1. Complex Analysis

A simple algebraic equation like $x^2 = -1$ may not have a real solution. Introducing complex numbers validates the existence of 'root' for every polynomial with a positive degree . Which then proves the fundamental theorem of algebra. The idea of complex numbers are widely used in Physics and Mathematics.

Definition 1.0.1 A number of the form $x+iy$, where x and y are real numbers and $i = \sqrt{(-1)}$, is called a complex number.

Real Part : x is called the real part of the complex number, $x+iy$ and is written as, $R(x+iy)$.

Imaginary Part : y is called the imaginary part of the complex number and is written as, $I(x+iy)$.

1.1 Representation of a Complex number

The point whose cartesian coordinates are (x,y) uniquely represents the complex number, $z = x+iy$ on the complex plane z. The diagram in which this representation is carried out is called the Argand's diagram. It's shown in the figure 1.1. Since x is the real part of z we call the x -axis the real axis. Likewise, the y -axis is the imaginary axis.

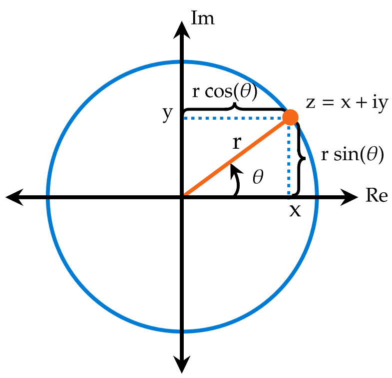


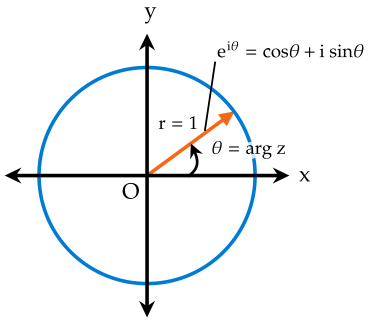
Figure 1.1: Argand Diagram

In terms of the polar coordinates , we have

$$x = r\cos\theta, \quad y = r\sin\theta \tag{1.1}$$

$$z = x + \text{ iy } = re^{i\theta}$$
$$= r(\cos\theta + i\sin\theta) \tag{1.2}$$

Then, the equation 1.2 is known as, Euler's formula



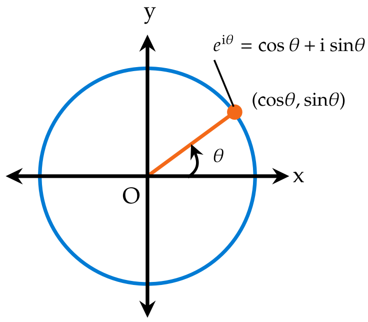


Figure 1.2: Polar representation

1.1.1 Absolute Value

We define the absolute value of a complex number $x+iy$ to be the length r of the vector from the origin to $P(x,y)$.

$$r = |x+iy| = \sqrt{x^2+y^2}$$

Properties:

- $|z_1+z_2| \leq |z_1|+|z_2|$
- $|z_1-z_2| \geq |z_1|-|z_2|$
- $|z_1 z_2| = |z_1|\,|z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

1.1.2 Argument of

The polar angle θ is called the argument of z and it is written as,

$$\theta = \arg z$$

Any integer multiple of 2π may be added to θ to produce another appropriate angle.
From the figure 1.1,

$$\theta = \arg z = \tan^{-1}\left(\frac{y}{x}\right)$$

Properties:

- $\text{Arg}\,(z_1 z_2 \cdot z_3 \ldots\ldots z_n) = \text{Arg}\,(z_1) + \text{Arg}\,(z_2) + \text{Arg}\,(z_3) + \ldots\ldots.. + \text{Arg}\,(z_n)$
- $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}\,(z_1) - \text{Arg}\,(z_2)$

Exercise 1.1 Find the modulus and principal argument of the complex number $\frac{1+2i}{1-(1-i)^2}$ ■

Solution:

$$\frac{1+2i}{1-(1-i)^2} = \frac{1+2i}{1-(1-1-2i)} = \frac{1+2i}{1+2i}$$
$$= 1 = 1+0i$$

$$\therefore \quad \left|\frac{1+2i}{1-(1-i)^2}\right| = |1+0i| = \sqrt{1^2} = 1$$

$$\text{Principal argument of } \frac{1+2i}{1-(1-i)^2} = \text{Principal argument of} \quad (1+0i)$$

$$\tan^{-1}\frac{0}{1} = \tan^{-1} 0$$

$$= 0^\circ$$

1.1.3 Conjugate of a Complex number

The conjugate of a complex number z is represented by,

$$\bar{z} = x - iy$$

Note

$$\frac{z+\bar{z}}{2} = Re\{z\}$$

$$\frac{z-\bar{z}}{2i} = Im\{z\}$$

$$z \cdot \bar{z} = |z|^2$$

1.2 Algebra of Complex numbers

For two Complex numbers, $a+ib$ and $c+id$

Equality:

$$a+ib = c+id$$

Two complex numbers (a,b) and (c,d) are equal if and only $a=c$ and $b=d$.

Addition:

$$(a+ib)+(c+id) \quad = (a+c)+i(b+d)$$

Multiplication:

$$(a+ib)(c+id) = (ac-bd)+i(ad+bc)$$

$$c(a+ib) = ac+i(bc)$$

Polar form:

$$\text{Let,} \quad z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1.z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$= r_1r_1[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)],$$

Division:

$$\frac{c+id}{a+ib} = \frac{(c+id)(a-ib)}{(a+ib)(a-ib)}$$

$$= \frac{(ac+bd)+i(ad-bc)}{a^2+b^2}$$

$$\text{Where,} \quad x = \frac{ac+bd}{a^2+b^2}, \quad \text{and} \quad y = \frac{ad-bc}{a^2+b^2}$$

1.3 Important Identities

1.3.1 Circular functions of Complex numbers

- $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$
- $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$
- $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$
- $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

1.3.2 Hyperbolic functions of Complex numbers

- $\sinh x = \dfrac{e^x - e^{-x}}{2}$
- $\cosh x = \dfrac{e^x + e^{-x}}{2}$
- $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
- $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
- $\mathrm{sech} x = \dfrac{2}{e^x + e^{-x}}$
- $\mathrm{cosech} x = \dfrac{2}{e^x - e^{-x}}$
- $\cosh x + \sinh x = \dfrac{e^x + e^{-x}}{2} + \dfrac{e^x - e^{-x}}{2} = e^x$

Note **Relation between Circular and Hyperbolic functions:**

- $\sin ix = i \sinh x$
- $\sinh ix = i \sin x$
- $\cos ix = \cosh x$
- $\cosh ix = \cos x$
- $\tan ix = i \tanh x$
- $\tanh ix = i \tan x$

Theorem 1.3.1 De Moivre's Theorem:

1. For any integer n, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
2. If n is a fraction, then $(\cos n\theta + i\sin n\theta)$ is one of the values .

Exercise 1.2 Express $\frac{(\cos\theta+i\sin\theta)^8}{(\sin\theta+i\cos\theta)^4}$ in the form $(x+iy)$ ■

Solution:

$$\begin{aligned}
\frac{(\cos\theta + i\sin\theta)^8}{(\sin\theta + i\cos\theta)^4} &= \frac{(\cos\theta + i\sin\theta)^8}{(i)^4\left(\cos\theta + \frac{1}{i}\sin\theta\right)^4} \\
&= \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta - i\sin\theta)^4} = \frac{(\cos\theta + i\sin\theta)^8}{[\cos(-\theta) + i\sin(-\theta)]^4} \\
&= \frac{(\cos\theta + i\sin\theta)^8}{[(\cos\theta + i\sin\theta)^{-1}]^4} = \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta + i\sin\theta)^{-4}} = (\cos\theta + i\sin\theta)^{12} \\
&= \cos 12\theta + i\sin 12\theta
\end{aligned}$$

Note Series expansion of different functions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$
\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\
\tan x &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \\
\tan^{-1}(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots
\end{aligned}
$$

1.4 Function of a Complex Variable

1.4.1 Basic Representation

$$
\begin{aligned}
W &= f(z) = v(x,y) + iv(x,y) & \text{Real part (u)}, \left(x^2 - y^2\right)^2 \\
f(z) &= z^2 = (x+iy)^2 = \left(x^2 - y^2\right)^2 + i2xy & \text{Imaginary part (v)}, 2xy
\end{aligned}
$$

1.4.2 Existance of $\lim_{z \to z_0} f(z)$:

The limit will exists only if the limiting value is independent of the path along which z approaches z_0

Exercise 1.3 Find whether the limit $\lim_{z \to 0} \frac{z}{|z|}$ exist or not. ■

Solution:

$$
\begin{aligned}
&z \to 0 \text{ means } x \to 0 \,\&\, y \to 0 \\
&\qquad \text{For } \mathrm{z} = 0, \text{ we have to choose a path passing through a origin.}
\end{aligned}
$$

Therefore, we have choosen a straight line passing through the origin i.e. $y = mx$

$$
\lim_{z \to 0} \frac{z}{|z|} = \lim_{\substack{z \to 0 \\ y \to 0}} \frac{x+iy}{\sqrt{x^2+y^2}} = \lim_{x \to 0} \frac{x+imx}{\sqrt{x^2+m^2x^2}} = \frac{1+im}{\sqrt{1+m^2}}
$$

Therefore, the limit depends on m i.e. slope of the straight line. Thus, the limiting values is dependent on the path and the limit does not exists.

