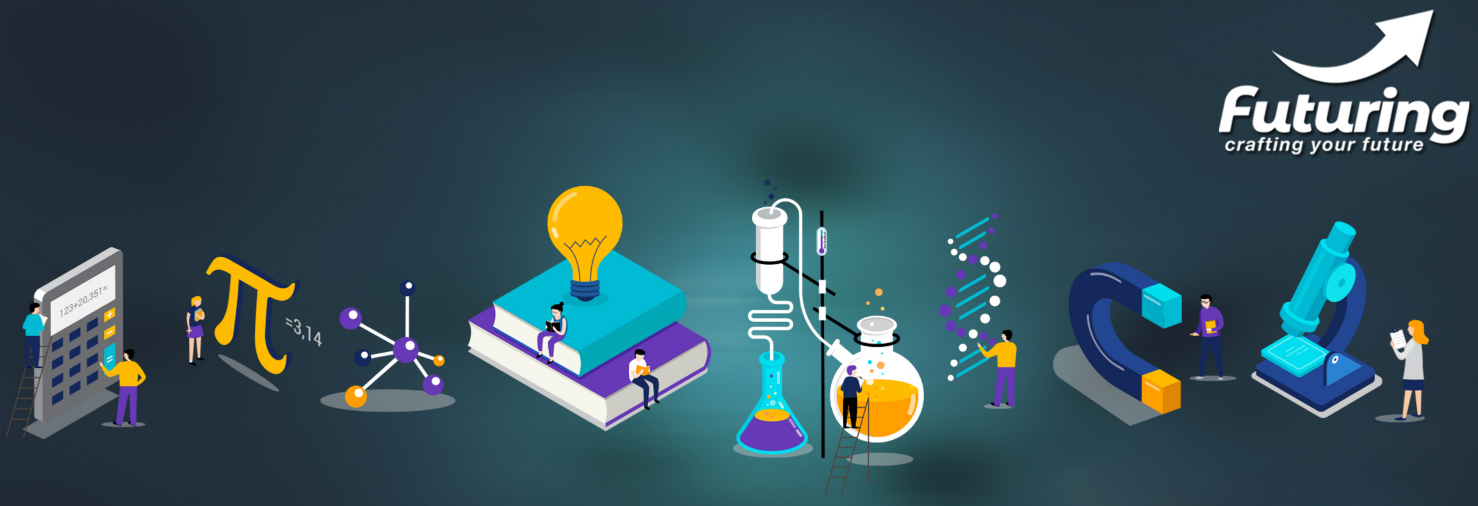


1. Hamiltonian Equation of Motion

Hamiltonian method providing a framework for theoretical basis for further developments like Hamiltonian Jacobi theory, perturbation approaches and chaos. Outside classical mechanics Hamiltonian formulation provides much of the language with which present-day statistical mechanics and quantum mechanics is constructed. Throughout the chapter we are assuming mechanical systems are holonomic and the forces are monogenic.

- In Hamiltonian formulation there can be no constraint equation among the coordinates.
- If n coordinates are not independent, a reduced set of m coordinates, with $m < n$, must be used for the formulation of the problem

1.1 Hamiltonian Formulation

- Describe the motion in terms of first order equations of motion
- Number of initial conditions determining the motion are stil be $2n$
- $2n$ independent first order equations expressed in terms of $2n$ independent variables
- $2n$ equations of motion describe behavior of the system point in a phase space
- Out of $2n$ independent quantities half of them are n generalized coordinates and other half set to be the generalized or conjugate momenta p_i

  $$p_i = \frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_j} \tag{1.1}$$

  where j index shows the set of q's and $\dot{q}$'s
- The quantities (p, q) are known as the canonical variables

1.2 Legendre Transformation

Consider a function of only two variables $f(x, y)$ so that a differential of f has the form

$$df = udx + vdy$$

where

$$u = \frac{\partial f}{dx}, \quad v = \frac{\partial f}{dy}$$

To change the basis of description from x, y to a new distinct set of variables u, y so that differential quantities are expressed in terms of du and dy

Let g be a function u and y defined by the equation

$$g = f - ux$$

A differential of g is then given as

$$dg = df - udx - xdu$$

or

$$dg = vdy - xdu$$

and we get

$$x = \frac{-\partial g}{\partial u}, \quad v = \frac{\partial g}{\partial y}$$

1.3 Hamilton Equation of Motion

Mathematically the transition from Lagrangian to Hamiltonian formulation corresponds to changing the variables in our mechanical functions from $(q, \dot{q}_j, t)$ to (q, p, t) when p is related to q and $\dot{q}$ by

$$p_i = \frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_i}$$

The proceedure for switching variables in this manner is provided by the 'Legendre transformation'.

Consider Lagrangian $L(q, \dot{q}, t)$ then

$$dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_1} + \frac{\partial L}{\partial t} dt \tag{1.2}$$

The canonical momentum was defined as $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$

Substituting it in Lagrange equation we obtain$\dot{p}_i = \dfrac{\partial L}{\partial q_i}$

$$\therefore dL = \dot{p}_i dq_i + p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt \tag{1.3}$$

The Hamiltonian $H(q, p, t)$ is generated by the Legendre transformation

$$H(q, p, t) = \dot{q}_i p_i - L(q, \dot{q}, t) \tag{1.4}$$

and has the differential

$$dH = \dot{q}_1 dp_i - \dot{p}_1 dq_1 - \frac{\partial L}{\partial t} dt \tag{1.5}$$

Where the term $p_i d\dot{q}_i$ removed by Legendre transformation since dH can also written as

$$dH = \frac{\partial H}{\partial q_i} dq_1 + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{dt} dt \tag{1.6}$$

We obtain $2n+1$ relations

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ -\dot{p}_i &= \frac{\partial H}{\partial q_i} \end{aligned}\right\} \tag{1.7}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

Equation 1.7 are known as the canonical equations of Hamiltonian

1.4 Steps to construct Hamiltonian

Hamiltonian for each problem must be constructed via the Lagrangian formulation.

