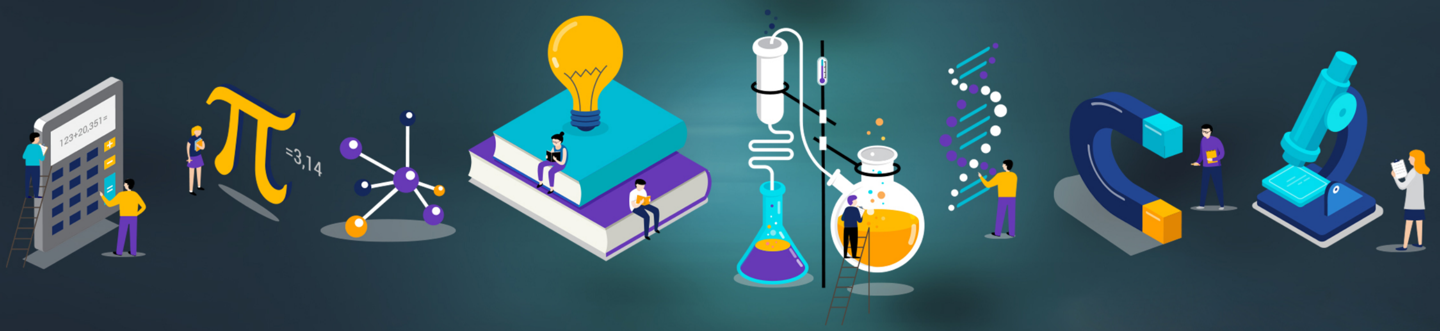


1. Power series Solution and Special functions

1.1 Series Solution Method

Series expansion is a method of obtaining one solution of the linear, second-order, homogeneous ODE. The method, will always work, provided the point of expansion is no worse than a regular singular point.In physics this very gentle condition is almost always satisfied. A linear, second-order, homogeneous ODE can be written in the form

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0 \tag{1.1}$$

The most general solution of the equation 1.1 may be written as,

$$y(x)=c_1y_1(x)+c_2y_2(x) \tag{1.2}$$

But a physical problem may lead to a nonhomogeneous, linear, second-order ODE

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=F(x) \tag{1.3}$$

Hence the most general solution to the equation will be of the form,

$$y(x)=c_1y_1(x)+c_2y_2(x)+y_p(x) \tag{1.4}$$

The constants c_1 and c_2 will eventually be fixed by boundary conditions.

There are two series solution method for differential equation,

1. **Simple series expansion method**
2. **Frobenious Method**

1.1.1 Simple Power Series Expansion Method

The simple series expansion method works for differential equations whose solutions are well-behaved at the expansion point $x=0$. This method can be illustrated by Linear classical oscillator problem

1.1.2 Classical Linear Oscillator

$$\frac{d^2y}{dx^2} + \omega^2 y = 0 \tag{1.5}$$

$$\text{with known solutions } y = \sin\omega x, \cos\omega x \tag{1.6}$$

$$\text{We try } y(x) = x^k\left(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\right) \tag{1.7}$$

$$= \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}, \quad a_0 \neq 0 \tag{1.8}$$

with the exponent k and all the coefficients a_λ still undetermined. Note that k need not be an integer. By differentiating twice, we obtain

$$\frac{dy}{dx} = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)x^{k+\lambda-1} \tag{1.9}$$

$$\frac{d^2y}{dx^2} = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda-2} \tag{1.10}$$

By substituting into equation.1.5, we have

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda-2} + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0 \tag{1.11}$$

The coefficients of each power of x on the left-hand side of equation.1.11 must vanish individually.The lowest power of x appearing in equation.1.11 is x^{k-2}, for $\lambda = 0$ in the first summation. The requirement that the coefficient vanish yields,

$$a_0k(k-1) = 0 \tag{1.12}$$

We had chosen a_0 as the coefficient of the lowest nonvanishing terms of the series 1.8, hence, by definition, $a_0 \neq 0$. Therefore we have,

$$k(k-1) = 0 \tag{1.13}$$

This equation, coming from the coefficient of the lowest power of x, we call the indicial equation. The indicial equation and its roots are of critical importance to our analysis.
From equation.1.13, $k = 0$ or $k = 1$
The only way a power series can be zero is, it's coefficients must be equal to zero. But here the power of x in the equation do not match up. The Coefficent of x in the first term is,$k+\lambda-2$ and for the second term it is,$k+\lambda$, to make them equal, we can replace λ by $\lambda+2$ in the first term. Then we get,

$$\sum_{\lambda=2}^{\infty} a_{\lambda+2}(k+\lambda+2)(k+\lambda+1)x^{k+\lambda} + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0 \tag{1.14}$$

$$\sum_{\lambda=2}^{\infty} a_{\lambda+2}(k+\lambda+2)(k+\lambda+1) + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda = 0 \tag{1.15}$$

