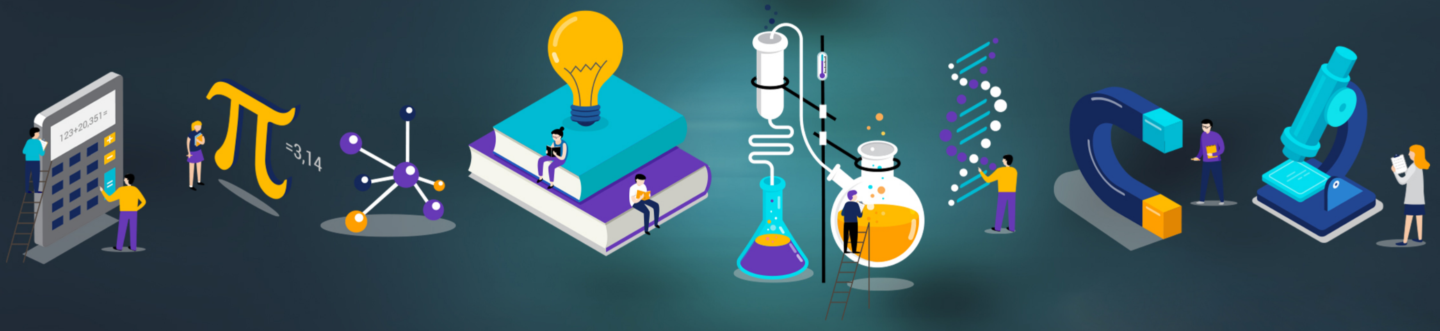
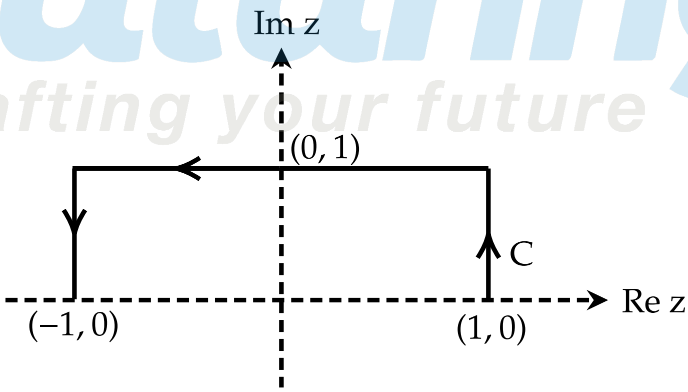


1. Complex Problem Set

Practise Set-1

1. The value of the integral $\int_C dz z^2 e^z$, where C is an open contour in the complex z-plane as shown in the figure below, is:

[NET/JRF(JUNE-2011)]



A. $\frac{5}{e}+e$ **B.** $e-\frac{5}{e}$ **C.** $\frac{5}{e}-e$ **D.** $-\frac{5}{e}-e$

Solution:

If we complete the contour, then by Cauchy integral theorem

$$\int_{-1}^{1} dz z^2 e^z + \int_C dz z^2 e^z = 0 \Rightarrow \int_C dz z^2 e^z = -\int_{-1}^{1} dz z^2 e^z$$
$$= -\left[z^2 e^z - 2ze^2 + 2e^2\right]_{-1}^{1} = \frac{5}{e} - e$$

So the correct answer is **Option (C)**

2. Which of the following is an analytic function of the complex variable $z = x + iy$ in the domain $|z| < 2$? **[NET/JRF(JUNE-2011)]**

A. $(3+x-iy)^7$
B. $(1+x+iy)^4(7-x-iy)^3$
C. $(1-x-iy)^4(7-x+iy)^3$
D. $(x+iy-1)^{1/2}$

Solution: Put $z = x+iy$. If $\bar{z} = x-iy$ appears in any of the expressions then that expression is non-analytic. For option (D) we have a branch point singularity as the power is $\frac{1}{2}$ which is fractional. Hence only option (B) is analytic.

So the correct answer is **Option (B)**

3. The first few terms in the Laurent series for $\frac{1}{(z-1)(z-2)}$ in the region $1 \le |z| \le 2$ and around $z = 1$ is **[NET/JRF(JUNE-2012)]**

A. $\frac{1}{2}\left[1+z+z^2+\ldots\right]\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\ldots.\right]$
B. $\frac{1}{1-z} - z - (1-z)^2 + (1-z)^3 + \ldots.$
C. $\frac{1}{z^2}\left[1+\frac{1}{z}+\frac{1}{z^2}+\ldots.\right]\left[1+\frac{2}{z}+\frac{4}{z^2}+\ldots..\right]$
D. $2(z-1)+5(z-1)^2+7(z-1)^3+\ldots$

Solution:

$$\begin{aligned}
\frac{1}{(z-1)(z-2)} &= \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{1-z} + \frac{1}{(z-1)-1} \\
&= \frac{1}{1-z} - (1+(1-z))^{-1} \\
&= \frac{1}{1-z} - \left[1+(1-z)+\frac{(-1)(-2)}{2!}(1-z)^2+\frac{(-1)(-2)(-3)}{3!}(1-z)^3\ldots\right] \\
&= \frac{1}{1-z} - \left[z+(1-z)^2-(1-z)^3+\ldots..\right]
\end{aligned}$$

So the correct answer is **Option (B)**

4. Let $u(x,y) = x + \frac{1}{2}\left(x^2-y^2\right)$ be the real part of analytic function $f(z)$ of the complex variable $z = x+iy$. The imaginary part of $f(z)$ is

[NET/JRF(JUNE-2012)]

A. $y+xy$
B. xy
C. y
D. y^2-x^2

Solution:

$$u(x,y) = x + \frac{1}{2}\left(x^2-y^2\right), v(x,y) = ?$$

$$\text{Check } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial y} = 1+x,$$

$$v = y + xy + f(x)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial v}{\partial x} = +y,$$

$$v = yx + f(y)$$
$$y + xy + f(x) = yx + f(y)$$
$$\text{If } f(x) = 0 \qquad f(y) = y$$
$$v = xy + y$$

So the correct answer is **Option (A)**

5. The value of the integral $\int_C \frac{z^3 dz}{(z^2-5z+6)}$, where C is a closed contour defined by the equation $2|z| - 5 = 0$, traversed in the anti-clockwise direction, is

[NET/JRF(DEC-2012)]

A. $-16\pi i$ **B.** 16πi **C.** $8\pi i$ **D.** $2\pi i$

Solution:

$$z^2 - 5z + 6 = 0 \Rightarrow z^2 - 2z - 3z + 6$$
$$= 0 \Rightarrow z(z-2) - 3(z-2) = 0 \Rightarrow z = 3, 2$$
$$2|z| = 5 \Rightarrow |z| = 2.5, \text{ only 2 will be inside.}$$
$$\text{Residue } = (z-2)\frac{z^3}{(z-3)(z-2)}\bigg|_{z=2} = \frac{8}{2-3}$$
$$= -8 \Rightarrow \int \frac{z^3 dz}{z^2 - 5z + 6} = 2\pi i(-8) = -16\pi i$$

So the correct answer is **Option (A)**

6. With $z = x + iy$, which of the following functions $f(x,y)$ is NOT a (complex) analytic function of z ?

[NET/JRF(JUNE-2013)]

