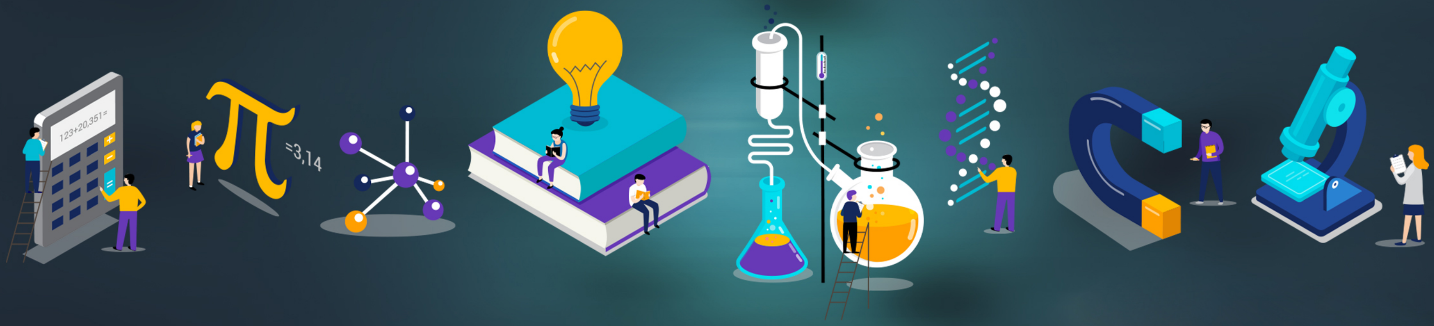


1. Single Variable Calculus

1.1 Limits and Continuity

1.1.1 Limits

Definition 1.1.1 Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. If $f(x)$ gets arbitrarily close to l for all x sufficiently close to x_0, We say that $f(x)$ approaches the limit l as x approaches x_0 , and write,

$$\lim_{x \to x_0} f(x) = l$$

If, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - l| < \varepsilon$$

Left-Hand and Right-Hand Limits

Left-Hand Limits

If the values of a function $f(x)$ at $x = c$ can be made as close as desired to the number l_1 at points close to and on the left of c, then l_1 is called left-hand limit. It is denoted by

$$\lim_{x \to c^-} f(x) = l_1$$

Right-Hand Limits

If the values of a function $f(x)$, at $x = c$ can be made as close as desired to the number l_2 at points close to and on the right of c, then l_2 is called right-hand limit. It is denoted by,

$$\lim_{x \to c^+} f(x) = l_2$$

Properties of Limits

Suppose we have $\lim_{x \to c} f(x) = a, \lim_{x \to c} g(x) = b$

- $\lim_{x \to c}(f(x) \pm g(x)) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x) = a \pm b$
- $\lim_{x \to c}(f(x) \cdot g(x)) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = a \cdot b$
- $\lim_{x \to c}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{a}{b}$, where $b \neq 0$
- $\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$, where k is a constant.

- $\lim_{x\to c}|f(x)| = |\lim_{x\to c} f(x)| = |a|$
- $\lim_{x\to c}[f(x)]^{g(x)} = \lim_{x\to c}[f(x)]^{\lim_{x\to c}[g(x)]} = a^b$
- If $\lim_{x\to c} f(x) = \pm\infty$, then $\lim_{x\to c} \frac{1}{f(x)} = 0$

Important Limits

- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim_{x\to 0} \frac{\tan x}{x} = 1$
- $\lim_{x\to 0} \cos x = 1$
- $\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$
- $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$
- $\lim_{x\to 0} \frac{(a^x-1)}{x} = \log a$, if $(a > 0)$
- $\lim_{x\to 0} \frac{x^n-a^n}{x-a} = na^{n-1}$
- $\lim_{x\to\infty} \frac{\log x}{x^m} = 0$, if $(m > 0)$

L'Hôpital's rule

L'Hôpital's rule is a method finding the limits of indeterminant forms.

$$\begin{aligned} &x\ln x &&\text{as } x\to 0^+, \\ &xe^{-x} &&\text{as } x\to\infty \\ &\frac{\ln x}{x} &&\text{as } x\to\infty \end{aligned}$$

Definition 1.1.2 If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$, $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ exists and $g'(x)\neq 0$ for all x, then,

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

Exercise 1.1 Evaluate $\lim_{x\to 0}\frac{\sin 5x}{\sin 2x}$ ■

Solution: Here $f(x) = \sin 5x$, $g(x) = \sin 2x$, and $a = 0$. Since $f(a) = g(a) = \sin 0 = 0$, we can apply L'Hôpital's rule and find this limit:

$$\begin{aligned} \lim_{x\to 0}\frac{\sin 5x}{\sin 2x} &= \lim_{x\to 0}\frac{5\cos 5x}{2\cos 2x} \qquad \text{(l'Hop)} \\ &= \lim_{x\to 0}\frac{5\cos(5\cdot 0)}{2\cos(2\cdot 0)} \\ &= \frac{5}{2} \end{aligned}$$

1.2 Continuous Functions

Definition 1.2.1 The function $f(x)$ is continuous at $x = c$ if

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) \text{ i.e. } \lim_{x\to c} f(x) \text{ exists and equals } f(c)$$

The function f is said to be continuous on its domain, if it is continuous at each point in it's domain. If f is not continuous at a particular value c, we say that f is discontinuous at c or that f has a discontinuity at c.

Continuity test:
A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x\to c} f(x) = f(c)$ (The limit equals the function value)

1.3 Differentiability

Definition 1.3.1 A real-valued function $f(x)$ defined on an open interval (a,b). The function is said to be differentiable for $x = c$, if

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \quad \text{exists for every} \quad c \in (a,b)$$

A function is always continuous at a point if the function is differentiable at the same point. However, the converse is not always true.

1.4 Tangents and Normals

1.4.1 Tangents

The tangent line to a curve at a given point is the straight line that touches the curve at that point. The point is called pont of tangency.