1. With chosen set of generalized coordinates, q_i the Lagrangian $L(\dot{q}, \dot{q}_i, t)$=T-V is constructed
2. The conjugate momenta are defined as function of $q_i, \dot{q}_i$ and t by equation
$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$
3. Legendre transformation used to form the Hamiltonian. At this stage we have some mixed functions of $q_i, \dot{q}_i, p_i$ and t
4. $p_i = \frac{\partial L}{\partial \dot{q}_i}$ then converted to obtain $\dot{q}_i$ as functions of (q, p, t) $(ie \quad \dot{q}_i = \frac{\partial H}{\partial p_i})$
5. The result of the previous steps are then applied to eliminate $\dot{q}_i$ from H so to express it solely as a function of (q, p, t)

Note In many problems Lagrangian is the sum of functions each homogeneous in the generalized velocities of degree $0, 1$ and 2 respectively in that are

$$H = \dot{q}_1 p_i - L = \dot{q}_i p - [L_0(q,t) + L_i(q,t)\dot{q}_k + L_1(q_i,t)\dot{q}_k\dot{q}_m]$$

If equations defining generalized coordinates don't depend on time then $L_2\dot{q}_k\dot{q}_m = T(K.E)$
If forces are derivable from a consevative potential V (ie work is independent of path), then $L_0 = -V$ When both there conditions are satisfied the Hamiltonian is automatically the total energy

$$H = T + V = E$$

Exercise 1.1 A particle of inass m moves inside a bowl under gravity. If the surface of the bowl is given by the equation $z = \frac{1}{2}a\left(x^2 + y^2\right)$, where a is a constant.
(A) Write down Lagrangian of the system in cylindrical co-ordinate.
(a) Identified the cyclic coordinate and law of conservation of momentum.
(b) Write down hamiltonion of the system in cylindrical coordinate system. ■

Solution:

$$\text{(A) } T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + a^2r^2\dot{r}^2\right) \quad \because z = \frac{1}{2}ar^2 \Rightarrow \quad \dot{z} = ar\dot{r}$$

$$V = mgz = \frac{1}{2}mgar^2$$

$$L = \frac{m}{2}\left[\dot{r}^2\left(1 + a^2r^2\right) + r^2\dot{\theta}^2 - agr^2\right]$$

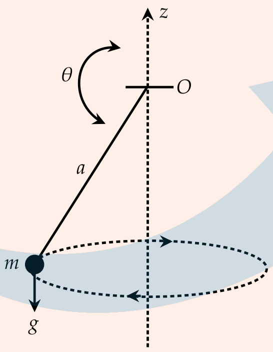
(a) θ is cyclic coordinate

$$\because \frac{\partial L}{\partial \theta}=0, \Rightarrow \dot{p}_\theta=0 \Rightarrow P_\theta= \text{ constant}$$

(b) Hamiltonian

$$H=\frac{p_r^2}{2m\left(1+a^2r^2\right)}+\frac{p_\theta^2}{2mr^2}+\frac{1}{2}magr^2$$
$$\because \frac{\partial L}{\partial \dot{r}}=p_r=m\left(1+a^2r^2\right)\dot{r} \text{ and } \frac{\partial L}{\partial \dot{\theta}}=p_\theta=mr^2\dot{\theta}$$

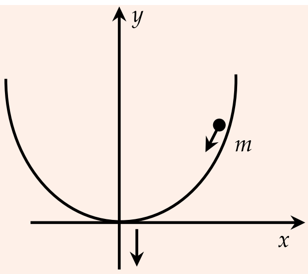
Exercise 1.2 A particle of mass m is attached to fixed point O by a weightless inextensible string of length a. It is rotating under the gravity as shown in the figure.
(a) Write down The Lagrangian of the system in spherical co-ordinate.
(b) write down Hamiltonian of the system.



Solution:

$$L=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)-[mg(-z)]$$
$$L=\frac{1}{2}m\left(a^2\dot{\theta}^2+a^2\sin^2\theta\dot{\phi}^2\right)+mga\cos(\pi-\theta)$$
$$L=\frac{1}{2}m\left(a^2\dot{\theta}^2+a^2\sin^2\theta\dot{\phi}^2\right)-mga\cos(\theta)$$
$$H=\sum\dot{q}_1p_1-L$$
$$H=\frac{p_\theta^2}{2ma^2}+\frac{p_\phi^2}{2ma^2\sin^2\theta}+mag\cos\theta \because \frac{\partial L}{\partial\dot{\theta}}=p_\theta=ma^2\dot{\theta} \text{ and } \frac{\partial L}{\partial\dot{\phi}}=p_\phi=ma^2\sin^2\theta\dot{\phi}$$

Exercise 1.3 Particle of mass m slides under the gravity without friction along the parabotic path $y=ax^2$ axis shown in the figure. Here a is a constant
(a) Write down Lagrangian of the system .
(b) Write down Lagranges equation of motion.
(c) write down Hamiltonian of the system.



Solution:

$$
\begin{aligned}
& y = ax^2 \\
& \text{(a) } \dot{y} = 2ax\dot{x} \\
& L = \frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right) - mgy = \frac{m}{2}\left(\dot{x}^2+4a^2x^2\dot{x}^2\right) - mgax^2 \\
& L = \frac{m}{2}\left(1+4a^2x^2\right)\dot{x}^2 - mgax^2 \\
& \text{(b) } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \\
& \frac{d}{dt}\left[m\left(1+4a^2x^2\right)\dot{x}\right] - \left[4ma^2\dot{x}^2x - 2\ \mathrm{mg}\ \ \mathrm{a}\ \ x\right] = 0 \\
& m\ddot{x} + 4ma^2\ddot{x}x^2 + 8ma^2\dot{x}x\dot{x} - 4ma^2\dot{x}^2x + 2mgax = 0 \\
& m\ddot{x} + 4ma^2x^2\ddot{x} + 4ma^2x\dot{x}^2 + 2mgax = 0 \\
& \text{(c) } H = \sum \dot{x}p_x - L \\
& H = \frac{p_x^2}{2m\left(1+4a^2x^2\right)} + mgax^2 \quad \because \frac{\partial L}{\partial \dot{x}} = p_x = m\left(1+4a^2x^2\right)\dot{x}
\end{aligned}
$$

Exercise 1.4 The Lagrangian of a particle of mass m moving in one dimension is $L = \exp(\alpha t)\left[\frac{m\dot{x}^2}{2} - \frac{kx^2}{2}\right]$, where α and k are positive constants.
(a) Find the Lagranges equation of motion of the particle.
(b) Write down Hamiltonian of the system.