A. $f(x,y) = (x+iy-8)^3\left(4+x^2-y^2+2ixy\right)^7$

B. $f(x,y) = (x+iy)^7(1-x-iy)^3$

C. $f(x,y) = \left(x^2-y^2+2ixy-3\right)^5$

D. $f(x,y) = (1-x+iy)^4(2+x+iy)^6$

Solution:

$$f(x,y) = (1-x+iy)^4(2+x+iy)^6$$
$$= \{1-(x-iy)\}^4(2+x+iy)^6$$
$$\text{Due to present of } \bar{z} = (x-iy)$$

So the correct answer is **Option (D)**

7. Which of the following functions cannot be the real part of a complex analytic function of $z = x + iy$?

[NET/JRF(DEC-2013)]

A. x^2y **B.** $x^2 - y^2$ **C.** $x^3 - 3xy^2$ **D.** $3x^2y - y - y^3$

Solution:

Let x^2y be real part of a complex function. Use Milne Thomson's method to write analytic complex function. The real part of that function should be (1) but that is not the case. So this cannot be real part of an analytic function. Also,

$$\begin{aligned} z^2 &= (x+iy)^2 = x^2 - y^2 + 2ixy, \text{ Real part option (2)} \\ z^3 &= (x+iy)^3 = x^3 - iy^3 + 3ixy(x+iy) \\ &= x^3 - iy^3 + 3ix^2y - 3xy^2, \text{ Real part option (3)} \end{aligned}$$

So the correct answer is **Option (A)**

8. Given that the integral $\int_0^\infty \frac{dx}{y^2+x^2} = \frac{\pi}{2y}$, the value of $\int_0^\infty \frac{dx}{(y^2+x^2)^2}$ is

[NET/JRF(DEC-2013)]

A. $\frac{\pi}{y^3}$ **B.** $\frac{\pi}{4y^3}$ **C.** $\frac{\pi}{8y^3}$ **D.** $\frac{\pi}{2y^3}$

Solution:

$$\int_0^\infty \frac{dx}{(y^2+x^2)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(y^2+x^2)^2}, \text{ pole is of } 2^{\text{nd}} \text{ order at } x = iy, \text{ residue } = 1/\left(4iy^3\right)$$

$$\text{Integral } = \left(\frac{1}{2}\right)(2\pi i)\frac{1}{4iy^3} = \frac{\pi}{(4y^3)}$$

9. If C is the contour defined by $|z| = \frac{1}{2}$, the value of the integral

$$\oint_C \frac{dz}{\sin^2 z}$$

is

[NET/JRF(JUNE-2014)]

A. ∞ **B.** $2\pi i$ **C.** 0 **D.** πi

Solution:

$$f(z) = \frac{1}{\sin^2 z} \quad \left(|z| = \frac{1}{2}\right)$$

$$\sin z = z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5}\ldots \Rightarrow \frac{1}{\sin^2 z} = \frac{1}{\left(z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5}\cdots\right)^2}$$

$$\Rightarrow \frac{1}{\sin^2 z} = \frac{1}{z^2}\left[1 - \frac{z^2}{\lfloor 3} + \frac{z^4}{\lfloor 5}\ldots\right]^{-2} \Rightarrow \oint_C \frac{dz}{\sin^2 z} = 0$$

So the correct answer is **Option (C)**

10. The principal value of the integral $\int_{-\infty}^\infty \frac{\sin(2x)}{x^3}dx$ is

[NET/JRF(DEC-2014)]

A. -2π **B.** $-\pi$ **C.** π **D.** 2π

Solution:

$$\text{Let } f(z) = \frac{e^{i2z}}{z^3}$$

$$\lim_{2\to 0}(z-0)^3 f(z)=\lim_{z\to 0}(z-0)^3\frac{e^{i2z}}{z^3}$$
$$=1(\text{ finite and } \neq 0)\Rightarrow z=0\text{is pole of order } 3.$$
$$\text{Residue } R=\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left[(z-0)^3\frac{e^{i2z}}{z^3}\right]=-2$$
$$\Rightarrow\int_{-\infty}^{\infty} f(x)dx=\pi i\Sigma R=\pi i(-2)=-2\pi i\Rightarrow \text{Im. Part}$$
$$=-2\pi\Rightarrow\int_{-\infty}^{\infty} f(x)dx=-2\pi$$

So the correct answer is **Option (A)**

11. The Laurent series expansion of the function $f(z)=e^2+e^{1/2}$ about $z=0$ is given by

[NET/JRF(DEC-2014)]

A. $\sum_{n=-\infty}^{\infty}\frac{z^n}{n!}$ for all $|z|<\infty$

B. $\sum_{n=0}^{\infty}\left(z^n+\frac{1}{z^n}\right)\frac{1}{n!}$ only if $0<|z|<1$

C. $\sum_{n=0}^{\infty}\left(z^n+\frac{1}{z^n}\right)\frac{1}{n!}$ for all $0<|z|<\infty$

D. $\sum_{n=-\infty}^{\infty}\frac{z^n}{n!}$ only if $|z|<1$

Solution:

$$e^z=\left(1+z+\frac{z^2}{2!}+\ldots\right)=\sum_{n=0}^{\infty}\frac{z^n}{n!}\text{ and } e^{1/z}$$
$$=1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+\ldots.=\sum_{n=0}^{\infty}\frac{1}{z^n n!}$$
$$\Rightarrow f(z)=\left(e^z+e^{1/2}\right)=\sum_{n=0}^{\infty}\left(z^n+\frac{1}{z^n}\right)\frac{1}{n!},\text{ for all } 0<|z|<\infty$$

So the correct answer is **Option (C)**

12. Consider the function $f(z)=\frac{1}{z}\ln(1-z)$ of a complex variable $z=re^{i\theta}(r\geq 0,\quad -\infty<\theta<\infty)$. The singularities of $f(z)$ are as follows:

[NET/JRF(DEC-2014)]

A. Branch points at $z=1$ and $z=\infty$; and a pole at $z=0$ only for $0\leq\theta<2\pi$

B. Branch points at $z=1$ and $z=\infty$; and a pole at $z=0$ for all θ other than $0\leq\theta<2\pi$

C. Branch points at $z=1$ and $z=\infty$; and a pole at $z=0$ for all θ

D. Branch points at $z=0, z=1$ and $z=\infty$.

Solution:

$$\text{For } f(z)=\frac{1}{z}\ln(1-z)=\frac{1}{z}\left(-z-\frac{z^2}{2}-\frac{z^3}{3}-\ldots..\right)$$
$$=-1-\frac{z}{2}-\frac{z^2}{3}-\ldots.$$

There is no principal part and when $z\to 0, f(z)=-1$. So there is removable singularity at $z=0$. Also $z=1$ and $z=\infty$ is Branch point.