Here the coefficients are independent summations and λ is a dummy index. Then we get,

$$a_{\lambda+2}(k+\lambda+2)(k+\lambda+1) + \omega^2 a_\lambda = 0 \tag{1.16}$$

$$a_{\lambda+2} = -a_\lambda \frac{\omega^2}{(k+\lambda+2)(k+\lambda+1)} \tag{1.17}$$

For this example, if we start with a_0, Equation.1.17 leads to the even coefficients a_2, a_4, and so on, and ignores a_1, a_3, a_5, and so on. Since a_1 is arbitrary if $k = 0$ and necessarily zero if $k = 1$,

$$a_3 = a_5 = a_7 = \cdots = 0$$

and all the odd-numbered coefficients vanish. The odd powers of x will actually reappear when the second root of the indicial equation is used.

$$a_{\lambda+2} = -a_\lambda \frac{\omega^2}{(\lambda+2)(\lambda+1)} \tag{1.18}$$

which leads to

$$a_2 = -a_0 \frac{\omega^2}{1 \cdot 2} = -\frac{\omega^2}{2!} a_0 \tag{1.19}$$

$$a_4 = -a_2 \frac{\omega^2}{3 \cdot 4} = +\frac{\omega^4}{4!} a_0 \tag{1.20}$$

$$a_6 = -a_4 \frac{\omega^2}{5 \cdot 6} = -\frac{\omega^6}{6!} a_0, \quad \text{and so on.} \tag{1.21}$$

By inspection (and mathematical induction),

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!} a_0 \tag{1.22}$$

and our solution is

$$y(x)_{k=0} = a_0 \left[1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} - \frac{(\omega x)^6}{6!} + \cdots \right] \tag{1.23}$$

$$= a_0 \cos \omega x \tag{1.24}$$

If we choose the indicial equation root $k = 1$ Equation.1.17, the recurrence relation becomes

$$a_{j+2} = -a_j \frac{\omega^2}{(j+3)(j+2)} \tag{1.25}$$

Substituting in $j = 0, 2, 4$, successively, we obtain

$$a_2 = -a_0 \frac{\omega^2}{2 \cdot 3} = -\frac{\omega^2}{3!} a_0 \tag{1.26}$$

$$a_4 = -a_2 \frac{\omega^2}{4 \cdot 5} = +\frac{\omega^4}{5!} a_0 \tag{1.27}$$

$$a_6 = -a_4 \frac{\omega^2}{6 \cdot 7} = -\frac{\omega^6}{7!} a_0, \quad \text{and so on.} \tag{1.28}$$

Again, by inspection and mathematical induction,

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n+1)!} a_0 \tag{1.29}$$

For this choice, $k = 1$, we obtain

$$y(x)_{k=1} = a_0 x \left[1 - \frac{(\omega x)^2}{3!} + \frac{(\omega x)^4}{5!} - \frac{(\omega x)^6}{7!} + \cdots \right] \tag{1.30}$$

$$= \frac{a_0}{\omega} \left[(\omega x) - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} - \frac{(\omega x)^7}{7!} + \cdots \right] \tag{1.31}$$

$$= \frac{a_0}{\omega} \sin \omega x \tag{1.32}$$

Power Series Solution (About an Ordinary Point)

Find the power series solution of $\left(1-x^2\right)y''-2xy'+2y=0$ about $x=0$

Since $x=0$ is an ordinary point of the given differential equation, the solution can be written as

$$
\begin{aligned}
y &= \sum_{k=0}^{\infty} a_k x^k \\
\frac{dy}{dx} &= \sum_{k=0}^{\infty} k a_k x^{k-1} \\
\frac{d^2y}{dx^2} &= \sum_{k=0}^{\infty} a_k k(k-1)x^{k-2}
\end{aligned}
$$