Let $y = f(x)$ be a given curve and $P(x,y)$ and $Q(x+\delta x, y+\delta y)$ be two neighbouring points on it. Equation of the line PQ is,

$$Y - y = \frac{y+\delta y - y}{x+\delta x - x}(X - x) \Longrightarrow Y - y = \frac{\delta y}{\delta x}(X - x) \tag{1.1}$$

This line will be tangent to the given curve at P, if $Q \to P$ which in turn means that $\delta x \to 0$ and we know that

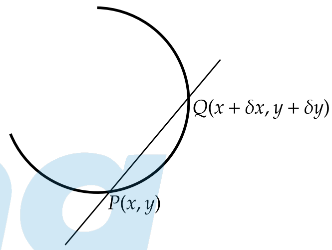


Figure 1.1: tangent to a curve.

$$\lim_{\delta x\to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx} \tag{1.2}$$

Therefore the equation of the tangent is,

$$Y - y = \frac{dy}{dx}(X - x) \tag{1.3}$$

1.4.2 Normals

The normal at a point (x,y) is the line perpendicular (at right angles) to the tangent at that point. It's slope is given by,

$$\text{Slope} \ = \frac{-1}{dy/dx} \tag{1.4}$$

And hence equation of the normal is ,

$$Y - y = \frac{-1}{dy/dx}(X - x) \tag{1.5}$$

Geometrical meaning

dy/dx represents the slope of the tangent to the given curve $y = f(x)$ at any point (x,y).

$$\frac{dy}{dx} = \tan\theta$$

(θ=The angle which the tangent to the curve makes with +ve direction of x -axis.)

1.5 Monotonic Functions

A monotonic function is a function which is either entirely nonincreasing or nondecreasing on an interval (a,b). A function is monotonic if its first derivative (which need not be continuous) does not change sign.

1. A function $f(x)$ is called monotonically increasing (also increasing or non-decreasing) on an interval (a,b), if,
$$x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2) \text{ For all values of } x_1, x_2 \in (a,b)$$
And if,
$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ For all values of } x_1, x_2 \in (a,b)$$
2. A function $f(x)$ is said to be decreasing on an interval (a,b) if,
$$x_1 \le x_2 \Rightarrow f(x_1) \ge f(x_2) \text{ For all values of } x_1, x_2 \in (a,b)$$
A function $f(x)$ is said to be strictly decreasing on an interval (a,b) if,
$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \text{ For all values of } x_1, x_2 \in (a,b)$$

Conditions for Increasing and Decreasing Functions

Consider $f(x)$ to be continous on $[a,b]$ and differentiable on (a,b) then,

- If $f'(x) > 0$ for each x in (a,b) then f is (strictly) increasing on (a,b).
- If $f'(x) < 0$ for each x in (a,b) then f is (strictly) decreasing on (a,b).
- If $f'(x) = 0$ for each x in (a,b) then f is a constant function on (a,b).
  - If $f'(x^+) > 0$ and $f'(x^-) > 0$, then increasing at x for all $x \in (a,b)$.
  - If $f'(x^+) < 0$ and $f'(x^-) < 0$, then decreasing at x for all $x \in (a,b)$.
  - Otherwise neither increasing nor decreasing.
- If $f'(x) = g'(x)$ for each x in (a,b) then there exists a constant c such that $f(x) = g(x) + c$ for each x in (a,b).

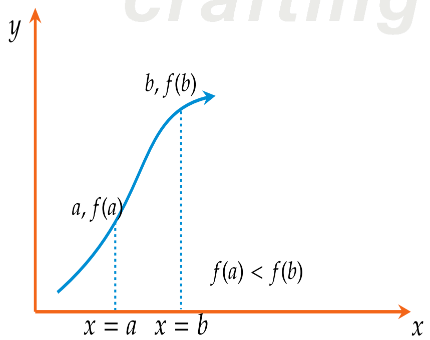


Figure 1.2: Increasing Function

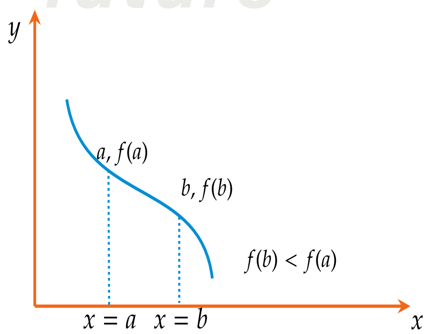


Figure 1.3: Decreasing Function

Exercise 1.2 Let $f(x) = x^3 - 3x^2 + 3x + y$. Examine the nature of function at $x = 0$ and 1 .

Solution:

$$f(x) = x^3 - 3x^2 + 3x + 4$$
$$f'(x) = 3x^2 - 6x + 3$$
$$\text{At} \quad x = 0, \; f'(0) = 3 > 0.$$

Hence $f(x)$ is increasing at $x = 0$

$$\text{At} \quad x = 1, f'(1) = 3 - 6 + 3 = 0.$$

Hence, let's check $f'(1^+)$ and $f(1^-)$

$$\text{Clearly,} \quad f'\left(1^+\right) = f(x = 1.001) > 0$$
$$\text{And,} \quad f'\left(1^-\right) = f(x = 0.999) > 0$$

Hence $f(x)$ is increasing at $x = 1$.

1.6 Maxima and Minima of Functions

For a continuous and differentiable function $f(x)$ a stationary point x^* is a point at which the slope of the function vanishes, i.e. $f'(x) = 0$ at $x = x^*$, where x^* belongs to its domain of definition. A stationary point may be a minimum, maximum or a saddle point(inflection point).

1.6.1 maximum value

A function $f(x,y)$ is said to have a maximum value at $x = a, y = b$, if there exists a small neighbourhood of (a,b) such that,

$$f(a,b) > f(a+h, b+k) \text{ for all values of } (h,k)$$

1.6.2 Minimum value

A function $f(x,y)$ is said to have a minimum value for $x = a, y = b$, if there exists a small neighbourhood of (a,b) such that

$$f(a,b) < f(a+h, b+k) \text{for all values of } (h,k)$$

.

The maximum and minimum values of a function are also called extreme or extremum values of the function.

Note

Saddle point: It is a point where function is neither maximum nor minimum.

If $f(a+h, b+k) - f(a,b)$ remains of the same sign for all values (positive or negative) of (h,k), then $f(a,b)$ is said to be extremum value of $f(x,y)$ at (a,b).

- If $f(a+h, b+k) - f(a,b) < 0$, Then $f(a,b)$ is maximum.
- If $f(a+h, b+k) - f(a,b) > 0$, Then $f(a,b)$ is minimum.

Rules to find Extremum values

1. Differentiate $y = f(x)$ and find out $\frac{dy}{dx}$
2. Put $\frac{dy}{dx} = 0$ and solve these equations for x . Let one root of $\frac{dy}{dx} = 0$ is at $x = a$.
3. If $\frac{d^2y}{d^2x} = -ve$ for $x = a$, then maxima is at $x = a$.
4. If $\frac{d^2y}{d^2x} = +ve$ for $x = a$, then minima is at $x = a$.
5. If $\frac{d^2y}{d^2x} = 0$ at $x = a$, then find $\frac{d^3y}{d^3x}$.

   If $\frac{d^3y}{d^3x} \neq 0$ at $x = a$, neither maximum nor minimum at $x = a$.