None of the above is correct

13. The value of integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$

[NET/JRF(JUNE-2015)]

A. $\frac{\pi}{\sqrt{2}}$ **B.** $\frac{\pi}{2}$ **C.** $\sqrt{2}\pi$ **D.** 2π

Solution:

$$\int_{-\infty}^{\infty} \frac{dz}{1+z^4} \quad \because |z| = R$$

$$\text{Now, pole } z = e^{(2n+1)\frac{\pi}{4}}$$

$$n = 0, \quad \Rightarrow z_0 = e^{\frac{i\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, n$$

$$= 2 \Rightarrow z_2 = \frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

$$n = 1 \Rightarrow z_1 = e^{\frac{i3\pi}{4}} = \frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, n$$

$$= 3 \Rightarrow z_3 = +\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

only z_0 and z_1 lies in contour

$$\text{i.e., residue at } \left(z = e^{\frac{i\pi}{4}}\right) = \frac{1}{4}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$\text{residue at } \left(z = e^{\frac{i3\pi}{4}}\right) = \frac{1}{4}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$\text{now } \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\pi i \Sigma \operatorname{Re} S = \frac{\pi}{\sqrt{2}}$$

So the correct answer is **Option (A)**

14. The function $\frac{Z}{\sin \pi z^2}$ of a complex variable z has

[NET/JRF(DEC-2015)]

A. A simple pole at 0 and poles of order 2 at $\pm\sqrt{n}$ for $n = 1, 2, 3\ldots$

B. A simple pole at 0 and poles of order 2 at $\pm\sqrt{n}$ and $\pm i\sqrt{n}$ for $n = 1, 2, 3\ldots$

C. Poles of order 2 at $\pm\sqrt{n}, n = 0, 1, 2, 3\ldots$

D. Poles of order 2 at $\pm n, n = 0, 1, 2, 3\ldots$

Solution:

$$f(z) = \frac{z}{\sin \pi z^2} = \frac{z}{\pi z^2 \frac{\sin \pi z^2}{\pi z^2}}$$

$$\text{at } z = 0, \text{ it is a simple pole since,} \lim_{z\to 0} \frac{\sin \pi z^2}{\pi z^2} = 1$$

$$\text{Also, } \sin \pi z^2 = \sin n\pi \Rightarrow \pi z^2$$

$$= \pm n\pi, z = \pm\sqrt{n}, \pm i\sqrt{n}$$

$$\lim_{z\to\sqrt{n}} (z-\sqrt{n})^2 \cdot \frac{z}{\sin \pi z^2}, \text{exists. So its pole of order 2}$$

So the correct answer is **Option (B)**

15. The value of the contour integral $\frac{1}{2\pi i}\oint_C \frac{e^{4z}-1}{\cosh(z)-2\sinh(z)}dz$ around the unit circle C traversed in the anti-clockwise direction, is

[NET/JRF(JUNE-2016)]

A. 0 **B.** 2 **C.** $\frac{-8}{\sqrt{3}}$ **D.** $-\tanh\left(\frac{1}{2}\right)$

Solution:

$$f(z)=\frac{e^{4z}-1}{\cosh z-2\sinh z}=\frac{e^{4z}-1}{\frac{e^2+e^{-z}}{2}-\left(e^z-e^{-z}\right)}$$

$$=\frac{e^{42}-1}{-\frac{e^z}{2}+\frac{3}{2}e^{-z}}$$

$$\Rightarrow f(z)=\frac{2e^2\left(e^{4z}-1\right)}{\left(3-e^{2z}\right)}=\frac{2\left(e^{5z}-e^z\right)}{\left(3-e^{2z}\right)}$$

For pole at $z=z_0, 3-e^{2\xi_0}=0\Rightarrow e^{2z_0}$

$$=3\Rightarrow z_0=\frac{\ln 3}{2}$$

It has simple pole at z_0

$$\operatorname{Re}\left(z_0\right)=\lim_{z\to z_0}\left(z-z_0\right)f(z)=\lim_{2\to z_0}\left(z-z_0\right)\frac{2\left(e^{5z}-e^2\right)}{3-e^{22}}$$

$$=\lim_{z\to z_0}\frac{\left(z-z_0\right)\times 2\left(5e^{5z}-e^z\right)+2\left(e^{5z}-e^z\right)\times 1}{-2e^{2z}}$$

$$=-\left(\frac{e^{5z_0}-e^{z_0}}{e^{2z_0}}\right)$$

$$=-\left(\frac{(\sqrt{3})^5-\sqrt{3}}{3}\right)=-\left(\frac{9\sqrt{3}-\sqrt{3}}{3}\right)=-\frac{8}{\sqrt{3}}$$

$$\frac{1}{2\pi i}\oint f(z)dz=\frac{1}{2\pi i}\times 2\pi i\sum \text{Residue }=-\frac{8}{\sqrt{3}}$$

So the correct answer is **Option (C)**

16. Let $u(x,y)=e^{ax}\cos(by)$ be the real part of a function $f(z)=u(x,y)+iv(x,y)$ of the complex variable $z=x+iy$, where a,b are real constants and $a\neq 0$. The function $f(z)$ is complex analytic everywhere in the complex plane if and only if

[NET/JRF(JUNE-2017)]

A. $b=0$ **B.** $b=\pm a$ **C.** $b=\pm 2\pi a$ **D.** $b=a\pm 2\pi$

Solution:

The function $f(z)$ will be analytic everywhere in the complex plane if and only if it satisfies the Cauchy Riemann equation in that region.

$$\Rightarrow \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\text{ and }\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$$\text{Hence } ae^{ax}\cos(by)=\frac{\partial v}{\partial y} \qquad \text{(i)}$$

$$\text{and } be^{ax}\sin(by)=\frac{\partial v}{\partial x} \qquad \text{(ii)}$$

From equation (i)

$$v(x,y) = \frac{ae^{ax}\sin(by)}{b} + c(y) \qquad \text{(iii)}$$

Differentiating partially with x gives

$$\frac{\partial v}{\partial x} = \frac{a^2e^{ax}\sin(by)}{b} \qquad \text{(iv)}$$

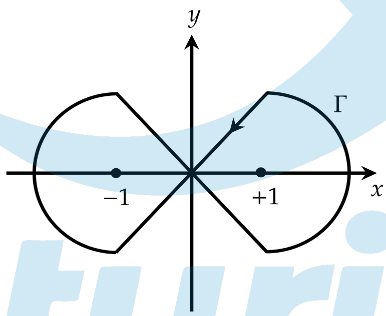
From equation (iii) and (iv)

$$be^{ax}\sin(by) = \frac{a^2e^{ax}\sin(by)}{b}$$
$$\Rightarrow b^2 = a^2 \Rightarrow b = \pm a$$

So the correct answer is **Option (B)**

17. The integral $\oint_\Gamma \frac{ze^{i\pi z/2}}{z^2-1}dz$ along the closed contour Γ shown in the figure is

[NET/JRF(JUNE-2017)]



A. 0 **B.** 2π **C.** -2π **D.** $4\pi i$

Solution:

$$f(z) = \frac{ze^{iz\pi/2}}{(z+1)(z-1)}$$

For $z = +1$ anti-clockwise

$$I = 2\pi i \lim_{z\to 1}\frac{ze^{i\pi z/2}}{(z+1)} = \frac{2\pi i}{2}e^{i\pi/2} = \pi i e^{i\pi/2}$$

For $z = -1$

$$I = -2\pi i \lim_{z\to -1}\frac{ze^{i\pi z/2}}{(z-1)} = -2\pi i \times \frac{(-1)e^{-i\pi/2}}{(-2)} = -\pi i e^{-i\pi/2}$$

$$\text{Integral } = \pi i\frac{\left(e^{i\pi/2} - e^{-i\pi/2}\right)}{2i} \times 2i = 2\pi i^2 \sin\frac{\pi}{2} = -2\pi$$

