y^2 + 2x^2 = 8x^6$

5. Find the solution to $9y'' + 6y' + y = 0$ for $y(0) = 4$ and $y'(0) = -1/3$.

   a. $y = (4+x)e^{-x/3}$

   b. $y = (4-x)e^{-x/3}$

   c. $y = (8-2x)e^{x/3}$

   d. $y = (1-x)e^{-x/3}$

6. The solution of the differential equation $\frac{dy}{dx} + y^2 = 0$ is

   a. $y = \frac{1}{x+c}$

   b. $y = \frac{-x^3}{3} + c$

   c. ce^x

   d. unsolvable as equation is non-linear.

7. Solution of the differential equation $x^2\frac{dy}{dx} + 2xy = y^3$ is

   a. $\frac{1}{5x} + cx^4$

   b. $\frac{2}{5x} + cx^4$

   c. $5x + cx^4$

   d. $5x - cx^{-4}$

8. Find the orthogonal trajectories of the family of curves $x^2 + y^2 = C$

9. The solution of the differential equation $\frac{dx}{dt} = x^2$ with the initial condition $x(0) = 1$ will blow up as t tend to

   a. ∞

   b. 2

   c. $\frac{1}{2}$

   d. 1

10. Consider the linear differential equation $\frac{dy}{dx} = xy$. If $y = 2$ at $x = 0$, then the value of y at $x = .2$ is

    a. e^{-2}

    b. $2e^{-2}$

    c. $2e^2$

    d. e^2

11. Biotransformation of an organic compound having concentration (x) can be modeled using an ordinary differential equation $\frac{dx}{dt} + kx^2 = 0$, where k is the reaction rate constant. If $x = a$ at $t = 0$, the solution of the equation is

a. $x = ae^{-kt}$ **b.** $\frac{1}{x} = \frac{1}{a} + kt$ **c.** $x = a\left(1 - e^{-kt}\right)$ **d.** $x = a + kt$

12. The solution for the differential equation $\frac{dy}{dx} = x^2 y$ with the condition that $y = 1$ at $x = 0$ is

a. $y = e^{\frac{1}{2x}}$ **b.** $\ln(y) = \frac{x^3}{3} + 4$ **c.** $\ln(y) = \frac{x^2}{2}$ **d.** $y = e^{\frac{x^3}{3}}$

13. It is given that $y'' + 2y' + y = 0, y(0) = 0, y(1) = 0$. What is $y(0.5)$?

a. 0 **b.** 0.62 **c.** 0.37 **d.** 1.13

14. The solution to the ordinary differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$ is

a. $y = c_1 e^{3x} + c_2 e^{-2x}$ **b.** $y = c_1 e^{3x} + c_2 e^{2x}$ **c.** $y = c_1 e^{-3x} + c_2 e^{2x}$ **d.** $y = c_1 e^{-3x} + c_2 e^{-2x}$

15. The solution of the differential equation $\frac{d^2y}{dt^2} + \frac{2dy}{dt} + y = 0$ with $y(0) = y'(0) = 1$ is

a. $(2 - t)e^t$ **b.** $(1 + 2t)e^{-t}$ **c.** $(2 + t)e^{-t}$ **d.** $(1 - 2t)e^t$

Answer key

Q.No.	Answer	Q.No.	Answer
1	**a**	2	**d**
3	**NAT-0**	4	**a**
5	**a**	6	**a**
7	**b**	8	**c**
9	**d**	10	**c**
11	**b**	12	**d**
13	**a**	14	**c**
15	**b**		

Problem Set -3

1. The order and degree of the differential equation $y+\frac{dy}{dx}=\frac{1}{4}\int y\cdot dx$ are

a. order $=2$ and degree $=1$
b. order $=1$ and degree $=2$
c. order $=1$ and degree $=1$
d. order $=2$ and degree $=2$

Solution: We have

$$y+\frac{dy}{dx}=\frac{1}{4}\int y\cdot dx$$
$$\Rightarrow \frac{dy}{dx}+\frac{d^2y}{dx^2}=\frac{1}{4}y \quad \text{[on differentiating w.r.t. } x\text{]}$$

Differential equation is of order 2 and degree 1.

