Practise Set-1

1. The Fourier transform of the derivative of the Dirac $\delta-$ function, namely $\delta'(x)$, is proportional to **[NET/JRF(DEC-2013)]**

A. 0 **B.** 1 **C.** $\sin k$ **D.** ik

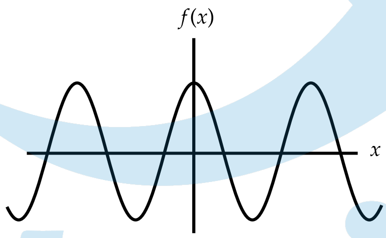
Solution:

Fourier transform of $\delta'(x)$

$$H(K) = \int_{-\infty}^{\infty} \delta'(x) e^{ikx} dx = ike^{(k\cdot 0)} = ik$$

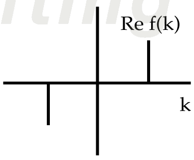
So the correct answer is **Option (D)**

2. $f(x)$ for the range $[-\infty, \infty]$ is shown below **[NET/JRF(DEC-2014)]**

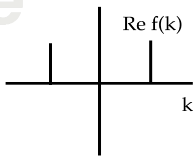


Which of the following graphs represents the real part of its Fourier transform?

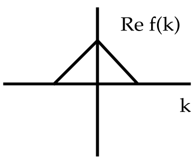
A.



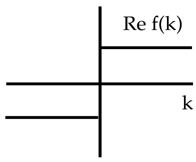
B.



C.



D.



Solution:

This is cosine function

$$f(x) = A\cos x$$

$$F(k) = \frac{A}{2}\left[\delta(k-k_0) + \delta(k+k_0)\right]$$

So the correct answer is **Option (B)**

3. The Fourier transform of $f(x)$ is $\tilde{f}(k) = \int_{-\infty}^{+\infty} dx e^{ikx} f(x)$. If $f(x) = \alpha\delta(x) + \beta\delta'(x) + \gamma\delta''(x)$, where $\delta(x)$ is the Dirac delta-function (and prime denotes derivative), what is $\tilde{f}(k)$?

[NET/JRF(DEC-2015)]

A. $\alpha + i\beta k + i\gamma k^2$ **B.** $\alpha + \beta k - \gamma k^2$

C. $\alpha - i\beta k - \gamma k^2$ **D.** $i\alpha + \beta k - i\gamma k^2$

Solution:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx e^{ikx} \left(\alpha\delta(x) + \beta\delta'(x) + \gamma\delta''(x)\right)$$

$$\int_{-\infty}^{\infty} \alpha\delta(x) e^{ikx} dx = \alpha$$

$$\int_{-\infty}^{\infty} \beta\delta'(x) e^{ikx} dx = \beta\left[e^{ikx}\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} ike^{ikx}\delta(x) dx\right] = -i\beta k$$

$$\int_{-\infty}^{\infty} \gamma\delta''(x) e^{ikx} dx = -\gamma k^2$$

So the correct answer is **Option (C)**

4. What is the Fourier transform $\int dx e^{ilx} f(x)$ of

$$f(x) = \delta(x) + \sum_{n=1}^{\infty} \frac{d^n}{dx^n}\delta(x)$$

where $\delta(x)$ is the Dirac delta-function?

[NET/JRF(JUNE-2016)]

A. $\frac{1}{1-ik}$ **B.** $\frac{1}{1+ik}$ **C.** $\frac{1}{k+i}$ **D.** $\frac{1}{k-i}$

Solution:

$$f(x) = \delta(x) + \sum_{n=1}^{\infty} \frac{d^n}{dx^n}\delta(x)$$

$$= \sum_{n=0}^{\infty} \frac{d^n}{dx^n}\delta(x) = \sum_{n=0}^{\infty} \delta^{(n)}(x)$$