   If $\frac{d^3y}{d^3x} = 0$ at $x = a$, then find $\frac{d^4y}{d^4x}$ and investigate further.

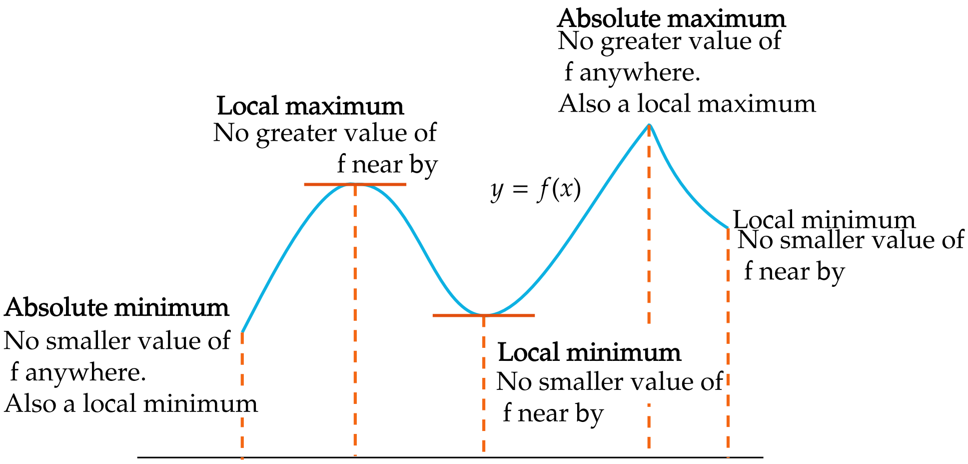


Figure 1.4: Minima and Maxima of functions

Exercise 1.3 Find the equilibrium points and find it's nature.
1. $f(x) = x^3 - 3x + 9$
2. $f(x) = 2x^3 - 21x^2 + 36x$

Solution: 1.

$$f'(x) = \frac{df}{dx} = 3x^2 - 3$$

$$\text{Equating} \quad \frac{df}{dx} = 0$$

$$\text{we get,} x = 1, -1$$

$$\frac{d^2 f}{dx^2} = 6x$$

$$\left.\frac{d^2 f}{dx^2}\right|_{x=1} = 6 > 0 \quad \text{(Local minima)}$$

$$\left.\frac{d^2 f}{dx^2}\right|_{x=1} = -6 < 0 \quad \text{(Local maxima)}$$

2.

$$f'(x) = 6x^2 - 42x + 36$$

$$\text{Equating} \quad \frac{df}{dx} = 0$$

$$6x^2 - 42x + 36 = 0$$

$$(x-1)(x-6) = 0 \rightarrow x = 1, 6$$

$$\frac{d^2 f}{dx^2} = 12x - 42$$

$$\left.\frac{d^2 f}{dx^2}\right|_{x=1} = -30 < 0 \quad \text{(Local maxima.)}$$

$$\left.\frac{d^2 f}{dx^2}\right|_{x=1} = 30 > 0 \quad \text{(Local minima.)}$$

1.7 Differentiation

Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable.Then the differential dy is given by,

$$dy = f'(x)dx$$
$$\frac{dy}{dx} = f'(x)$$

1.7.1 Important Properties of Differentiation

1. $\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)$
2. $\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}f(x)-\frac{d}{dx}g(x)$
3. $\frac{d}{dx}(f(x)\cdot g(x)) = f(x)\cdot\frac{d}{dx}(g(x))+\frac{d}{dx}(f(x))\cdot g(x)$
4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}f(x)\cdot g(x)-f(x)\frac{d}{dx}g(x)}{(g(x))^2}$

(* The section 1.7.2 can be omitted without losing continuity.)

1.7.2 Partial Differentiation

Definition 1.7.1 The partial derivative of $f(x,y)$ with respect to y at the point (x_0,y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{d}{dy}f(x_0,y)\right|_{y=y_0} = \lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h},$$

provided the limit exists.

If a derivative of function of several independent variable, $f(x,y,z..)$ be found with respect to any one of them, keeping the others as constants, it is said to be partial derivative .And the operation of finding is called partial differentiation.

The partial derivative of $f(x,y)$ with respect to x and y are generally denoted by,

$$f_x = \frac{\partial f}{\partial x} \quad ; \quad f_y = \frac{\partial f}{\partial y}$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial x^2} \equiv f_{xx} \quad ; \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \equiv \frac{\partial^2 f}{\partial y^2} \equiv f_{yy}$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \equiv \frac{\partial^2 f}{\partial x\partial y} \equiv f_{xy} \quad ; \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial y\partial x} \equiv f_{yx} \qquad \text{it is given that } f_{xy} = f_{yx}$$

Exercise 1.4 Find $\partial f/\partial y$ if $f(x,y) = y\sin xy$. ■

Solution: We treat x as a constant and f as a product of y and $\sin xy$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y\sin xy) = y\frac{\partial}{\partial y}\sin xy + (\sin xy)\frac{\partial}{\partial y}(y)$$
$$= (y\cos xy)\frac{\partial}{\partial y}(xy) + \sin xy$$
$$= xy\cos xy + \sin xy.$$

Exercise 1.5 Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4,-5)$ if $f(x,y)=x^2+3xy+y-1$ ■

Solution: To find $\partial f/\partial x$, we treat y as a constant and differentiate with respect to x

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(x^2+3xy+y-1\right)\\ &= 2x+3\cdot 1\cdot y+0-0=2x+3y\\ \partial f/\partial x|_{(4,5)} &= (4)+3(-5)=-7\end{aligned}$$

To find $\partial f/\partial y$, we treat x as a constant and differentiate with respect to y :

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x^2+3xy+y-1\right)=0+3\cdot x\cdot 1+1-0=3x+1\\ \partial f/\partial y|_{(4,5)} &= 3(4)+1=13\end{aligned}$$

Theorem 1.7.3 **Euler's theorem:**
If u is a homogenous function of degree ' n ' in x and y then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

For any number of variables,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}+\ldots\ldots\ldots+w\frac{\partial u}{\partial w}=nu$$

Note

$$\text{If, } u=f(x,y)\text{but,} \quad x=\phi(s,t); \quad y=\phi(s,t);$$
$$\text{Then, } \frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} \quad\text{and}\quad \frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Exercise 1.6 Find $\partial z/\partial x$ if the equation

$$yz-\ln z=x+y$$

defines z as a function of the two independent variables x and y and the partial derivative exists. ■