Correct answer is **option(a)**

2. The following differential equation has

$$3\left(\frac{d^2y}{dt^2}\right)+4\left(\frac{dy}{dt}\right)^3+y^2+2=x$$

a. Degree $=2$, order $=1$
b. Degree $=1$, order $=2$
c. Degree $=4$, order $=3$
d. Degree $=2$, order $=3$

Solution: The highest derivative term of the equation is 2, hence order $=2$. The power of highest derivative term is 1 , hence degree $=1$

Correct answer is **option (b)**.

3. If $y(x)$ is the solution of the differential equation $-x\frac{dy}{dx}+y=y^2\log x$ with $y(1)=-1$ then

a. $y(x)$ is defined and finite in the range $-\infty<x<0$
b. $y(x)$ is defined and finite in the range $0<x<3$
c. $y(x)$ is defined and finite in the range $x\geq 3$
d. $y(x)$ blows up at $x=e$

Solution:

$$-x\frac{dy}{dx}+y=y^2\log x$$
$$\Rightarrow -\frac{dy}{dx}+\frac{y}{x}=y^2\frac{\log x}{x}$$
$$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{x}\times\frac{1}{y}=\frac{\log}{x}$$
$$\text{Let } \frac{1}{y}=z\Rightarrow -\frac{1}{y^2}\frac{dy}{dx}=\frac{dz}{dx}$$
$$\text{Therefore, } \frac{dz}{dx}+\frac{z}{x}=\frac{\log x}{x}\Rightarrow \frac{dz}{dx}+\frac{1}{x}z=\frac{\log x}{x}$$
$$\text{Here, I.F. }=e^{\int\frac{1}{x}dx}=e^{\log x}=x$$

Therefore, required solution will be

$$zx = \int \frac{\log x}{x} \times x dx + c \Rightarrow \frac{x}{y} = \int \log x + c \Rightarrow \frac{x}{y} = (x \log x - x) + c$$

Now, $y(1) = -1$, put $x = 1, y = -1 \quad \Rightarrow -1 = 0 - 1 + c \Rightarrow c = 0$

$$\text{Therefore, } \frac{x}{y} = x \log x - x \Rightarrow y = \frac{1}{\log x - 1}$$

Therefore, for y to be defined $\log x$ must be defined i.e., $x > 0$

The solution will not be defined when $(\log x - 1) = 0 \Rightarrow \log x = 1 \Rightarrow x = e \simeq 2.73$

So in the range $0 < x < 3$ [option (b)], there will be a point $x = 2.73$ at which y is not defined.

Then Correct answers are **option (c)** and **option (d)**.

4. The initial velocity of an object is 40 m/s. The acceleration a of the object is given by the following expression:

$$a = -0.1v$$

where v is the instantaneous velocity of the object. The velocity of the object after 3 s will be

Solution: We are given the following expression:

$$ca = -0.1v$$

$$\frac{dv}{dt} = -0.1v$$

$$\frac{dv}{v} = -0.1dt$$

On integration we get,

$$\ln v = -0.1t + \ln k$$

$$v = ke^{-0.1t}$$

$$\text{at } t = 0; v = 40 \Rightarrow k = 40$$

$$v = 40e^{-0.1t}$$

$$\text{At } t = 3 \text{ s}, V = 40e^{-0.1 \times 3} = 29.6327 \text{ m/s}$$

5. Solve $ydx - xdy + \left(1 + x^2\right) dx + x^2 \sin y dy = 0$

Solution:

Dividing each term of the given equation by x^2, we get

$$\frac{ydx - xdy}{x^2} + \frac{1 + x^2}{x^2} dx + \sin y dy = 0$$

$$\Rightarrow -\frac{xdy - ydx}{x^2} + \left(\frac{1}{x^2} + 1\right) dx + \sin y dy = 0$$

$$\Rightarrow -d\left(\frac{y}{x}\right) + \left(1 + \frac{1}{x^2}\right) dx + \sin y dy = 0$$

Integrating,

$$-\left(\frac{y}{x}\right)+x-\frac{1}{x}=\cos y+c$$
$$\Rightarrow -y+x^2-1-x\cos y=cx$$