Solution:

$$L = e^{\alpha t}\left(\frac{m\dot{x}^2}{2} - \frac{kx^2}{2}\right)$$

$$\text{(a) } \frac{d}{dt}\left(e^{\alpha t}m\dot{x}\right) - e^{\alpha t}kx = 0 \Rightarrow e^{\alpha t}m\ddot{x} + m\dot{x}e^{\alpha t}\cdot\alpha - e^{\alpha t}kx = 0 \Rightarrow e^{\alpha t}\left[m\ddot{x} + \alpha m\dot{x} - kx\right] = 0$$

$$\text{(b) } H = e^{-\alpha t}\frac{p_x^2}{2m} + e^{\alpha t}\frac{kx^2}{2} \because \frac{\partial L}{\partial \dot{x}} = p_x = e^{\alpha t}m\dot{x}$$

1.5 Cyclic Coordinate and Conservation Theorem

If q_j a cyclic coordinate then, its conjugate momentum p_j is constant.
From Lagrangian and Hamiltonian equations of motions:

$$\dot{p}_j = \frac{\partial L}{\partial q_j} = -\frac{\partial H}{\partial qj}$$

$\therefore$ coordinate that is cyclic will thus also be absent from the Hamiltonian. Conversely if a generalized coordinate does not occur in H the conjugate momentum is conserved.

- If L is not explicit function of t then H is a constant of motion

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t}$$

$$\text{substituting} \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i \quad \text{and} \quad \frac{\partial H}{\partial p_i} = \dot{q}_i \quad \text{we get}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = \frac{-\partial L}{\partial t}$$

Therefore if t doesn't appear explicity in L, it will also not be present in H and H will be constant in time

- If the equations of transformations that define the generalized coordinates

$$r_m = r_m(q_1, \ldots .q_n, t) \tag{1.8}$$

do not depend explicity upon the time, and if potential is velocity independent then H is the total energy $H = T + V$

- The identification of H as a constant of motion and as the total energy are two seperate matters. If equation 1.8 involve time explicity but H doesnot, then H is constant of motion but it is not the total energy.

- Unlike L, using different set of generalized coordinates in the definition of H may leads to an entirely different quantity for Hamiltonian. It may be that for one set of generalized coordinates H is conserved, but that for another it varies in time.

- Explicity first order equations in the 2^{nd} dynamical variables

$$\text{If } H = H(q,p), \text{ ray(autonomous)}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p}$$

$$= \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} = 0$$

Hamiltonian is a constant of motion as long as it is not explicity dependent on time.
Hniltonian governs the time evolution of a system so we call it infinitesimal generator of time translations

1.6 Other Constants of Motion

Suppose F is a constant of motion

$$F = F(q,p,t), \text{it could be explicity time dependent}$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial p}\dot{p} + \frac{\partial F}{\partial t}$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial F}{\partial t}$$

for n degrees of freedom

$$\frac{dF}{dt} = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{dF}{dt}$$

This constructed out of 2 functions of F and H is called the poisson bracket of F with H

$$\frac{dF}{dt} \equiv \underbrace{\{F,H\}}_{poissonbracket} + \frac{\partial F}{\partial t}$$

It plays the same role in classical mechanics as the commutators of matrices or operators would in quantum mechanics
F is a constant of motion if $\frac{dF}{dt}$ vanishes identically
F= constant of motion if and only if

$$\{F,H\} + \frac{\partial F}{\partial t} = 0$$

it's poisson commute with H or to say F and H are said to be in involution with each other. Where the Hamiltonian itself is a constant of motion for autonomous systems.

1.7 Poisson Bracket

You could define poisson bracket of any two functions of phase variables

$$\{A,B\} = \frac{\partial A}{\partial \dot{q}} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial q}$$

- $\{A,B\} = -\{B,A\}$ (antisymmetry)
- $\{A+B,C\} = \{A,C\} + \{B,C\}$
- $\{\alpha A,B\} = \alpha \{A,B\}$
- $\{A,BC\} = B\{A,C\} + \{A,B\}C$ (follow the order)

This is the way you could find poisson brackets of various complicated functions of the phase space variable given a few elementary poisson brackets

$$\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0$$

1.8 Conjugate Momentum

We have independent variable $(q_i \ldots q_N, p_i \ldots p_N)$

$$\{q_k,q_l\} = \sum_{i-1}^{n} \left(\frac{\partial q_k}{\partial q_i} \frac{\partial q_l}{\partial p_i} - \frac{\partial q_k}{\partial p_i} \frac{\partial q_l}{\partial q_i} \right) = 0 \text{Since } p \text{ and } q \text{ are independent}$$

$$\{p_k,p_l\} = 0$$

$$\{q_k,p_l\} = \sum_{i-1}^{n} \left(\frac{\partial q_k}{\partial q_i} \frac{\partial p_l}{\partial p_i} - \frac{\partial q_k}{\partial p_i} \frac{\partial p_l}{\partial q_i} \right)$$

$$\{q_k,p_l\} = \sum_i \delta_{ik} \delta_{il} = \delta_{kl}$$

Canonical Poisson Bracket Relations
$q_k, p_l = \delta_{kl}$
$q_k, q_l = 0$
$p_k, p_l = 0$
once there relations are satisfied
momentum P_K is conjugate to the generalized coordinates $q_k (q_k, p_k)$ form conjugate pair
for the poisson commutes with all other $p's$ except it's conjugate $q_k, p_k = 1$

Practice set-1

1. The Hamiltonian of a system with n degrees of freedom is given by $H\left(q_1, \ldots . q_n ; p_1, \ldots \ldots . p_n ; t\right)$, with an explicit dependence on the time t. Which of the following is correct?

   [NET/JRF (June-2011)]

   a. Different phase trajectories cannot intersect each other.

   b. H always represents the total energy of the system and is a constant of the motion.

   c. The equations $\dot{q}_i=\partial H / \partial p_i, \dot{p}_i=-\partial H / \partial q_i$ are not valid since H has explicit time dependence.

   d. Any initial volume element in phase space remains unchanged in magnitude under time evolution.