Substituting these values in the given equation we get,

$$
\begin{aligned}
(1-x^2)\sum_k a_k k(k-1)x^{k-2} - 2x\sum_k a_k(k)x^{k-1} + 2\sum_k a_k x^k = 0 \\
\sum_{k=2} a_k k(k-1)x^{k-2} - \sum\left(k^2+k-2\right)a_k x^k = 0
\end{aligned}
$$

now equating the coefficient of x^k then

$$
\begin{aligned}
(k+2)(k+1)a_{k+2} - \left(k^2+k-2\right)a_k &= 0 \\
a_{k+2} &= \frac{k-1}{(k+1)}a_k \\
\text{For } k=0 &\Rightarrow a_2 = -a_0 \\
k=1 &\Rightarrow a_3 = 0 \\
k=2 &\Rightarrow a_4 = \frac{a_2}{3} = \frac{-a_0}{3} \\
k=3 &\Rightarrow a_5 = \frac{2}{4}a_3 = 0 \\
\text{Therefore, solution } y &= a_0 + a_1 x + a_2 x^2 + \ldots\ldots \\
&= a_0\left[1 - x^2 - \frac{x^4}{3}\ldots\ldots\right] + a_1 x
\end{aligned}
$$

1.1.3 Frobenious Method

Even though the simple power series expansion method works for many functions there are some whose behaviour precludes the simple series method like the Bessel's function. The need of Frobenious method lies under the fact that, **any functions involving negative or fractional powers would not be amenable to power series solution method**. The Frobenious method extends the simple power series solution method to include negative and fractional powers, and it also allows a natural extension involving logarithm terms.
The basic idea of the Frobenius method is to look for solutions of the form

$$
\begin{aligned}
y(x) &= a_0 x^{\lambda} + a_1 x^{\lambda+1} + a_2 x^{\lambda+2} + a_3 x^{\lambda+3} + \ldots \\
&= x^{\lambda}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\right) \\
&= x^{\lambda}\sum_{k=0}^{\infty} a_k x^k \\
&= \sum_{k=0}^{\infty} a_k x^{k+\lambda}
\end{aligned}
$$

The extension of the simple power series method is all in the factor x^{λ}. The power c must now be determined, as well as the coefficients a_k. Since λ may be negative, positive, and possibly non-integral, this extends considerably the range of functions which may be treated. Note that a_0 is the lowest non-zero coefficient, so by definition it cannot be zero.

1.1.4 Bessel Function

Problem Set -3

1. Let $p_n(x)$ (where $n = 0, 1, 2, \ldots\ldots$) be a polynomial of degree n with real coefficients, defined in the interval $2 \le n \le 4$. If $\int_2^4 p_n(x) p_m(x) dx = \delta_{nm}$, then

[NET/JRF(JUNE-2011)]

A. $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{\frac{3}{2}}(-3-x)$ **B.** $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{3}(3+x)$

C. $p_0(x) = \frac{1}{2}$ and $p_1(x) = \sqrt{\frac{3}{2}}(3-x)$ **D.** $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{\frac{3}{2}}(3-x)$

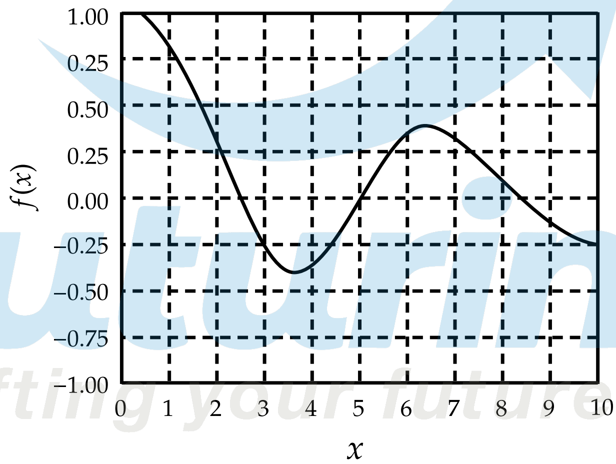
2. The generating function $F(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n$ for the Legendre polynomials $P_n(x)$ is $F(x,t) = \left(1 - 2xt + t^2\right)^{-1/2}$. The value of $P_3(-1)$ is

[NET/JRF(DEC-2011)]

A. $5/2$ **B.** $3/2$ **C.** $+1$ **D.** -1

3. The graph of the function $f(x)$ shown below is best described by

[NET/JRF(DEC-2012)]