Exercise 1.4 Calculate the vahue $\lim_{z \to \infty} \frac{iz^3+iz-1}{(2z+3i)(z-i)^2}$ ■

Solution:

$$
\lim_{z \to \infty} \frac{iz^3+iz-1}{(2z+3i)(z-i)^2} = \lim_{z \to \infty} \frac{z^3\left(i + \frac{i}{z^2} - \frac{1}{z^3}\right)}{z\left(2+\frac{3i}{z}\right) z^2 \left(1 - \frac{i}{z}\right)^2} = \frac{i}{2}
$$

1.4.3 Differentiability of Complex Function

$$
\mathrm{f}'(\mathrm{z}) = \underset{\delta \mathrm{z} \to 0}{\mathrm{Lt}} \frac{[\mathrm{f}(\mathrm{z}+\delta \mathrm{z}) - \mathrm{f}(\mathrm{z})]}{\delta \mathrm{z}}
$$

The function will be differentiable if limit should exists and it is independent of path along with $\delta z \to 0$.

Ex: $\mathrm{f}(z) = (4x+y)+i(4y-x) \Rightarrow u = (4x+y)$ and $v = (4y-x)$

$$\Rightarrow f(z+\delta z) = 4(x+\delta x)+(y+\delta y)+i[4(y+\delta y)-(x+\delta x)]$$
$$\Rightarrow f(z+\delta z)-f(z) = 4\delta x+\delta y+i(4\delta y-\delta x)$$
$$\Rightarrow \frac{f(z+\delta z)-f(z)}{\delta z} = \frac{4\delta x+\delta y+i(4\delta y-\delta x)}{\delta z} \Rightarrow \frac{\delta f}{\delta z} = \frac{4\delta x+\delta y-i\delta x+4i\delta y}{\delta x+i\delta y}$$

Along real axis : $\delta x = \delta z, \delta y = 0, \Rightarrow \dfrac{\delta f}{\delta z} = 4-i$

Along imaginary axis : $i\delta y = \delta z, \delta x = 0, \Rightarrow \dfrac{\delta f}{\delta z} = 4-i$

Along a line : $y = x, \delta y = \delta x, \delta z = (1+i)\delta x, \Rightarrow \dfrac{\delta f}{\delta z} = \dfrac{5\delta x+3i\delta x}{(1+i)\delta \dot{x}} = \dfrac{5+3i}{1+i} = 4-i$

1.5 Complex Analysis Function

A function $f(z)$ is said to be analytic at a point $z = z_0$ if it is single valued and has the derivative at every point in some neighbourhood of z_0. The function $f(z)$ is said to be analytic in a domain D if it is single valued and is differentiable at every point of domain D.

1.5.1 Cauchy Reamann Equations

For a function $f(z) = u+iv$ to be analytic at all points in some region ' R ', the necessary conditions are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Sufficient Condition: $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are continuous functions of x and y.

Derivative of f(z) : $f'(\mathrm{z}) = \dfrac{\partial u}{\partial x}+i\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y}-i\dfrac{\partial u}{\partial y}$

Exercise 1.5 Check whether $f(z) = \sin z$ is analytic or not. ■

Solution:

$$f(z) = \sin z = \sin(x+iy) = \sin x\cdot\cos(iy)+\cos x\cdot\sin(iy) = \sin x\cdot\cosh y+i\cos x\cdot\sinh y$$

Therefore, $u = \sin x\cdot\cosh y$ and $v = \cos x.sinhy$

$$\frac{\partial u}{\partial x} = \cos x\cdot\cosh y; \frac{\partial u}{\partial y} = \sin x\cdot\sinh y; \frac{\partial v}{\partial x} = -\sin x\cdot\sinh y; \frac{\partial v}{\partial y} = \cos x\cdot\cosh y$$

So, C-R equation is satisfied, given f(z) is analytic.

Exercise 1.6 If the real part of a complex analytic function is $u(x,y) = x+\frac{1}{2}\left(x^2-y^2\right)$, find the corresponding imaginary part. ■

Solution:

$$\frac{\partial u}{\partial x} = x+1 = \frac{\partial v}{\partial y} \Rightarrow v(x,y) = y+xy+f(x)$$

$$\frac{\partial u}{\partial y} = -y = -\frac{\partial v}{\partial x} \Rightarrow v(x,y) = xy + g(y)$$

Therefore, the imaginary part will be $v(x,y) = y + xy + C$ ($C =$ numerical constant)

Exercise 1.7 Example-7: If $f(x,y) = (1+x+y)(1+x-y) + a\left(x^2-y^2\right) - 1 + 2iy(1-x-ax)$ is a complex analytic function then find the value of ' a '. ■

Solution:

$$u(\text{x},\text{y}) = (1+x+y)(1+x-y) + a\left(x^2-y^2\right) - 1 \quad \Rightarrow \frac{\partial u}{\partial x} = 2x + 2 + 2ax$$

$$v(x,y) = 2y(1-x-ax) \quad \Rightarrow \frac{\partial v}{\partial y} = 2(1-x-ax)$$

$$\text{According to Cauchy Reamann equation,} \frac{\partial \text{u}}{\partial \text{x}} = \frac{\partial \text{v}}{\partial \text{y}} \Rightarrow 4\text{x} = -4\text{ax} \Rightarrow \text{a} = -1$$

Exercise 1.8 The harmonic conjugate function of $u(x,y) = 2x(1-y)$ corresponding to a complex analytic function $\omega = u(x,y) + iv(x,y)$ is given $v(x,y) = \alpha x^2 + \beta y + \gamma y^2$ (Taking the integration constant to be zero). Which of the following statement is true ?

a. $\alpha - \gamma = \beta$

b. $\alpha + \gamma + \beta = 0$

c. $\alpha + \gamma = \beta$

d. $\alpha\gamma\beta = 1$. ■

Solution:

$$u(x,y) = 2x(1-y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2(1-y) = \frac{\partial v}{\partial y} \Rightarrow v = 2y - y^2 + f_1(x)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -2x = -\frac{\partial v}{\partial x} \Rightarrow v = x^2 + f_2(y)$$

Therefore, the imaginary part of the complex function $v = x^2 - y^2 + 2y$

Comparing with the question, $\alpha = 1, \beta = 2, \gamma = -1 \quad \Rightarrow \alpha - \gamma = \beta$

1.5.2 Method for Finding Conjugate Function

Case 1: $f(z) = u + iv$, and u is known.

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \Rightarrow v = -\int \frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy$$

Case 2: $f(z) = u + iv$, and v is known

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy \Rightarrow u = \int \frac{\partial v}{\partial y}dx - \int \frac{\partial v}{\partial x}dy$$

Exercise 1.9 Find the imaginary part of the complex analytic function whose real part is $u(x,y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ ■

Solution:

$$
\begin{aligned}
&\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x = \frac{\partial v}{\partial y} \Rightarrow v = 3x^2y - y^3 + 6xy + f_1(x) \\
&\frac{\partial u}{\partial y} = -6xy - 6y = -\frac{\partial v}{\partial x} \Rightarrow v = 3x^2y + 6xy + f_2(y) \\
&v(x,y) = 3x^2y - y^3 + 6xy + C
\end{aligned}
$$

1.5.3 Milne-Thomson Method : (To find Analytic function if either ' u ' or ' v ' is given)

Case 1: When 'u' is given,

(1) Find $\frac{\partial u}{\partial x} = \phi_1(x,y)$and $\frac{\partial u}{\partial y} = \phi_2(x,y)$

(2) Replace x by z and y by 0 in $\phi_1(x,y)$ and $\phi_2(x,y)$ to get $\phi_1(z,0)$ and $\phi_2(z,0)$.

(3) Find $f(z) = \int \{\phi_1(z,0) - i\phi_2(z,0)\}\, dz + c$

Case 2: When ' v ' is given,

(1) Find $\frac{\partial v}{\partial x} = \psi_2(x,y)$ and $\frac{\partial v}{\partial y} = \psi_1(x,y)$

(2) Replace x by z and y by 0 in $\psi_1(x,y)$ and $\psi_2(x,y)$ to get $\psi_1(z,0)$ and $\psi_1(z,0)$.