Solution: We differentiate both sides of the equation with respect to x, holding y constant and treating z as a differentiable function of x :

$$\begin{aligned}\frac{\partial}{\partial x}(yz)-\frac{\partial}{\partial x}\ln z &= \frac{\partial x}{\partial x}+\frac{\partial y}{\partial x}\\ y\frac{\partial z}{\partial x}-\frac{1}{z}\frac{\partial z}{\partial x} &= 1+0\\ \left(y-\frac{1}{z}\right)\frac{\partial z}{\partial x} &= 1\\ \frac{\partial z}{\partial x} &= \frac{z}{yz-1}\end{aligned}$$

With y constant,
$\frac{\partial}{\partial x}(yz)=y\frac{\partial z}{\partial x}$

1.8 Integration

1.8.1 Definite and Indefinite Integrals

1.8.2 Definite Integrals

If a function $f(x)$ is defined in the interval $[a,b]$, then the definite integral of the function is given by

$$\int_a^b f(x)\cdot dx = [F(x)]_a^b = F(b) - F(a)$$

Where $F(x)$ is an integral of $f(x), a$ is called the lower limit and b is the upper limit of the integral. Geometrically, a definite integral represents the area bounded by a curve, $y = f(x), x$ -axis and the lines $x = a$ and $x = b$.

Properties

1. $\int_a^b f(x)\cdot dx = \int_a^b f(y)\cdot dy$
2. $\int_a^b f(x)\cdot dx = -\int_b^a f(x)\cdot dx$
3. $\int_a^b f(x)\cdot dx = \int_a^c f(x)\cdot dx + \int_c^b f(x)\cdot dx \quad$ if $a < c < b$
4. $\int_0^{2a} f(x)\cdot dx = \int_0^a f(x)\cdot dx + \int_0^a f(2a-x)\cdot dx$
5. $\int_0^a f(x)\cdot dx = \int_0^a f(a-x)\cdot dx$
6. $\int_{-a}^a f(x)\cdot dx = 2\int_0^a f(x)\cdot dx$, If the function is even. $\int_{-a}^a f(x)\cdot dx = 0$, if the function is odd.
7. $\int_0^{na} f(x)\cdot dx = n\int_0^a f(x)\cdot dx$ if $f(x) = f(x+a)$

1.8.3 Indefinite Integrals

The set of all antiderivatives of the function f is called the indefinite integral of f with respect to x, and is symbolized by,

$$\int f(x)dx$$

An indefinite integral $\int f(x)dx$ is a function plus an arbitrary constant C.

1.8.4 Methods of Integration

1.8.5 Integration by Substitution

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Method of solving

1. Substitute $u = g(x)$ and $du = g'(x)dx$ to obtain the integral

$$\int f(u)du$$

2. Integrate with respect to u.
3. Replace u by $g(x)$ in the result.

Exercise 1.7 Integrate

1. $\int x^2 \sin\left(x^3\right) dx$
2. $\int \frac{1}{\cos^2 2x} dx$
3. $\int \sin^2 x dx$

Solution: 1.

$$\begin{aligned}
\int x^2 \sin\left(x^3\right) dx &= \int \sin\left(x^3\right)\cdot x^2 dx \\
&= \int \sin u \cdot \frac{1}{3} du \\
&= \frac{1}{3}\int \sin u du \\
&= \frac{1}{3}(-\cos u) + C \\
&= -\frac{1}{3}\cos\left(x^3\right) + C
\end{aligned}$$

Let,$u = x^3$
$du = 3x^2 dx$
$(1/3)du = x^2 dx$
Integrate with respect to u

Replace u by x^3.

2.

$$\begin{aligned}
\int \frac{1}{\cos^2 2x} dx &= \int \sec^2 2x dx \\
&= \int \sec^2 u \cdot \frac{1}{2} du \\
&= \frac{1}{2}\int \sec^2 u du \\
&= \frac{1}{2}\tan u + C \\
&= \frac{1}{2}\tan 2x + C
\end{aligned}$$

$\frac{1}{\cos 2x} = \sec 2x$
$u = 2x$
$du = 2dx$
$dx = (1/2)du$

$\frac{d}{du}\tan u = \sec^2 u$
$u = 2x$

3.

$$\begin{aligned}
\int \sin^2 x dx &= \int \frac{1-\cos 2x}{2} dx \quad \sin^2 x = \frac{1-\cos 2x}{2} \\
&= \frac{1}{2}\int (1-\cos 2x) dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x dx \\
&= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C
\end{aligned}$$

$\because \sin^2 x = \frac{1-\cos 2x}{2}$

1.8.6 Integration by Parts

Integration by parts is a technique for performing integration (definite and indefinite) by expanding the differential of a product of function $d(uv)$ and expressing the original integral in terms of a known integral.

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)dx$ and $dv = g'(x)dx$. Then,

$$\boxed{\int u dv = uv - \int v du}$$

Note
For definite integral, $\int_a^b f(x)g'(x)dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)dx$

Tabular Integration

Tabular integration is a special technique for integration by parts. It can be applied to certain functions in the form $f(x)g(x)$, where one of $f(x)$ or $g(x)$ can be differentiated multiple times with ease, while the other function can be integrated multiple times with ease.

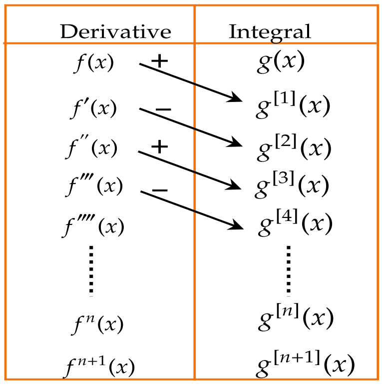


Figure 1.5: Tabular integration

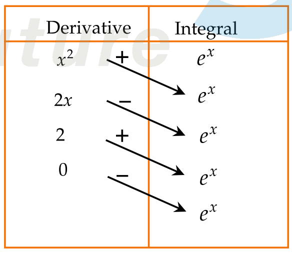
Method of solving

1. In the product comprising the function , identify the polynomial and denote it $f(x)$. Denote the other function in the product by $g(x)$.
2. Create a table of $f(x)$ and $g(x)$, and successively differentiate $f(x)$ until you reach 0 . Successively integrate $g(x)$ the same amount of times.
3. Construct the integral by taking the product of $f(x)$ and the first integral of $g(x)$, then add the product of $f'(x)$ times the second integral of $g(x)$, then add the product of $f''(x)$ times the third integral of $g(x)$, etc...

Exercise 1.8 Integrate $\int x^2 e^x dx$ ■

Solution: .