6. One of the possible solutions of the differential equation $y\sqrt{(1+x^2)}dy+x\sqrt{(1+y^2)}dx=0$ (where c is some constant) is

a. $\left(\sqrt{1+y^2}\right)\left(\sqrt{1+x^2}\right)=c$

b. $\frac{\sqrt{1+y^2}}{\sqrt{1+x^2}}=c$

c. $\sqrt{1+y^2}+\sqrt{1+x^2}=c$

d. $\sqrt{1+y^2}-\sqrt{1+x^2}=c$

Solution:

$$y\sqrt{(1+x^2)}dy+x\sqrt{(1+y^2)}dx=0$$
$$\frac{y}{\sqrt{1+y^2}}dy+\frac{x}{\sqrt{1+x^2}}dx=0$$
$$\int\frac{y}{\sqrt{1+y^2}}dy+\int\frac{x}{\sqrt{1+x^2}}dx=c$$

Let, $1+y^2=t$ and $1+x^2=z$

$$\int\frac{1}{2}t^{-1/2}dt+\int\frac{1}{2}z^{-1/2}dz=c$$
$$\sqrt{t}+\sqrt{z}=c$$
$$\sqrt{1+y^2}+\sqrt{1+x^2}=c$$

Correct answer is **option (c)**.

7. The solution of the differential equation $(e^y+2)\sin x dx-e^y\cos x dy=0$ (where c is some constant) is

a. $(e^y+2)\sin x=c$

b. $(e^y+2)\cos x=c$

c. $(e^y+2)\operatorname{cosec} x=c$

d. $(e^y+2)\sec x=c$

Solution:

$$M=(e^y+2)\sin x \quad N=-e^y\cos x$$
$$\frac{\partial M}{\partial y}=e^y\sin x \quad \frac{\partial N}{\partial x}=e^y\sin x$$
$$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$
$$\int(e^y+2)\sin x dx+0=c$$
$$(e^y+2)\cos x=-c=c$$
$$(e^y+2)\cos x=c$$

Correct answer is **option (b)**.

8. Solve $x(y-x)\frac{dy}{dx}=y(y+x)$

Solution:

$$\text{Let, } y=vx; \quad \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$
\begin{aligned}
\mathrm{v}+\mathrm{x}\frac{dv}{dx} &= \frac{v^2x^2+vx^2}{vx^2-x^2}=\frac{v^2+v}{v-1}\\
\mathrm{x}\frac{dv}{dx} &= \frac{2v}{v-1}\\
\frac{v-1}{2v}dv &= \frac{1}{x}dx\\
\int\frac{1}{2}dv-\int\frac{1}{2v}dv &= \int\frac{1}{x}dx\\
\int\frac{1}{2}dv-\frac{1}{2}\int\frac{1}{v}dv &= \int\frac{1}{x}dx\\
\frac{1}{2}v-\frac{1}{2}\log v &= \log x+C\\
v-\log v &= 2\log x+C\\
v &= \frac{y}{x}\\
\text{Then, }\frac{y}{x}-\log\frac{y}{x}x^2 &= C\\
\frac{y}{x}-\log xy &= C
\end{aligned}
$$

9. Determine the order and degree of

$$\frac{\left[1+(dy/dx)^2\right]^{3/2}}{d^2y/dx^2}=K$$

a. Order =1 and Degree = 2

b. Order =2 and Degree = 2

c. Order =2 and Degree = 1

d. Order =2 and Degree = 3

Solution: The given differential equation when written as a polynomial in derivatives becomes

$$K^2\left(\frac{d^2y}{dx^2}\right)^2=\left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

The highest order differential coefficient in this equation is $\frac{d^2y}{dx^2}$ and its power is 2 .
The order is 2 and degree is 2 .

The correct answer is option **b** .

10. Consider a linear ordinary differential equation: $\frac{dy}{dx}+p(x)y=r(x)$. Functions $p(x)$ and $r(x)$ are defined and have a continuous first derivative. The integrating factor of this equation is non-zero. Multiplying this equation by its integrating factor converts this into a:

a. Homogeneous differential equation

b. Non-linear differential equation

c. Second-order differential equation

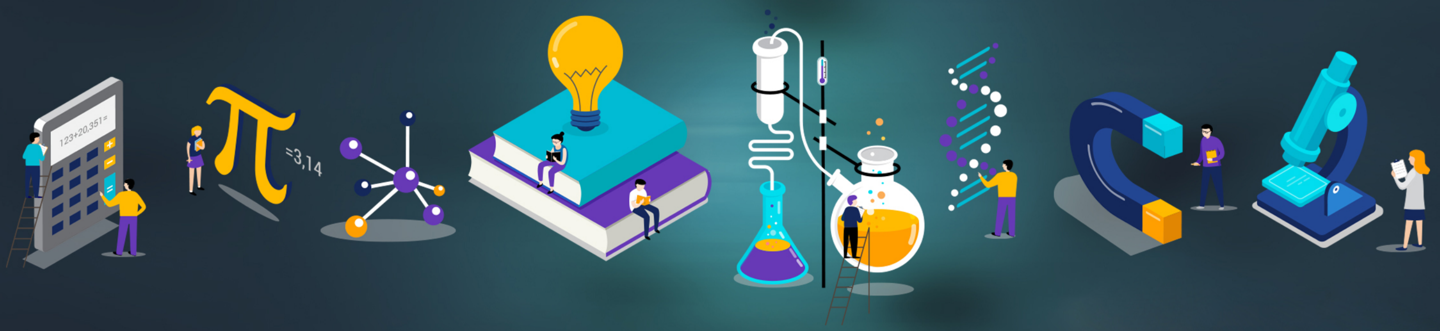
d. Exact differential equation

Solution: Linear differential equation

$$y'+p(x)y=r(x)$$

Multiplying above equation by integrating factor $e^{\int p(x)dx}$ makes the equation exact.

The correct answer is option **d** .



2. Dirac Delta Function

In mathematical models of physical systems we often come across functions that have finite or infinite discontinuities (Potential barriers, Impulse functions). Even though they dont belongs to the generalm definition of functions we can represent them as generalised function or distributions. The most common among them are the step function and the dirac delta function.

2.1 The Step Function

Let's start with the definition of the unit step function, $\theta(x)$:

$$\theta(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}$$

We do not define $\theta(x)$ at $x = 0$. Rather, at $x = 0$ we think of it as in transition between 0 and 1 .The function is called the unit step function because it takes a unit step at $x = 0$. It is sometimes called the **Heaviside function**. The graph of $\theta(x)$ is simple. It is obvious that $\theta(x)$ has a finite jump at $x = 0$. It is sometimes convenient to define $\theta(0)$ to be the average value $\frac{1}{2}$, but this is not always necessary.

$$\text{The sum } \; \theta(x) + \theta(-x) = 1$$
$$\text{The difference } \theta(x) - \theta(-x) = \varepsilon(x)$$

Where, $\varepsilon(x)$ is the signum function.

$$\varepsilon(x) = \begin{cases} +1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \end{cases} \tag{2.1}$$

The function $\varepsilon(x)$ looks like the limit of a tanh (or hyperbolic tangent) function as the 'kink' in the function becomes more and more steep, i.e., as the slope at the origin tends to infinity, as shown in Figure.2.1 In fact, we could define $\varepsilon(x)$ as the limit of a continous sequence of functions $\tanh\left(\frac{x}{\varepsilon(x)}\right)$.

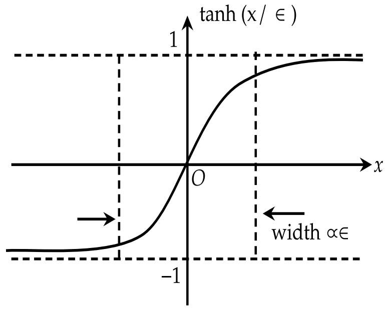


Figure 2.1: The function of $\tanh(x/\varepsilon)$

2.2 Dirac Delta Function

2.2.1 Kronecker delta δ

let us consider a sequence $(a_1, a_2, \ldots) = \{a_j \mid j = 1, 2, \ldots\}$. How do we select a particular member a_i from the sequence? We do so by summing over all members of the sequence with a selector called the **Kronecker delta**, denoted by δ_{ij} and defined as $\delta_{ij} \overset{\text{def.}}{=} \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ It follows immediately that

$$\sum_j \delta_{ij} a_j = a_i$$

$$\sum_j \delta_{ij} = a_i \qquad \text{For each value of } i$$

$$\delta_{ij} = \delta_{ji} \qquad \text{Symmetry property.}$$

Now if we have a continuos function, we must replace the summation over j by an integration over x. The role of the specified index i is played by the specified point a. The analog of the Kronecker delta is written like a function, retaining the same symbol δ for it. So we seek a 'function' $\delta(x-a)$ such that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a) \tag{2.2}$$