A. The Bessel function $J_0(x)$ **B.** $\cos x$

C. $e^{-x} \cos x$ **D.** $\frac{1}{x} \cos x$

4. Given that $\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{-t^2+2tx}$ the value of $H_4(0)$ is

[NET/JRF(JUNE-2013)]

A. 12 **B.** 6 **C.** 24 **D.** -6

5. Given $\sum_{n=0}^{\infty} P_n(x) t^n = \left(1 - 2xt + t^2\right)^{-1/2}$, for $|t| < 1$, the value of $P_5(-1)$ is

[NET/JRF(JUNE-2014)]

A. 0.26 **B.** 1 **C.** 0.5 **D.** -1

6. The function $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1}$, satisfies the differential equation

[NET/JRF(DEC-2014)]

A. $x^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} + \left(x^2 + 1\right) f = 0$ **B.** $x^2 \frac{d^2 f}{dx^2} + 2x \frac{df}{dx} + \left(x^2 - 1\right) f = 0$

C. $x^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} + \left(x^2 - 1\right) f = 0$ **D.** $x^2 \frac{d^2 f}{dx^2} - x \frac{df}{dx} + \left(x^2 - 1\right) f = 0$

7. The Hermite polynomial $H_n(x)$, satisfies the differential equation

$$\frac{d^2H_n}{dx^2} - 2x\frac{dH_n}{dx} + 2nH_n(x) = 0$$

The corresponding generating function $G(t,x) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n$, satisfies the equation

[NET/JRF(DEC-2015)]

A. $\frac{\partial^2 G}{\partial x^2} - 2x\frac{\partial G}{\partial x} + 2t\frac{\partial G}{\partial t} = 0$

B. $\frac{\partial^2 G}{\partial x^2} - 2x\frac{\partial G}{\partial x} - 2t^2\frac{\partial G}{\partial t} = 0$

C. $\frac{\partial^2 G}{\partial x^2} - 2x\frac{\partial G}{\partial x} + 2\frac{\partial G}{\partial t} = 0$

D. $\frac{\partial^2 G}{\partial x^2} - 2x\frac{\partial G}{\partial x} + 2\frac{\partial^2 G}{\partial x \partial t} = 0$

8. A stable asymptotic solution of the equation $x_{n+1} = 1 + \frac{3}{1+x_n}$ is $x = 2$. If we take $x_n = 2 + \varepsilon_n$ and $x_{n+1} = 2 + \varepsilon_{n+1}$, where ε_n and ε_{n+1} are both small, the ratio $\frac{\varepsilon_{n+1}}{\varepsilon_n}$ is approximately

[NET/JRF(DEC-2016)]

A. $-\frac{1}{2}$ **B.** $-\frac{1}{4}$ **C.** $-\frac{1}{3}$ **D.** $-\frac{2}{3}$

9. The Green's function satisfying

$$\frac{d^2}{dx^2} g(x, x_0) = \delta(x - x_0)$$

with the boundary conditions $g(-L, x_0) = 0 = g(L, x_0)$, is

[NET/JRF(JUNE-2017)]

A. $\begin{cases} \frac{1}{2L}(x_0 - L)(x + L), & -L \leq x < x_0 \\ \frac{1}{2L}(x_0 + L)(x - L), & x_0 \leq x \leq L \end{cases}$

B. $\begin{cases} \frac{1}{2L}(x_0 + L)(x + L), & -L \leq x < x_0 \\ \frac{1}{2L}(x_0 - L)(x - L), & x_0 \leq x \leq L \end{cases}$

C. $\begin{cases} \frac{1}{2L}(L - x_0)(x + L), & -L \leq x < x_0 \\ \frac{1}{2L}(x_0 + L)(L - x), & x_0 \leq x \leq L \end{cases}$

D. $\frac{1}{2L}(x - L)(x + L), \quad -L \leq x \leq L$

10. The generating function $G(t,x)$ for the Legendre polynomials $P_n(t)$ is

$$G(t,x) = \frac{1}{\sqrt{1 - 2xt + x^2}} = \sum_{n=0}^{\infty} x^n P_n(t), \text{ for } |x| < 1$$

If the function $f(x)$ is defined by the integral equation $\int_0^x f(x')\,dx' = xG(1,x)$, it can be expressed as

[NET/JRF(DEC-2017)]