(3) Find $f(z) = \int \{\psi_1(z,0) + i\psi_2(z,0)\}\, dz + c$

Exercise 1.10 Find the analytical function whose imaginary part is $v(x,y) = e^x(x\cos y - y\sin y)$ ■

Solution:

$$\frac{\partial v}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y = \Psi_2(x,y) \quad \Rightarrow \frac{\partial v}{\partial y} = -e^x x\sin y - e^x(\sin y + y\cos y) = \Psi_1(x,y)$$

$$\Psi_1(z,0) = 0 \text{ and } \Psi_1(z,0) = e^z z + e^z \quad \Rightarrow f(z) = \int 0 + i\,[e^z z + e^z]\, dz = ize^z + C$$

Exercise 1.11 If the real part of a complex analytic function f(z) is given as, $u(x,y) = e^{-2xy}\sin\left(x^2 - y^2\right)$, then f(z) can be written as

a. $ie^{i^2} + C$

b. $-ie^{ix^2} + C$

c. $-ie^{-iz^2} + C$

d. $ie^{-ii^2} + C$

■

Solution:

$$
\begin{aligned}
&u(x,y) = e^{-2xy}\sin\left(x^{-2} - y^2\right) \\
&\Rightarrow \frac{\partial u}{\partial x} = e^{-2xy}(-2y)\sin\left(x^2 - y^2\right) + e^{-2xy}\cos\left(x^2 - y^2\right)2x = \phi_1(x,y) \\
&\Rightarrow \frac{\partial u}{\partial y} = e^{-2xy}(-2x)\sin\left(x^2 - y^2\right) + e^{-2xy}\cos\left(x^2 - y^2\right)(-2y) = \phi_2(x,y) \\
\therefore \quad &\phi_1(z,0) = \cos z^2 \cdot 2z, \phi_2(z,0) = \sin z^2(-2z)
\end{aligned}
$$

$$\therefore \quad f(z) = \int \left(\cos z^2 \cdot 2z - i\sin z^2 \cdot (-2z)\right) dz + c = 2\int \left(\cos z^2 + i\sin z^2\right) \cdot z dz + c$$
$$= 2\int e^{iz^2} \cdot z dz + c = -ie^{iz^2} + c$$

So the correct answer is **option (b)**

1.5.4 Harmonic Function

Any function which satisfies the Laplace's equation, is known as harmonic function. If $u+iv$ is an analytic function, then u, v are conjugate harmonic functions i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Exercise 1.12 Find the values of m, n such that $f(x,y) = x^2 + mxy + ny^2$ is harmonic in nature. ■

Solution:

$$\text{Since, } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{dy^2} = 0 \Rightarrow 2n + 2 = 0 \Rightarrow n = -1; 'm' \text{ can take any value.}$$

1.6 Cauchy's Integral Theorem

If a funciion $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve ' C ', then $\oint_C f(z)dz = 0$

1.6.1 Cauchy's Integral Formula

If $f(z)$ is analytic within or on a closed curve C and if ' a ' is any point within C, where $\frac{f(z)}{z-a}$ is not analytic at $z = a$ then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz \qquad \text{and } f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}dz$$

$$\text{Similarly,} \qquad f''(\mathrm{a}) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}dz \quad \text{and } \mathrm{f}^{\mathrm{n}}(\mathrm{a}) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz$$

Exercise 1.13 Evaluate the integral $\frac{1}{2\pi i}\oint \frac{ze^z}{(z-a)^3}$ ■

Solution:

$$\frac{1}{2\pi i}\oint \frac{ze^z}{(z-a)^3} = \frac{1}{2\pi i} 2\pi i \frac{1}{2!}\frac{d^2}{dz^2}\left(ze^z\right)\bigg|_{z=a}$$
$$= \frac{ae^a + 2e^a}{2}$$
$$= \frac{1}{2}(a+2)e^a$$

1.7 Power Series Expansion of Complex Function

Every analytic function which is analytic at $z = z_0$ can be expanded into power series about $z = z_0$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \ldots\ldots$$

where, z_0 is the centre of power series.

1.7.1 Radius of Convergence

Imagine a circle of centre z_0 and radius r, then $|z - z_0| = R$, The power series is convergent in the region $|z - z_0| < R$ (i.e. within the circle) and divergent $|z - z_0| > R$ (outside the circle). Therefore, R is known as the radius of convergence of power series and defined as

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Exercise 1.14 Find the radius of convergence of the series

$$\frac{z}{2} + \frac{1.3}{2.5} z^2 + \frac{1.3.5}{2.5.8} z^3 + \ldots\ldots$$

Solution: The coefficient of z^n of the given power series is given by

$$a_n = \frac{1.3.5\ldots(2n-1)}{2.5.8\ldots(3n-1)}$$
$$a_{n+1} = \frac{1.3.5\ldots(2n-1)(2n+1)}{25.8\ldots(3n-1)(3n+2)}$$
$$\text{So } \frac{a_{n+1}}{a_n} = \frac{2n+1}{3n+2} = \frac{2}{3} \cdot \frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{2}{3n}\right)}$$
$$\text{Therefore, } \frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{2}{3} \cdot \frac{(1+0)}{(1+0)} = \frac{2}{3} \Rightarrow R = \frac{3}{2}$$

Taylor Series Expansion

If a function f(z) is analytic at all points inside and on a circle C, with its center at the point ' a ' and radius ' r ', then at each point z inside C, the function $f(z)$ can be expanded as,

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots + \frac{f^n(a)}{n!}(z-a)^n + \ldots$$
$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!} (z - z_0)^n$$

Exercise 1.15 Expand the function $\ln(\cosh x)$ about the point $x = 0$.

Solution:

$$f(x) = f(0) + (x-0)f'(0) + \frac{1}{2!}(x-0)^2 f''(0) + \ldots\ldots$$
$$f'(x) = \frac{1}{\cosh x} \cdot \sinh x = \tanh x, f''(x) = \sec h^2 x$$
$$f'''(x) = 2\,\mathrm{sech}\,x(-\mathrm{sech}\,x \tanh x) = -2\sec h^2 x \tanh x$$
$$f'''(x) = 4\,\mathrm{sech}\,h^2 x \cdot \tanh^2 x - 2\sec h^4 x$$
$$\text{Therefore, } f(x) = 0 + 0 + \frac{x^2}{2} + 0 - \frac{x^4}{12} + \ldots\ldots = \frac{x^2}{2} - \frac{x^4}{12} + \ldots..$$

1.8 Laurent Series

Let C_1 and C_2 be two circles with center at z_0. Let $f(z)$ be analytic in the region R between the circles. Then $f(z)$ can be expanded in a series of the form,

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots \tag{1.3}$$

convergent in R. Such a series is called a Laurent series. The "b" series in equation 1.8 is called the principal part of the Laurent series.
Consider the Laurent series in equation.

$$\begin{aligned} f(z) = 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots + \left(\frac{z}{2}\right)^n + \cdots \\ + \frac{2}{z} + 4\left(\frac{1}{z^2} - \frac{1}{z^3} + \cdots + \frac{(-1)^n}{z^n} + \cdots\right) \end{aligned} \tag{1.4}$$

Let us see where this series converges. First consider the series of positive powers; by the ratio test (see Chapters 1 and 2), this series converges for $|z/2| < 1$, that is, for $|z| < 2$. Similarly, the series of negative powers converges for $|1|z| < 1$, that is, $|z| > 1$. Then both series converge (and so the Laurent series converges) for $|z|$ between 1 and 2 , that is, in a ring between two circles of radii 1 and 2 .
We expect this result in general. The "a" series is a power series, and a power series converges inside some circle (say C_2 in Figure .1.3). The "b" series is a series of inverse powers of z, and so converges for $|1/z| <$ some constant. Thus the "b" series converges outside some circle. Then a Laurent series converges between two circles (if it converges at all). (Note that the inner circle may be a point and the outer circle may have infinite radius).

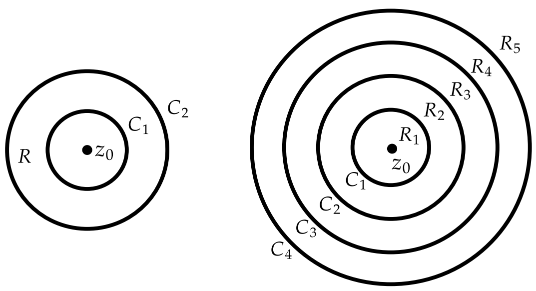


Figure 1.3: Laurent series

The formulas for the coefficients are

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)^{n+1}}, \quad b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)^{-n+1}}$$

Where C is any simple closed curve surrounding z_0 and lying in R. However, this is not usually the easiest way to find a Laurent series. Like power series about a point, the Laurent series (about z_0) for a function in a given annular ring (about z_0) where the function is analytic, is unique, and we can find it by any method we choose. (See examples below.) If $f(z)$ has several isolated singularities, there are several annular rings, $R_1, R_2, \cdots$, in which $f(z)$ is analytic; then there are several different Laurent series for $f(z)$, one for each ring. The Laurent series which we usually want is the one that converges near z_0. If you have any doubt about the ring of convergence of a Laurent series, you can find out by testing the "a" series and the "b" series separately.

1.9 Singularity of Complex Function

Singular Point of an Analytic Function

A point at which the function ceases to be analytic is called a singular point.

