1. EM waves

1.1 Electrodynamics Before Maxwell

Till now in electromagnetic theory we found out four equations which contribute to the foundation of Electromagnetic theory, and they are,

$$2\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \Rightarrow \text{Gauss' Law}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \Rightarrow \text{Gauss' Law for magnetism}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \Rightarrow \text{Faraday's Law}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}\right) \qquad \Rightarrow \text{Ampere's Law}$$

There is a fatal inconsistency in these formulas. It has something to do with the old rule that divergence of curl is always zero $(\nabla.(\nabla\times E) = 0)$. If you apply the divergence to Faraday's law, everything works out:

$$\nabla\cdot(\nabla\times\mathbf{E}) = \nabla\cdot\left(-\frac{\partial\mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla\cdot\mathbf{B})$$

The left side is zero because divergence of curl is zero; the right side is zero by virtue of Gauss law for magnetism. But when you do the same thing to Ampere's law, you get into trouble:

$$\nabla\cdot(\nabla\times\mathbf{B}) = \mu_0(\nabla\cdot\mathbf{J})$$

The left side must be zero, but the right side, in general, is not. For steady currents, the divergence of **J** is zero, but evidently when we go beyond magnetostatics Ampère's law cannot be right.

1.1.1 How Maxwell Fixed Ampère's Law

Applying the continuity equation and Gauss's law, in Ampere's law the offending term can be rewritten as:

$$\begin{aligned}
\nabla\cdot\mathbf{J} &= -\frac{\partial\rho}{\partial t} \\
&= -\frac{\partial}{\partial t}(\varepsilon_0\nabla\cdot\mathbf{E}) \\
&= -\nabla\cdot\left(\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)
\end{aligned}$$

If we were to combine $\varepsilon_0(\partial \mathbf{E}/\partial t)$ with $\mathbf{J}$, in Ampère's law, it would be just right to kill off the extra divergence:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Such a modification changes nothing, as far as magnetostatics is concerned: when $\mathbf{E}$ isconstant, we still have $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$.
Just as a changing magnetic field induces an electric field (Faraday's law), so **A changing electric field induces a magnetic field**. Maxwell called his extra term the displacement current:

Displacement Current

$$\boxed{\mathbf{J}_d \equiv \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}}$$

It's a misleading name, since $\varepsilon_0(\partial \mathbf{E}/\partial t)$ has nothing to do with current, except that it adds to $\mathbf{J}$ in Ampère's law.

1.1.2 Maxwell's Equations

(i)	$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho$	(Gauss's law)
(ii)	$\nabla \cdot \mathbf{B} = 0$	(No name.)
(iii)	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(Faraday's law)
(iv)	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	(Ampère's law with Maxwell's correction)

Every phenomenon in electricity and magnetism can be derived from these equations. Many of our most important tools for various analyses come from the integral version of these equations, which are.

(i)	$\iint_S \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{A}} = \frac{Q}{\varepsilon_0}$	(Gauss's law)
(ii)	$\iint_S \overrightarrow{\mathbf{B}} \cdot d\overrightarrow{\mathbf{A}} = 0$	(No name.)
(iii)	$\oint \overrightarrow{\mathbf{E}} \cdot d\overrightarrow{\mathbf{s}} = -\frac{d\phi_{\mathbf{B}}}{dt}$	(Faraday's law)
(iv)	$\oint \overrightarrow{\mathbf{B}} \cdot d\overrightarrow{\mathbf{s}} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$	(Ampère's law with Maxwell's correction)

Together with the force law, $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, they summarize the entire theoretical content of classical electrodynamics. Even the continuity equation, $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$ which is the mathematical expression of conservation of charge, can be derived from Maxwell's equations by applying the divergence to Ampere's law.

1.1.3 Maxwell's Equations In Free Space

An extremely important limit of Maxwell's equations is found when there are no sources: $\rho = 0, \vec{J} = 0$. The equations become,

(i)	$\nabla \cdot \mathbf{E} = 0$	(Gauss's law)
(ii)	$\nabla \cdot \mathbf{B} = 0$	(No name.)
(iii)	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(Faraday's law)
(iv)	$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	(Ampère's law with Maxwell's correction)

1.1.4 Maxwell's Equations in Matter

For inside polarized matter there will be accumulations of "bound" charge and current over which you exert no direct control. It would be nice to reformulate Maxwell's equations in such a way as to make explicit reference only to those sources we control directly: the "free" charges and currents.
We have already learned, from the static case, that an electric polarization **P** produces a bound charge density

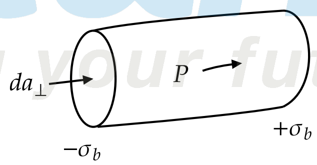
$$\rho_b = -\nabla \cdot \mathbf{P}$$

Likewise, a magnetic polarization (or "magnetization") **M** results in a bound current

$$\mathbf{J}_b = \nabla \times \mathbf{M}$$

There's just one new feature to consider in the nonstatic case: Any change in the electric polarization involves a flow of (bound) charge (call it $\mathbf{J}_p$), which must be included in the total current. For suppose we examine a tiny chunk of polarized material The polarization introduces a charge density $\sigma_b = P$ at one end and $-\sigma_b$ at the other. If P now increases a bit, the charge on each end increases accordingly.
giving a net current

$$dI = \frac{\partial \sigma_b}{\partial t} da_{\perp} = \frac{\partial P}{\partial t} da_{\perp}$$



The current density, therefore, is

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}$$

This polarization current has nothing whatever to do with the bound current $\mathbf{J}_b$. The latter is associated with magnetization of the material and involves the spin and orbital motion of electrons; $\mathbf{J}_p$, by contrast, is the result of the linear motion of charge when the electric polarization changes. If **P** points to the right and is increasing, then each plus charge moves a bit to the right and each minus charge to the left; the cumulative effect is the polarization current $\mathbf{J}_p$

In fact, $\mathbf{J}_p$ is essential to account for the conservation of bound charge.
In view of all this, the total charge density can be separated into two parts:

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P}$$

and the current density into three parts:

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}$$

Gauss's law can now be written as

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\left(\rho_f - \nabla \cdot \mathbf{P}\right)$$

or

$$\nabla \cdot \mathbf{D} = \rho_f$$

where $\mathbf{D}$, as in the static case, is given by

$$\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P}$$

Meanwhile, Ampère's law (with Maxwell's term) becomes

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}\right) + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

or

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

where, as before,

$$\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

Faraday's law and $\nabla \cdot \mathbf{B} = 0$ are not affected by our separation of charge and current into free and bound parts, since they do not involve ρ or $\mathbf{J}$.
In terms of free charges and currents, then, Maxwell's equations read

$$\boxed{\begin{array}{llll} \textbf{(i)} & \nabla \cdot \mathbf{D} = \rho_f & \textbf{(iii)} & \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\ \textbf{(ii)} & \nabla \cdot \mathbf{B} = 0 & \textbf{(iv)} & \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{array}}$$

for linear media

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \quad \textit{and} \quad \mathbf{M} = \chi_m \mathbf{H}$$

$$\mathbf{D} = \varepsilon \mathbf{E} \quad \textit{and} \quad \mathbf{H} = \tfrac{1}{\mu}\mathbf{B}$$

where $\varepsilon \equiv \varepsilon_0 (1 + \chi_e)$ and $\mu \equiv \mu_0 (1 + \chi_m)$ D is called the electric "displacement"; that's why the second term in the Ampère/Maxwelf equation (iv) is called the displacement current,
In integral form Maxwell's equations in matter can be written as

$$\begin{array}{ll} \text{(i)} & \oint_s D \cdot da = \theta_{fenc} \\ \text{(ii)} & \oint_s B \cdot da = 0 \\ \text{(iii)} & \oint_p E \cdot dl = \frac{-d}{dt}\int_s B \cdot da \\ \text{(iv)} & \oint_p H \cdot dl = I_{end} + \frac{d}{dt}\int_s D \cdot da \end{array}$$

P is the closed loop enclosing the surface S

1.2 Boundary conditions

In general the fields E,B,D, and H will be discontineous at a boundary between two different media or at surface that carries charge density σ or current density K. The explicit form of the discontinities can be deduced from Maxwell's equations,in their integral form

(i) $\oint_{\mathscr{S}} \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$

(ii) $\oint_{\mathscr{S}} \mathbf{B} \cdot d\mathbf{a} = 0$

$$\left.\begin{array}{ll|l} \text{(iii)} & \oint_{\mathscr{P}} \mathbf{E} \cdot d\mathbf{l} & = -\frac{d}{dt}\int_{\mathscr{S}} \mathbf{B} \cdot d\mathbf{a} \\ \text{(iv)} & \oint_{\mathscr{P}} \mathbf{H} \cdot d\mathbf{l} & = I_{f_{\text{enc}}} + \frac{d}{dt}\int_{\mathscr{S}} \mathbf{D} \cdot d\mathbf{a} \end{array}\right\} \text{for any surface } \mathscr{S} \text{ bounded by the closed loop } \mathscr{P}.$$

Applying (i) to a tiny, wafer-thin Gaussian pillbox extending just slightly into the material on either side of the boundary, we obtain

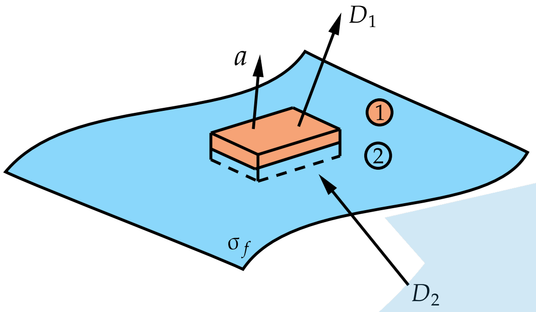


Figure 1.1

$$\mathbf{D}_1 \cdot \mathbf{a} - \mathbf{D}_2 \cdot \mathbf{a} = \sigma_f a$$

Thus ,the component of D that is perperdicular to the interface is discontineous in the amount

$$D_1^{\perp} - D_2^{\perp} = \sigma_f$$

Identical reasoning,applied to equation(ii) yields

$$B_1^{\perp} - B_2^{\perp} = 0$$

consider equation(iii) a very thin Amperian loop straddling the surface gives

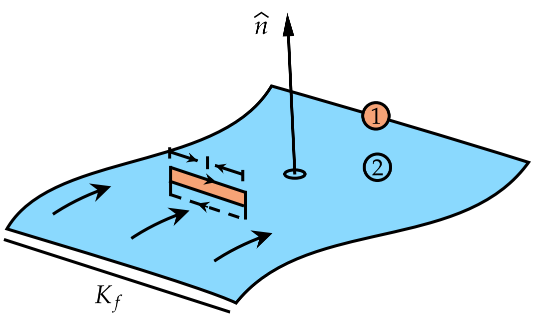


Figure 1.2

$$E_1 \cdot L - E_2 \cdot L = -\frac{d}{dt}\int_s B \cdot da$$

But in the limit as the width of the loop goes to zero ,the flux vanishes.Therefore

$$E_1^{\|} - E_2^{\|} = 0$$

That is the comonents of E parallel to the interface are contineous across the boundary.
equation (iv) implies

$$H_1 \cdot L - H_2 \cdot L = I_{f_{enc}}$$

Where $I_{f_{enc}}$ is the free current passing through the Amperian loop.No volume current density will contribute but a surface current can . In fact if $\hat{n}$ is a unit vector perpendicular to the interface so that $(\hat{n} \times L)$ is normal to the Amperian loop then,

$$I_{f_{enc}} = K_f \cdot (\hat{n} \times L) = (K_f \times \hat{n}) \cdot L$$

And hence

$$H_1^{\|} - H_2^{\|} = K_f \times \hat{n}$$

So the parallel components H are discontinuous by an amount proportional to the free surface charge density.
In the case of linear media they can be expressed in terms of E and B alone
(i) $\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f$,
(iii) $\mathbf{E}_1^{\|} - \mathbf{E}_2^{\|} = 0$,
(ii) $B_1^{\perp} - B_2^{\perp} = 0$,
(iv) $\frac{1}{\mu_1}\mathbf{B}_1^{\|} - \frac{1}{\mu_2}\mathbf{B}_2^{\|} = \mathbf{K}_f \times \hat{\mathbf{n}}$.
In particular, if there is no free charge or free current at the interface, then
(i) $\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = 0$,
(iii) $\mathbf{E}_1^{\|} - \mathbf{E}_2^{\|} = 0$
(ii) $B_1^{\perp} - B_2^{\perp} = 0$,
(iv) $\frac{1}{\mu_1}\mathbf{B}_1^{\|} - \frac{1}{\mu_2}\mathbf{B}_2^{\|} = 0$.