$$\because F[\delta(x)] = 1 \Rightarrow F\left[\delta^{(n)}(x)\right]$$

$$= (-ik)^n F[\delta(x)] = (-ik)^n$$

$$\because f(x) = \sum_{n=0}^{\infty} \delta^{(n)}(x)$$

$$\Rightarrow F[f(x)] = \sum_{n=0}^{\infty} (-ik)^n = 1 - ik + (ik)^2 - (ik)^3 + \ldots.$$

$$= \frac{1}{1-(-ik)} = \frac{1}{1+ik}$$

So the correct answer is **Option (B)**

5. The Fourier transform $\int_{-\infty}^{\infty} dx f(x) e^{ikx}$ of the function $f(x) = \frac{1}{x^2+2}$ is

[NET/JRF(DEC-2016)]

A. $\sqrt{2}\pi e^{-\sqrt{2}|||}$ **B.** $\sqrt{2}\pi e^{-\sqrt{2}k}$ **C.** $\frac{\pi}{\sqrt{2}} e^{-\sqrt{2}k}$ **D.** $\frac{\pi}{\sqrt{2}} e^{-\sqrt{2}|k|}$

Solution:

$$\text{Fourier transform of } f(x) = \frac{1}{x^2+a^2},$$
$$a > 0 \text{ is } \int \frac{1}{x^2+a^2} e^{ikx} dx = \frac{\pi}{a} e^{-a|k|}$$
$$\text{Hence } \int \frac{1}{x^2+a^2} e^{ikx} dx = \frac{\pi}{\sqrt{2}} e^{-\sqrt{2}|k|}$$

So the correct answer is **Option (D)**

6. The Fourier transform $\int_{-\infty}^{\infty} dx f(x) e^{ikx}$ of the function $f(x) = e^{-|x|}$

[NET/JRF(JUNE-2018)]

A. $-\frac{2}{1+k^2}$ **B.** $-\frac{1}{2(1+k^2)}$ **C.** $\frac{2}{1+k^2}$ **D.** $\frac{2}{(2+k^2)}$

Solution:

$$\int_{-\infty}^{+\infty} dx e^{-|x|} e^{ikx} = \int_{-\infty}^{+\infty} dx e^{-|x|} \cos kx dx \text{ odd functions in } kx \text{ vanishes}$$
$$\Rightarrow 2\int_0^{\infty} e^{-x} \cos kx dx = 2\frac{e^{-x}}{1+k^2}[-\cos kx + k \sin kx]_0^{\infty}$$
$$\because \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx]$$
$$\Rightarrow 2\int_0^{\infty} e^{-x} \cos kx dx = 2\frac{e^0}{1+k^2} = \frac{2}{1+k^2}$$

So the correct answer is **Option (C)**

7. The Heaviside function is defined as $H(t) = \begin{cases} +1, & \text{for } t > 0 \\ -1, & \text{for } t < 0 \end{cases}$ and its Fourier transform is given by $-2i/\omega$. The Fourier transform of $\frac{1}{2}[H(t+1/2) - H(t-1/2)]$ is

[GATE 2015]

A. $\frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}$ **B.** $\frac{\cos\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}$ **C.** $\sin\left(\frac{\omega}{2}\right)$ **D.** 0

Solution:

$$H(f) = \int_{-\infty}^{\infty} H(t) e^{-i2\pi f t} dt, \text{ for a function } H(t), H(f) = -\frac{2i}{\omega}$$
$$\text{For } H(t-t_0), \text{ Fourier Transform is } e^{-i2\pi f_0} H(f)$$
$$\text{Shifting Theorem}$$
$$\text{For } \frac{1}{2}\left[H\left(t+\frac{1}{2}\right) - H\left(t-\frac{1}{2}\right)\right] = \frac{1}{2}\left[e^{i\frac{\omega}{2}} - e^{-i\frac{\omega}{2}}\right]\frac{-2i}{\omega}$$
$$= \frac{1}{2i}\left[e^{i\frac{\omega}{2}} - e^{-i\frac{\omega}{2}}\right]\frac{-2i}{\omega} \times i$$
$$\text{The Fourier transform of } \frac{1}{2}[H(t+1/2) - H(t-1/2)] = \frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}$$