2. A If the Lagrangian of a particle moving in one dimensions is given by $L=\frac{\dot{x}^2}{2 x}-V(x)$ the Hamiltonian is

   [NET/JRF (June-2012)]

   a. $\frac{1}{2} x p^2+V(x)$ **b.** $\frac{\dot{x}^2}{2 x}+V(x)$

   c. $\frac{1}{2} \dot{x}^2+V(x)$ **d.** $\frac{p^2}{2 x}+V(x)$

3. B The Hamiltonian of a relativistic particle of rest mass m and momentum p is given by $H=\sqrt{p^2+m^2}+V(x)$, in units in which the speed of light $c=1$. The corresponding Lagrangian is

   [NET/JRF (DEC-2013)]

   a. $L=m \sqrt{1+\dot{x}^2}-V(x)$ **b.** $L=-m \sqrt{1-\dot{x}^2}-V(x)$

   c. $L=\sqrt{1+m \dot{x}^2}-V(x)$ **d.** $L=\frac{1}{2} m \dot{x}^2-V(x)$

4. B A particle of mass m and coordinate q has the Lagrangian $L=\frac{1}{2} m \dot{q}^2-\frac{\lambda}{2} q \dot{q}^2$, where λ is a constant. The Hamiltonian for the system is given by

   [NET/JRF (June-2014)]

   a. $\frac{p^2}{2 m}+\frac{\lambda q p^2}{2 m^2}$ **b.** $\frac{p^2}{2(m-\lambda q)}$

   c. $\frac{p^2}{2 m}+\frac{\lambda q p^2}{2(m-\lambda q)^2}$ **d.** $\frac{p \dot{q}}{2}$

5. C The Hamiltonian of a one-dimensional system is $H=\frac{x p^2}{2 m}+\frac{1}{2} k x$, where m and k are positive constants. The corresponding Euler-Lagrange equation for the system is

   [NET/JRF (June-2018)]

   a. $m \ddot{x}+k=0$ **b.** $m \ddot{x}+2 \dot{x}+k x^2=0$

   c. $2 m x \ddot{x}-m \dot{x}^2+k x^2=0$ **d.** $m x \ddot{x}+2 m \dot{x}^2+k x^2=0$

6. D The Hamiltonian of a particle of unit mass moving in the xy-plane is given to be: $H=x p_x-y p_y-\frac{1}{2} x^2+\frac{1}{2} y^2$ in suitable units. The initial values are given to be $(x(0), y(0))=(1,1)$ and $\left(p_x(0), p_y(0)\right)=\left(\frac{1}{2},-\frac{1}{2}\right)$. During the motion, the curves traced out by the particles in the xy-plane and the $p_x p_y-$ plane are

   [NET/JRF (June-2011)]

   a. Both straight lines

   b. A straight line and a hyperbola respectively

   c. A hyperbola and an ellipse, respectively

   d. Both hyperbolas

7. C If the Lagrangian of a dynamical system in two dimensions is $L=\frac{1}{2}m\dot{x}^2+m\dot{x}\dot{y}$, then its Hamiltonian is **[NET/JRF (June-2015)]**

a. $H=\frac{1}{m}p_xp_y+\frac{1}{2m}p_y^2$ **b.** $H=\frac{1}{m}p_xp_y+\frac{1}{2m}p_x^2$

c. $H=\frac{1}{m}p_xp_y-\frac{1}{2m}p_y^2$ **d.** $H=\frac{1}{m}p_xp_y-\frac{1}{2m}p_x^2$

8. A The Hamiltonian of a system with generalized coordinate and momentum (q,p) is $H=p^2q^2$.A solution of the Hamiltonian equation of motion is (in the following A and B are constants) **[NET/JRF (June-2016)]**

a. $p=Be^{-2At},\quad q=\frac{A}{B}e^{2At}$ **b.** $p=Ae^{-2At},\quad q=\frac{A}{B}e^{-2At}$

c. $p=Ae^{At},\quad q=\frac{A}{B}e^{-At}$ **d.** $p=2Ae^{-A^2t},\quad q=\frac{A}{B}e^{A^2t}$

9. A The Hamiltonian for a system described by the generalised coordinate x and generalised momentum p is

$$H=\alpha x^2p+\frac{p^2}{2(1+2\beta x)}+\frac{1}{2}\omega^2x^2$$

where α,β and ω are constants. The corr **[NET/JRF (JUNE-2017)]**

a. esponding Lagrangian is $\frac{1}{2}\left(\dot{x}-\alpha x^2\right)^2(1+2\beta x)-\frac{1}{2}\omega^2x^2$

b. $\frac{1}{2(1+2\beta x)}\dot{x}^2-\frac{1}{2}\omega^2x^2-\alpha x^2\dot{x}$

c. $\frac{1}{2}\left(\dot{x}^2-\alpha^2x\right)^2(1+2\beta x)-\frac{1}{2}\omega^2x^2$

d. $\frac{1}{2(1+2\beta x)}\dot{x}^2-\frac{1}{2}\omega^2x^2+\alpha x^2\dot{x}$

10. C A point mass m, is constrained to move on the inner surface of a paraboloid of revolution $x^2+y^2=az$ (where $a>0$ is a constant). When it spirals down the surface, under the influence of gravity (along $-z$ direction), the angular speed about the z - axis is proportional to **[NET/JRF (JUNE-2020)]**

a. 1 (independent of z) **b.** z

c. z^{-1} **d.** z^{-2}

11. B The Poisson bracket $\{|\vec{r}|,|\vec{p}|\}$ has the value **[NET/JRF (June-2012)]**

a. $|\vec{r}||\vec{p}|$ **b.** $\hat{r}\cdot\hat{p}$ **c.** 3 **d.** 1

12. D Let A,B and C be functions of phase space variables (coordinates and momenta of a mechanical system). If , represents the Poisson bracket, the value of $\{A,\{B,C\}\}-\{\{A,B\},C$ is given by **[NET/JRF (DEC-2013)]**

a. 0 **b.** $\{B,\{C,A\}\}$ **c.** $\{A,\{C,B\}\}$ **d.** $\{\{C,A\},B\}$

13. D The coordinates and momenta $x_i,p_i(i=1,2,3)$ of a particle satisfy the canonical Poisson bracket relations $\{x_i,p_j\}=\delta_{ij}$. If $C_1=x_2p_3+x_3p_2$ and $C_2=x_1p_2-x_2p_1$ are constants of motion, and if $C_3=\{C_1,C_2\}=x_1p_3+x_3p_1$, then