With $f(x) = x^2$ and $g(x) = e^x$, we list:



We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C$$

1.8.7 Integration by Partial fraction

The method of integrating rational functions (of the form $\frac{P(x)}{Q(x)}$) as a sum of simpler fractions is called the method of partial fraction.This method works by algebraically splitting $P(x)/Q(x)$ into pieces that are easier to integrate.

Method of solving

1. Let $x-r$ be a linear factor of $Q(x)$. Suppose that $(x-r)^m$ is the highest power of $x-r$ that divides $Q(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}$$

Do this for each distinct linear factor of $Q(x)$

2. Let x^2+px+q be a quadratic factor of $Q(x)$. Suppose that $\left(x^2+px+q\right)^n$ is the highest power of this factor that divides $Q(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x+C_1}{x^2+px+q}+\frac{B_2x+C_2}{\left(x^2+px+q\right)^2}+\cdots+\frac{B_nx+C_n}{\left(x^2+px+q\right)^n}$$

   Do this for each distinct quadratic factor of $Q(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $P(x)/Q(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Exercise 1.9 Integrate $\int \frac{6x+7}{(x+2)^2}dx$ ■

Solution: Express the integrand as a sum of partial fractions with undetermined coefficients.

$$\begin{aligned}\frac{6x+7}{(x+2)^2} &= \frac{A}{x+2}+\frac{B}{(x+2)^2} \qquad (\rightarrow \text{Multiply both sides by}(x+2)^2)\\ 6x+7 &= A(x+2)+B\\ &= Ax+(2A+B)\end{aligned}$$

Equating coefficients of corresponding powers of x gives

$$A=6 \quad \text{and} \quad 2A+B=12+B=7, \quad \text{or} \quad A=6 \quad \text{and} \quad B=-5.$$

$$\begin{aligned}\therefore \int \frac{6x+7}{(x+2)^2}dx &= \int\left(\frac{6}{x+2}-\frac{5}{(x+2)^2}\right)dx\\ &= 6\int\frac{dx}{x+2}-5\int(x+2)^{-2}dx\\ &= 6\ln|x+2|+5(x+2)^{-1}+C\end{aligned}$$

Exercise 1.10 Integrate $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx$ ■

Solution: The given function is an improper fraction.First we need to divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r} 2x \\ x^2-2x-3\overline{\smash{)}\,2x^3-4x^2-x-3}\\ \underline{2x^3-4x^2-6x}\\ 5x-3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3}=2x+\frac{5x-3}{x^2-2x-3}$$

and then,

$$\begin{aligned}\int\frac{2x^3-4x^2-x-3}{x^2-2x-3}dx &= \int 2xdx+\int\frac{5x-3}{x^2-2x-3}dx\\ &= \int 2xdx+\int\frac{2}{x+1}dx+\int\frac{3}{x-3}dx\end{aligned}$$

$$= x^2 + 2\ln|x+1| + 3\ln|x-3| + C$$

1.8.8 Trigonometrtic Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions.

Trigonmetric Substitution

Trigonometric substitution is used to simplify certain integrals containing radical expressions. Depending on the function we need to integrate, we substitute one of the following expressions:

- For $\sqrt{a^2 - x^2}$, use $x = a\sin\theta$.
- For $\sqrt{a^2 + x^2}$, use $x = a\tan\theta$.
- For $\sqrt{x^2 - a^2}$, use $x = a\sec\theta$.

Powers of sines and cosines

For the integrals of the form

$$\int \sin^m x \cos^n x dx$$

- If , **m=n=odd**
  Write m as $2k+1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain
  $$\sin^m x = \sin^{2k+1} x = \left(\sin^2 x\right)^k \sin x = \left(1 - \cos^2 x\right)^k \sin x$$
  Then combine the single $\sin x$ with dx in the integral and set $\sin x dx$ equal to $-d(\cos x)$.
- If , **m=even and n=odd**
  Write n as $2k+1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain
  $$\cos^n x = \cos^{2k+1} x = \left(\cos^2 x\right)^k \cos x = \left(1 - \sin^2 x\right)^k \cos x.$$
  Then combine the single $\cos x$ with dx and set $\cos x dx$ equal to $d(\sin x)$.
- If , **m=n=even**
  Substitute
  $$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}$$
  To reduce the integrand to one in lower powers of $\cos 2x$.

Note

$$\sin mx \sin nx = \tfrac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

$$\sin mx \cos nx = \tfrac{1}{2}[\sin(m-n)x + \sin(m+n)x]$$

$$\cos mx \cos nx = \tfrac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

Exercise 1.11 Evaluate $\int \cos^5 x dx$ ■

Solution:

$$\begin{aligned}\int \cos^5 x dx = \int \cos^4 x \cos x dx &= \int \left(1-\sin^2 x\right)^2 d(\sin x) \\ &= \int \left(1-u^2\right)^2 du \\ &= \int \left(1 - 2u^2 + u^4\right) du\end{aligned}$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

Exercise 1.12 Evaluate $\int_0^{\pi/4} \sqrt{1+\cos 4x}dx$ ■

Solution: To eliminate the square root we use the identity

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \text{or} \quad 1+\cos 2\theta = 2\cos^2\theta$$

With $\theta = 2x$, this becomes

$$1+\cos 4x = 2\cos^2 2x$$

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}dx$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x| dx = \sqrt{2}\int_0^{\pi/4} \cos 2x dx \qquad \cos 2x \geq 0 \text{ on } [0, \pi/4]$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1-0] = \frac{\sqrt{2}}{2}$$

Exercise 1.13 Evaluate, $\int \frac{x^2dx}{\sqrt{9-x^2}}$ ■

Solution:

$$\text{Let, } x = 3\sin\theta, \quad dx = 3\cos\theta d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$9-x^2 = 9 - 9\sin^2\theta = 9\left(1-\sin^2\theta\right) = 9\cos^2\theta$$

Then

$$\int \frac{x^2dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta d\theta}{|3\cos\theta|}$$

$$= 9\int \sin^2\theta d\theta \quad \cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= 9\int \frac{1-\cos 2\theta}{2} d\theta$$

$$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C \qquad |\sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C$$

Exercise 1.14 Evaluate $\int \sin 3x \cos 5x dx$ ■

Solution: We know that , $\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$

m = 3 and n = 5 we get

$$\begin{aligned}\int \sin 3x \cos 5x dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x]dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x)dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C\end{aligned}$$

1.9 Summation and Serises

Sum of Arithmatic series

$$S = \sum_{n=0}^{+N}[a+(n-1)d]$$

$$S_N = \frac{N}{2}[a+l] \rightarrow (l = \text{Last term})$$

$$S_N = \frac{N}{2}[2a+(n-1)d]$$

Sum of Geometric series

$$S_n = \frac{a(1-r^n)}{1-r} \qquad (r<1)$$

$$S_n = \frac{a(r^n-1)}{r-1} \qquad (r>1)$$

Sum of Infinite series

$$S_n = \frac{a}{r-1}$$

1.10 Binomial Theorem

Theorem 1.10.1 For any positive integer n, the n^{th} power of the sum of two numbers a and b may be expressed as the sum of $n+1$ terms of the form,

$$(a+b)^n = \sum_{n=o}^{r} {}^nC_r a^{n-r} b^r \tag{1.6}$$

1.11 Taylor and Maclaurin series

Definition 1.11.1 Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor series generated by f at $x=a$ is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The Maclaurin series generated by f is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

The Taylor series generated by f at $x=0$.