■ **Example 1.1** $f(z) = \frac{1}{(z-2)}$ has a singularity at z = 2. ■

Different kinds of singularities exist.they are,

1.9.1 Isolated Singularity

A point $z = z_0$ is said to be isolated singularity of $f(z)$ if,

(a) $f(z)$ is not analytic at $z = z_0$.

(b) $f(z)$ is analytic in the neighbourhood of $z = z_0$ i.e. there exists a neighbourhood of $z = z_0$, containing no other singularity.

■ **Example 1.2**

(i) Function $f(z) = \frac{1}{z}$ is analytic everywhere except at $z = 0$, therefore $z = 0$ is an isolated singularity.

(ii) The function $f(z) = \frac{z+2}{(z-1)(z-2)(z-3)}$ has three isolated singularities at $z = 1, 2$ and 3 . ■

1.9.2 Non-isolated Singularity

A singular point z_0 is said to be an non-isolated singularity if z_0 is not an isolated singular point.

■ **Example 1.3**

$f(z) = \frac{1}{\left[\sin \frac{\pi}{z}\right]}$

$$f(z) = \frac{1}{\left[\sin \frac{\pi}{z}\right]} \text{ is not analytic when } \sin\frac{\pi}{z} = 0$$

$$\frac{\pi}{z} = n\pi$$

$$z = \frac{1}{n}(n = 0, 1, 2, 3 \ldots\ldots\ldots\ldots)$$

Thus,$z = 0$ is an non-isolated singularity of $f(z)$ surrounded by a infinite number of other singularity $z = \frac{1}{n}$

$$f(z) = \frac{1}{\sin \pi/z} \text{ has non-isolated singularity at } z = 0$$

■

1.9.3 Types of Isolated Singularity

If f(z) is an isolated singular point at $z = a$, the we can expand $f(z)$ about $z = a$ into laurent series as:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} = \left[a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots..\right] + \left[\frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \ldots..\right]$$

Therefore, three types of singularity exists and they are as follows.

1. Removable Singularity

If the principal part of the Laurrent series expansion of $f(z)$ about $z=a$ contains no term i.e. if $b_n=0$ for all ' n ', then f(z) has a removable singularity at z = a. In this case, Laurrent series expansion is $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$

■ Example 1.4

Suppose $f(z)=\frac{\sin z}{z}$, then $\lim_{z\to 0}\left(\frac{\sin z}{z}\right)=1$, therefore, $z=0$ is a removable singularity of $f(z)$.
Again, $\frac{\sin z}{z}=\frac{1}{z}\left(z-\frac{z^3}{3!}+\ldots\ldots\ldots\right)=1-\frac{z^2}{3!}+\frac{z^4}{5!}\ldots\ldots$
Since, there is no negative term in the laurent series expansion of $f(z)$ about $z=0$, hence $z=0$ is a removable singularity of $f(z)$. ■

2. Non-essential singularity or Pole:

If the principal part of the Laurrent series expansion of $f(z)$ about $z=$ a contains a finite number of terms, saym, i.e. $b_n=0$ for all $n>m$, then $f(z)$ has a non-essential singularity or a pole of order ' m ' at $z=a$. A polc of order one is also known as simple pole.
Thus if z = a is pole of order mof function f(z), then f(z) will have the Laurent series expansion of the form

$$f(z)=\sum_{n=0}^{\infty} a_n\left(z-z_0\right)^n+\sum_{n=1}^{m} b_n\left(z-z_0\right)^{-n}$$

■ Example 1.5 $f(z)=\frac{z}{(z-1)(z+2)^2}$ has a simple pole at $z=1$ and a pole of order 2 at $z=-2$. ■

3. Essential singularity:

If the principal part of the Laurrent series expansion of $f(z)$ about $z=a$, contains infinite number of terms i.e. $b_n\neq 0$ for infinitely many values of n, then $f(z)$ has an essential singularity at $z=a$.

■ Example 1.6 $f(z)=e^{1/2}$ has an essential singularity at $z=0$, since the expansion $e^{\frac{1}{z}}=1+\frac{1}{z}+\frac{1}{z^2}2!+\ldots\ldots$ is an infinite series of-ve powers of z. ■

Exercise 1.16 Examine the nature of singularity of the functions: (a) $\sin\left(\frac{1}{1-z}\right)$, (b) $(z-3)\sin\left(\frac{1}{z+2}\right)$. ■

Solution:

(a) $$\sin\left(\frac{1}{1-z}\right)=\frac{1}{1-z}-\frac{1}{(1-z)^3\cdot 3!}+\frac{1}{(1-z)^5\cdot 5!}-.$$

so, $z=1$ is an isolated essential singular point.

(b) $$(z-3)\sin\left(\frac{1}{z+2}\right)=(z-3)\left[\frac{1}{z+2}-\frac{1}{(z+2)^3\cdot 3!}+\frac{1}{(z+2)^5.5!}-\ldots\ldots.\right]$$

so, z = −2 is an isolated essential point

1.10 Zero of an Analytic Function

A zero of an analytic function $f(z)$ is a value of z such that

$$f(z)=0$$

An analytic function f(z) is said to have a zero of order ' m ' at $z=z_0$ if f(z) is expressible as,

$$f(z)\left(z-z_0\right)^m\phi(z)$$

where $\phi(z)$ is analytic and $\phi\left(z_0\right)\neq 0$. For $m=1, f(z)$ is said to have a simple zero at $z=z_0$

1.11 Residue of Complex Fuction

1.11.1 Definition of residue at a pole:

Let, z = a be a pole of order ' m ' of f(z) and C_1 is a circle of radius ' r ' with center at z = a which does not contain singularities except z = a, then f(z) is analytic within the annular region r < |z − a| < R can be expanded into Laurrent series within the annulur region as:

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$$

Co-efficient b_n is known as residue of f(z) at z = a i.e.

$$\mathrm{Res.f(z=a)} = \mathrm{b}_n = \frac{1}{2\pi i}\oint f(z)dz \tag{1.5}$$

1.11.2 Methods of Finding Residues

(a)

Method 1: Res. $f(z=a) = \underset{z\to a}{Lt}(z-a)f(z)$

Method 2: If $f(z) = \frac{\phi(z)}{\psi(z)}$ where $\psi(a) = 0$ but $\phi(a) \neq 0$, then

$$\mathrm{Res}\,.f(z=a) = \frac{\phi(a)}{\psi'(a)}$$

(b) Residue at a pole of order ' n '

Method 1: Res. $f(z=a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}_{z=a}$

Method 2: First put $z+a=t$ and expand it into series, then Res. $f(z=a) =$ co-efficient of $1/t$

(c) Residue at z = ∞ : Res. $f(z=\infty)\ \underset{z\to\infty}{Lt}[-zf(z)]$

Exercise 1.17 Find the singular points of the following function and the corresponding residues:
(a) $f(z) = \frac{1-2z}{z(z-1)(z-2)}$ (b) $f(z) = \frac{z^2}{z^2+a^2}$ (c) $f(z) = z^2e^{1/z}$ ■

Solution:

$$\text{(a)}f(z) = \frac{1-2z}{z(z-1)(z-2)} \Rightarrow \text{ Poles } : z=0, z=1, z=2$$

$$\text{Res. } f(z=0) = \mathrm{Lt}_{z\to 0}(z-0)f(z) = \mathrm{Lt}_{z\to 0}\frac{1-2z}{(z-1)(z-2)} = \frac{1}{2}.$$

$$\text{Res. } f(z=1) = L_{z\to 1}(z-1)f(z) = L_{z\to 1}\frac{1-2z}{z(z-2)} = 1$$

$$\text{Res. } f(z=2) = \underset{z\to 2}{Lt}(z-2)f(z) = \underset{z\to 2}{Lt}\frac{1-2z}{z(z-1)} = -\frac{3}{2}$$

$$\text{(b) } f(z) = \frac{z^2}{z^2+a^2} \Rightarrow \text{Poles : } z = ia, z = -ia$$

$$\text{Res. } f(z=ia) = \left(\frac{z^2}{2z}\right)_{z=ia} = \frac{1}{2}ia; \mathrm{Res}\,.f(z=-ia) = \left(\frac{z^2}{2z}\right)_{z=-ia} = -\frac{1}{2}ia$$

$$\text{(c) } f(z) = z^2e^{1/z} = z^2\left[1+\frac{1}{z}+\frac{1}{z^2\cdot 2!}+\frac{1}{z^3\cdot 3!}+\ldots\ldots\right] \Rightarrow \text{ Poles } : z=0$$

$$\text{Res. } f(z=0) = \text{ Coefficient of } \frac{1}{z} = \frac{1}{3!} = \frac{1}{6}$$

1.12 Cauchy's Residue Theorem

If f(z) in single-valued and analytic in a closed curve ' C ', except at a finite number of poles within ' C, then,

$$\oint_C \mathrm{f(z)dz} = 2\pi \mathrm{i} \text{ (Sum of the residues at poles within 'C')} \tag{1.6}$$

Exercise 1.18 Evaluate the integral: $\oint_C \frac{4-3z}{z(z-1)(z-3)}dz$ where $|z| = \frac{3}{2}$ ■

Solution:

$$\hat{f}(z) = \frac{4-3z}{z(z-1)(z-3)} \Rightarrow \text{ Poles : } z=0, z=1, z=3$$

But, the given contour is circle centered at the origin and radius $3/2$ units.