1.3 Electromagnetic Wave

Waves
Classical wave equation can be represents as

$$\frac{d^2 f}{dz^2} = \frac{1}{V^2} \frac{d^2 f}{dt^2} \tag{1.1}$$

Where V is the velocity of propagation

$$V = \sqrt{\frac{T}{\mu}} \text{in the case of waves in a string}$$
$$T = \text{Tension}$$
$$\mu = \text{man per unit length}$$

Equation.1.1 has a general solution in the form.

$$f(z,t) = A\cos[k(z - vt) + \delta]$$
$$\text{where } k = \frac{2\pi}{\lambda} \text{is clled wave number}$$
$$\text{If } T \text{ is the time period} T = \frac{1}{x}$$
$$\text{And } \omega \text{ is the angular frequency } \omega = 2\pi x$$
$$f(z,t) = A\cos(kz - \omega t + \delta)$$

By using complex notation,

$$f(z,t) = Ae^{i(kz-\omega t)}$$
$$f(z,t) = Re[f(z,t)]$$

$f(z,t)$is the actual wave function

1.4 Electromagnetic Waves in Vaccum

The wave equation for $\vec{E}$ and $\vec{B}$ is obtained from Maxwell's equations. The four Maxwell's equations in vaccum where there is no charge or current $(\rho = 0$ and $\vec{J} = 0)$,is given by,

(i) $\nabla \cdot E = 0$ **(iii)** $\nabla \times \vec{E} = \frac{-dB}{dt}$

(ii) $\nabla \cdot B = 0$ **(iv)** $\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$

Applying curl to **(iii)** $\nabla \times \vec{E} = \frac{-dB}{dt}$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

Applying $\nabla \cdot E = 0$ and $\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$, we get,

$$\Rightarrow \quad -\nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\Rightarrow \quad \nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\boxed{\nabla^2 E - \mu_0 \varepsilon_0 \frac{d^2 E}{dt^2} = 0}$$

Similiarly applying curl to **(iv)** $\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$ we get,

$$\nabla \times (\nabla \times B) = \nabla \times (\mu_0 \varepsilon_0 \frac{\partial E}{\partial t})$$

$$\Rightarrow \nabla(\nabla \cdot B) - \nabla^2 B = \nabla \times (\mu_0 \varepsilon_0 \frac{\partial E}{\partial t})$$

$$\Rightarrow \quad -\nabla^2 B = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times E)$$

$$= -\mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}$$

$$\Rightarrow \quad \nabla^2 B = \mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}$$

$$\boxed{\nabla^2 B - \mu_0 \varepsilon_0 \frac{d^2 B}{dt^2} = 0}$$

Thus both equations obey the general wave equation,

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

We can write the speed of electromagnetic wave propagation in free space is

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 m/s$$

Which is speed of light in free space. This says that the light is an electromagnetic wave.

Plane Wave Solution

Let us look at "plane wave" solutions to these wave equations for $\vec{E}$ and $\vec{B}$. A plane wave is one for which the surfaces of constant phases, are planes. Time harmonic solutions are of the form $\sin\left(\vec{k}\cdot\vec{r}-\omega t\right)$ or $\cos\left(\vec{k}\cdot\vec{r}-\omega t\right)$. However, mathematically it turns out to be simple to consider an exponential form and take, at the end of calculations, the real or the imaginary part.

We take the solutions to be of the form,

$$\vec{E}=\vec{E}_0e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\vec{B}=\vec{B}_0e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Note that the surfaces of constant phase are given by,

$$\vec{k}\cdot\vec{r}-\omega t=\text{ constant}$$

$\vec{E}_0,\vec{B}_0$ are the complex amplitude, and $\vec{k}$ is the wave vector. $\vec{k}$ is defined as,

$$\begin{aligned}\vec{k}&=\frac{2\pi}{\lambda}\hat{k}\\ &=\frac{2\pi\nu}{c}\hat{k} && \text{Since, } c=\nu\lambda\\ &=\frac{\omega}{c}\hat{k} && \text{Since, } 2\pi\nu=\omega\end{aligned}$$

($\hat{k}$ is unit vector along propagation direction.)

1.4.1 Directions of $\vec{E},\vec{H},\vec{k}$

We have solution of wave equation,

$$\vec{E}(r,t)=\vec{E}_0e^{i(\hat{k}\cdot\vec{r}-\omega t)}$$
$$\vec{H}(r,t)=\vec{H}_0e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

We have four Maxwell's equations as,

$$\nabla\cdot E=0 \qquad \nabla\times\vec{E}=\frac{-dB}{dt}$$
$$\nabla\cdot B=0 \qquad \nabla\times\vec{B}=\mu_0\varepsilon_0\frac{\partial E}{\partial t}$$

When we substitute the solutions of the wave quations, in Maxwell's equations we get,

$$\begin{aligned}&\vec{\nabla}\cdot\vec{E}=0 && \Rightarrow i\vec{k}\cdot\vec{E}=0 && \therefore\vec{k}\perp\vec{E}\\ &\vec{\nabla}\cdot\vec{B}=0 && \Rightarrow i\vec{k}\cdot\vec{B}=0 && \therefore\vec{k}\perp\vec{B}\\ &\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} && \Rightarrow i\vec{k}\times\vec{E}=i\omega\vec{B} && \therefore\vec{k}\times\vec{E}=\omega\vec{B}\\ &\vec{\nabla}\times\vec{B}=\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} && \Rightarrow i\vec{k}\times\vec{B}=-i\omega\mu_0\varepsilon_0\vec{E} && \therefore\vec{k}\times\vec{B}=\omega\mu_0\varepsilon_0\vec{E}\end{aligned}$$

From these equations we can say $\vec{k},\vec{E},\vec{B}$ are mutually perpendicular to each other.

1.4.2 Poynting Vector in Electromagnetic waves

The poynting's vector for the plane electromagnetic wave in free space . Energy per unit volume stored in em field is

$$\begin{aligned}
\vec{S} &= \frac{1}{\mu_0}(\vec{E}\times\vec{B}) \qquad \text{Since, } \vec{k}\times\vec{E} = \omega\vec{B} && (1.2)\\
&= \frac{1}{\mu_0\omega}\vec{E}\times(\vec{k}\times\vec{E})\\
&= \frac{1}{\mu_0\omega}[\vec{k}(\vec{E}\cdot\vec{E}) - E(E\cdot k)]\\
\vec{S} &= \frac{E^2}{\mu_0\omega}\vec{k} && (1.3)
\end{aligned}$$

Thus, the energy flow is in the direction of wave propagation.

We know that $\vec{E}$ is normal to $\vec{k}$,

$$\vec{k}\times\vec{E} = \omega\vec{B} \tag{1.4}$$

We can write in terms of magnitude,

$$\begin{aligned}
kE &= \omega B \qquad \Rightarrow \qquad \frac{k}{\omega}E = B\\
\frac{1}{c}E &= B \qquad \Rightarrow \qquad \sqrt{\mu_0\varepsilon_0}E = B\\
\sqrt{\varepsilon_0}E &= \frac{1}{\sqrt{\mu_0}}B && (1.5)
\end{aligned}$$

Squaring both sides, and multiplying by $\frac{1}{2}$ we get,

$$\varepsilon_0 E^2 = \frac{1}{\mu_0}B^2 \tag{1.6}$$

This shows that in case of electromagnetic waves in free space electromagnetic energy is equally shared between electric and magnetic fields.