Taylor polynomial

Let f be a function with derivatives of order k for $k = 1,2,\ldots,N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the Taylor polynomial of order n generated by f at $x=a$ is the polynomial,

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Exercise 1.15 Find the Taylor series and the Taylor polynomials generated by $f(x) = e^x$ at $x = 0$. ■

Solution:

$$f(x) = e^x, \quad f'(x) = e^x, \quad \ldots, \quad f^{(n)}(x) = e^x, \quad \ldots$$

we have

$$f(0) = e^0 = 1, \quad f'(0) = 1, \quad \ldots, \quad f^{(n)}(0) = 1, \quad \ldots$$

The Taylor series generated by f at $x = 0$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$
$$= 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Exercise 1.16 In the Taylor's series expansion of e^x about $x = 2$, the coefficient of $(x-2)^4$ is ■

Solution: $f(x)$ in the neighborhood of a is given by

$$f(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$$

$$\text{where} b_n = \frac{f^n(a)}{n!}$$

$$f^4(x) = e^x \cdot f^4(2) = e^2$$

$$\text{Therefore, coefficient of}(x-2)^4 = b_4 = \frac{f^4(2)}{4!} = \frac{e^2}{4!}$$

Derivatives	**Integrals**
$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$
$\frac{d}{dx}(x) = 1$	$\int dx = x + C$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dx}(-\cos x) = \sin x$	$\int \sin x dx = -\cos x + C$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x dx = \tan x + C$
$\frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x$	$\int \operatorname{cosec}^2 x dx = -\cot x + C$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x dx = \sec x + C$
$\frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x$	$\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$
$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{\mathrm{dx}}{\sqrt{1-x^2}} = \sin^{-1} x + C$
$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$	$\int -\frac{dx}{\sqrt{1-x^2}} = \cos^{-1} x + C$
$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
$\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$	$\int -\frac{dx}{1+x^2} = \cot^{-1} x + C$
$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int \frac{dx}{\lvert x\rvert\sqrt{x^2-1}} = \sec^{-1} x + C$
$\frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right) = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int \frac{dx}{-\lvert x\rvert\sqrt{x^2-1}} = \operatorname{cosec}^{-1} x + C$
$\frac{d}{dx}\left(e^x\right) = e^x$	$\int e^x dx = e^x + C$
$\frac{d}{dx}(\log\lvert x\rvert) = \frac{1}{x}$	$\int \frac{dx}{x} = \log\lvert x\rvert + C$
$\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x$	$\int a^x dx = \frac{a^x}{\log a} + C$
	$\int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}\frac{(2n-1)!}{(2\alpha)^n}$
	$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$
	$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert + C$
	$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left\lvert x + \sqrt{x^2 \pm a^2}\right\rvert + C$
	$\int x \sin nx dx = \frac{1}{n^2}(\sin nx - nx\cos nx) + C$
	$\int e^{ax} \sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C$
	$\int e^{ax} \cos bx dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + C$

Table 1.1: List of important Derivatives and Integrals

Basic Trigonometric identities

Pythogorean identity

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Angle Sum and Differences

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Double Angle

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ &= \frac{2\tan\theta}{1+\tan^2\theta} \end{aligned}$$

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \\ &= \frac{1-\tan^2\theta}{1+\tan^2\theta} \end{aligned}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Half Angle

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\begin{aligned} \tan\frac{\theta}{2} &= \csc\theta - \cot\theta \\ &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\ &= \frac{\sin\theta}{1+\cos\theta} \\ &= \frac{1-\cos\theta}{\sin\theta} \end{aligned}$$

Product to Sum

$$\sin\theta\sin\varphi = \frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{2}$$

$$\sin\theta\cos\varphi = \frac{\sin(\theta+\varphi)+\sin(\theta-\varphi)}{2}$$

$$\cos\theta\sin\varphi = \frac{\sin(\theta+\varphi)-\sin(\theta-\varphi)}{2}$$

$$\cos\theta\cos\varphi = \frac{\cos(\theta-\varphi)+\cos(\theta+\varphi)}{2}$$

Sum to Product

$$\sin\theta \pm \sin\varphi = 2\sin\left(\frac{\theta\pm\varphi}{2}\right)\cos\left(\frac{\theta\mp\varphi}{2}\right)$$

$$\cos\theta + \cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$$

$$\cos\theta - \cos\varphi = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$$

Practice Set-1

1. For what value of λ is the function $f(x)$ continuous at $x=3$? $f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, x\neq 3\\ \lambda, x=3\end{cases}$

   a. $\lambda=6$ **b.** $\lambda=3$
   c. $\lambda=2$ **d.** $\lambda=9$

2. $f(x)=(1+3x)^{1/3}$ when $\neq 0 f(0)=e^3$ is continuous for $x=0$

   a. 0 **b.** 1
   c. e^3 **d.** e^{-3}

3. Let $f(x)=(x-2)^{17}(x+5)^{24}$. Then

   a. f does not have a critical point at 2
   b. f has a minimum at 2
   c. f has a maximum at 2
   d. f has neither a minimum nor a maximum at 2

4. Find the intervals in which the function $f(x)=x^4-\frac{x^3}{3}$ is decreasing.

   a. $\left(-\infty,\frac{1}{4}\right)$. **b.** $\left(+\infty,\frac{1}{4}\right)$.
   c. $\left(-\frac{1}{4},+\frac{1}{4}\right)$. **d.** $\left(-\infty,\frac{1}{2}\right)$.