So the correct answer is **Option (C)**

18. What is the value of a for which $f(x,y) = 2x + 3\left(x^2 - y^2\right) + 2i(3xy + ay)$ is an analytic function of complex variable $z = x + iy$

[NET/JRF(JUNE-2018)]

A. 1 **B.** 0 **C.** 3 **D.** 2

Solution:

$$f(x,y)=2x+3\left(x^2-y^2\right)+2i(3xy+\alpha y)$$
$$u=2x+3\left(x^2-y^2\right), v=2(3xy+\alpha y)$$
$$\text{C-R conditions: } u_x=v_y, u_y=-v_x$$
$$2+3(2x)=2(3x+\alpha)\Rightarrow \alpha=1\Rightarrow -6y=-6y$$

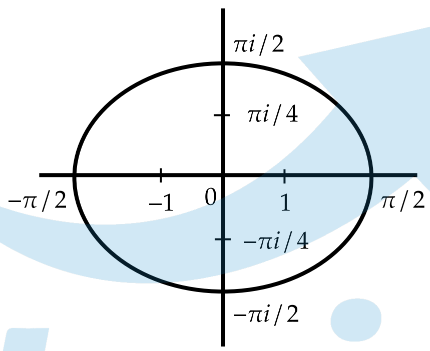
So the correct answer is **Option (A)**

19. The value of the integral $\oint_C \frac{dz}{z}\frac{\tanh 2z}{\sin \pi z}$, where C is a circle of radius $\frac{\pi}{2}$, traversed counter-clockwise, with centre at $z=0$, is

[NET/JRF(DEC-2018)]

A. 4 **B.** $4i$ **C.** $2i$ **D.** 0

Solution:



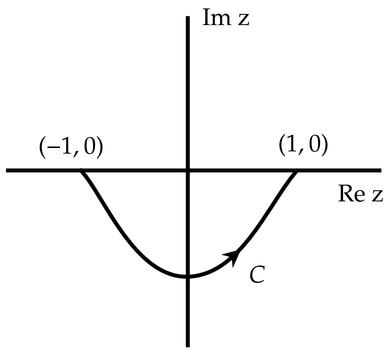
$$\oint_C \frac{dz}{z}\frac{\tanh 2z}{\sin \pi z}dz$$
$$z=0,1,-1,\frac{\pi i}{4},\frac{-\pi i}{4}$$
$$f(z)=\frac{2z-\frac{1}{3}(2z)^3+\frac{2}{15}(2z)^5\ldots.}{z\left(\pi z-\frac{\pi^3 z^3}{3!}+\ldots\right)}$$
$$\frac{2}{\pi z}\left(1-\frac{1}{2}z^2+\ldots\right)\left(1-\frac{\pi^2 z^2}{2!}+\ldots\right)$$
$$b_1=\frac{2}{\pi}$$
$$\text{As Re } z=1,\frac{\tanh^2}{-\pi} \text{ and Re} z=-1,\frac{\tanh^2}{-\pi}$$
$$\text{Re} z=\frac{i\pi}{4}=-\frac{1}{\pi}\left(2\operatorname{cosec} h\frac{\pi^2}{4}\right)$$
$$\text{Re} z=\frac{-i\pi}{4}=-\frac{1}{\pi}\left(2\operatorname{cosech}\frac{\pi^2}{4}\right)$$

$I=2\pi i\Sigma R=4i$ only when 0 lies inside, otherwise wrong question.

So the correct answer is **Option (B)**

20. The integral $I = \int_C e^z dz$ is evaluated from the point $(-1,0)$ to $(1,0)$ along the contour C, which is an arc of the parabola $y = x^2 - 1$, as shown in the figure. The value of I is

[NET/JRF(DEC-2018)]



A. 0 **B.** $2\sinh 1$ **C.** $e^{2i}\sinh 1$ **D.** $e + e^{-1}$

Solution:

$$\int_C f(z)dz = 2\pi i \Sigma R$$

$$\int_C f(z)dz + \int_1^{-1} e^x dx = 0$$

$$\int_C f(z)dz = -\int_1^{-1} e^x dx = \int_1^{-1} e^x dx$$

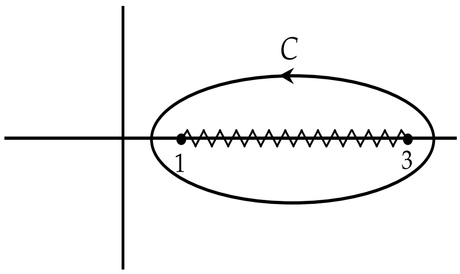
$$= \frac{(e^1 - e^{-1})}{2} \cdot 2 = 2\sinh 1$$

So the correct answer is **Option (B)**

21. The contour C of the following integral

$$\oint_C dz \frac{\sqrt{(z-1)(z-3)}}{(z^2-25)^3}$$

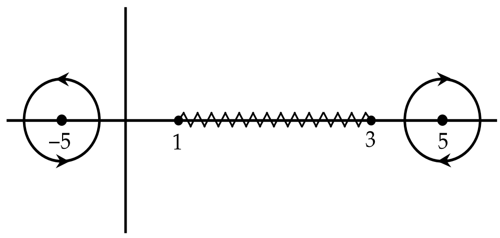
in the complex z plane is shown in the figure below.



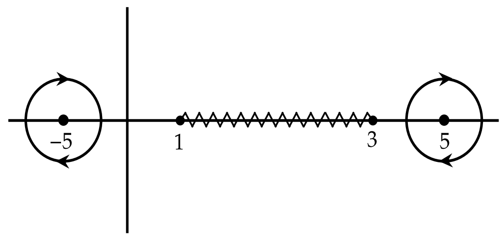
This integral is equivalent to an integral along the contours

[NET/JRF(DEC-2018)]

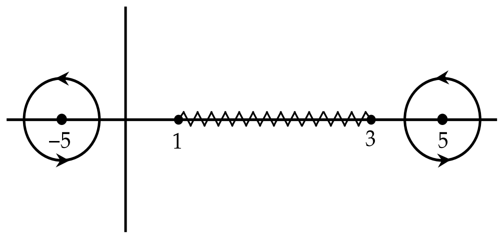
A.



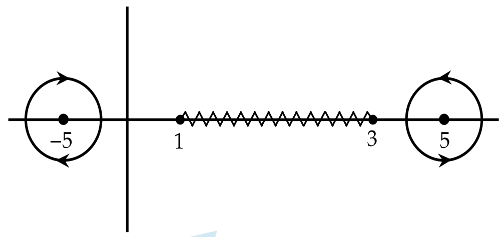
B.



C.



D.