Then total energy density of an EM wave in free space can be written as,

$$\begin{aligned}
u &= u_{em} + u_m && (1.7)\\
u &= \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2 && (1.8)\\
u &= \varepsilon_0 E^2 && (1.9)
\end{aligned}$$

$$\boxed{u = \varepsilon_0 E^2}$$

From equation.1.3 we know that ,

$$\vec{S} = \frac{E^2}{\mu_0\omega}\vec{k} \tag{1.10}$$

substituting equation. 1.9 and taking only magnitude we get,

$$S = \frac{u}{\varepsilon_0}\frac{k}{\omega\mu_0} \tag{1.11}$$

$$= \frac{u}{\mu_0\varepsilon_0}\frac{k}{\omega} \tag{1.12}$$

$$= uc^2\frac{1}{c} \tag{1.13}$$

$$S = uc \tag{1.14}$$

In vector form,

$$\vec{S} = uc\hat{k} \qquad \text{Where } \hat{k} \text{ is the propogation direction.} \tag{1.15}$$

$$\boxed{\begin{aligned} \vec{S} &= \frac{1}{\mu_0}(\vec{E}\times\vec{B}) \\ \vec{S} &= \frac{E^2}{\mu_0\omega}\vec{k} \\ \vec{S} &= uc\hat{k} \end{aligned}}$$

Momentum

The momentum density stored in the electromagnetic field is given by,

$$\vec{P} = \frac{\vec{S}}{c^2} \qquad \text{But,} \quad \vec{S} = uc\hat{k}$$

$$\vec{P} = \frac{u}{c}\hat{k}$$

$$\text{Or,} \qquad \vec{P} = \frac{1}{c}\varepsilon_0 E^2\,\hat{k}$$

Avarage Values

If we find the average values of electric field E and magnetic field B we get the values as,

$$\langle E\rangle = \frac{E_0}{\sqrt{2}} \tag{1.16}$$

$$\text{And} \quad \langle B\rangle = \frac{B_0}{\sqrt{2}} \qquad \text{(Here, average value equals the rms value.)} \tag{1.17}$$

Where E_0 and B_0 are the amplitudes.

The average value of $\vec{S}$,

$$\vec{S} = \frac{1}{\mu_0}(E\times B)\hat{k} \tag{1.18}$$

$$\langle\vec{S}\rangle = \frac{1}{\mu_0}\langle(E\times B)\rangle\hat{k} = \frac{1}{\mu_0}\frac{E_0}{\sqrt{2}}\cdot\frac{B_0}{\sqrt{2}}\hat{k} \tag{1.19}$$

$$= \frac{1}{2\mu_0}E_0B_0\hat{k} \tag{1.20}$$

We know that, $E = cB$ and $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$

Then equation. 1.20 becomes,

$$\langle \vec{S} \rangle = \frac{1}{2} c\varepsilon_0 E_0^2 \hat{k} \tag{1.21}$$

Average energy density,

We know that, $u = \varepsilon_0 E^2$. Then,

$$\langle u \rangle = \frac{1}{2} \varepsilon_0 {E_0}^2$$

Average momentum density

We know that, $\vec{P} = \frac{1}{c} \varepsilon_0 E^2 \; \hat{k}$, then

$$\langle \vec{p} \rangle = \frac{1}{2c} \varepsilon_0 {\varepsilon_0}^2 \hat{k}$$

Intensity of Electromagnetic wave

The magnitude of the time average of the Poynting's vector is called the intensity of radiation (I). Thus, the intensity.

$$\begin{aligned} I &= |\langle s \rangle| \\ &= \frac{1}{2} c\varepsilon_0 E^2 \end{aligned}$$

But, we found that, $\langle u \rangle = \frac{1}{2} \varepsilon_0 {E_0}^2$, Then,

$$I = \langle u \rangle c$$

Exercise 1.1 Compute the intensity of the standing electromagnetic wave given by

$$E_y(x,t) = 2E_0 \cos kx \cos \omega t, \quad B_z(x,t) = 2B_0 \sin kx \sin \omega t$$

Solution: The Poynting vector for the standing wave is

$$\begin{aligned} \overrightarrow{\mathbf{S}} &= \frac{\overrightarrow{\mathbf{E}} \times \overrightarrow{\mathbf{B}}}{\mu_0} = \frac{1}{\mu_0} \left(2E_0 \cos kx \cos \omega t \hat{\mathbf{j}}\right) \times \left(2B_0 \sin kx \sin \omega t \hat{\mathbf{k}}\right) \\ &= \frac{4E_0 B_0}{\mu_0} (\sin kx \cos kx \sin \omega t \cos \omega t) \hat{\mathbf{i}} \\ &= \frac{E_0 B_0}{\mu_0} (\sin 2kx \sin 2\omega t) \hat{\mathbf{i}} \end{aligned}$$

The time average of S is

$$\langle S \rangle = \frac{E_0 B_0}{\mu_0} \sin 2kx \langle \sin 2\omega t \rangle = 0$$

The result is to be expected since the standing wave does not propagate. Alternatively, we may say that the energy carried by the two waves traveling in the opposite directions to form the standing wave exactly cancel each other, with no net energy transfer.

1.5 Electromagnetic Waves in Matter

The Maxwell's equation inside matter where there is no free charge or free current are given by,

(i) $\nabla \cdot \vec{D} = 0$ **(iii)** $\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$

(ii) $\nabla \cdot \vec{B} = 0$ **(iv)** $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$

If the medium is linear, $D = \varepsilon E$, $H = \frac{B}{\mu}$

If the medium is homogeneous so that ε and μ do not vary point to point. The Maxwell's equations becomes,

(i) $\nabla \cdot \vec{E} = 0$ **(iii)** $\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$

(ii) $\nabla \cdot \vec{B} = 0$ **(iv)** $\nabla \times \vec{B} = \mu\varepsilon\frac{\partial \vec{E}}{\partial t}$

The electromagnetic waves propgates through a linear homogeneous medium at a speed,

$$V = \frac{1}{\sqrt{\varepsilon\mu}}$$
$$= \frac{c}{n} \qquad \text{Where, } n = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \Rightarrow \text{refractive index of the substance.}$$

Since, for most materials μ is very close to μ_0

$$n \approxeq \sqrt{\varepsilon_r} \qquad \Rightarrow \varepsilon_r = \sqrt{\frac{\varepsilon}{\varepsilon_0}}$$

ε_r is dielectric constant which almost always greater than 1. So speed of light in matter $(V = \frac{c}{n})$ should always be less than C.

