EXAM-PHY001
VECTOR CALCULUS

SECTION A (MCQ)

[Q. No. 1-10 (3.5 Marks)]

1. A curve is given by $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$. The unit vector of the tangent to the curve at $t = 1$ is

A. $\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ **B.** $\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}$ **C.** $\frac{\hat{i}+2\hat{j}+2\hat{k}}{3}$ **D.** $\frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{14}}$

Solution: Let $\hat{n}$ be a unit vector tangent to the curve at t.
By definition

$$\hat{n} = \frac{d\vec{r}/dt}{|d\vec{r}/dt|}$$

$$\text{So } \hat{n} = \frac{d\vec{r}/dt}{|d\vec{r}/dt|} = \frac{\hat{i}+2t\hat{j}+3t^2\hat{k}}{\sqrt{1+4t^2+9t^4}}$$

$$\text{At } t = 1, \hat{n} = \frac{\hat{i}+2j+3\hat{k}}{\sqrt{14}}$$

So the correct answer is **Option d**

2. The divergence of $\vec{c} \times (\vec{r} \times \vec{c})$ where $\vec{c}$ is a numerical vector is given by

A. zero **B.** $|\vec{c}|^2$ **C.** $2|\vec{c}|^2$ **D.** $-2|\vec{c}|^2$

Solution: According to vector triple product,

$$\vec{c} \times (\vec{r} \times \vec{c}) = \vec{r}(\vec{c}\cdot\vec{c}) - \vec{c}(\vec{c}\cdot\vec{r})$$

$$\therefore \quad \vec{\nabla}\cdot[\vec{c} \times (\vec{r} \times \vec{c})] = c^2\vec{\nabla}\cdot\vec{r} - \vec{\nabla}\cdot[\vec{c}(\vec{c}\cdot\vec{r})]$$

$$= 3c^2 - [\vec{\nabla}(\vec{c}\cdot\vec{r})\cdot\vec{c} + (\vec{c}\cdot\vec{r})(\vec{\nabla}\cdot\vec{c})]$$

$$= 3c^2 - [\vec{c}\cdot\vec{c} + 0] = 2c^2$$

Correct option is (c)

3. Find the equation to the tangent plane to the surface , $x^2 + y^2 - z^2 = 7$ at the point $(2, 2, 1)$.

a. $4(x-1) - 4(y-1) + 2(z-1) = 0.$ **b.** $2(x-4) + 2(y-4) - (z-2) = 0.$

c. $2(x-2) + 2(y-2) - (z-1) = 0.$ **d.** $4(x-2) + 4(y-2) - 2(z-1) = 0.$

Solution:

$$(x-x_0)\frac{\partial\phi}{\partial x} + (y-y_0)\frac{\partial\phi}{\partial y} + (z-z_0)\frac{\partial\phi}{\partial z} = 0$$

$$\text{Here, } \phi \Rightarrow x^2 + y^2 - z^2 = 7$$

$$\frac{\partial \phi}{\partial x}|_{(x=2)} = 2x = 4$$
$$\frac{\partial \phi}{\partial y}|_{(y=2)} = 2y = 4$$
$$\frac{\partial \phi}{\partial z}|_{(z=1)} = -2z = -2$$

Thus the equation of tangent plane is,

$$4(x-2)+4(y-2)-2(z-1)=0.$$

Correct answer is **option (d)**.

4. Let the position vector be given by, $r = xi + yj + zk$. With, $r = |r| = \sqrt{x^2+y^2+z^2}$. Compute the gradient of the scalar field $\phi(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ in terms of $\vec{r}$ and r.

a. $-\frac{\vec{r}}{r^3}$ **b.** $-\frac{\vec{r}}{r^2}$ **c.** $\frac{\vec{r}}{r^{3/2}}$ **d.** $\frac{\vec{r}}{r}$

Solution:

$$\text{Let, } \phi(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}.$$

The gradient is given by,

$$\begin{aligned} \nabla\phi &= \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) \\ &= -\frac{x}{(x^2+y^2+z^2)^{3/2}}i - \frac{y}{(x^2+y^2+z^2)^{3/2}}j - \frac{z}{(x^2+y^2+z^2)^{3/2}}k \\ &= -\frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{3/2}} \end{aligned}$$

In terms of position vector,

$$\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$$

Correct answer is **option (a)**.