Solution:

$z = 1, 3$ are branch points ∞ is not a branch point 1 branch cut 3

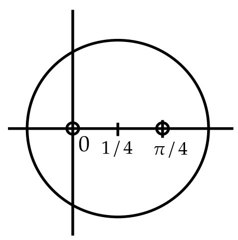
So the correct answer is **Option (C)**

22. Let C be the circle of radius $\frac{\pi}{4}$ centered at $z = \frac{1}{4}$ in the complex z-plane that is traversed counter-clockwise. The value of the contour integral $\oint_C \frac{z^2}{\sin^2 4z} dz$ is

[NET/JRF(DEC-2019)]

A. 0 **B.** $\frac{i\pi^2}{4}$ **C.** $\frac{i\pi^2}{16}$ **D.** $\frac{i\pi}{4}$

Solution:



$$f(z) = \left(\frac{\pi}{\sin 4z}\right)^2$$

$$z_0 = 0, \frac{\pi}{4} \text{ are poles}$$

$$4z = n\pi, z = 0, \frac{\pi}{4}$$

Others are outside the contour.

$$\text{Residue at } z=0 \text{ is } \left[\frac{\pi}{4z-\frac{4^3z^3}{3!}+\ldots}\right]^2$$

$$=\left[\frac{1}{4-\frac{4^3z^2}{3!}+\ldots.}\right]^2 \qquad \text{No terms for } \frac{1}{z}, b_1=0$$

$$=\left[4-\frac{4^3z^2}{3!}+\ldots.\right]^{-2}$$

$$\text{Residue for } z=\frac{\pi}{4}$$

$$z-\frac{\pi}{4}=t$$

$\sin(4t+\pi)=-\sin 4t$ (But square so no effect)

$$\left[\frac{t+\frac{\pi}{4}}{\sin 4\left(t+\frac{\pi}{4}\right)}\right]^2$$

$$\left(\frac{t+\frac{\pi}{4}}{\sin 4t}\right)^2=\frac{t^2+\frac{\pi^2}{4}+2t\cdot\frac{\pi}{4}}{\sin^2 4t}$$

$$\frac{\pi}{2}\frac{t}{16t^2[1-\ldots.]^2}=\frac{\pi}{32t}[1-\ldots.]^{-2}(\text{from first term})$$

$$b_1=\frac{\pi}{32}$$

$$\oint_C \frac{z^2}{\sin^2 4z}dz=2\pi i\left[0+\frac{\pi}{32}\right]=\frac{i\pi^2}{16}$$

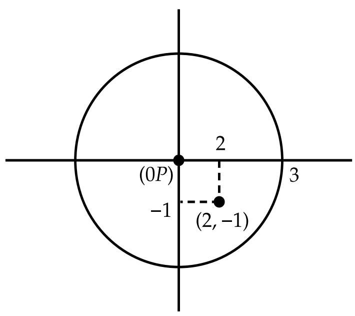
So the correct answer is **Option (C)**

23. A function of a complex variable z is defined by the integral $f(z)=\oint_\Gamma \frac{w^2-2}{w-z}dw$, where Γ is a circular contour of radius 3 , centred at origin, running counter-clockwise in the w - plane. The value of the function at $z=(2-i)$ is

[NET/JRF(JUNE-2020)]

A. 0 **B.** $1-4i$ **C.** $8\pi+2\pi\mathrm{i}$ **D.** $-\frac{2}{\pi}-\frac{i}{2\pi}$

Solution:



$$f(z)=\oint_\Gamma \frac{w^2-2}{w-z}dw$$

$$\omega = z \text{ is a simple pole.}$$

$$\text{Residue } \lim_{\omega\to z}(\omega - z)\frac{(\omega^2-2)}{(\omega-z)} = (2-i)^2 - 2$$

$$= 4 - 1 - 4i - 2 = (1-4i)$$

$$f(z) = \oint_\Gamma \frac{w^2-2}{w-z}dw = 2\pi i(1-4i) = 2\pi i + 8\pi$$

So the correct answer is **Option (C)**

Answer key			
Q.No.	Answer	Q.No.	Answer
1	**C**	2	**B**
3	**B**	4	**A**
5	**A**	6	**D**
7	**A**	8	**-**
9	**C**	10	**A**
11	**C**	12	**-**
13	**A**	14	**B**
15	**C**	16	**B**
17	**C**	18	**A**
19	**B**	20	**B**
21	**C**	22	**C**
23	**C**		

Practise Set-2

1. The value of the integral $\oint_C \frac{e^z \sin(z)}{z^2} dz$, where the contour C is the unit circle: $|z-2| = 1$, is **[GATE 2010]**

A. $2\pi i$ **B.** $4\pi i$ **C.** πi **D.** 0

Solution:

$|z-2| = 1 \Rightarrow 1 < z < 3$ i.e. the pole $z = 0$ does not lie inside the contour.

$$\therefore \quad \oint_C \frac{e^z \sin z}{z^2} dz = 2\pi i \times 0 = 0.$$

So the correct answer is **Option (D)**

2. Which of the following statements is TRUE for the function $f(z) = \frac{z \sin z}{(z-\pi)^2}$? **[GATE 2011]**

A. $f(z)$ is analytic everywhere in the complex plane

B. $f(z)$ has a zero at $z = \pi$

C. $f(z)$ has a pole of order 2 at $z = \pi$

D. $f(z)$ has a simple pole at $z = \pi$

Solution:

$$f(z) = \frac{z \sin z}{(z-\pi)^2} \text{ has a pole of order 2 at } z = \pi$$

So the correct answer is **Option (C)**

3. For the function $f(z) = \frac{16z}{(z+3)(z-1)^2}$, the residue at the pole $z = 1$ is (your answer should be an integer)——- **[GATE 2013]**

Solution:

$$\text{At } z = 1, \text{ pole is of order 2 .}$$

$$\text{So, residue is } \frac{1}{\lfloor -1} \frac{d^{2-1}}{dz^{2-1}} \left[\frac{(z-1)^2 16z}{(z+3)(z-1)^2} \right]_{z=1} = 3$$

4. The value of the integral

$$\oint_C \frac{z^2}{e^z + 1} dz$$

where C is the circle $|z| = 4$, is **[GATE 2014]**

A. $2\pi i$ **B.** $2\pi^2 i$ **C.** $4\pi^3 i$ **D.** $4\pi^2 i$

Solution:

$$\text{Pole } e^z = -1 \Rightarrow e^z = e^{i(2m+1)\pi} \text{ where } m = 0,1,2,3\ldots\ldots$$

$$\begin{aligned}\text{For } z = i\pi,\ \text{Res } &= \lim_{z=i\pi} \frac{\phi(z)}{\phi'(z)} \\ &= -\frac{\pi^2}{e^{i\pi}} = \pi^2 \\ \text{Similarly, for } z = -i\pi,\ \text{Res } &= \pi^2 \\ \therefore I = 2\pi i\left(\pi^2+\pi^2\right) &= 4\pi^3 i\end{aligned}$$

So the correct answer is **Option (C)**

5. Consider a complex function $f(z) = \frac{1}{z\left(z+\frac{1}{2}\right)\cos(z\pi)}$. Which one of the following statements is correct?