1.5.1 Boundary condition

The boundary conditions at the interface can be written as,

(i) $\varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp}$ **(iii)** $E_1^{\parallel} = E_2^{\parallel}$

(ii) $B_1^{\perp} = B_2^{\perp}$ **(iv)** $\frac{B_1^{\parallel}}{\mu_1} = \frac{1}{\mu_2} B_2^{\parallel}$

1.5.2 Reflection and Transmission at Normal incidence

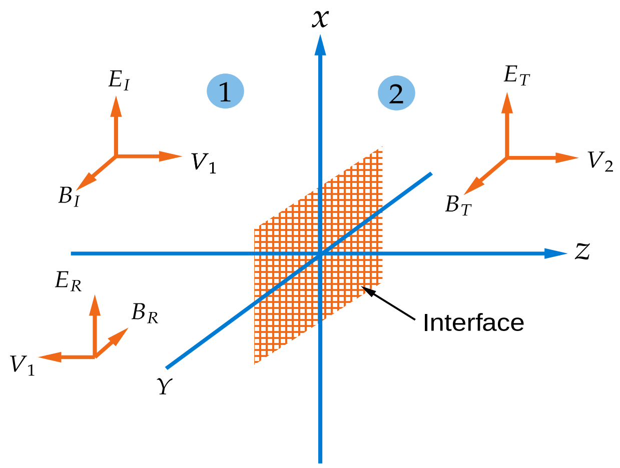


Figure 1.3: Reflection and Transmission at Normal incidence

Suppose the xy plane forms the boundary between two linear media. A plane wave of frequency(ω)travelling in the $z$direction and polarized in x direction approaches the interface from the left.

Incident wave :

$$\left.\begin{aligned} \vec{\mathbf{E}}_I(z,t) &= \vec{E}_{0_I} e^{i(k_1 z-\omega t)}\hat{i} \\ \vec{\mathbf{B}}_I(z,t) &= \tfrac{1}{v_1}\vec{E}_{0_I} e^{i(k_1 z-\omega t)}\hat{j} \end{aligned}\right\} \tag{1.22}$$

v_1 = Velocity in first medium

Reflected wave :

$$\left.\begin{aligned} \vec{\mathbf{E}}_R(z,t) &= \vec{E}_{0_R} e^{i(-k_1 z-\omega t)}\hat{i} \\ \vec{\mathbf{B}}_R(z,t) &= -\tfrac{1}{v_1}\vec{E}_{0_R} e^{i(-k_1 z-\omega t)}\hat{j} \end{aligned}\right\} \tag{1.23}$$

Transmitted wave :

$$\left.\begin{aligned} \vec{\mathbf{E}}_T(z,t) &= \vec{E}_{0_T} e^{i(k_2 z-\omega t)}\hat{i} \\ \vec{\mathbf{B}}_T(z,t) &= \tfrac{1}{v_2}\vec{E}_{0_T} e^{i(k_2 z-\omega t)}\hat{j} \end{aligned}\right\} \tag{1.24}$$

v_2 = Velocity in second medium

At $z = 0$, the combined field on the left $\vec{E}_I + \vec{E}_R$ and $\tilde{B}_I + \vec{B}_R$, must join the fields on the right $\vec{E}_T \& \vec{B}_T$, in accordance with the boundary conditions in section 1.5.1.

There are no electric components in perpendicular direction,Then the third boundary condition $E_1^{\parallel} = E_2^{\parallel}$ gives,

$$\vec{E}_{01} + \vec{E}_{0R} = \vec{E}_{0T} \tag{1.25}$$

Then the fourth boundary condition $\frac{1}{\mu_1}B_1^{\parallel} = \frac{1}{\mu_2}B_2^{\parallel}$ gives,

$$\frac{1}{\mu_1}\left(\frac{1}{v_1}\vec{E}_{0I} - \frac{1}{v_1}\vec{E}_{0R}\right) = \frac{1}{\mu_2}\left(\frac{1}{v_1}\vec{E}_{0I} - \frac{1}{v_2}\vec{E}_{0T}\right) \tag{1.26}$$

$$\text{Since} \quad \vec{B}_{0I} = \frac{1}{v_1}\vec{E}_{0I} \quad : \quad \vec{B}_{0R} = \frac{1}{v_1}\vec{E}_{0R} \quad : \quad \vec{B}_{0T} = \frac{1}{v_2}\vec{E}_{0T}$$

$$n_2 = \frac{c}{v_2} \quad \text{and} \quad n_1 = \frac{c}{v_1}$$

Then the equation.1.26 becomes,

$$\vec{E}_{01} - \vec{E}_{0R} = \beta\vec{E}_{0T} \tag{1.27}$$

$$\text{Where,} \quad \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

By solving equation.1.25 and equation. 1.27 we will get,

$$\vec{E}_{0R} = \left(\frac{1-\beta}{1+\beta}\right)\vec{E}_{0I} \quad : \quad \vec{E}_{0T} = \left(\frac{2}{1+\beta}\right)\vec{E}_{0I}$$

If $\mu_1 = \mu_2 = \mu_0 \Rightarrow \quad \beta = \frac{v_1}{v_2} = \frac{n_2}{n_1}$ (For non-magnetic medium). Then,

$$\vec{E}_{0R} = \left(\frac{v_2 - v_1}{v_1 + v_2}\right)\vec{E}_{0I} \quad : \quad \vec{E}_{0T} = \left(\frac{2v_2}{v_1 + v_2}\right)\vec{E}_{0I} \tag{1.28}$$

The reflected wave $\tilde{E}_{0R}$ are in phase with incident wave if $v_2 > v_1$ and out of phase if $v_2 < v_1$

$$,\therefore E_{0R} = \left|\frac{v_2 - v_1}{v_1 + v_2}\right| E_{0I} \quad : \quad E_{0T} = \left(\frac{2v_2}{v_1 + v_2}\right) E_{0I} \tag{1.29}$$

In terms of reflective indices $n_1$and n_2

$$E_{0R} = \left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{0I} \quad : \quad E_{0T} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{0I} \tag{1.30}$$

Reflection coefficient,the function of incident is reflected can be find out,

$$R = \frac{I_R}{I_I} \tag{1.31}$$

The intensity in general can be written as

$$I = \frac{1}{2}\varepsilon v E_0^2 \tag{1.32}$$

if $\mu_1 = \mu_2 = \mu_0$ the **Reflection coefficient**,

$$R = \frac{(E_{0R})^2}{(E_{0I})^2} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \tag{1.33}$$

And **Transmission coefficient**,

$$T = \frac{I_I}{I_I} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\frac{(E_{0I})^2}{(E_{0I})^2} = \frac{4n_1 n_2}{(n_1 + n_2)^2} \tag{1.34}$$

And from these equations we can show that $R + T = 1$ ie. energy is conserved.

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

$$T = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

$$R + T = 1$$

Note When light passes from air $(n_1 = 1)$ and to glass$(n_2 = 1.5)$

$$R = 0.04, \quad 4\%$$
$$T = 0.96, \quad 96\%$$

Most of the light are transmitted.

1.5.3 Reflection and transmission at oblique incidence

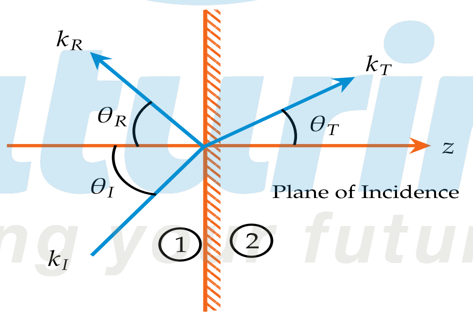


Figure 1.4

Suppose the light is travelling from $-z$ axis.There is a medium at xy plane.Light get reflected and transmitted from the plane.suppose the incident light make an angle θ_I with the normal ie z axis.Let ω be the frequency of the travelling wave.