a. $\{C_2,C_3\}=C_1$ and $\{C_3,C_1\}=C_2$

b. $\{C_2,C_3\}=-C_1$ and $\{C_3,C_1\}=-C_2$

c. $\{C_2,C_3\}=-C_1$ and $\{C_3,C_1\}=C_2$

d. $\{C_2,C_3\}=C_1$ and $\{C_3,C_1\}=-C_2$

14. A The Hamiltonian of a simple pendulum consisting of a mass m attached to a massless string of length l is $H = \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta)$. If L denotes the Lagrangian, the value of $\frac{dL}{dt}$ is:
[NET/JRF (DEC-2012)]

a. $-\frac{2g}{l} p_\theta \sin\theta$ **b.** $-\frac{g}{l} p_\theta \sin 2\theta$
c. $\frac{g}{l} p_\theta \cos\theta$ **d.** $l p_\theta^2 \cos\theta$

15. A A particle moves in one dimension in the potential $V = \frac{1}{2}k(t)x^2$, where $k(t)$ is a time dependent parameter. Then $\frac{d}{dt}\langle V\rangle$, the rate of change of the expectation value $\langle V\rangle$ of the potential energy is
[NET/JRF (June-2015)]

a. $\frac{1}{2}\frac{dk}{dt}\langle x^2\rangle + \frac{k}{2m}\langle xp+px\rangle$ **b.** $\frac{1}{2}\frac{dk}{dt}\langle x^2\rangle + \frac{1}{2m}\langle p^2\rangle$
c. $\frac{k}{2m}\langle xp+px\rangle$ **d.** $\frac{1}{2}\frac{dk}{dt}\langle x^2\rangle$

16. A A particle in two dimensions is in a potential $V(x,y) = x+2y$. Which of the following (apart from the total energy of the particle) is also a constant of motion?
[NET/JRF (DEC-2016)]

a. $p_y - 2p_x$ **b.** $p_x - 2p_y$ **c.** $p_x + 2p_y$ **d.** $p_y + 2p_x$

17. A The Hamiltonian of a classical one-dimensional harmonic oscillator is $H = \frac{1}{2}(p^2+x^2)$, in suitable units. The total time derivative of the dynamical variable $(p+\sqrt{2}x)$ is
[NET/JRF (JUNE-2018)]

a. $\sqrt{2}p - x$ **b.** $p - \sqrt{2}x$ **c.** $p + \sqrt{2}x$ **d.** $x + \sqrt{2}p$

18. D A system is governed by the Hamiltonian

$$H = \frac{1}{2}(p_x - ay)^2 + \frac{1}{2}(p_x - bx)^2$$

where a and b are constants and p_x, p_y are momenta conjugate to x and y respectively. For what values of a and b will the quantities $(p_x - 3y)$ and $(p_y + 2x)$ be conserved?
[NET/JRF (June-2013)]

a. $a=-3, b=2$ **b.** $a=3, b=-2$
c. $a=2, b=-3$ **d.** $a=-2, b=3$

19. A The Lagrangian of a system moving in three dimensions is

$$L = \frac{1}{2}m\dot{x}_1^2 + m\left(\dot{x}_2^2+\dot{x}_3^2\right) - \frac{1}{2}kx_1^2 - \frac{1}{2}k(x_2+x_3)^2$$

The independent constants of motion is/are
[NET/JRF (June-2016)]

a. Energy alone
b. Only energy, one component of the linear momentum and one component of the angular momentum
c. Only energy, one component of the linear momentum
d. Only energy, one component of the angular momentum

20. A The Hamiltonian of a system with two degrees of freedom is $H = q_1p_1 - q_2p_2 + aq_1^2$, where $a>0$ is a constant. The function $q_1q_2 + \lambda p_1p_2$ is a constant of motion only if λ is
[NET/JRF (DEC-2019)]

a. 0 **b.** 1
c. $-a$ **d.** a

Answer key			
Q.No.	Answer	Q.No.	Answer
1	**A**	2	**A**
3	**B**	4	**B**
5	**C**	6	**D**
7	**C**	8	**A**
9	**A**	10	**C**
11	**B**	12	**D**
13	**D**	14	**A**
15	**A**	16	**A**
17	**A**	18	**D**
19	**A**	20	**A**

Practice set-2

1. Consider the Lagrangian $L = a\left(\frac{dx}{dt}\right)^2 + b\left(\frac{dy}{dt}\right)^2 + cxy$, where a, b and c are constants. If p_x and p_y are the momenta conjugate to the coordinates x and y respectively, then the Hamiltonian is

[GATE- 2020]

   a. $\frac{p_x^2}{4a} + \frac{p_y^2}{4b} - cxy$ **b.** $\frac{p_x^2}{2a} + \frac{p_y^2}{2b} - cxy$

   c. $\frac{p_x^2}{2a} + \frac{p_y^2}{2b} + cxy$ **d.** $\frac{p_x^2}{a} + \frac{p_y^2}{b} + cxy$

2. If Hamiltonion is given by $H = \frac{P_\theta^2}{2ml^2} + mgl(1-\cos\theta)$ Hamilton's equations are then given by

[GATE- 2010]

   a. $\dot{p}_\theta = -mgl\sin\theta; \quad \dot{\theta} = \frac{p_\theta}{ml^2}$ **b.** $\dot{p}_\theta = mgl\sin\theta; \quad \dot{\theta} = \frac{p_\theta}{ml^2}$

   c. $\dot{p}_\theta = -m\ddot{\theta}; \quad \dot{\theta} = \frac{p_\theta}{m}$ **d.** $\dot{p}_\theta = -\left(\frac{g}{l}\right)\theta; \quad \dot{\theta} = \frac{p_\theta}{ml}$

3. A particle of mass m is attached to a fixed point O by a weightless inextensible string of length a. It is rotating under the gravity as shown in the figure. The Lagrangian of the particle is $L(\theta,\phi) = \frac{1}{2}ma^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - mga\cos\theta$ where θ and ϕ are the polar angles. The Hamiltonian of the particles is