[GATE 2015]

A. $f(z)$ has simple poles at $z = 0$ and $z = -\frac{1}{2}$

B. $f(z)$ has second order pole at $z = -\frac{1}{2}$

C. $f(z)$ has infinite number of second order poles

D. $f(z)$ has all simple poles

Solution:

$$f(z) = \frac{1}{z\left(z+\frac{1}{2}\right)\cos(z\pi)}$$

$$\begin{aligned}\text{For } n^{th} \text{ order pole, Res. } &= \lim_{z\to a}(z-a)^n f(z) = \text{ finite} \\ &\text{At } z = 0, \lim_{z\to 0} zf(z) = \text{ finite} \\ &\Rightarrow z = 0 \text{ is a simple pole.} \\ &\text{At } z = -\frac{1}{2},\ \lim_{z\to -\frac{1}{2}} \frac{\left(z+\frac{1}{2}\right)^2}{z\left(z+\frac{1}{2}\right)\cos z\pi} = \lim_{z\to -\frac{1}{2}} \frac{\left(z+\frac{1}{2}\right)}{z\cos z\pi} \\ &= \lim_{z\to -\frac{1}{2}} \frac{1}{1.\cos z\pi + z.\pi(-\sin z\pi)} \\ &= \lim_{z\to -\frac{1}{2}} \frac{1}{\cos z\pi - z\pi\sin z\pi} = \frac{1}{-\frac{\pi}{2}} \\ &= -\frac{2}{\pi} = \text{ finite} \\ \Rightarrow f(z) \text{ has second order pole at } z = -\frac{1}{2}\end{aligned}$$

So the correct answer is **Option (A)**

6. Consider $w = f(z) = u(x,y) + iv(x,y)$ to be an analytic function in a domain D. Which one of the following options is NOT correct?

[GATE 2015]

A. $u(x,y)$ satisfies Laplace equation in D

B. $v(x,y)$ satisfies Laplace equation in D

C. $\int_1^{z_2} f(z)dz$ is dependent on the choice of the contour between z_1 and z_2 in D

D. $f(z)$ can be Taylor expended in D

Solution:

$w = f(z) = u(x,y) + iv(x,y)$ to be an analytic function in a domain D, $\int_{z_1}^{z_2} f(z)dz$ is independent of the choice of the contour between z_1 and z_2 in D.

So the correct answer is **Option (C)**

7. A function $y(z)$ satisfies the ordinary differential equation $y'' + \frac{1}{z}y' - \frac{m^2}{z^2}y = 0$, where $m = 0, 1, 2, 3, \ldots$. Consider the four statements P, Q, R, S as given below.
P : z^m and z^{-m} are linearly independent solutions for all values of m
Q: z^m and z^{-m} are linearly independent solutions for all values of $m > 0$
R : $\ln z$ and 1 are linearly independent solutions for $m = 0$
S: z^m and $\ln z$ are linearly independent solutions for all values of m
The correct option for the combination of valid statements is

[GATE 2015]

A. P, R and S only **B.** P and R only **C.** Q and R only **D.** R and S only

Solution:

$$y'' + \frac{1}{z}y' - \frac{m^2}{z^2}y = 0 \Rightarrow z^2y'' + zy' - m^2y$$
$$= 0, m = 0, 1, 2, 3, \ldots, \quad z = e^x, D = \frac{d}{dx}$$
$$\text{If } m = 0; \quad z^2y'' + zy' = 0, [D(D-1) + D]y$$
$$= 0 \Rightarrow \left[D^2 - D + D\right]y = 0$$
$$D^2y = 0 \Rightarrow y = c_1 + c_2x \Rightarrow y$$
$$= c_1 + c_2 \ln z \quad (R \text{ is correct})$$
$$\text{And if } m \neq 0, m > 0, \text{ then } m \neq 0, \text{ then } \left(D^2 - m^2\right)y$$
$$= 0 \Rightarrow D = \pm m$$
$$y = c_1e^{mx} + c_2e^{-mx} = c_1e^{m\log z} + c_2e^{-m\log z}$$
$$= c_1z^m + c_2z^{-m}$$
$$\text{or if } m \neq 0, m > 0, \text{ then}$$
$$y = c_1\cosh(m\log(z)) + ic_2\sinh(m\log(x)), \quad m > 0$$

So the correct answer is **Option (C)**

8. Which of the following is an analytic function of z everywhere in the complex plane?

[GATE 2016]

A. z^2 **B.** $(z^*)^2$ **C.** $|z|^2$ **D.** $\sqrt{z}$

Solution:

$$z^2 = (x+iy)^2 = x^2 - y^2 + i(2xy) \Rightarrow u$$
$$= x^2 - y^2 \text{ and } v = 2xy$$
$$\text{Cauchy Riemann equations } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y \text{ satisfies.}$$

So the correct answer is **Option (A)**

9. The contour integral $\oint \frac{dz}{1+z^2}$ evaluated along a contour going from $-\infty$ to $+\infty$ along the real axis and closed in the lower half-plane circle is equal to.............. (up to two decimal places).

[GATE 2017]

Solution:

$$\oint_c \frac{1}{1+z^2}dz = \int_{-\infty}^{+\infty} \frac{1}{1+x^2}dx + \oint_c \frac{1}{1+z^2}dz$$

Poles, $1+z^2 = 0 \Rightarrow z = \pm i, \quad z = -i$ is inside C

$$\therefore \operatorname{Res}(z=-i) = \lim_{z\to -i}(z+i)\frac{1}{(z-i)(z+i)}$$
$$= \frac{1}{-i-i} = \frac{1}{-2i}$$
$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}dx = -\frac{1}{2i} \times -2\pi i = \pi$$

(Since, here we use lower half plane i.e., we traversed in clockwise direction, hence we have to take $-2\pi i$)

10. The imaginary part of an analytic complex function is $v(x,y) = 2xy+3y$. The real part of the function is zero at the origin. The value of the real part of the function at $1+i$ is .... (up to two decimal places)

[GATE 2017]

Solution:

Solution: The imaginary part of the given analytic function is $v(x,y) = 2xy+3y$. From the Cauchy - Riemann condition

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x+3$$

Integrating partially gives

$$u(x,y) = x^2+3x+g(y)$$

From the second Cauchy - Riemann condition

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \text{ we obtain} \frac{\partial u}{\partial y}$$
$$= -2y, \mu(x,y) = -y^2+g(x)$$
$$\frac{dg(y)}{dy} = -2y \Rightarrow g(y) = -y^2+c$$
$$\text{Hence, } u(x,y) = x^2+3x-y^2+c$$

Since, the real part of the analytic function is zero at the origin.

$$\text{Hence, } 0 = 0+0-0+c \Rightarrow c = 0$$
$$\text{Thus, } u(x,y) = x^2+3x-y^2$$
$$\therefore f(z) = \left(x^2+3x-y^2\right)+i(2xy+3y)$$

Thus, the value of real part when

$$z = 1 + i, i.e. x = 1 \text{ and } y = 1 \text{ is } u(x,y)$$
$$= (1)^2 + 3(1) - 1 = 3$$

11. The absolute value of the integral

$$\int \frac{5z^3 + 3z^2}{z^2 - 4} dz$$

over the circle $|z - 1.5| = 1$ in complex plane, is … (up to two decimal places).