Incident wave:

$$E_I(r,t) = E_{0I}e^{(k_I.r-\omega t)}$$

$$B_I(r,t) = \frac{1}{v_1}(k_I \times E_I)$$

Reflected wave:

$$E_R(r,t) = E_{0R}e^{(k_R.r-\omega t)}$$

$$B_R(r,t) = \frac{1}{v_1}(k_R \times E_R)$$

Where v_1 be the velocity of first medium.

Transmitted wave

$$E_T(r,t) = E_{0T}e^{(k_T.r-\omega t)}$$

$$B_T(r,t) = \frac{1}{v_2}(k_T \times E_T)$$

Where v_2 be the velocity of the second medium.
The frequency of all the waves are same.so we can relate,

$$k_I v_1 = k_R v_1 = k_T v_2$$
$$\implies k_I = k_R$$
$$\implies k_I = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T$$

Where n_1 and n_2 are the refractive index of the first and second medium.
At $z = 0$ the combined field of electric and magnetic components are equal
ie

$$E_I + E_R = E_T$$

and

$$B_I + B_R = B_T$$

Which give rise to

$$E_I(r,t) = E_{0I} e^{(k_I.r-\omega t)} + E_R(r,t) = E_{0R} e^{(k_R.r-\omega t)} = E_T(r,t) = E_{0T} e^{(k_T.r-\omega t)}$$

Because the boundary conditions must hold at all points on the plane and for all time ,these exponential factors must be equal at z=0.Which implies

$$k_I \cdot r = k_R \cdot r = k_T \cdot r \quad atz = 0$$

after taking the dot product

$$x(k_I)_x + y(k_I)_y = x(k_R)_x + y(k_R)_y = x(k_T)_x + y(k_T)_y$$

if x=0

$$(k_I)_y = (k_R)_y = (k_T)_y$$

suppose the incident ray lies in the xz plane then $(k_I)_y = 0$
Then

$$(k_R)_y = (k_T)_y = 0$$

There will be no ray along y axis. Which means that all the rays lies in the same plane.
conclusion
First law:The incident ,reflected, and transmitted wavevectors form a plane.(called plane of incidence),which also include the normal to the surface(z axix)
Here the common plane is xz plane
if y=o

$$(k_I)_x = (k_R)_x = (k_T)_x$$

Which implies that

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

Where θ_I is the angle of incidence,θ_R is the angle of reflection,and θ_T is called angle of transmission also known as angle of refraction.
Second law

$$k_I \sin\theta_I = k_R \sin\theta_R$$
$$k_I = k_R$$

Then

$$\theta_I = \theta_R$$

The angle of incidence is the angle of reflection

$$k_I = k_R \neq k_T$$

Then

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{k_T}{k_R}$$

Third law

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{n_1}{n_2}$$

This is the **law of refraction or Snell's law.**

1.5.4 polarization

Consider the boundary conditions at the surface

$$(i) \quad \varepsilon_1(E_{0_I} + E_{0_R})_z = \varepsilon_2(E_{0_T})_z$$

$$(ii) \quad (B_{O_I} + B_{0_R})_z = (B_{0_T})_z$$

$$(iii) \quad (E_{0_I} + E_{0_R})_{x,y} = (E_{0_T})_{x,y}$$

$$(iv) \quad \frac{1}{\mu_1}(B_{0_I} + B_{0_R})_{x,y} = \frac{1}{\mu_2}(B_{0_T})_{x,y}$$

Suppose the polarization of the incident wave is parallel to the plane of incidenceie in the xz plane. Consider the first boundary condition

$$\varepsilon_1(-E_{0_I}\sin\theta_I + E_{0_R}\sin\theta_R) = \varepsilon_2(-E_{0_T}\sin\theta_T)$$

Since magnetic field has no z components (ii) add nothing.
(iii)equation become

$$E_{0_I}\cos\theta_I + E_{0_R}\cos\theta_R = E_{0_T}\cos\theta_T$$

(iv) Become

$$\frac{1}{\mu_1 v_1}(E_{0_I} - E_{0_R}) = \frac{1}{\mu_2 v_2}E_{0_T}$$

Combining with $\varepsilon_1(-E_{0_I}\sin\theta_I + E_{0_R}\sin\theta_R) = \varepsilon_2(-E_{0_T}\sin\theta_T)$ and using laws of reflection and refraction,we will get

$$E_{0_I} - E_{0_R} = \beta E_{0_T}$$

Where

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

Equation

$$E_{0_I}\cos\theta_I + E_{0_R}\cos\theta_R = E_{0_T}\cos\theta_T$$

$$E_{0_I} + E_{0_R} = \alpha E_{0_T}$$

Where

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I}$$

solving $E_{0_I} + E_{0_R} = \alpha E_{0_T}, \quad E_{0_I} - E_{0_R} = \beta E_{0_T}$ this equation for reflected and transmitted amplitude we obtain
Amplitude of reflected wave is

$$E_{0_R} = \frac{\alpha - \beta}{\alpha + \beta}E_{0_I}$$

Amplitude of the transmitted wave is

$$E_{0_T} = \left(\frac{2}{\alpha + \beta}\right)E_{0_I}$$

These are known as **Fresnel's equations**

Note The transmitted wave is always in phase with the incident one;reflected wave is either in phase if $\alpha > \beta$ or 180 out of phase if $\alpha < \beta$

The amplitude of the transmitted and reflected wave depends on the angle of incidence because α is a function of θ_I

$$\alpha = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-[(n_1/n_2)\sin\theta_I]^2}}{\cos\theta_I}$$

Brewster's angle

At normal incidenace most of the light are transmitted.At $\theta_I = 90^\circ$ the wave is totally reflected.Between these two angle there is some angle called Brewster's angle at which the reflected ray is completely disappered.

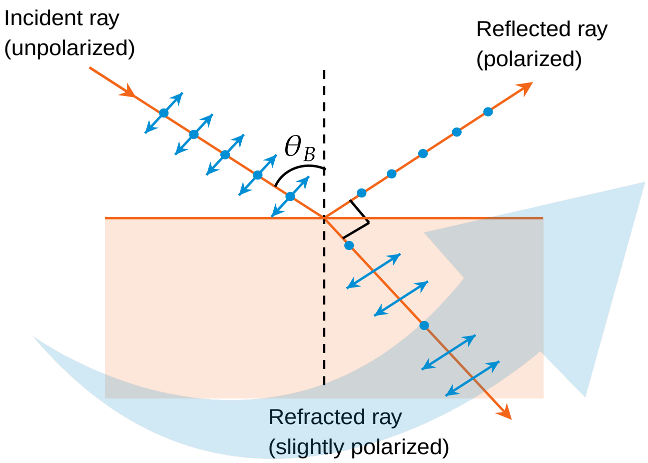


Figure 1.5

This occurs when $\alpha = \beta$

$$\sin^2\theta_B = \frac{1-\beta^2}{(n_1/n_2)^2-\beta^2}$$

For the typical case $\mu_1 \approx \mu_2$ so $\beta \approx n_2, \sin^2\theta_B \approx \frac{\beta^2}{1+\beta^2}$ and hence

$$\tan\theta_B \approx \frac{n_2}{n_1}$$

Note If the incident light is unpolarized the reflected ray will be totally polarized parallel to the interface at Brewster's angle.In the above condition we use plane polarized light as incident light,that is why there is no reflected rays.