5. Find the intervals in which the function $f(x)=2x^3+9x^2+12x+24$ is increasing.

   a. $(-\infty,-2)\cup(-1,\infty)$ **b.** $(-2,-1)$
   c. $(-\infty,-1)\cup(2,\infty)$ **d.** $(-1,\infty)$

6. In the Taylor series expansion of e^x about $x=2$ the coefficient of $(x-2)^4$

   a. $\frac{1}{4!}$ **b.** $\frac{2^4}{4!}$ **c.** $\frac{e^2}{4!}$ **d.** $\frac{e^4}{4!}$

7. For the function e^{-x}, the linear approximation around $x=2$ is

   a. $(3-x)e^{-2}$ **b.** $1-x$
   c. $[3+2\sqrt{2}-(1+\sqrt{2})x]e^{-2}$ **d.** e^{-2}

8. What is the value of $\lim_{x\to 0}\frac{e^x-\left(1+x+\frac{x^2}{2}\right)}{x^3}$?

   a. 0 **b.** $1/6$ **c.** $1/3$ **d.** 1

9. The infinite series $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$ corresponds to

   a. $\sec x$ **b.** e^x **c.** $\cos x$ **d.** $1+\sin^2 x$

10. The volume of solid of revolution when rotated about x -axis is given as

    a. $\int_a^b \pi y^2 dy$ **b.** $\int_a^b \pi x^2 dx$ **c.** $\int_a^b \pi y^2 dx$ **d.** $\int_a^b \pi x^2 dy$

11. The expansion of $e^{\sin(x)}$ is?

a. $1+x+x^2/2+x^4/8+\ldots$

b. $1+x-x^2/2+x^4/8+\ldots.$

c. $1+x+x^2/2-x^4/8+\ldots$

d. $1+x+x^3/6-x^5/10+\ldots$

12. The necessary condition for the maclaurin expansion to be true for function f(x) is

a. f(x) should be continuous.

b. f(x) should be differentiable.

c. f(x) should exists at every point.

d. f(x) should be continuous and differentiable.

13. Let $F(x)=\int_0^x \left(t^2-3t+2\right)dt$, then F has

a. A local maximum at $x=1$ and a local minimum at $x=2$

b. A local minimum at $x=1$ and a local maximum at $x=2$

c. Local maxima at $x=1$ and $x=2$

d. Local minima at $x=1$ and $x=2$

14. Find the Taylor series generated by $f(x)=1/x$ at $a=2$.

a. $\frac{1}{2}+\frac{(x-2)}{2^2}+\frac{(x-2)^2}{2^3}+\cdots+(-1)^{n+1}\frac{(x-2)^n}{2^{n+1}}+\cdots$

b. $\frac{1}{2}-\frac{(x-2)}{2^2}+\frac{(x-2)^2}{2^3}-\cdots+(-1)^n\frac{(x-2)^n}{2^{n+1}}+\cdots$

c. $\frac{(x-2)}{2^2}-\frac{(x-2)^2}{2^3}-\cdots+(-1)^n\frac{(x-2)^n}{2^{n+1}}-\cdots$

d. $1-\frac{(x)}{2^2}+\frac{(x)^2}{2^3}-\cdots+(-1)^n\frac{(x-)^n}{2^{n+1}}+\cdots$

15. Let $A(t)$ denote the area bounded by the curve $y=e^{-|x|}$, the x -axis and the straight lines $x=-t$ and $x=t$. Then $\lim_{t\to\infty}A(t)$ is equal to

a. 2 **b.** -2

c. 1 **d.** -1

Answer key			
Q.No.	Answer	Q.No.	Answer
1	**a**	2	**c**
3	**d**	4	**a**
5	**a**	6	**c**
7	**a**	8	**b**
9	**b**	10	**c**
11	**c**	12	**d**
13	**a**	14	**b**
15	**a**		

Practice Set-2

1. If a function is given by
$$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & x \neq 0 \\ 2, & x = 0 \end{cases}$$
Find out whether or not $f(x)$ is continuous at $x = 0$.

Solution: We have L.H.L. at $x = 0$

$$\begin{aligned} \text{L.H.L.} &= \lim_{x\to 0} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) \\ &= \lim_{h\to 0}\left[\frac{\sin(-h)}{-h} + \cos(-h)\right] = 1+1 = 2 \end{aligned}$$

R.H.L. at $x = 0$

$$\begin{aligned} \text{R.H.L.} &= \lim_{x\to 0} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) \\ &= \lim_{h\to 0}\left[\frac{\sin h}{h} + \cos h\right] = 1+1 = 2 \end{aligned}$$

Also, we know that $f(0) = 2$. Thus, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$
Hence, $f(x)$ is continuous at $x = 0$.

2. Discuss the continuity of $f(x) = 2x - |x|$ at $x = 0$.

Solution: We have
$$f(x) = 2x - |x| = \begin{cases} 2x - x, \text{ if } x \geq 0 \\ 2x - (-x), \text{ if } \quad x < 0 \end{cases}$$
$$\Rightarrow f(x) = \begin{cases} x, \text{ if } x \geq 0 \\ 3x, \text{ if } \quad x < 0 \end{cases}$$

L.H.L. at x=0
$$= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3x = 3\times 0 = 0$$
R.H.L. at $x = 0$
$$= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$$
and $f(0) = 0$
$$= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$
So, $f(x)$ is continuous at $x = 0$.

3. Find all local maxima and minima for $f(x,y) = x^2 - y^2$.

Solution: Here, $f_x = 2x, f_y = -2y, f_{xx} = 2, f_{yy} = -2, f_{xy} = 0$

Applying $f_x = 0, f_y = 0$, we get $x = 0$ and $y = 0$.

So, the critical point of the functions are $(0,0)$

Here, $Df(x,y)(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = (2\times -2) - 0 = -4 < 0$

So, there is neither a maximum nor minimum at $(0,0)$, and so there are no local maxima or minima of the function.

Note **Classifying stationary points of a function of two variables**

Consider a function of two variables, $f(xy)$. Let (a,b) be a stationary point, so that $f_x = 0$ and $f_y = 0$ at (a,b). Then:

- If $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a,b) then (a,b) is a saddle point.
- If $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b) then (a,b) is either a maximum or a minimum. Distinguish between these as follows:
  - If $f_{xx} < 0$ and $f_{yy} < 0$ at (a,b) then (a,b) is a maximum point.
  - If $f_{xx} > 0$ and $f_{yy} > 0$ at (a,b) then (a,b) is a minimum point.
- If $f_{xx}f_{yy} - f_{xy}^2 = 0$ then anything is possible. More advanced methods are required to classify the stationary point properly.