Therefore, only $z=0$ and $z=1$ within the contour.

$$I = 2\pi i[\mathrm{Re}s. f(z=0) + \mathrm{Re}s. f(z=1)] = 2\pi i\left[\frac{4}{3} - \frac{1}{2}\right] = \frac{5\pi i}{3}$$

Exercise 1.19 Evaluate the integral: $\oint_C \frac{e^{2z}+z^2}{(z-1)^5}dz$ where $|z| = 2$ ■

Solution:

$$f(z) = \frac{e^{2z}+z^2}{(z-1)^5} \Rightarrow \text{ Poles } : z = 1 (\text{ order } 5)$$

$$I = 2\pi i \times \mathrm{Re}s. f(z=1) = 2\pi i \times \frac{1}{4!}\frac{d^4}{dz^4}\left[e^{2z}+z^2\right]_{z=1} = 2\pi i \times \frac{2e^2}{3} = \frac{4\pi i e^2}{3}.$$

1.12.1 Definite Integrals of Trigonometric Functions of $\cos\theta$ and $\sin\theta$: (Integration round the unit circle)

Method: Consider the contour to be a circle centered at the origin and having radius one unit i.e. $|z| = 1$

$$\text{Assume, } z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta$$

$$d\theta = \frac{dz}{iz}$$

$$\text{Therefore, } \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$\text{and } \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

And the limit will be changed from $0 \to 2\pi$ to $\oint_C$

The replacements regarding $\cos\theta$ and $\sin\theta$ is to be done only in the denominator of the given integral.

Exercise 1.20 Evaluate the integral: $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}; \quad a > b > 0$

Solution:

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{dz/iz}{a+b\left(\frac{z^2+1}{2z}\right)}$$

$$= \int_0^{2\pi} \frac{2dz}{i\,(bz^2+2az+b)}$$

$$\text{The singular points are at } (z=\alpha) = \frac{-a+\sqrt{a^2-b^2}}{b}$$

$$\text{The singular points are at } (z=\beta) = \frac{-a-\sqrt{a^2-b^2}}{b}$$

The singular point $z=\beta$ will lie outside the unit circle as $a > b > 0$ while the singular point $z=\alpha$ will lie inside the unit circle which is a simple pole.

$$\text{Res. } f(z=\alpha) = \lim_{z\to\alpha}(z-\alpha)f(z)$$

$$= \lim_{z\to\alpha} \frac{2}{ib} \frac{(z-\alpha)}{(z-\alpha)(z-\beta)} = \frac{2}{ib(\alpha-\beta)}$$

$$= \frac{2}{ib} \times \frac{b}{2\sqrt{a^2-b^2}}$$

$$= \frac{1}{i\sqrt{a^2-b^2}}$$

$$\text{Therefore,by cauchy Residue theorem. } I = 2\pi i \times \text{ Residue}$$

$$= 2\pi\mathrm{i} \times \frac{1}{i\sqrt{a^2-b^2}}$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

1.12.2 Evaluation of improper integrals between the limit $-\infty$ to $+\infty$:

Theorem I:

If $f(x)$ contain only polynomial terms

Then $f(x) = f(x)$

$\to$ find singular points

$\to$ check point lie in upper half

A. If singular point lie on imaginary axis

$$\int_{-\infty}^{\infty} f(z)dz = \int_{-\infty}^{\infty} f(x)dx = 2\pi i[\Sigma \text{ Res}] - \left(z \overset{\lim}{\to} \infty \quad zf(x)\right) x\pi i$$

B. If singular point lie on real axis

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(z)dz = \pi i[\Sigma \text{ Res}] - \pi i \left[\lim_{z \to \infty} z f(z) \right]$$

Theorem II:

If $f(x)$ contains sine and cosine function along with polynomial function rule is same, except second term which is 0 in this case.

A. If singular point lie on imaginary axis

$$\int_{-\infty}^{\infty} f(z)dz = \int_{-\infty}^{\infty} f(x)dx = 2\pi i[\Sigma \text{ Res}]$$

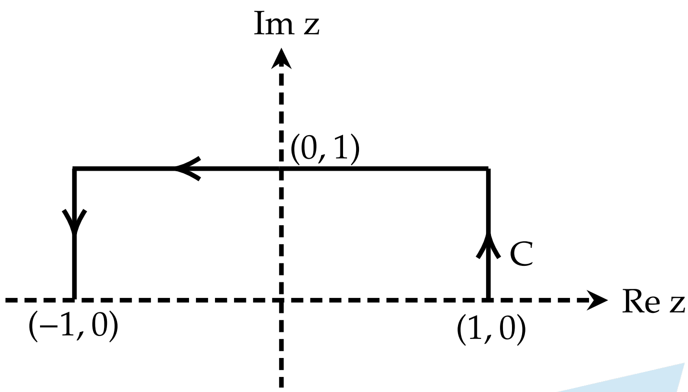
B. If singular point lie on real axis

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(z)dz = \pi i[\Sigma \text{ Res}]$$

Practise Set-1

1. The value of the integral $\int_C dz z^2 e^z$, where C is an open contour in the complex z-plane as shown in the figure below, is:

[NET/JRF(JUNE-2011)]



A. $\frac{5}{e}+e$ **B.** $e-\frac{5}{e}$ **C.** $\frac{5}{e}-e$ **D.** $-\frac{5}{e}-e$

2. Which of the following is an analytic function of the complex variable $z=x+iy$ in the domain $|z|<2$?

[NET/JRF(JUNE-2011)]

A. $(3+x-iy)^7$ **B.** $(1+x+iy)^4(7-x-iy)^3$
C. $(1-x-iy)^4(7-x+iy)^3$ **D.** $(x+iy-1)^{1/2}$

3. The first few terms in the Laurent series for $\frac{1}{(z-1)(z-2)}$ in the region $1\leq|z|\leq 2$ and around $z=1$ is

[NET/JRF(JUNE-2012)]

A. $\frac{1}{2}\left[1+z+z^2+\ldots\right]\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\ldots.\right]$
B. $\frac{1}{1-z}-z-(1-z)^2+(1-z)^3+\ldots.$
C. $\frac{1}{z^2}\left[1+\frac{1}{z}+\frac{1}{z^2}+\ldots.\right]\left[1+\frac{2}{z}+\frac{4}{z^2}+\ldots..\right]$
D. $2(z-1)+5(z-1)^2+7(z-1)^3+\ldots$

4. Let $u(x,y)=x+\frac{1}{2}\left(x^2-y^2\right)$ be the real part of analytic function $f(z)$ of the complex variable $z=x+iy$. The imaginary part of $f(z)$ is

[NET/JRF(JUNE-2012)]

A. $y+xy$ **B.** xy **C.** y **D.** y^2-x^2

5. The value of the integral $\int_C \frac{z^3 dz}{(z^2-5z+6)}$, where C is a closed contour defined by the equation $2|z|-5=0$, traversed in the anti-clockwise direction, is

[NET/JRF(DEC-2012)]

A. $-16\pi i$ **B.** 16πi **C.** $8\pi i$ **D.** $2\pi i$

6. With $z=x+iy$, which of the following functions $f(x,y)$ is NOT a (complex) analytic function of z ?

[NET/JRF(JUNE-2013)]

A. $f(x,y)=(x+iy-8)^3\left(4+x^2-y^2+2ixy\right)^7$
B. $f(x,y)=(x+iy)^7(1-x-iy)^3$
C. $f(x,y)=\left(x^2-y^2+2ixy-3\right)^5$
D. $f(x,y)=(1-x+iy)^4(2+x+iy)^6$

7. Which of the following functions cannot be the real part of a complex analytic function of $z = x + iy$?
[NET/JRF(DEC-2013)]

A. x^2y **B.** $x^2 - y^2$ **C.** $x^3 - 3xy^2$ **D.** $3x^2y - y - y^3$

8. Given that the integral $\int_0^\infty \frac{dx}{y^2+x^2} = \frac{\pi}{2y}$, the value of $\int_0^\infty \frac{dx}{(y^2+x^2)^2}$ is
[NET/JRF(DEC-2013)]

A. $\frac{\pi}{y^3}$ **B.** $\frac{\pi}{4y^3}$ **C.** $\frac{\pi}{8y^3}$ **D.** $\frac{\pi}{2y^3}$

9. If C is the contour defined by $|z| = \frac{1}{2}$, the value of the integral

$$\oint_C \frac{dz}{\sin^2 z}$$

is
[NET/JRF(JUNE-2014)]

A. ∞ **B.** $2\pi i$ **C.** 0 **D.** πi

10. The principal value of the integral $\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^3} dx$ is
[NET/JRF(DEC-2014)]