[GATE 2018]

Solution:

$$f(z) = \frac{5z^3 + 3z^2}{(z-2)(z+2)}$$
$$\text{Pole, } z = 2, -2$$
$$z = -2 \text{ is outside the center}$$
$$|-2 - 1.5| > 1 \text{So, will not be considered}$$
$$\text{Now, } \mathrm{Re}\,s(2) = \lim_{z \to 2}(z-2)\frac{(5z^3 + 3z^2)}{(z-2)(z+2)}$$
$$= \frac{52^3 + 32^2}{4} = \frac{40 + 12}{4} = 13$$
$$I = 2\pi i \times residue = 2\pi i \times 13$$
$$= 26 \times 3.14 \Rightarrow I = 81.64$$

12. The pole of the function $f(z) = \cot z$ at $z = 0$ is

[GATE 2019]

A. A removable pole

B. An essential singularity

C. A simple pole

D. A second order pole

Solution:

$$f(z) = \cot z \text{ at } z = 0$$
$$f(z) = \frac{1}{\tan z} \quad z = 0 \text{ is a simple pole}$$
$$f(z) = \frac{1}{z}\left[1 - \frac{1}{3}z^2 + \dots\right]$$

So the correct answer is **Option (C)**

13. The value of the integral $\int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2+a^2} dx$, where $k > 0$ and $a > 0$, is

[GATE 2019]

A. $\frac{\pi}{a} e^{-ka}$

B. $\frac{2\pi}{a} e^{-ka}$

C. $\frac{\pi}{2a} e^{-ka}$

D. $\frac{3\pi}{2a} e^{-ka}$

Solution:

$$\int_{-\infty}^{\infty} \frac{\cos kx}{x^2 + a^2} dx$$
$$f(z) = \frac{e^{ikx}}{z^2 + a^2} = \frac{e^{ikz}}{(z + ia)(z - ia)}$$

$$I = \text{Re}.2\pi i \times \frac{e^{ik(ia)}}{2ia} = \frac{\pi e^{-ka}}{a}$$

So the correct answer is **Option (A)**

14. The value of the integral $\int_0^\infty \frac{\ln x}{(x^2+1)^2} dx$ is

[JEST 2012]

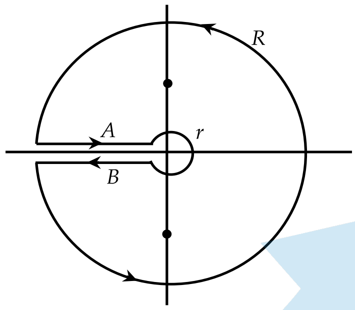
a. 0

b. $\frac{-\pi}{4}$

c. $\frac{-\pi}{2}$

d. $\frac{\pi}{2}$

Solution:



$$\int_0^\infty \frac{\ln x}{(x^2+1)^2} dx = \int_0^\infty \frac{\ln z}{(z^2+1)^2} dz$$

Let us consider new function $f(z) = \left(\frac{\ln z}{z^2+1}\right)^2$, then $I = \int_0^\infty \left(\frac{\ln z}{z^2+1}\right)^2 dz$

Pole at $z = \pm i$ is simple pole of second order.

Residue at $z = i$ is

$$= \frac{d}{dz}(z-i)^2 \frac{(\ln z)^2}{(z-i)^2(z+i)^2} = \frac{d}{dz}\frac{(\ln z)^2}{(z+i)^2}$$

$$= \frac{(z+i)^2 2(\ln z)\cdot\frac{1}{z} - (\ln z)^2 \cdot 2(z+i)}{(z+i)^4} = \frac{(z+i)2\ln(z)\frac{1}{z} - (\ln z)^2 \cdot 2}{(z+i)^3}$$

$$= \frac{(2i)2 \times \frac{1}{i}\ln i - (\ln i)^2 \cdot 2}{(2i)^3} = \frac{4\frac{i\pi}{2} - \left(\frac{i\pi}{2}\right)^2 \times 2}{-8i} = \frac{2\pi i + \frac{\pi^2}{2}}{-8i}$$

$$\Rightarrow \text{Res}|_{z=i} = \frac{-\pi}{4} + \frac{\pi^2}{16} i$$

Similarly, at $z = -i$; $\text{Res}|_{z=-i} = \frac{-\pi}{4} - \frac{\pi^2}{16} i$

$$I = \int_0^\infty \left(\frac{\ln z}{z^2+1}\right)^2 dz = 2\pi i \left(\frac{-\pi}{4} + \frac{\pi^2}{16} i - \frac{\pi}{4} - \frac{\pi^2}{16} i\right) = -\pi^2 i$$

$$-\pi^2 i = \left(\int_R \int_A \int_B\right) f(z) dz = \left(\iint_{AB}\right) f(z) dz; \qquad \left[\because \int_{AB} \text{ vanish}\right]$$

Along path A; $\quad z = -x + i\varepsilon$ and along path B; $\quad z = -x - i\varepsilon$

$$\text{Thus } -\pi^2 i = \left(\iint_{AB}\right) f(z) dz = -\int_{-\infty}^0 \left[\frac{\ln(-x+i\varepsilon)}{(-x+i\varepsilon)^2+1}\right] dx - \int_0^\infty \left[\frac{\ln(-x-i\varepsilon)}{(-x-i\varepsilon)^2+1}\right] dx$$

$$\Rightarrow -\pi^2 i = \int_0^\infty \left[\frac{\ln(-x+i\varepsilon)}{(-x+i\varepsilon)^2+1}\right]^2 dx - \int_0^\infty \left[\frac{\ln(-x-i\varepsilon)}{(-x-i\varepsilon)^2+1}\right]^2 dx$$

$$\Rightarrow -\pi^2 i = \int_0^\infty \left[\frac{\ln(x)+i\pi}{1+x^2}\right]^2 dx - \int_0^\infty \left[\frac{\ln(x)-i\pi}{1+x^2}\right]^2 dx; \quad \varepsilon \to 0$$

$$\Rightarrow -\pi^2 i = \int_0^\infty \frac{(\ln(x)+i\pi)^2 - (\ln(x)-i\pi)^2}{(1+x^2)^2} dx$$

$$= 4\pi i \int_0^\infty \frac{\ln x}{(x^2+1)^2} \Rightarrow \int_0^\infty \frac{\ln x}{(x^2+1)^2} = \frac{-i\pi^2}{4\pi i} = \frac{-\pi}{4}$$

So the correct answer is **Option (B)**

15. Compute $\lim_{z\to 0} \frac{\mathrm{Re}(z^2)+\mathrm{Im}(z^2)}{z^2}$

[JEST 2013]

a. The limit does not exist **b.** 1

c. $-i$ **d.** -1

Solution:

$$\lim_{z\to 0} \frac{\mathrm{Re}\left(z^2\right)+\mathrm{Im}\left(z^2\right)}{z^2} = \lim_{z\to 0} \frac{x^2-y^2+2xy}{x^2-y^2+2ixy} = \lim_{\substack{y=0\\x\to 0}} \frac{x^2-y^2+2xy}{x^2-y^2+2ixy} = 1$$

$$\lim_{\substack{x=0\\y\to 0}} \frac{x^2-y^2+2xy}{x^2-y^2+2ixy} = 1 \text{ and } \lim_{\substack{y=x\\x\to 0}} \frac{x^2-y^2+2xy}{x^2-y^2+2ixy} = -i$$

So the correct answer is **Option (A)**

16. The value of limit

$$\lim_{z\to i} \frac{z^{10}+1}{z^6+1}$$

is equal to

[JEST 2014]

a. 1 **b.** 0

c. $\frac{-10}{3}$ **d.** $\frac{5}{3}$

Solution:

$$\lim_{z\to i} \frac{z^{10}+1}{z^6+1} = \lim_{z\to i} \frac{10z^9}{6z^5} = \lim_{z\to i} \frac{10z^4}{6} = \frac{10}{6} = \frac{5}{3}$$