5. The directional derivative of the scalar function $f(x,y,z) = x^2+2y^2+z$ at point $P=(1,1,2)$ in the direction of the vector $\vec{a} = 3\hat{i} - 4\hat{j}$ is,

Solution:

$$\text{We know that,} \nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k = 2x\hat{i} + 4y\hat{j} + \hat{k}$$
$$\text{At point } P(1,1,2)\ \nabla f = 2\hat{i} + 4\hat{j} + \hat{k}$$

Now directional derivative of f at $P(1,1,2)$ in the direction of vector $a = 3\hat{i} - 4\hat{j}$ is given by,

$$\begin{aligned}\frac{a}{|a|}\ \text{grad}\ f &= \left(\frac{3\hat{i}-4\hat{j}}{\sqrt{25}}\right)\cdot(2\hat{i}+4\hat{j}+\hat{k})\\ &= \frac{1}{5}(6-16+0)\\ &= -2\end{aligned}$$

6. Let $u = y\hat{i} + x\hat{j}$. The value of $\oint_C u \cdot dr$, where C is the unit circle centered at the origin, is given by,

Solution:

$$\begin{aligned}\nabla \times u &= \nabla \times (yi + xj)\\ &= 0 \Rightarrow \text{u is a conservative field.}\end{aligned}$$

Then the line integral around any closed curve is zero.

$$\oint_C u \cdot dr = 0$$

7. Find the line integral of the vector field $\vec{F} = \left(x^2+y^2\right)\hat{\imath} - 2xy\hat{\jmath}$ along a path $y = \sqrt{x}$ from$(0,0)$ to $(1,1)$.

Solution:

$$\begin{aligned}\int_0^1 F.dr &= \int_0^1 \left(x^2+y^2\right)\hat{\imath} - 2xy\hat{\jmath}\cdot\left(dx\hat{\imath}+dy\hat{\jmath}\right)\\ &= \int_0^1 \left(x^2+y^2\right)dx - 2xydy\end{aligned}$$

Substituting $y = \sqrt{x}, x = y^2$ the line integral can be expressed as,

$$\begin{aligned}\int_0^1 F.dr &= \int_0^1 \left(x^2+x\right)dx - 2\int_0^1 y^3 dy\\ &= \frac{1}{3}\\ &= 0.33.\end{aligned}$$

8. Let $u = -x^2yi + xy^2j$ Compute $\oint_C u \cdot dr$ for a unit square in the first quadrant with vertex at the origin. Here, it is simpler to compute an area integral. The answer is

Solution:

$$\begin{aligned}u &= -x^2y\hat{i} + xy^2\hat{j}\\ \text{We have },\nabla \times u &= \left(x^2+y^2\right)\hat{k}.\end{aligned}$$

According to Stokes theorem

$$\begin{aligned},\oint_C u \cdot dr &= \int_S (\nabla \times u)\cdot dS \Rightarrow dS = dxdy\hat{k}\\ &= \int_0^1\int_0^1 \left(x^2+y^2\right)dxdy\end{aligned}$$

$$= \int_0^1 x^2 dx \int_0^1 dy + \int_0^1 dx \int_0^1 y^2 dy$$
$$= \frac{2}{3} = 0.666$$

9. Find the directional derivative of the function $f(x,y) = 3x^2y$ at a point $(-2,1)$ along the direction $4\hat{i} + 3\hat{j}$.

Solution:

The unit vector along $4\hat{i} + 3\hat{j}$ *is*,

$$\hat{u} = \frac{4i + 3\hat{j}}{5}$$

Gradient of the given function is

$$\nabla f = 6xy\hat{i} + 3x^2\hat{j}$$

Thus the directional derivative at (x,y) is

$$\nabla f \cdot \hat{u} = \left(\frac{24}{5}\right)xy + \left(\frac{9}{5}\right)x^2$$

At $(-2,1)$ it's value is $\frac{-12}{5} = -2.4.$

10. The directional derivative of $f(x,y,z) = 2x^2 + 3y^2 + z^2$ at point $P(2,1,3)$ in the direction of the vector $a = i - 2k$ is $\cdots$

Solution:

$$\begin{aligned}
\text{We have } f &= 2x^2 + 3y^2 + z^2, P(2,1,3) \\
a &= i - 2k \\
\nabla f &= i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} \\
&= 4xi + 6yj + 2zk \\
\text{at } P(2,1,3) \quad \nabla f &= 4 \times 2 \times i + 6 \times 1 \times j + 2 \times 3 \times k \\
&= 8i + 6j + 6k
\end{aligned}$$