A. -2π **B.** $-\pi$ **C.** π **D.** 2π

11. The Laurent series expansion of the function $f(z) = e^2 + e^{1/2}$ about $z = 0$ is given by
[NET/JRF(DEC-2014)]

A. $\sum_{n=-\infty}^{\infty} \frac{z^n}{n!}$ for all $|z| < \infty$

B. $\sum_{n=0}^{\infty} \left(z^n + \frac{1}{z^n}\right) \frac{1}{n!}$ only if $0 < |z| < 1$

C. $\sum_{n=0}^{\infty} \left(z^n + \frac{1}{z^n}\right) \frac{1}{n!}$ for all $0 < |z| < \infty$

D. $\sum_{n=-\infty}^{\infty} \frac{z^n}{n!}$ only if $|z| < 1$

12. Consider the function $f(z) = \frac{1}{z}\ln(1-z)$ of a complex variable $z = re^{i\theta}(r \geq 0, \quad -\infty < \theta < \infty)$. The singularities of $f(z)$ are as follows:
[NET/JRF(DEC-2014)]

A. Branch points at $z = 1$ and $z = \infty$; and a pole at $z = 0$ only for $0 \leq \theta < 2\pi$

B. Branch points at $z = 1$ and $z = \infty$; and a pole at $z = 0$ for all θ other than $0 \leq \theta < 2\pi$

C. Branch points at $z = 1$ and $z = \infty$; and a pole at $z = 0$ for all θ

D. Branch points at $z = 0, z = 1$ and $z = \infty$.

13. The value of integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$
[NET/JRF(JUNE-2015)]

A. $\frac{\pi}{\sqrt{2}}$ **B.** $\frac{\pi}{2}$ **C.** $\sqrt{2}\pi$ **D.** 2π

14. The function $\frac{Z}{\sin \pi z^2}$ of a complex variable z has
[NET/JRF(DEC-2015)]

A. A simple pole at 0 and poles of order 2 at $\pm\sqrt{n}$ for $n = 1, 2, 3\ldots$

B. A simple pole at 0 and poles of order 2 at $\pm\sqrt{n}$ and $\pm i\sqrt{n}$ for $n = 1, 2, 3\ldots$

C. Poles of order 2 at $\pm\sqrt{n}, n = 0, 1, 2, 3\ldots$

D. Poles of order 2 at $\pm n, n = 0, 1, 2, 3\ldots$

15. The value of the contour integral $\frac{1}{2\pi i}\oint_C \frac{e^{4z}-1}{\cosh(z)-2\sinh(z)} dz$ around the unit circle C traversed in the anti-clockwise direction, is
[NET/JRF(JUNE-2016)]

A. 0 **B.** 2 **C.** $\frac{-8}{\sqrt{3}}$ **D.** $-\tanh\left(\frac{1}{2}\right)$

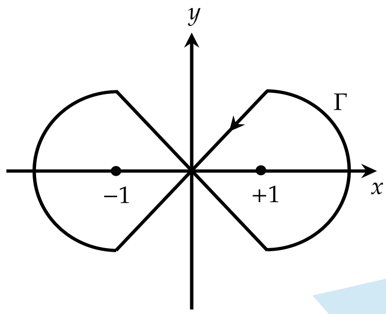
16. Let $u(x,y) = e^{ax}\cos(by)$ be the real part of a function $f(z) = u(x,y) + iv(x,y)$ of the complex variable $z = x + iy$, where a, b are real constants and $a \neq 0$. The function $f(z)$ is complex analytic everywhere in the complex plane if and only if

[NET/JRF(JUNE-2017)]

A. $b = 0$ **B.** $b = \pm a$ **C.** $b = \pm 2\pi a$ **D.** $b = a \pm 2\pi$

17. The integral $\oint_\Gamma \frac{ze^{i\pi z/2}}{z^2-1}dz$ along the closed contour Γ shown in the figure is

[NET/JRF(JUNE-2017)]



A. 0 **B.** 2π **C.** -2π **D.** $4\pi i$

18. What is the value of a for which $f(x,y) = 2x + 3\left(x^2 - y^2\right) + 2i(3xy + ay)$ is an analytic function of complex variable $z = x + iy$

[NET/JRF(JUNE-2018)]

A. 1 **B.** 0 **C.** 3 **D.** 2

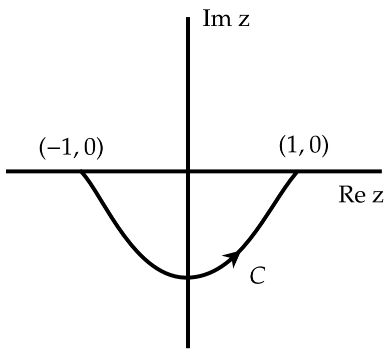
19. The value of the integral $\oint_C \frac{dz}{z}\frac{\tanh 2z}{\sin \pi z}$, where C is a circle of radius $\frac{\pi}{2}$, traversed counter-clockwise, with centre at $z = 0$, is

[NET/JRF(DEC-2018)]

A. 4 **B.** $4i$ **C.** $2i$ **D.** 0

20. The integral $I = \int_C e^z dz$ is evaluated from the point $(-1,0)$ to $(1,0)$ along the contour C, which is an arc of the parabola $y = x^2 - 1$, as shown in the figure. The value of I is

[NET/JRF(DEC-2018)]

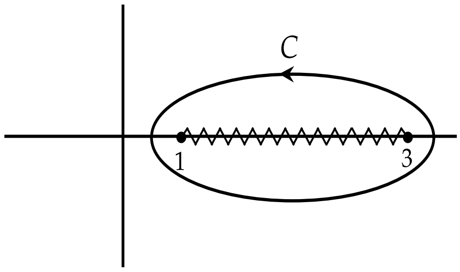


A. 0 **B.** $2\sinh 1$ **C.** $e^{2i}\sinh 1$ **D.** $e + e^{-1}$

21. The contour C of the following integral

$$\oint_C dz \frac{\sqrt{(z-1)(z-3)}}{\left(z^2-25\right)^3}$$

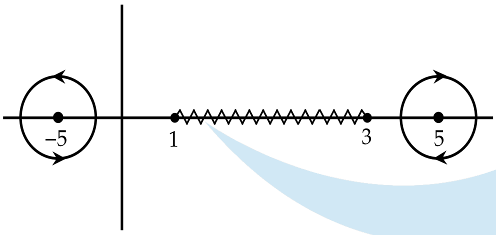
in the complex z plane is shown in the figure below.



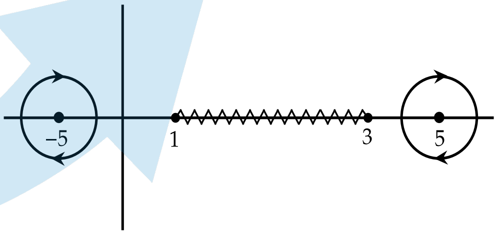
This integral is equivalent to an integral along the contours

[NET/JRF(DEC-2018)]

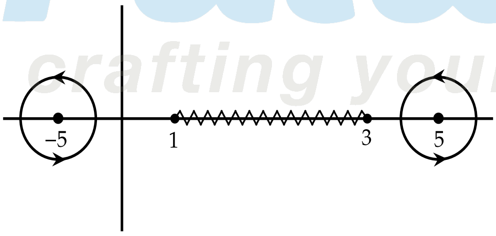
A.



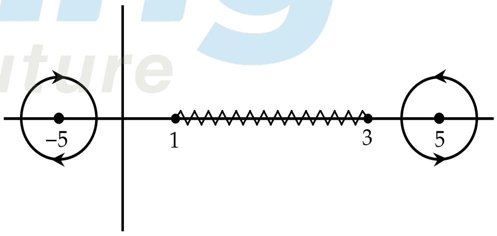
B.



C.



D.