So the correct answer is **Option (A)**

8. The Fourier transform of the function $\frac{1}{x^4+3x^2+2}$ up to proportionality constant is

[JEST 2017]

A. $\sqrt{2}\exp\left(-k^2\right)-\exp\left(-2k^2\right)$ **B.** $\sqrt{2}\exp(-|k|)-\exp\left(-\sqrt{2}|k|\right)$

C. $\sqrt{2}\exp\left(-\sqrt{|k|}\right)-\exp\left(-\sqrt{2|k|}\right)$ **D.** $\sqrt{2}\exp\left(-\sqrt{2}k^2\right)-\exp\left(-2k^2\right)$

Solution:

$$f(x)=\frac{1}{(x^4+3x^2+2)}=\frac{1}{(x^2+1)}-\frac{1}{\left[x^2+(\sqrt{2})^2\right]}$$

Now, Fourier transform of f(x) is,

$$\begin{aligned}
F(p) &= A\int_{-\infty}^{\infty} f(x)e^{-1kx}dx \\
&= A\int_{-\infty}^{\infty}\left[\frac{1}{(x^2+1)}-\frac{1}{x^2+(\sqrt{2})^2}\right]e^{-ikx}dx \\
&= A\left[\int_{-\infty}^{\infty}\frac{1}{(x^2+1)}\times e^{-ikx}dx-\int_{-\infty}^{\infty}\frac{e^{-ikx}}{x^2+(\sqrt{2})^2}dx\right]
\end{aligned}$$

$$\because \int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)}e^{-ikx}dx=\sqrt{\frac{\pi}{2}}\frac{e^{-a|k|}}{a}$$

$$\begin{aligned}
F(k) &= A\left[\sqrt{\frac{\pi}{2}}\frac{e^{-|k|}}{1}-\sqrt{\frac{\pi}{2}}\frac{e^{-\sqrt{2}|k|}}{\sqrt{2}}\right] \\
&= \frac{A\sqrt{\pi}}{2}[\sqrt{2}\exp(-|k|)-\exp\left(-\sqrt{2}|k|\right)]
\end{aligned}$$

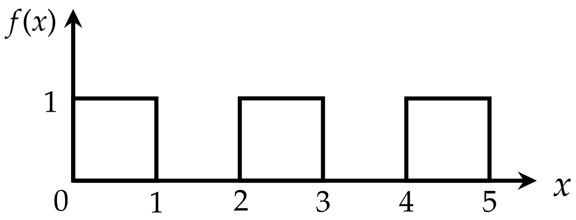
Correct option is (**option(B)**)

Answer key			
Q.No.	Answer	Q.No.	Answer
1	**D**	2	**B**
3	**C**	4	**B**
5	**D**	6	**C**
7	**A**	8	**B**

Problem Set-2

1. The graph of the function $f(x)=\begin{cases} 1 & \text{for } 2n\le x\le 2n+1 \\ 0 & \text{for } 2n+1\le x\le 2n+2 \end{cases}$ where $n=(0,1,2,\ldots\ldots)$ is shown below. Its Laplace transform $\tilde{f}(s)$ is

[NET/JRF(DEC-2011)]



A. $\frac{1+e^{-s}}{s}$ **B.** $\frac{1-e^{-s}}{s}$ **C.** $\frac{1}{s(1+e^{-s})}$ **D.** $\frac{1}{s(1-e^{-s})}$

Solution:

$$
\begin{aligned}
L(f(x)) &= \int_0^\infty e^{-sx} f(x)dx \\
&= \int_0^1 e^{-sx}\cdot 1dx + \int_1^2 e^{-sx}\cdot 0dx + \int_2^3 e^{-sx}\cdot 1dx + \ldots\ldots \\
&= \left[\frac{e^{-sx}}{-s}\right]_0^1 + 0 + \left[\frac{e^{-sx}}{-s}\right]_2^3 + \ldots\ldots \\
&= \frac{1}{-s}\left[e^{-s}-1\right] + \frac{1}{-s}\left[e^{-3s}-e^{-2s}\right] + \ldots\ldots \\
&= \frac{1}{-s}\left[-1+e^{-s}-e^{-2s}+e^{-3s}+\ldots\ldots\right] \\
&= \frac{1}{s}\left[1-e^{-s}+e^{-2s}-e^{-3s}+\ldots.\right]
\end{aligned}
$$

$$
\text{Since } S_\infty = \frac{a}{1-r} \text{ where } r=-e^{-s} \text{ and } a
$$

$$
=1 \Rightarrow S_\infty = \frac{1}{s}\left[\frac{1}{(1+e^{-s})}\right]
$$

So the correct answer is **Option(C)**

2. The inverse Laplace transforms of $\frac{1}{s^2(s+1)}$ is

[NET/JRF(JUNE-2013)]

A. $\frac{1}{2}t^2e^{-t}$ **B.** $\frac{1}{2}t^2+1-e^{-t}$ **C.** $t-1+e^{-t}$ **D.** $\frac{1}{2}t^2\left(1-e^{-t}\right)$

Solution:

$$
\begin{aligned}
f(s) = \frac{1}{s+1} \Rightarrow f(t) = e^{-t} \Rightarrow L^{-1}\left[\frac{1}{s(s+1)}\right] \\
= \int_0^t e^{-t}dt = \left(-e^{-t}\right)_0^t = \left(-e^{-t}+1\right) \\
\Rightarrow L^{-1}\left[\frac{1}{s^2(s+1)}\right] = \int_0^t \left(-e^{-t}+1\right)dt \\
= \left(e^{-t}+t\right)_0^t = e^{-t}+t-1
\end{aligned}
$$

So the correct answer is **Option (C)**

3. The Laplace transform of

$$f(t)=\begin{cases}\frac{t}{T}, & 0<t<T\\ 1 & t>T\end{cases}$$

is

[NET/JRF(DEC-2016)]

A. $\frac{-\left(1-e^{-sT}\right)}{s^2T}$ **B.** $\frac{\left(1-e^{-sT}\right)}{s^2T}$ **C.** $\frac{\left(1+e^{-sT}\right)}{s^2T}$ **D.** $\frac{\left(1-e^{sT}\right)}{s^2T}$

Solution:

we can write

$$\begin{aligned} f(t) &= [u_0(t)-u_T(t)]\frac{t}{T}+u_T(t)\\ &= [1-u_T(t)]\frac{t}{T}+u_T(t)=\frac{t}{T}-u_T(t)\frac{t}{T}+u_T(t)\end{aligned}$$

Hence the transform of $f(t)$ is

$$\begin{aligned} L\{f(t)\} &= L\left\{\frac{t}{T}\right\}-L\left\{u_T(t)\left[\frac{(t-T)+T}{T}\right]\right\}+L\{u_T(t)\}\\ &= \frac{1}{s^2T}-\frac{e^{-sT}}{T}\left(\frac{1}{s^2}+\frac{T}{s}\right)+\frac{e^{-sT}}{s}\\ &= \frac{1-e^{-sT}}{s^2T}\end{aligned}$$

So the correct answer is **Option (B)**

4. Consider the differential equation $\frac{dy}{dt}+ay=e^{-bt}$ with the initial condition $y(0)=0$. Then the Laplace transform $Y(s)$ of the solution $y(t)$ is

[NET/JRF(DEC-2017)]