A. $\sum_{n,m=0}^{\infty} x^{n+m} P_n(1) P_m\left(\frac{1}{2}\right)$

B. $\sum_{n,m=0}^{\infty} x^{n+m} P_n(1) P_m(1)$

C. $\sum_{n,m=0}^{\infty} x^{n-m} P_n(1) P_m(1)$

D. $\sum_{n,m=0}^{\infty} x^{n-m} P_n(0) P_m(1)$

11. In the function $P_n(x)e^{-x^2}$ of a real variable $x, P_n(x)$ is polynomial of degree n. The maximum number of extrema that this function can have is

[NET/JRF(JUNE-2018)]

A. $n+2$ **B.** $n-1$ **C.** $n+1$ **D.** n

12. The Green's function $G(x,x')$ for the equation $\frac{d^2y(x)}{dx^2} + y(x) = f(x)$, with the boundary values $y(0) = y\left(\frac{\pi}{2}\right) = 0$, is

[NET/JRF(JUNE-2018)]

A. $G(x,x') = \begin{cases} x\left(x' - \frac{\pi}{2}\right), & 0 < x < x' < \frac{\pi}{2} \\ \left(x - \frac{\pi}{2}\right)x', & 0 < x' < x < \frac{\pi}{2} \end{cases}$

B. $G(x,x') = \begin{cases} -\cos x' \sin x, & 0 < x < x' < \frac{\pi}{2} \\ -\sin x' \cos x, & 0 < x' < x < \frac{\pi}{2} \end{cases}$

C. $G(x,x') = \begin{cases} \cos x' \sin x, & 0 < x < x' < \frac{\pi}{2} \\ \sin x' \cos x, & 0 < x' < x < \frac{\pi}{2} \end{cases}$

D. $G(x,x') = \begin{cases} x\left(\frac{\pi}{2} - x'\right), & 0 < x < x' < \frac{\pi}{2} \\ x'\left(\frac{\pi}{2} - x\right), & 0 < x' < x < \frac{\pi}{2} \end{cases}$

13. The polynomial $f(x) = 1 + 5x + 3x^2$ is written as linear combination of the Legendre polynomials $\left(P_0(x) = 1, P_1(x), P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)\right)$ as $f(x) = \sum_n c_n P_n(x)$. The value of c_0 is

[NET/JRF(DEC-2018)]

A. $\frac{1}{4}$ **B.** $\frac{1}{2}$ **C.** 2 **D.** 4

14. The Green's function $G(x,x')$ for the equation $\frac{d^2y(x)}{dx^2} = f(x)$, with the boundary values $y(0) = 0$ and $y(1) = 0$, is

[NET/JRF(DEC-2018)]

A. $G(x,x') = \begin{cases} \frac{1}{2}x(1-x'), & 0 < x < x' < 1 \\ \frac{1}{2}x'(1-x) & 0 < x' < x < 1 \end{cases}$

B. $G(x,x') = \begin{cases} x(x'-1), & 0 < x < x' < 1 \\ x'(1-x) & 0 < x' < x < 1 \end{cases}$

C. $G(x,x') = \begin{cases} -\frac{1}{2}x(1-x'), & 0 < x < x' < 1 \\ \frac{1}{2}x'(1-x) & 0 < x' < x < 1 \end{cases}$

D. $G(x,x') = \begin{cases} x(x'-1), & 0 < x < x' < 1 \\ x'(x-1) & 0 < x' < x < 1 \end{cases}$

15. The Green's function for the differential equation $\frac{d^2x}{dt^2} + x = f(t)$, satisfying the initial conditions $x(0) = \frac{dx}{dt}(0) = 0$ is

$$G(t,\tau) = \begin{cases} 0 & \text{for} \quad 0 < t < \tau \\ \sin(t-\tau) & \text{for} \quad t > \tau \end{cases}$$

The solution of the differential equation when the source $f(t) = \theta(t)$ (the Heaviside step function) is

[NET/JRF(JUNE-2020)]

A. $\sin t$ **B.** $1 - \sin t$ **C.** $1 - \cos t$ **D.** $\cos^2 t - 1$

Answer key

Q.No.	Answer	Q.No.	Answer
1	**D**	2	**D**
3	**A**	4	**A**
5	**D**	6	**C**
7	**A**	8	**C**
9	**A**	10	**B**
11	**C**	12	**B**
13	**C**	14	**D**
15	**C**		