So the correct answer is **Option (D)**

17. The value of integral

$$I = \oint \frac{\sin z}{2z-\pi} dz$$

with c a circle $|z| = 2$, is

[JEST 2014]

a. 0 **b.** $2\pi i$

c. πi **d.** $-\pi i$

Solution:

$$I = \oint_C \frac{\sin z}{2z-\pi}, \text{ for pole } 2z-\pi = 0 \Rightarrow z = \frac{\pi}{2}$$

$$\text{Residue at } z=\frac{\pi}{2} \because |z|=2, \text{ so pole will lie within the contour}$$
$$I=\oint_C \frac{e^{iz}}{2\left(z-\frac{\pi}{2}\right)}=\sum R\times 2\pi i$$
$$\operatorname{Res}|_{z=\frac{\pi}{2}}=\frac{\left(z-\frac{\pi}{2}\right)e^{iz}}{2\left(z-\frac{\pi}{2}\right)}=\frac{e^{i\pi/2}}{2}=\frac{i}{2} \text{ (taking imaginary part); Residue } =\frac{1}{2}$$
$$\text{Now, } I=\frac{1}{2}\times 2\pi i=\pi i$$

So the correct answer is **Option (C)**

18. Given an analytic function $f(z)=\phi(x,y)+i\psi(x,y)$, where $\phi(x,y)=x^2+4x-y^2+2y$ If C is a constant, which of the following relations is true?

[JEST 2015]

a. $\psi(x,y)=x^2y+4y+C$

b. $\psi(x,y)=2xy-2x+C$

c. $\psi(x,y)=2xy+4y-2x+C$

d. $\psi(x,y)=x^2y-2x+C$

Solution:

$$u=\phi(x,y)=x^2+4x-y^2+2y, v=\psi$$
$$\text{From C.R. equation, } \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \Rightarrow \frac{\partial \phi}{\partial x}=\frac{\partial \psi}{\partial y}, \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\Rightarrow \frac{\partial \phi}{\partial y}=-\frac{\partial \psi}{\partial x}$$
$$\text{Now, } \frac{\partial \phi}{\partial x}=2x+4=\frac{\partial \psi}{\partial y}$$
$$\Rightarrow \psi=2xy+4y+f(x) \tag{1.1}$$
$$\text{and } \frac{\partial \phi}{\partial y}=-2y+2\Rightarrow \frac{\partial \psi}{\partial x}=+2y-2$$
$$\psi=2xy+2x+f(y) \tag{1.2}$$
$$\text{From (1.1) and (1.2), } 2xy+4y+f(x)=2xy-2x+f(y)$$
$$f(x)=-2x, \quad f(y)=4y$$
$$\psi=2xy+4y-2x+c$$

So the correct answer is **Option (C)**

19. Which one is the image of the complex domain $\{z \mid xy\geq 1, x+y>0\}$ under the mapping $f(z)=z^2$, if $z=x+iy$?

[JEST 2017]

a. $\{z \mid xy\geq 1, x+y>0\}$

b. $\{z \mid x\geq 2, x+y>0\}$

c. $\{z \mid y\geq 2\forall x\}$

d. $\{z \mid y\geq 1\forall x\}$

20. The integral $I=\int_1^\infty \frac{\sqrt{x-1}}{(1+x)^2}dx$ is

[JEST 2017]

a. $\frac{\pi}{\sqrt{2}}$

b. $\frac{\pi}{2\sqrt{2}}$

c. $\frac{\sqrt{\pi}}{2}$

d. $\sqrt{\frac{\pi}{2}}$

Solution:

$$I=\int_1^\infty \frac{\sqrt{x-1}}{(1+x)^2}dx$$

$$\text{Put,} x = \left(1+z^2\right), dx = 2z dz$$
$$\text{Hence, } I = \int_0^\infty \frac{2z^2 dz}{\left(2+z^2\right)^2}$$
$$\text{Here poles, } \left(2+z^2\right) = 0 \Rightarrow (z+i\sqrt{2})(z-i\sqrt{2}) = 0$$
$$\text{Only } (z = i\sqrt{2}) \text{ poles is allowed}$$
$$\text{Then } R(i\sqrt{2}) = \lim_{z\to i\sqrt{2}} \frac{1}{\sqrt{2-1}} \frac{d}{dz}\left[\frac{2z^2(z-i\sqrt{2})^2}{(z-i\sqrt{2})^2(z+i\sqrt{2})^2}\right]$$
$$= \lim_{z\to i\sqrt{2}} \left[\frac{(z+i\sqrt{2})^2\cdot 4z - 2z^2\cdot 2(z+i\sqrt{2})}{(z+i\sqrt{2})^4}\right]$$
$$= \frac{(2i\sqrt{2})^2 \times 4(i\sqrt{2}) - 2(i\sqrt{2})^2\cdot 2(2i\sqrt{2})}{(2i\sqrt{2})^4}$$
$$= -\frac{32\sqrt{2}i + 16\sqrt{2}i}{64} = -\frac{16\sqrt{2}i}{64} = -\frac{i}{2\sqrt{2}}$$
$$\text{Hence, } \int_{-\infty}^\infty \frac{2z^2}{\left(2+z^2\right)^2} dz = 2\pi i\left(-\frac{i}{2\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}$$
$$\Rightarrow \int_0^\infty \frac{2z^2}{\left(2+z^2\right)^2} dz = \frac{\pi}{2\sqrt{2}} \Rightarrow \int_i^\infty \frac{\sqrt{x-1}}{(1+x)^2} dx = \frac{\pi}{2\sqrt{2}}$$

So the correct answer is **Option (B)**

21. The integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} dx \text{ is}$$

[JEST 2018]

a. $\frac{\pi}{e}$

b. πe^{-2}

c. π

d. zero

Solution:

$$f(z) = \frac{e^{iz}}{z^2+1} = \frac{e^{iz}}{(z+i)(z-i)}$$
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} dx = \operatorname{Re} 2\pi i \times \frac{e^{i\cdot i}}{zi} = \frac{\pi}{e}$$

So the correct answer is **Option (A)**

22. Consider the function $f(x,y) = |x| - i|y|$. In which domain of the complex plane is this function analytic?
[JEST 2019]

a. First and second quadrants

b. Second and third quadrants

c. Second and fourth quadrants

d. Nowhere

Solution:

$$f(x,y) = |x| - i|y|$$
$$f(x,y) = x - iy = \bar{z}$$
$$f(x,y) = -x - iy = -z$$

$$f(x,y) = -x + iy = -\bar{z}$$
$$f(x,y) = x + iy = z$$

We know $\bar{z}$ is not analytic and z and $-z$ are analytic.

So the correct answer is **Option (C)**

Answer key			
Q.No.	Answer	Q.No.	Answer
1	**D**	2	**C**
3	**3(NAT)**	4	**C**
5	**A**	6	**C**
7	**C**	8	**A**
9	**π(NAT)**	10	**3(NAT)**
11	**81.64(NAT)**	12	**C**
13	**A**	14	**B**
15	**A**	16	**D**
17	**C**	18	**C**
19	**-**	20	**B**
21	**A**	22	**C**

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