22. Let C be the circle of radius $\frac{\pi}{4}$ centered at $z=\frac{1}{4}$ in the complex z-plane that is traversed counter-clockwise. The value of the contour integral $\oint_C \frac{z^2}{\sin^2 4z} dz$ is

[NET/JRF(DEC-2019)]

A. 0 **B.** $\frac{i\pi^2}{4}$ **C.** $\frac{i\pi^2}{16}$ **D.** $\frac{i\pi}{4}$

23. A function of a complex variable z is defined by the integral $f(z)=\oint_\Gamma \frac{w^2-2}{w-z} dw$, where Γ is a circular contour of radius 3 , centred at origin, running counter-clockwise in the w - plane. The value of the function at $z=(2-i)$ is

[NET/JRF(JUNE-2020)]

A. 0 **B.** $1-4i$ **C.** $8\pi+2\pi\mathrm{i}$ **D.** $-\frac{2}{\pi}-\frac{i}{2\pi}$

Answer key			
Q.No.	Answer	Q.No.	Answer
1	**C**	2	**B**
3	**B**	4	**A**
5	**A**	6	**D**
7	**A**	8	**-**
9	**C**	10	**A**
11	**C**	12	**-**
13	**A**	14	**B**
15	**C**	16	**B**
17	**C**	18	**A**
19	**B**	20	**B**
21	**C**	22	**C**
23	**C**		

Practise Set-2

1. The value of the integral $\oint_C \frac{e^z \sin(z)}{z^2} dz$, where the contour C is the unit circle: $|z-2| = 1$, is **[GATE 2010]**

   A. $2\pi i$ **B.** $4\pi i$ **C.** πi **D.** 0

2. Which of the following statements is TRUE for the function $f(z) = \frac{z \sin z}{(z-\pi)^2}$? **[GATE 2011]**

   A. $f(z)$ is analytic everywhere in the complex plane
   B. $f(z)$ has a zero at $z = \pi$
   C. $f(z)$ has a pole of order 2 at $z = \pi$
   D. $f(z)$ has a simple pole at $z = \pi$

3. For the function $f(z) = \frac{16z}{(z+3)(z-1)^2}$, the residue at the pole $z = 1$ is (your answer should be an integer)——- **[GATE 2013]**

4. The value of the integral
$$\oint_C \frac{z^2}{e^z + 1} dz$$
where C is the circle $|z| = 4$, is **[GATE 2014]**

   A. $2\pi i$ **B.** $2\pi^2 i$ **C.** $4\pi^3 i$ **D.** $4\pi^2 i$

5. Consider a complex function $f(z) = \frac{1}{z\left(z+\frac{1}{2}\right)\cos(z\pi)}$. Which one of the following statements is correct? **[GATE 2015]**

   A. $f(z)$ has simple poles at $z = 0$ and $z = -\frac{1}{2}$
   B. $f(z)$ has second order pole at $z = -\frac{1}{2}$
   C. $f(z)$ has infinite number of second order poles
   D. $f(z)$ has all simple poles

6. Consider $w = f(z) = u(x,y) + iv(x,y)$ to be an analytic function in a domain D. Which one of the following options is NOT correct? **[GATE 2015]**

   A. $u(x,y)$ satisfies Laplace equation in D
   B. $v(x,y)$ satisfies Laplace equation in D
   C. $\int_{1}^{z_2} f(z)dz$ is dependent on the choice of the contour between z_1 and z_2 in D
   D. $f(z)$ can be Taylor expended in D

7. A function $y(z)$ satisfies the ordinary differential equation $y'' + \frac{1}{z}y' - \frac{m^2}{z^2}y = 0$, where $m = 0, 1, 2, 3, \ldots$. Consider the four statements P, Q, R, S as given below.
P : z^m and z^{-m} are linearly independent solutions for all values of m
Q: z^m and z^{-m} are linearly independent solutions for all values of $m > 0$
R : $\ln z$ and 1 are linearly independent solutions for $m = 0$
S: z^m and $\ln z$ are linearly independent solutions for all values of m
The correct option for the combination of valid statements is **[GATE 2015]**

A. P, R and S only **B.** P and R only **C.** Q and R only **D.** R and S only

8. Which of the following is an analytic function of z everywhere in the complex plane?

[GATE 2016]

A. z^2 **B.** $(z^*)^2$ **C.** $|z|^2$ **D.** $\sqrt{z}$

9. The contour integral $\oint \frac{dz}{1+z^2}$ evaluated along a contour going from $-\infty$ to $+\infty$ along the real axis and closed in the lower half-plane circle is equal to.............. (up to two decimal places).

[GATE 2017]

10. The imaginary part of an analytic complex function is $v(x,y) = 2xy + 3y$. The real part of the function is zero at the origin. The value of the real part of the function at $1+i$ is .... (up to two decimal places)

[GATE 2017]

11. The absolute value of the integral

$$\int \frac{5z^3+3z^2}{z^2-4}dz$$

over the circle $|z-1.5| = 1$ in complex plane, is ... (up to two decimal places).

[GATE 2018]

12. The pole of the function $f(z) = \cot z$ at $z = 0$ is

[GATE 2019]

A. A removable pole **B.** An essential singularity

C. A simple pole **D.** A second order pole

13. The value of the integral $\int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2+a^2}dx$, where $k > 0$ and $a > 0$, is

[GATE 2019]

A. $\frac{\pi}{a}e^{-ka}$ **B.** $\frac{2\pi}{a}e^{-ka}$ **C.** $\frac{\pi}{2a}e^{-ka}$ **D.** $\frac{3\pi}{2a}e^{-ka}$

14. The value of the integral $\int_0^{\infty} \frac{\ln x}{(x^2+1)^2}dx$ is

[JEST 2012]

a. 0 **b.** $\frac{-\pi}{4}$

c. $\frac{-\pi}{2}$ **d.** $\frac{\pi}{2}$

15. Compute $\lim_{z\to 0} \frac{\mathrm{Re}(z^2)+\mathrm{Im}(z^2)}{z^2}$

[JEST 2013]

a. The limit does not exist **b.** 1

c. $-i$ **d.** -1

16. The value of limit

$$\lim_{z\to i} \frac{z^{10}+1}{z^6+1}$$

is equal to

[JEST 2014]

a. 1 **b.** 0

c. $\frac{-10}{3}$ **d.** $\frac{5}{3}$

17. The value of integral

$$I = \oint \frac{\sin z}{2z - \pi} dz$$

with c a circle $|z| = 2$, is

[JEST 2014]

a. 0 **b.** $2\pi i$

c. πi **d.** $-\pi i$

18. Given an analytic function $f(z) = \phi(x,y) + i\psi(x,y)$, where $\phi(x,y) = x^2 + 4x - y^2 + 2y$ If C is a constant, which of the following relations is true?

[JEST 2015]

a. $\psi(x,y) = x^2y + 4y + C$ **b.** $\psi(x,y) = 2xy - 2x + C$

c. $\psi(x,y) = 2xy + 4y - 2x + C$ **d.** $\psi(x,y) = x^2y - 2x + C$

19. Which one is the image of the complex domain $\{z \mid xy \geq 1, x + y > 0\}$ under the mapping $f(z) = z^2$, if $z = x + iy$?

[JEST 2017]

a. $\{z \mid xy \geq 1, x + y > 0\}$ **b.** $\{z \mid x \geq 2, x + y > 0\}$

c. $\{z \mid y \geq 2\forall x\}$ **d.** $\{z \mid y \geq 1\forall x\}$

20. The integral $I = \int_1^\infty \frac{\sqrt{x-1}}{(1+x)^2} dx$ is

[JEST 2017]

a. $\frac{\pi}{\sqrt{2}}$ **b.** $\frac{\pi}{2\sqrt{2}}$

c. $\frac{\sqrt{\pi}}{2}$ **d.** $\sqrt{\frac{\pi}{2}}$

21. The integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} dx \text{ is}$$

[JEST 2018]

a. $\frac{\pi}{e}$ **b.** πe^{-2}

c. π **d.** zero

22. Consider the function $f(x,y) = |x| - i|y|$. In which domain of the complex plane is this function analytic?
[JEST 2019]

a. First and second quadrants **b.** Second and third quadrants

c. Second and fourth quadrants **d.** Nowhere

Answer key

Q.No.	Answer	Q.No.	Answer
1	**D**	2	**C**
3	**3(NAT)**	4	**C**
5	**A**	6	**C**
7	**C**	8	**A**
9	π**(NAT)**	10	**3(NAT)**
11	**81.64(NAT)**	12	**C**
13	**A**	14	**B**
15	**A**	16	**D**
17	**C**	18	**C**
19	**-**	20	**B**
21	**A**	22	**C**

Practise Set-3

1. The amplitude of $\frac{1+i\sqrt{3}}{\sqrt{3}+i}$ is

a. $\frac{\pi}{3}$ **b.** $-\frac{\pi}{3}$

c. $\frac{\pi}{6}$ **d.** $-\frac{\pi}{6}$

Solution:

$$\begin{aligned}\frac{1+i\sqrt{3}}{\sqrt{3}+i} &= \frac{(1+i\sqrt{3})(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)}\\ &= \frac{2\sqrt{3}+2i}{4}\\ &= \frac{\sqrt{3}}{2}+\frac{1}{2}i\end{aligned}$$

Since both the real and complex parts are greater than zero, hence the argument is the acute angle given by $\tan^{-1}\left|\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right| = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

So the correct answer is **Option (c)**

2. If $\frac{1-ix}{1+ix} = a+ib$, then a^2+b^2 is

a. 1 **b.** -1

c. 0 **d.** none of these

Solution:

$$a+ib = \frac{1-ix}{1+ix} \Rightarrow a-ib = \frac{1+ix}{1-ix}$$

$$\therefore \quad (a+ib)(a-ib) = \frac{1-ix}{1+ix}\cdot\frac{1+ix}{1-ix} \Rightarrow a^2+b^2 = \frac{1+x^2}{1+x^2} = 1$$

So the correct answer is **Option (a)**

3. If $z = 1-\cos\theta + i\sin\theta$, then $|z|$ equals

a. $2\sin\frac{\theta}{2}$ **b.** $2\cos\frac{\theta}{2}$

c. $2\left|\sin\frac{\theta}{2}\right|$ **d.** $2\left|\cos\frac{\theta}{2}\right|$