The directional derivative of f in direction of vector $a = i - 2k$ is the component of grad f in the direction of vector a and is given by $\frac{a}{|a|} \cdot$ grad f

$$= \left[\frac{i - 2k}{\sqrt{1^2 + (-2)^2}}\right] \cdot (8i + 6j + 6k)$$
$$= \frac{1}{\sqrt{5}}[1.8 + 0.6 + (-2)6] = \frac{-4}{\sqrt{5}}$$

SECTION B (MCQ)

[Q. No. 10-15 (5 Marks)]

11. What is the angle (in degrees) between the surfaces $y^2+z^2=2$ and $y^2-x^2=0$ at the point $(1,-1,1)$

Solution:

The equations of two surfaces are,

$$f(x,y,z)=2 \text{ and } g(x,y,z)=0$$
$$\text{where} f(x.y,z)=y^2+z^2 \text{ and } \quad g(x,y,z)=y^2-x^2$$

The normal to the first surfaces is

$$\overrightarrow{\nabla f}=\frac{\partial f}{\partial x}\hat{\imath}+\frac{\partial f}{\partial y}\hat{\jmath}+\frac{\partial f}{\partial z}\hat{k} \Rightarrow \overrightarrow{\nabla f}=2y\hat{\jmath}+2z\hat{k}$$
$$\overrightarrow{\nabla g}=\frac{\partial g}{\partial x}\hat{\imath}+\frac{\partial g}{\partial y}+\hat{\jmath}+\frac{\partial g}{\partial z}\hat{k} \Rightarrow \overrightarrow{\nabla g}=-2x\hat{\imath}+2y\hat{\jmath}$$
$$\text{At point}(1,-1,1), \overrightarrow{\nabla f}=-2\hat{\jmath}+2\hat{k} \text{ and } \quad \overrightarrow{\nabla g}=-2\hat{\imath}-2\hat{\jmath}$$

Hence the angle between the two surfaces is

$$\theta=\cos^{-1}\frac{\vec{\nabla}\vec{f}\cdot\overrightarrow{\nabla g}}{|\overrightarrow{\nabla f}||\overrightarrow{\nabla g}|}$$
$$=\cos^{-1}\frac{(-2\hat{\jmath}+2\hat{k})\cdot(-2\hat{\imath}-2\hat{\jmath})}{\sqrt{8}\sqrt{8}}$$
$$\text{Or} \quad \theta=\cos^{-1}\frac{4}{8}$$
$$=\cos^{-1}\frac{1}{2}$$
$$=60^\circ$$

Correct option is (C)

12. If $\vec{r}$ is position vector of a point, for what value of n, the vector $r^n\vec{r}$ is solenoidal?

Solution:

We know that,

$$\nabla\cdot f(r)\hat{r}=\frac{1}{r^2}\frac{\partial}{\partial r}r^2 f(r) \quad -(\text{radial component of divergence})$$
$$\text{Here, } f(r)\hat{r}=r^n\vec{r}$$
$$=r^n r\hat{r}$$
$$=r^{n+1}\hat{r}$$
$$\text{Then, } \nabla\cdot f(r)\hat{r}=\frac{1}{r^2}\frac{\partial}{\partial r}r^2 r^{n+1}$$
$$=\frac{1}{r^2}\frac{\partial}{\partial r}r^{n+3}=\frac{n+3}{r^2}r^{n+2}$$
$$=(n+3)r^n$$

To be solenoidal, $\nabla f(r)\hat{r} = 0$

$$\Rightarrow (n+3)r^n = 0$$
$$\Rightarrow \quad n = -3$$

Correct option is (B)

13. If a vector field is given by $F = \sin y\hat{i} + x(1+\cos y)\hat{j}$, then evaluate the line integral over a circular path given by $x^2 + y^2 = a^2, z = 0$.

a. $\frac{\pi}{2}a$ **b.** 2π **c.** $2\pi^2 a^2$ **d.** πa^2

Solution:

The particle moves in xy plane,$z = 0$

Now, let $R = x\hat{i} + y\hat{j}$

Then, $dR = dx\hat{i} + dy\hat{j}$

Also, path given is $x^2 + y^2 = a^2$

So, let $x = a\cos t$ and $y = a\sin t$,

t varies from 0 to 2π

$$\text{Therefore, } \oint_C F \cdot dR = \oint_C (\sin y\hat{i} + x(1+\cos y)\hat{j}) \cdot (dx\hat{i} + dy\hat{j})$$
$$= \oint_C [\sin y \cdot dx + x(1+\cos y)dy]$$
$$= \oint_C [[\sin y \cdot dx + x\cos y dy] + xdy]$$
$$= \oint_C [d(x\sin y) + xdy]$$

Now, substituting the values of x and y, we get,

$$\int_0^{2\pi} \left[d(a\cos t\sin(a\sin t)) + a^2\cos^2 t dt\right] = |a\cos t\sin(a\sin t)|_0^{2\pi} + \frac{a^2}{2}\left|t + \frac{\sin 2t}{2}\right|_0^{2\pi}$$

$$[\because \cos^2 t = 1 + \cos 2t]$$

$$= 0 + \frac{a^2}{2}[2\pi + 0 - (0+0)]$$
$$= \pi a^2$$

Correct answer is **option (d)**

14. Which one of the following vectors lie along the line of intersection of the two planes $x + 3y - z = 5$ and $2x - 2y + 4z = 3$?

A. $10\hat{i} - 2\hat{j} + 5\hat{k}$ **B.** $10\hat{i} - 6\hat{j} - 8\hat{k}$

C. $10\hat{i} + 2\hat{j} + 5\hat{k}$ **D.** $10\hat{i} - 2\hat{j} - 5\hat{k}$

Solution:

Unit vector normal to $x + 3y - z = 5$: $\hat{n}_1 = \dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \dfrac{\hat{i} + 3\hat{j} - \hat{k}}{\sqrt{1+9+1}}$

$$= \frac{\hat{i}+3\hat{j}-\hat{k}}{\sqrt{11}}$$

Unit vector normal to $2x-2y+4z=3$: $\hat{n}_2 = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2\hat{i}-2\hat{j}+4\hat{k}}{\sqrt{4+4+16}}$

$$= \frac{2\hat{i}-2\hat{j}+4\hat{k}}{\sqrt{24}}$$

Let's check for option (B)

$$\hat{n} = 10\hat{i}-6\hat{j}-8\hat{k}$$

$$\hat{n}_1 \cdot \hat{n} = \frac{10-18+8}{\sqrt{11}} = 0$$

$$\hat{n}_2 \cdot \hat{n} = \frac{20+12-32}{\sqrt{24}} = 0$$

Correct option is (B)

15. Four forces are given below in Cartesian and spherical polar coordinates

i. $\vec{F}_1 = K\exp\left(\frac{-r^2}{R^2}\right)\hat{r}$ **ii.** $\vec{F}_2 = K\left(x^3\hat{y}-y^3\hat{z}\right)$

iii. $\vec{F}_3 = K\left(x^3\hat{x}+y^3\hat{y}\right)$ **iv.** $\vec{F}_4 = K\left(\frac{\hat{\phi}}{r}\right)$.

where K is a constant Identify the correct option

A. (iii) and (iv) are conservative but (i) and (ii)are not.

B. (i) and (ii) are conservative but (iii) and (iv) are not.

C. (ii) and (iii) are conservative but (i) and (iv) are not.

D. (i) and (iii) are conservative but (ii) and (iv)-are not.

Solution:

$$\vec{\nabla}\times\vec{F}_1 = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\theta\theta} & \frac{\partial}{\partial\hat{\theta}} \\ k\exp\left(-\frac{r^2}{R^2}\right) & 0 & 0 \end{vmatrix} = 0$$

$$\vec{\nabla}\times\vec{F}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & kx^3 & -ky^3 \end{vmatrix} = \hat{i}\left(-3ky^2-0\right) = -3ky^2\hat{i}$$

$$\vec{\nabla}\times\vec{F}_3 = \begin{vmatrix} x & y & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ kx^3 & ky^3 & 0 \end{vmatrix} = 0$$

$$\vec{\nabla}\times\vec{F}_4 = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\theta\theta} & \frac{\partial}{\partial\hat{\theta}} \\ 0 & 0 & rsin\theta K/r \end{vmatrix} = Kr\cos\theta$$

Correct answer is option(D)

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