4. Find the absolute maximum and minimum values of $f(x) = \sin x + \frac{1}{2}\cos 2x$ in $\left[0, \frac{\pi}{2}\right]$

Solution: We have

$$f(x) = \sin x + \frac{1}{2}\cos 2x \quad \text{in} \quad \left[0, \frac{\pi}{2}\right]$$

Differentiating with respect to x, we get

$$f'(x) = \cos x - \sin 2x$$

For absolute maximum and absolute minimum,

$$\begin{aligned}
&f'(x) = 0 \\
&\Rightarrow \cos x - 2\sin x \cos x = 0 \\
&\Rightarrow \cos x(1 - 2\sin x) = 0 \\
&\Rightarrow \cos x = 0 \quad \text{or} \quad \sin x = \frac{1}{2} \\
&x = \frac{\pi}{2} \text{ or } \frac{\pi}{6}
\end{aligned}$$

$$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} + \frac{1}{2}\cos\frac{\pi}{3} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

Now, $f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \frac{1}{2}\cos\pi = 1 - \frac{1}{2} = \frac{1}{2}$

$$f(0) = \sin 0 + \frac{1}{2}\cos 0 = 0 + \frac{1}{2} = \frac{1}{2}$$

The absolute maximum value $= 3/4$

The absolute minimum value $= 1/2$.

5. Expand $e^{\sin x}$ by Maclaurin's series up to the term containing x^4.

Solution:

$$f(x) = e^{\sin x}$$

$$\begin{aligned}
f'(x) &= e^{\sin x}\cos x \cdot f(x)\cdot \cos x\\
f''(x) &= f'(x)\cos x - f(x)\sin x\\
f''(0) &= 1\\
f'''(x) &= f''(x)\cos x - 2f'(x)\sin x - f(x)\cos x\\
f'''(0) &= 0\\
f''''(x) &= f'''(x)\cos x - 3f'(x)\sin x - 3f'(x)\cos x \cdot f(x)\sin x\\
f''''(0) &= -3
\end{aligned}$$

and so on. Substituting the values of $f(0), f'(0)$, etc. in the Maclaurin's series, we get

$$\begin{aligned}
e^{(\sin x)} &= 1 + x\cdot 1 + \frac{x^2\cdot 1}{2!} + \frac{x^3\cdot 0}{3!} + \frac{x^4\cdot(-3)}{4!} + \cdots\\
&= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots
\end{aligned}$$

6. For what interval is $f(x) = \frac{x}{2} + \frac{2}{x}, x \neq 0$ increasing?

a. $(-\infty,-2)\cup(2,\infty)$

b. $(-\infty,-1)\cup(1,\infty)$

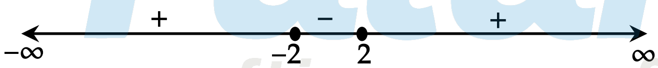
c. $(-2,2)$

d. $(0,\infty)$

Solution: We have

$$\begin{aligned}
f(x) &= \frac{x}{2} + \frac{2}{x}\\
\Rightarrow \quad f'(x) &= \frac{1}{2} - \frac{2}{x^2} = \frac{x^2-4}{2x^2}
\end{aligned}$$

For $f(x)$ to be increasing,



$$\begin{aligned}
& f'(x) > 0\\
\Rightarrow & \frac{x^2-4}{2x^2} > 0\\
\Rightarrow & x^2 - 4 > 0 \Rightarrow (x-2)(x+2) > 0\\
\Rightarrow & x < -2 \quad \text{or} \quad x > 2
\end{aligned}$$

So, $f(x)$ is increasing on $(-\infty,-2)\cup(2,\infty)$.

7. As x increased from $-\infty$ to ∞, the function $f(x) = \frac{e^x}{1+e^x}$ (a) monotonically increases (b) monotonically decreases (c) increases to a maximum value and then decreases

a. Monotonically increases

b. Monotonically decreases

c. Increases to a maximum value and then decreases

d. Decreases to a minimum value and then increases

Solution:

We have,

$$f(x) = \frac{e^x}{1+e^x}$$

Differentiating $f(x)$, we get

$$f'(x) = \frac{e^x(1+e^x) - e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$$

Since e^x is positive for all values of x, $f'(x)$ is positive for all values of x and hence $f(x)$ monotonically increases.

8. The Taylor's series expansion of $\frac{\sin x}{x-\pi}$ at $x = \pi$ is given by

a. $1 + \frac{(x-\pi)^2}{3!} + \cdots$ **b.** $-1 - \frac{(x-\pi)^2}{3!} + \cdots$

c. $1 - \frac{(x-\pi)^2}{3!} + \cdots$ **d.** $-1 + \frac{(x-\pi)^2}{3!} + \cdots$

Solution: Taylor's series expansion of $f(x)$ around $x = \pi$ is

$$f(x) = f(\pi) + \frac{x-\pi}{1!} f'(\pi) + \frac{(x-\pi)^2}{2!} f''(\pi) + \cdots$$

$$\text{Now, } f(\pi) = \lim_{x\to\pi} \frac{\sin x}{x-\pi} = \lim_{x\to\infty} \frac{\cos x}{1} = -1$$

Similarly, by using L'Hospital's rule, we can show that

$$f'(\pi) = 0$$

$$\text{and} \quad f''(\pi) = -\frac{1}{3}$$

$$\text{So, the expansion is} \quad f(x) = -1 + (-1/6)(x-\pi)^2 + \cdots$$

$$f(x) = -1 - \frac{(x'-\pi)^2}{3!} + \cdots$$

So the correct answer is **b**

9. Find $\frac{dz}{dt}$ when $z = xy^2 + x^2y, x = at^2, y = 2at$

Solution:

$$\text{We have, } z = xy^2 + x^2y \Rightarrow \frac{\partial z}{\partial x} - y^2 + 2xy \text{ and } \frac{\partial z}{\partial y} = 2xy + x^2$$

$$\text{Also, } x = at^2, y = 2at \Rightarrow \frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$$

$$\begin{aligned}
\text{Hence, } ,\frac{dz}{dt} &= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \left(y^2 + 2xy\right)(2at) + \left(2xy + x^2\right)2a \\
&= \left(4a^2t^2 + 2at^2 \cdot 2at\right)2at + \left(2at^2 2at + a^2t^4\right)2a \\
&= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^4t^3 \\
&= 16a^3t^3 + 8a^3t^4 + 2a^4t^3 \\
&= 16a^3t^3 + 10a^3t^4
\end{aligned}$$

10. If $x^y + y^x = c$, find the value of $\frac{dy}{dx}$

Solution:

$$\text{Let } f(x,y) = x^y + y^x$$

$$\text{Then, } \frac{\partial f}{\partial x} = yx^{y-1} + y^x \log_e y; \text{ Similarly } , \frac{\partial f}{\partial y} = x^y \log_e x + xy^{x-1}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{y^x \log_e y + y^{y-1}}{x^y \log_e x + xy^{x-1}}$$

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