Malus law

The power per unit area

Note

1. Incident intensity

$$I_I = \frac{1}{2}\varepsilon_1 v_1 E_{0_I}^2 \cos\theta_I$$

2. Reflected intencity

$$I_R = \frac{1}{2}\varepsilon_1 v_1 E_{0_R}^2 \cos\theta_R$$

3. Transmitted intencity

$$I_T = \frac{1}{2}\varepsilon_2 v_2 E_{0_T}^2 \cos\theta_T$$

Reflection and transmission coefficient

$$R = \frac{I_R}{I_I} = \left(\frac{E_{0_R}}{E_{O_I}}\right) = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$$

$$T = \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{E_{0_T}}{E_{0I}}\right)\frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2$$

1.6 Elctromagnetic waves in conductors

When we are talking about wave propagation through a vacuum or through a dielctric mterial we restricted our assumption that J_f and ρ_f are zeros.But in the case of conductors we donnot independently control the flow of charge, and in general J_f is not certainly zero.
According to ohm's law the current density in a conductor is proportional to the electric field.

$$J_f = \sigma E$$

Maxwell' equation for a linear media.
(i) $\nabla \cdot$ E=$\frac{\rho_f}{\varepsilon}$
(ii) $\nabla \cdot B = 0$
(iii) $\nabla \times$ =-$\frac{\partial B}{\partial t}$
(iv) $\nabla \times B = \mu\sigma E + \mu\varepsilon\frac{\partial E}{\partial t}$
Continuity equation ($\nabla \cdot J_f = \frac{\partial \rho_f}{\partial t}$) with Ohm's law The gauss's law become

$$\frac{\partial \rho_f}{\partial t} = -\sigma(\nabla \cdot E) = -\frac{\sigma}{\varepsilon}\rho_f$$

$$\rho_f(t) = e^{\frac{-\sigma}{\varepsilon}t}\rho_f(0)$$

which means that if you put some charges on a conductor,it will flow out of the edges.Donot mind this transient behaviour.consider the time up to which the accumulated free charges to disappear.From then $\rho_f = 0$
Now maxwell's quations become
(i) $\nabla \cdot$ E=0
(ii) $\nabla \cdot B = 0$
(iii) $\nabla \times$ =-$\frac{\partial B}{\partial t}$
(iv) $\nabla \times B = \mu\sigma E + \mu\varepsilon\frac{\partial E}{\partial t}$
Applying curl to (iii) and (iv) we will get modified wave equation for E and B.

$$\nabla^2 \mathbf{E} = \mu\varepsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma\frac{\partial \mathbf{E}}{\partial t}, \quad \nabla^2 \mathbf{B} = \mu\varepsilon\frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma\frac{\partial \mathbf{B}}{\partial t}$$

These equations still admit plane-wave solutions,

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(\tilde{k}z-\omega t)},$$

but this time the "wave number" $\tilde{k}$ is complex:

$$\tilde{k}^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega,$$

$$\bar{k} = k_{real} + ik_{img}$$

where

$$k_{real} \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2}$$

and

$$k_{img} \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2}$$

Now

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-k_{real}z} e^{i(k_{img}z-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{-k_{real}z} e^{i(k_{imag}z-\omega t)}$$

Skin depth

The distance it takes to reduce the amplitude by a factor of $1/e$ (about a third) is called the skin depth:

$$d \equiv \frac{1}{\kappa}$$

it is a measure of how far the wave penetrates into the conductor. Meanwhile, the real part of k determines the wavelength, the propagation speed, and the index of refraction, in the usual way:

$$\lambda = \frac{2\pi}{k}, \quad v = \frac{\omega}{k}, \quad n = \frac{ck}{\omega}.$$

1.6.1 Skin depth in a poor conductor:

For poor conductor

$$\sigma \ll \omega\varepsilon$$

$$\kappa \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2}$$

$$\approx \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[1+\frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2-1\right]^{1/2}$$

Then

$$\kappa \approx \omega\sqrt{\frac{\varepsilon\mu}{2}}\frac{1}{\sqrt{2}}\frac{\sigma}{\varepsilon\omega} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

So

$$d = \frac{1}{\kappa} \cong \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$$

1.6.2 Skin depth in a good conductor:

For good conductor

$$\sigma \gg \omega\varepsilon$$

$$\kappa \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2}$$

$$\approx \omega\sqrt{\frac{\varepsilon\mu}{2}}\left(\frac{\sigma}{\varepsilon\omega}\right)^{1/2} \approx \sqrt{\frac{\mu\sigma\omega}{2}}$$

So for a good conductor

$$d \cong \sqrt{\frac{2}{\mu\sigma\omega}}$$

For good conductor as $\sigma \gg \omega\varepsilon$ so, you can see

$$k \cong \kappa$$

or

$$\lambda = \frac{2\pi}{k} \cong \frac{2\pi}{\kappa} = 2\pi d, \text{ or } d = \frac{\lambda}{2\pi}$$

Phase shift
Like any complex number, $\tilde{k}$ can be expressed in terms of its modulus and phase:

$$\tilde{k} = Ke^{i\phi},$$

$$K \equiv |\tilde{k}| = \sqrt{k_{real}^2 + k_{img}^2} = \omega\sqrt{\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}}$$

$$\phi \equiv \tan^{-1}(k_{img}/k_{real})$$

The complex amplitudes $\tilde{E}_0 = E_0 e^{i\delta_E}$ and $\tilde{B}_0 = B_0 e^{i\delta_B}$ are related by

$$B_0 e^{i\delta_B} = \frac{Ke^{i\phi}}{\omega} E_0 e^{i\delta_E}$$

Evidently the electric and magnetic fields are no longer in phase; in fact,

$$\delta_B - \delta_E = \phi$$

the magnetic field lags behind the electric fleld. Meanwhile, the (real) amplitudes of **E** and B are related by

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}}.$$

The (real) electric and magnetic fields are, finally,

$$\left.\begin{array}{c}\mathbf{E}(z,t) = E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\,\hat{\mathbf{x}}, \\ \hline \mathbf{B}(z,t) = B_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\,\hat{\mathbf{y}}\end{array}\right\}$$