[GATE- 2012]

   a. $H = \frac{1}{2ma^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - mga\cos\theta$

   b. $H = \frac{1}{2ma^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) + mga\cos\theta$

   c. $H = \frac{1}{2ma^2}\left(p_\theta^2 + p_\phi^2\right) - mga\cos\theta$

   d. $H = \frac{1}{2ma^2}\left(p_\theta^2 + p_\phi^2\right) + mga\cos\theta$

4. The Hamiltonian for a particle of mass m is $H = \frac{p^2}{2m} + kqt$ where q and p are the generalized coordinate and momentum, respectively, t is time and k is a constant. For the initial condition, $q = 0$ and $p = 0$ at $t = 0, q(t) \propto t^\alpha$. The value of α is ———

[GATE - 2019]

5. Consider the Hamiltonian $H(q,p) = \frac{ap^2q^4}{2} + \frac{\beta}{q^2}$, where α and β are parameters with appropriate dimensions, and q and p are the generalized coordinate and momentum, respectively. The corresponding Lagrangian $L(q,\dot{q})$ is

[GATE - 2019]

   a. $\frac{1}{2\alpha}\frac{\dot{q}^2}{q^4} - \frac{\beta}{q^2}$ **b.** $\frac{1}{2\alpha}\frac{\dot{q}^2}{q^4} + \frac{\beta}{q^2}$

   c. $\frac{1}{\alpha}\frac{\dot{q}^2}{q^4} + \frac{\beta}{q^2}$ **d.** $-\frac{1}{2\alpha}\frac{\dot{q}^2}{q^4} + \frac{\beta}{q^2}$

6. The Lagrangian for a particle of mass m at a position $\vec{r}$ moving with a velocity $\vec{v}$ is given by $L = \frac{m}{2}\vec{v}^2 + C\vec{r}\cdot\vec{v} - V(r)$, where $V(r)$ is a potential and C is a constant. If $\vec{p}_c$ is the canonical momentum, then its Hamiltonian is given by

[GATE- 2015]

   a. $\frac{1}{2m}(\vec{p}_c + C\vec{r})^2 + V(r)$ **b.** $\frac{1}{2m}(\vec{p}_c - C\vec{r})^2 + V(r)$

   c. $\frac{p_c^2}{2m} + V(r)$ **d.** $\frac{1}{2m}p_c^2 + C^2r^2 + V(r)$

7. The Hamiltonian for a system of two particles of masses m_1 and m_2 at $\vec{r}_1$ and $\vec{r}_2$ having velocities $\vec{v}_1$ and $\vec{v}_2$ is given by $H = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{C}{(\vec{r}_1 - \vec{r}_2)^2}\hat{z}\cdot(\vec{r}_1 \times \vec{r}_2)$, where C is constant. Which one of the following statements is correct?
[GATE- 2015]

   a. The total energy and total momentum are conserved

   b. Only the total energy is conserved

   c. The total energy and the z - component of the total angular momentum are conserved

   d. The total energy and total angular momentum are conserved

8. The Hamilton's canonical equation of motion in terms of Poisson Brackets are
[GATE- 2014]

   a. $\dot{q} = \{q,H\}; \dot{p} = \{p,H\}$ **b.** $\dot{q} = \{H,q\}; \dot{p} = \{H,p\}$

   c. $\dot{q} = \{H,p\}; \dot{p} = \{H,p\}$ **d.** $\dot{q} = \{p,H\}; \dot{p} = \{q,H\}$

9. The Poisson bracket $[x, xp_y + yp_x]$ is equal to
[GATE- 2017]

   a. $-x$ **b.** y **c.** $2p_x$ **d.** p_y

10. If H is the Hamiltonian for a free particle with mass m, the commutator $[x,[x,H]]$ is
[GATE - 2018]

   a. $\hbar^2/m$ **b.** $-\hbar^2/m$ **c.** $-\hbar^2/(2m)$ **d.** $\hbar^2/(2m)$

11. The Poisson bracket between θ and $\dot{\theta}$ is
[GATE- 2010]

   a. $\{\theta,\dot{\theta}\} = 1$ **b.** $\{\theta,\dot{\theta}\} = \frac{1}{ml^2}$

   c. $\{\theta,\dot{\theta}\} = \frac{1}{m}$ **d.** $\{\theta,\dot{\theta}\} = \frac{g}{l}$

12. A dynamical system with two generalized coordinates q_1 and q_2 has Lagrangian $L = \dot{q}_1^2 + \dot{q}_2^2$. If p_1 and p_2 are the corresponding generalized momenta, the Hamiltonian is given by
[JEST-2014]

   a. $\left(p_1^2 + p_2^2\right)/4$ **b.** $\left(\dot{q}_1^2 + \dot{q}_2^2\right)/4$

   c. $\left(p_1^2 + p_2^2\right)/2$ **d.** $\left(p_1\dot{q}_1 + p_2\dot{q}_2\right)/4$

13. The Hamiltonian for a particle of mass m is given by $H = \frac{(p-\alpha q)^2}{(2m)}$, where α is a nonzero constant. Which one of the following equations is correct?
[JEST-2020]

   a. $p = m\dot{q}$ **b.** $\alpha\dot{p} = \dot{q}$

   c. $\ddot{q} = 0$ **d.** $L = \frac{1}{2}m\dot{q}^2 - \alpha q\dot{q}$

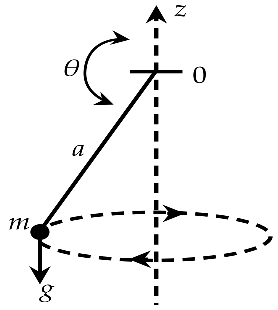
14. If the Poisson bracket $\{x,p\} = -1$, then the Poisson bracket $\left\{x^2 + p, p\right\}$ is ?
[JEST-2013]

   a. $-2x$ **b.** $2x$ **c.** 1 **d.** -1

Answer key			
Q.No.	Answer	Q.No.	Answer
1	**A**	2	**A**
3	**B**	4	**3**
5	**A**	6	**B**
7	**C**	8	**A**
9	**B**	10	**B**
11	**B**	12	**A**
13	**C**	14	**A**

Practice set-3

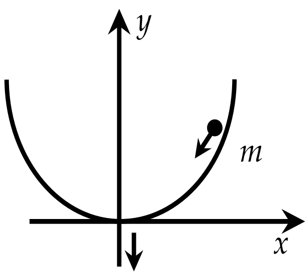
1. A particle of mass m is attached to fixed point O by a weightless inextensible string of length a. It is rotating under the gravity as shown in the figure.
(a) Write down The Lagrangian of the system in spherical co-ordinate.
(b) write down Hamiltonian of the system.