A. $\frac{1}{(s+a)(s+b)}$ **B.** $\frac{1}{b(s+a)}$ **C.** $\frac{1}{a(s+b)}$ **D.** $\frac{e^{-a}-e^{-b}}{b-a}$

Solution:

$$\text{Given } \frac{dy}{dt}+ay=e^{-bt}$$

Taking Laplace transform of both sides

$$\begin{aligned} &\text{We obtain}\\ L\left\{\frac{dy}{dt}\right\}+aL\{y(t)\}=L\left\{e^{-bt}\right\} &\Rightarrow sY(s)-y(0)+aY(s)=\frac{1}{s+b}\\ &\text{Since, } y(0)=0, \text{ we obtain}\\ &(s+a)Y(s)=\frac{1}{s+b}\end{aligned}$$

$$Y(s) = \frac{1}{(s+a)(s+b)}$$

So the correct answer is **Option (A)**

5. If $f(x) = \begin{cases} 0 & \text{for } x < 3, \\ x-3 & \text{for } x \geq 3 \end{cases}$ then the Laplace transform of $f(x)$ is

[GATE 2010]

A. $s^{-2}e^{3s}$ **B.** $s^{2}e^{3s}$ **C.** s^{-2} **D.** $s^{-2}e^{-3s}$

Solution:

$$\begin{aligned} L\{f(x)\} &= \int_0^\infty e^{-sx} f(x)dx \\ &= \int_0^3 e^{-sx} f(x)dx + \int_3^\infty e^{-sx} f(x)dx \\ &= \int_3^\infty (x-3)e^{-sx}dx \\ L\{f(x)\} &= (x-3)\frac{e^{-sx}}{-s}\Big|_3^\infty - \int_3^\infty 1\cdot\left(\frac{e^{-sx}}{-s}\right)dx \\ &= 0 + \frac{1}{s}\int_3^\infty e^{-sx}dx \\ &= \frac{1}{s}\left[\frac{e^{-sx}}{-s}\right]_3^\infty \\ &= s^{-2}e^{-3s} \end{aligned}$$

So the correct answer is **Option (D)**

6. The Laplace transform of $\frac{(\sin(at) - at\cos(at))}{(2a^3)}$ is

[JEST 2018]

A. $\frac{2as}{(s^2+a^2)^2}$ **B.** $\frac{s^2-a^2}{(s^2+a^2)^2}$ **C.** $\frac{1}{(s+a)^2}$ **D.** $\frac{1}{(s^2+a^2)^2}$

Solution:

$$\begin{aligned} Lf(t) &= L\left\{\frac{1}{2a^3}(\sin at - at\cos at)\right\} \\ &= \frac{1}{2a^3}(L\{\sin at\} - aL\{t\cos at\}) \\ &= \frac{1}{2a^3}\left(\frac{a}{s^2+a^2} - a\left(-\frac{d}{ds}\frac{s}{s^2+a^2}\right)\right) \\ &= \frac{1}{2a^2}\left(\frac{1}{s^2+a^2} + \left(\frac{a^2-s^2}{s^2+a^2}\right)\right) \\ &= \frac{1}{2a^2}\left(\frac{s^2+a^2}{(s^2+a^2)^2} + \left(\frac{a^2-s^2}{s^2+a^2}\right)\right) \\ &= \frac{1}{2a^2}\left(\frac{2a^2}{(s^2+a^2)^2}\right) \\ &= \frac{1}{(s^2+a^2)^2} \end{aligned}$$

So the correct answer is **Option(D)**

7. The Laplace transformation of $e^{-2t}\sin 4t$ is

[JEST 2014]

A. $\frac{4}{s^2+4s+25}$ **B.** $\frac{4}{s^2+4s+20}$ **C.** $\frac{4s}{s^2+4s+20}$ **D.** $\frac{4s}{2s^2+4s+20}$

Solution:

$$\because L\left[e^{-at}\sin bt\right] = \frac{b}{(s+a)^2+b^2}$$
$$\Rightarrow L\left[e^{-2t}\sin 4t\right] = \frac{4}{(s+2)^2+4^2}$$
$$= \frac{4}{s^2+4s+20}$$

Correct option is (**Option C**)

Answer key			
Q.No.	Answer	Q.No.	Answer
1	**C**	2	**C**
3	**B**	4	**A**
5	**D**	6	**D**
7	**C**		