Exactly as in the discrete case of the Kronecker delta, we impose the normalization and symmetry properties,

$$\int_{-\infty}^{\infty} \delta(x-a)\, dx = 1 \quad \text{and} \quad \delta(x-a) = \delta(a-x) \tag{2.3}$$

$\delta(x-a)$ is more like the kernel of an integral

$$\int_{-\infty}^{\infty} \delta(x-a)\, dx = 0 \quad \text{if} \quad x \neq a \quad \text{Or the limit of integration excludes the point '}a\text{'} \tag{2.4}$$

2.3 Various representations of Delta function

The delta function may be approximated by the sequences of functions,

$$\delta_n(x) = \begin{cases} 0, & x < -\frac{1}{2n} \\ n, & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0, & x > \frac{1}{2n} \end{cases}$$

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} \exp\left(-n^2 x^2\right)$$

$$\delta_n(x) = \frac{n}{\pi} \cdot \frac{1}{1+n^2x^2}$$

$$\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi} \int_{-n}^{n} e^{ixt} dt$$

These approximations have varying degrees of usefulness.

2.3.1 Rectangular Function

Let us consider a rectangular fuunction defined as,

$$R_\varepsilon = \begin{cases} \frac{1}{2\varepsilon} & a-\varepsilon < x < a+\varepsilon \\ 0 & \text{Otherwise} \end{cases} \tag{2.5}$$

We can consider this as a rectangular window of width 2ε and height $\frac{1}{2\varepsilon}$ so that the area is unity. Let us plot the function as shown in the figure below,

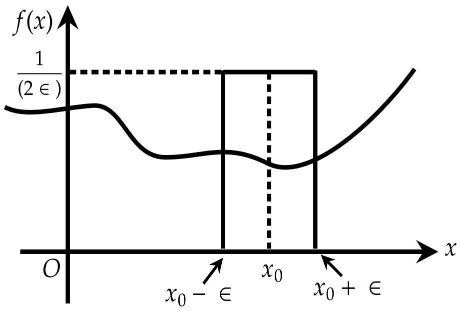


Figure 2.2: Rectangular function

When the window is centred at the choosen point x_0, the integral of $f(x)$ multiplied by this window function is,

$$\frac{1}{2\varepsilon}\int_{a-\varepsilon}^{a+\varepsilon} dx f(x)$$

if we take the limit,$\varepsilon \to 0$ The window becomes vanishingly small and it's height become arbitrarily large such that the area under the curve remains the same.

Then in the limit, $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0} R_\varepsilon = \delta(x-a)$$

An explicit form of the dirac delta function make sense only when it occures in an integral like,

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx \tag{2.6}$$

when it acts on ordinary function like $f(x)$ and an integration over x is carried out . Then the value of 2.6

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = \int_{a-\varepsilon}^{a+\varepsilon} \delta(x-a) f(x) dx \tag{2.7}$$

in the range $a-\varepsilon, a+\varepsilon$ as $\varepsilon \to 0$ the function is almost constant

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a) \lim_{\varepsilon\to 0} \int_{a-\varepsilon}^{a+\varepsilon} R_\varepsilon dx \tag{2.8}$$

$$= f(a) \tag{2.9}$$

2.4 Integral Representations for the Delta Function

Integral transforms, such as the Fourier integral

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(i\omega t) dt \tag{2.10}$$

Lead to the corresponding integral representations of Dirac's delta function. For example, take

$$\delta_n(t-x) = \frac{\sin n(t-x)}{\pi(t-x)} = \frac{1}{2\pi}\int_{-n}^{n} \exp(i\omega(t-x))d\omega \tag{2.11}$$

Using Equation 2.11. We have

$$f(x) = \lim_{n\to\infty}\int_{-\infty}^{\infty} f(t)\delta_n(t-x)dt,$$

where $\delta_n(t-x)$ is the sequence in Eq. (1.192) defining the distribution $\delta(t-x)$. Note that Eq. (1.193a) assumes that $f(t)$ is continuous at $t=x$. If we substitute Eq. (1.192) into Eq. (1.193a) we obtain

$$f(x) = \lim_{n\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\int_{-n}^{n} \exp(i\omega(t-x))d\omega dt.$$

Interchanging the order of integration and then taking the limit as $n\to\infty$, we have the Fourier integral theorem, Eq. (15.20).

With the understanding that it belongs under an integral sign, as in Eq. (1.193a), the identification