Solution:

$$L=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)-[mg(-z)]$$
$$L=\frac{1}{2}m\left(a^2\dot{\theta}^2+a^2\sin^2\theta\dot{\phi}^2\right)+mga\cos(\pi-\theta)$$
$$L=\frac{1}{2}m\left(a^2\dot{\theta}^2+a^2\sin^2\theta\dot{\phi}^2\right)-mga\cos(\theta)$$
$$H=\sum\dot{q}_1p_1-L$$
$$H=\frac{p_\theta^2}{2ma^2}+\frac{p_\phi^2}{2ma^2\sin^2\theta}+mag\cos\theta\because\frac{\partial L}{\partial\dot{\theta}}=p_\theta=ma^2\dot{\theta}\text{ and }\frac{\partial L}{\partial\dot{\phi}}=p_\phi=ma^2\sin^2\theta\dot{\phi}$$

2. Particle of mass m slides under the gravity without friction along the parabolic path $y=ax^2$ axis shown in the figure. Here a is a constant. (a) write down Lagrangian of the system . (b) write down Hamiltonian of the system .



Solution:

$$y=ax^2$$
$$\text{(a) }\dot{y}=2ax\dot{x}$$
$$L=\frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right)-mgy=\frac{m}{2}\left(\dot{x}^2+4a^2x^2\dot{x}^2\right)-mgax^2$$
$$L=\frac{m}{2}\left(1+4a^2x^2\right)\dot{x}^2-mgax^2$$
$$\text{(b) }H=\sum\dot{x}p_x-L$$

$$H = \frac{p_x^2}{2m\left(1+4a^2x^2\right)} + mgax^2 \quad \because \frac{\partial L}{\partial \dot{x}} = p_x = m\left(1+4a^2x^2\right)\dot{x}$$

3. The Lagrangian of a particle of mass m moving in one dimension is $L = \exp(\alpha t)\left[\frac{m\dot{x}^2}{2} - \frac{kx^2}{2}\right]$, where α and k are positive constants.
(a) Write down Hamiltonian of the system.

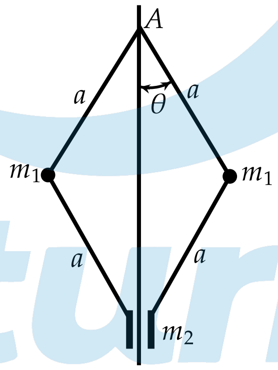
Solution:

$$L = e^{\alpha t}\left(\frac{m\dot{x}^2}{2} - \frac{kx^2}{2}\right)$$

$$(b) \quad H = e^{-\alpha t}\frac{p_x^2}{2m} + e^{\alpha t}\frac{kx^2}{2}$$

$$\because \frac{\partial L}{\partial \dot{x}} = p_x = e^{\alpha t}m\dot{x}$$

4. As shown in figure the particle of mass m_2 moves on a vertical axis and the whole system rotates about this axis with a constant angular velocity ω.
(b) Write down Lagrangian of the system in spherical polar co-ordinate.
(d) Write down Hamiltonian of the system.



Solution:

$$\text{(b) } L = m_1a^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) + 2m_2a^2\dot{\theta}^2\sin^2\theta + 2\left(m_1+m_2\right)ga\cos\theta$$

$$\text{(d) } H = \sum \dot{\theta}p_\theta - L$$

$$\frac{\partial L}{\partial \dot{\theta}} = p_\theta = 2m_1a^2\dot{\theta} + 4m_2a^2\dot{\theta}\sin^2\theta \Rightarrow \dot{\theta} = \frac{p_\theta}{2m_1a^2 + 4m_2a^2\sin^2\theta}$$

$$H = \frac{p_\theta^2}{4ma^2 + \theta m_2a^2\sin^2\theta} - m_1a^2\omega^2\sin^2\theta - 2\left(m_1+m_2\right)ga\cos\theta$$

5. A system is governed by the Hamiltonian

$$H = \frac{1}{2}\left(p_x - ay\right)^2 + \frac{1}{2}\left(p_y - bx\right)^2$$

where a and b are constants and p_x, p_y are momenta conjugate to x and y respectively. For what values of a and b will the quantities $(p_x - 3y)$ and $(p_y + 2x)$ be conserved.

Solution:

$$\text{Poisson bracket } \left[p_x - 3y, H\right] = 0 \text{ and } \left[p_y + 2y, H\right] = 0$$

$$p_y(b-3) + x\left(3b - b^2\right) = 0 \text{ and } p_x(a+2) - y\left(2a + a^2\right) = 0$$

$$\Rightarrow a=-2, b=3$$

6. A particle of mass m and coordinate q has the Lagrangian $L=\frac{1}{2}m\dot{q}^2-\frac{\lambda}{2}q\dot{q}^2$ where λ is a constant. Find the Hamiltonian of the system.

Solution:

$$H=\sum \dot{q}p-L \text{ where } L=\frac{1}{2}m\dot{q}^2-\frac{1}{2}\lambda q\dot{q}^2$$
$$\frac{\partial L}{\partial \dot{q}}=p=m\dot{q}-\lambda q\dot{q} \Rightarrow p=\dot{q}(m-\lambda q) \Rightarrow \dot{q}=\frac{p}{m-\lambda q}$$
$$\Rightarrow H=\dot{q}p-L=\frac{p^2}{(m-\lambda q)}-\frac{1}{2}m\frac{(p^2)}{(m-\lambda q)^2}+\frac{\lambda}{2}q\cdot\frac{p^2}{(m-\lambda q)^2}$$
$$\Rightarrow H=\frac{p^2}{(m-\lambda q)}-\frac{p^2}{2(m-\lambda q)^2}(m-\lambda q)=\frac{p^2}{(m-\lambda q)}-\frac{p^2}{2(m-\lambda q)}$$
$$\Rightarrow H=\frac{p^2}{2(m-\lambda q)}$$

7. The coordinates and momenta $x_i, \quad p_i (i=1,2,3)$ of a particle satisfy the canonical Poisson bracket relations $\{x_i, p_j\}=\delta_{ij}$. If $C_1=x_2p_3+x_3p_2, C_2=x_1p_2-x_2p_1$ and $C_3=x_1p_3+x_3p_1$. Then
(a) Find the value of $\{C_1, C_2\}$
(b) Find the relation between C_1, C_2 and C_3.