Solution:

$$\begin{aligned}|z| &= \sqrt{(1-\cos\theta)^2+\sin^2\theta}\\ &= \sqrt{2-2\cos\theta}\\ &= \sqrt{4\sin^2\frac{\theta}{2}}\end{aligned}$$

$$= 2\left|\sin\frac{\theta}{2}\right|$$

So the correct answer is **Option (c)**

4. If $z = \frac{1}{(2+3i)^2}$, then $|z|$ equals

a. $\frac{1}{13}$
b. $\frac{1}{15}$
c. $\frac{1}{12}$
d. none of these

Solution:

$$|z| = \frac{1}{|2+3i|^2} = \frac{1}{\left(\sqrt{2^2+3^2}\right)^2} \quad |z| = \frac{1}{13}$$

So the correct answer is **Option (a)**

5. If the number $\frac{z-1}{z+1}$ is purely imaginary, then

a. $|z| = 1$
b. $|z| > 1$
c. $|z| < 1$
d. $|z| > 2$

Solution:

we have: $\dfrac{z-1}{z+1}$ is purely imaginary

$\Rightarrow$ argument of $\dfrac{z-1}{z+1}$ is $\pm\dfrac{\pi}{2} \Rightarrow \arg\left(\dfrac{z-1}{z+1}\right) = \pm\dfrac{\pi}{2}$

$\Rightarrow$ z lies on a circle having $(1,0)$ and $(-1,0)$ as the end point of a diameter.

$\Rightarrow$ z lies on a circle with centre at the origin and radius are unit

$\Rightarrow$ z lies on $|z| = 1 \Rightarrow |z| = 1$

So the correct answer is **Option (a)**

6. The value of integral $I = \int_0^{\pi} \frac{2d\theta}{R-\cos\theta}$ is given by where R is real constant.

a. $\frac{-1}{2\sqrt{R^2-1}}$
b. $\frac{2\pi}{\sqrt{R^2-1}}$
c. $\frac{\pi}{\sqrt{1-R^2}}$
d. $\frac{\pi}{\sqrt{R^2-1}}$

Solution:

$$\int_0^{\pi} \frac{2d\theta}{R-\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{2d\theta}{R-\cos\theta} = \int_0^{2\pi} \frac{d\theta}{R-\cos\theta}$$

$$z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta = izd\theta \Rightarrow d\theta = \frac{dz}{iz}$$

$$\therefore \int_0^{\pi} \frac{2d\theta}{R-\cos\theta} = \oint_c \frac{dz/iz}{R-\frac{1}{2}(z+z^{-1})}$$

$$= \oint_C \frac{dz/iz}{R-\frac{1}{2}\left(\frac{z^2+1}{z}\right)}$$

$$= \oint_C \frac{dz/iz}{\frac{2Rz-(z^2+1)}{2z}}$$

$$= -\frac{2}{i}\oint_c \frac{dz}{z^2 - 2Rz + 1} \qquad \text{where } C\text{; unit circle}$$

$$\text{Poles are } : z^2 - 2Rz + 1 = 0$$

$$\Rightarrow z = \frac{-(-2R) \pm \sqrt{4R^2 - 4 \times 1 \times 1}}{2 \times 1}$$

$$\Rightarrow z = \frac{2R \pm \sqrt{4R^2 - 4}}{2}$$

$$\Rightarrow z = \frac{2R \pm 2\sqrt{R^2 - 1}}{2}$$

$$= R \pm \sqrt{R^2 - 1}$$

$$z_1 = R + \sqrt{R^2 - 1} z_2 = R - \sqrt{R^2 - 1} \quad (\text{ inside } C)$$

$$\operatorname{Res}(z = z_2) = \lim_{z \to z_2} (z - z_2) \frac{1}{(z - z_1)(z - z_2)}$$

$$= \frac{1}{z_2 - z_1} = \frac{1}{R - \sqrt{R^2 - 1 - R - \sqrt{R^2 - 1}}} = \frac{-1}{2\sqrt{R^2 - 1}}$$

$$\therefore \int_0^\pi \frac{2d\theta}{(R - \cos\theta)} = \frac{-2}{i} \times 2\pi i \times \frac{-1}{2\sqrt{R^2 - 1}}$$

$$= \frac{2\pi}{\sqrt{R^2 - 1}}$$

So the correct answer is **Option (b)**

7. The value of integral $\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2}$ is given by

a. $\frac{\pi}{2}$ **b.** π

c. $i\frac{\pi}{2}$ **d.** $\frac{1}{4i}$

Solution:

$$\oint_C \frac{dz}{(1+z^2)^2} = \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2} + \int_\Gamma \frac{dz}{(1+z^2)^2}$$

poles, $1 + z^2 = 0 \quad z = \pm i$ of order 2 $\quad z = i$ is inside c

$$\therefore \operatorname{Res}(z = i) = \lim_{z \to i} \frac{1}{\lfloor} \frac{d}{dz}\left[(z - i)^2 \frac{1}{(z - i)^2 (z + i)^2}\right] = \lim_{z \to i} \frac{-2}{(z + i)^3} = \frac{1}{4i}$$

$$\oint_C \frac{dz}{(1+z^2)^2} = 2\pi i \times \frac{1}{4i} = \frac{\pi}{2} \text{ also } \int_\Gamma \frac{dz}{(1+z^2)^2} = 0$$

$$\therefore \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$$

So the correct answer is **Option (a)**

8. The value of $\oint_C \frac{\sin 3z}{z^2} dz \; c : |z| = \pi$ is given by

a. $6\pi i$ **b.** $-6\pi i$

c. 0 **d.** 3

Solution:

$$\frac{\sin 3z}{z^2} = \frac{1}{z^2}\left[3z - \frac{(3z)^3}{\lfloor 3} + \ldots\right] = \frac{3}{z} - \frac{9}{2} z + \ldots$$

$$\text{Residue } = 3$$
$$\therefore \oint_C \frac{\sin 3z}{z^2}dz = 2\pi i \times 3$$
$$= 6\pi i$$

So the correct answer is **Option (a)**

9. Consider a complex function $f(z) = \frac{1}{z(z+\frac{1}{2})\cos(z\pi)}$. Which one of the following statements is correct?

a. $f(z)$ has simple poles at $z = 0$ and $z = -\frac{1}{2}$

b. $f(z)$ has second order pole at $z = -\frac{1}{2}$

c. $f(z)$ has infinite number of second order poles

d. $f(z)$ has all simple poles

Solution:

$$f(z) = \frac{1}{z\left(z+\frac{1}{2}\right)\cos(z\pi)}$$

For n^{th} order pole $\because \lim_{z\to a}(z-a)^n f(z) = \text{finite and} \neq 0$

At $z = 0$

$$\lim_{z\to 0} z f(z) = \text{ finite } \Rightarrow z = 0 \text{ is a simple pole.}$$

At $z = -\frac{1}{2}$

$$\lim_{z\to -\frac{1}{2}} \frac{\left(z+\frac{1}{2}\right)^2}{z\left(z+\frac{1}{2}\right)\cos z\pi} = \lim_{z\to -\frac{1}{2}} \frac{\left(z+\frac{1}{2}\right)}{z\cos z\pi}$$
$$= \lim_{z\to -\frac{1}{2}} \frac{1}{1\cdot \cos z\pi + z\cdot \pi(-\sin z\pi)}$$
$$= \lim_{z\to -\frac{1}{2}} \frac{1}{\cos z\pi - z\pi \sin z\pi}$$
$$= \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$
$$= \text{ finite}$$

$$\Rightarrow f(z) \text{ has second order pole at } z = -\frac{1}{2}$$

So the correct answer is **Option (b)**

10. The value of integral

$$I = \oint_c \frac{\sin z}{2z-\pi}dz$$

with c a circle $|z| = 2$, is

a. 0

b. $2\pi i$

c. πi

d. $-\pi i$

Solution:

$$I = \oint_C \frac{\sin z}{2z-\pi} \quad \text{pole } \Rightarrow 2z-\pi = 0 \Rightarrow z = \frac{\pi}{2}$$

$$\text{Residue at } z = \frac{\pi}{2} \quad \because |z| = 2 \text{ so it will be lies within the contour}$$

$$
\begin{aligned}
I &= \oint_C \frac{e^{iz}}{2\left(z-\frac{\pi}{2}\right)} = \sum Res \times 2\pi i \\
\text{Res}\,| &= \frac{\left(z-\frac{\pi}{2}\right)e^{iz}}{2\left(z-\frac{\pi}{2}\right)} \\
&= \frac{e^{i\pi/2}}{2} \\
&= \frac{i}{2} \text{ (taking imaginary part, Residue} = \frac{1}{2}) \\
\text{Now } I &= \frac{1}{2} \times 2\pi i \\
&= \pi i
\end{aligned}
$$

So the correct answer is **Option (c)**

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