Solution:

$$\text{(a) } \{C_1, C_2\}=\{x_2p_3+x_3p_2, x_1p_2-x_2p_1\}$$
$$\{C_1, C_2\}=\{x_2p_3, x_1p_2\}-\{x_2p_3, x_2p_1\}+\{x_3p_2, x_1p_2\}-\{x_3p_2, x_2p_1\}$$
$$=x_1\{x_2, p_2\}p_3+0+0-x_3\{p_2, x_2\}p_1$$
$$\Rightarrow \{C_1, C_2\}=x_1p_3+x_3p_1$$
$$\text{(b) } \{C_1, C_2\}=C_3$$

8. If Hamiltonian of the Harmonic oscillator is given by $H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2$, the quantity defined $u(x,p,t)=\ln(p+im\omega x)-i\omega t$ then check whether u is conserve during the motion.

Solution:

$$\text{If } u \text{ is conserve then } \frac{du}{dt}=[u,H]+\frac{\partial u}{\partial t}=0 \Rightarrow \frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{\partial H}{\partial p}-\frac{\partial u}{\partial p}\cdot\frac{\partial H}{\partial x}+\frac{\partial u}{\partial t}$$
$$\Rightarrow \frac{du}{dt}=\frac{im\omega}{p+im\omega x}\cdot\frac{p}{m}-\frac{1}{p+im\omega x}\cdot m\omega^2x-i\omega$$
$$=\frac{i\omega p}{p+im\omega x}-\frac{m\omega^2x}{p+im\omega x}-i\omega=\frac{i\omega(p+im\omega x)}{p+im\omega x}-i\omega=0$$

So u is conserve during the motion

9. The Hamiltonian of a particle of unit mass moving in the xy-plane is given to be: $H=xp_x-yp_y-\frac{1}{2}x^2+\frac{1}{2}y^2$ in suitable units. The initial values are given to be $(x(0), y(0))=(1,1)$ and $(p_x(0), p_y(0))=\left(\frac{1}{2}, \frac{1}{2}\right)$
(a) Discuss equation of motion
(b) Plot the curves traced out by the particles in the xy-plane and the p_xp_y- plane.

Solution:

(a) Solving Hamiltonian equation of motion

$$\frac{\partial H}{\partial x} = -\dot{p}_x \Rightarrow p_x - x = -\dot{p}_x \text{ and } \frac{\partial H}{\partial y} = -\dot{p}_y \Rightarrow -p_y + y = -\dot{p}_y$$
$$\frac{\partial H}{\partial p_x} = \dot{x} \Rightarrow x = \dot{x} \text{ and } \frac{\partial H}{\partial p_y} = \dot{y} \Rightarrow -y = \dot{y}$$

(b) After solving these four differential equation and eliminating time t and using boundary condition one will get $\Rightarrow x \propto \frac{1}{y}$ and $p_x \propto \frac{1}{p_y}$

10. The Hamiltonian of the system is given by $H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2(x^2+y^2)}{2}$. Check whether the following quantities are conserve during the motion or not.
(a) $S_1 = \frac{1}{2}(xp_y - yp_x)$
(b) $S_2 = \frac{1}{2m\omega}\left(p_x p_y + m^2\omega^2 xy\right)$
(c) $S_3 = \frac{1}{4m\omega}\left[p_x^2 - p_y^2 + m^2\omega^2\left(y^2 - x^2\right)\right]$

Solution:

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2\left(x^2+y^2\right) \quad S_1 = \frac{1}{2}(xp_y - yp_x)$$

(a) $[S_1, H] = 0$

$$\Rightarrow \frac{\partial S_1}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial S_1}{\partial p_x}\frac{\partial H}{\partial x} + \frac{\partial S_1}{\partial y}\frac{\partial H}{\partial p_y} - \frac{\partial S_1}{\partial p_y}\cdot\frac{\partial H}{\partial y}$$
$$\Rightarrow \frac{p_y}{2}\cdot\frac{p_x}{m} - \left(-\frac{y}{2}\right)\cdot m\omega^2 x + \frac{1}{2}(-p_x)\cdot\frac{p_y}{m} - \frac{1}{2}(x)\cdot m\omega^2 y = 0$$

(b) $[S_2, H] = 0 S_2 = \dfrac{1}{2m\omega}\left(p_x p_y + m^2\omega^2 xy\right)$

$$\Rightarrow \frac{\partial S_2}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial S_2}{\partial p_x}\frac{\partial H}{\partial x} + \frac{\partial S_2}{\partial y}\frac{\partial H}{\partial p_y} - \frac{\partial S_2}{\partial p_y}\frac{\partial H}{\partial y}$$
$$\Rightarrow \frac{1}{2m\omega}\left(m^2\omega^2 y\right)\left(\frac{p_x}{m}\right) - \frac{p_y}{2m\omega}\cdot\left(m\omega^2 x\right) + \frac{1}{2m\omega}m^2\omega^2 x\cdot\frac{p_y}{m} - \frac{p_x}{2m\omega}\cdot m\omega^2 y = 0$$

(c) $[S_3, H] = 0$

$$\frac{\partial S_3}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial S_3}{\partial p_x}\frac{\partial H}{\partial x} + \frac{\partial S_3}{\partial y}\frac{\partial H}{\partial p_y} - \frac{\partial S_3}{\partial p_y}\cdot\frac{\partial H}{\partial y}$$
$$\Rightarrow \frac{1}{4m\omega}\times m^2\omega^2(-2x)\cdot\frac{p_x}{m} - \frac{1}{4m\omega}\times 2p_x\cdot m\omega^2 x + \frac{m^2\omega^2}{4m\omega}\times 2y\cdot\frac{p_y}{m} - \frac{1}{4m\omega}\left(-2p_y\cdot m\omega^2 y\right)$$
$$\Rightarrow \frac{-p_x x\omega}{2} - \frac{p_x\omega x}{2} + \frac{p_y\cdot y\omega}{2} + \frac{p_y y\omega}{2} \neq 0$$