Exercise 1.2 (a) Find the skin depth of pure water. You are given that for pure water conductivity $\sigma = 1/\left(2.5\times 10^5\right)$, dielectric constant $k = 80.1$ and $\mu \approx \mu_0$.
(b) Find the skin depth (in nanometers) for a typical metal whose conductivity $\sigma \approx 10^7$, the frequency $\omega \approx 10^{15}$ with $\varepsilon \approx \varepsilon_0$ and $\mu \approx \mu_0$. ■

Solution: (a) Conductivity shows that the water is a poor conductor. Hence we will take approximation $d = \frac{1}{\kappa} \cong \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$.

$$d = (2)\left(2.5\times 10^5\right)\sqrt{\frac{(80.1)\left(8.85\times 10^{-12}\right)}{4\pi\times 10^{-7}}} = 1.19\times 10^4 \text{ m}$$

(b) We have proved that for good conductor $d \approx \sqrt{\frac{2}{\mu\sigma\omega}}$. Put the values to get

$$d = \frac{1}{8\times 10^7} = 1.3\times 10^{-8} = 13 \text{ nm}$$

So the fields do not penetrate far into a metal

1.6.3 Reflection at a conducting surface

Boundary conditions used to solve em waves at the conductor involves ρ_f and K_f
(i) $\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f$,
(iii) $\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0$,
(ii) $B_1^{\perp} - B_2^{\perp} = 0$,
(iv) $\frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}$,
Suppose that xy plane forms a boundary between a non conducting non linear medium 1 and a conductor 2.A monochromatic plane wave travelling in the z direction and polarized in the x direction ,approches from the left

as shown in figure.

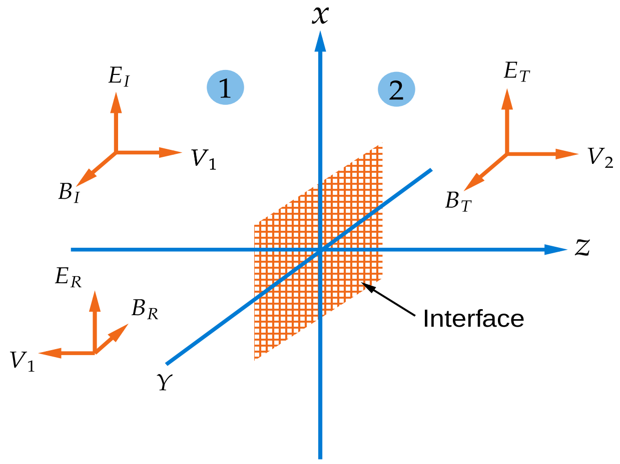


Figure 1.6: Reflection and Transmission at Normal incidence

Incident wave:

$$\tilde{\mathbf{E}}_I(z,t) = \tilde{E}_{0_I} e^{i(k_1 z - \omega t)} \hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}_I(z,t) = \frac{1}{v_1} \tilde{E}_{0_I} e^{i(k_1 z - \omega t)} \hat{\mathbf{y}}$$

Reflected wave:

$$\tilde{\mathbf{E}}_R(z,t) = \tilde{E}_{0_R} e^{i(-k_1 z - \omega t)} \hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}_R(z,t) = -\frac{1}{v_1} \tilde{E}_{0_R} e^{i(-k_1 z - \omega t)} \hat{\mathbf{y}}$$

Transmitted wave:

$$\tilde{\mathbf{E}}_T(z,t) = \tilde{E}_{0_T} e^{i(\tilde{k}_2 z - \omega t)} \hat{\mathbf{x}}$$

$$\tilde{\mathbf{B}}_T(z,t) = \frac{\tilde{k}_2}{\omega} \tilde{E}_{0_T} e^{i(\tilde{k}_2 z - \omega t)} \hat{\mathbf{y}}$$

Transmitted wave will get attinuated while entering the conducting material.
Now consider the boundary conditions one by one.
(i) Gives $\sigma_f = 0$ because there will be no perpendicular component of elctric field in two media($E^{\perp}$)=0
(ii)yields $B^{\perp} = 0$
(iii) gives

$$\tilde{E}_{0_1} + \tilde{E}_{0_R} = \tilde{E}_{0_\tau}$$

(iv) gives

$$\frac{1}{\mu_1 v_1}\left(\tilde{E}_{0_I} - \tilde{E}_{0_R}\right) - \frac{\tilde{k}_2}{\mu_2 \omega} \tilde{E}_{0_T} = 0,$$

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \tilde{\beta} \tilde{E}_{0_T},$$

$$\tilde{\beta} \equiv \frac{\mu_1 v_1}{\mu_2 \omega} \tilde{k}_2.$$

It follows that

$$\tilde{E}_{0_R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right) \tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{1+\tilde{\beta}}\right) \tilde{E}_{0_I}.$$

For a perfect conductor ($\sigma = \inf$),$k_2 = \infty$ so β is finite and

$$E_{0_R} = -E_{0_I}, \quad E_{0_T} = 0$$

In this case the wave is totally reflected with a 180° phase shift.That is why excellent conductors make good mirrors.

1.7 Guided waves

Untill now we have dealt with plane wave of infinite extent;Now we consider the em waves confines to the interior of a hollow pipe or wave guide.We will assume the wave guide is perfect conductor. so that E=0 and B=0 inside the material itself.Hence the boundary conditions at the boundary wall are

$$(i) \quad E^{\parallel} = 0$$
$$(ii) \quad B^{\perp} = 0$$

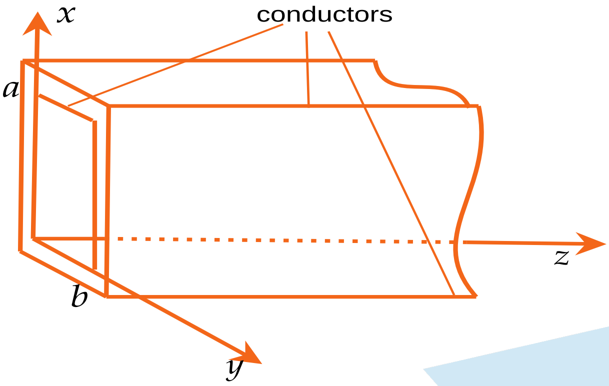


Figure 1.7

Let us take the medium is restricted by x and y and the wave is propagating along z direction.So E and B has the generic form.

$$(i)\tilde{\mathbf{E}}(x,y,z,t) = \tilde{\mathbf{E}}_0(x,y)e^{i(kz-\omega t)}$$
$$(ii)\tilde{\mathbf{B}}(x,y,z,t) = \tilde{\mathbf{B}}_0(x,y)e^{i(kz-\omega t)}$$

The maxwell's equation in the interior of the waveguide

$$(i)\nabla\cdot\mathbf{E} = 0$$
$$(ii)\nabla\cdot\mathbf{B} = 0$$
$$(iii)\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$
$$(iv)\nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$$

Boundary conditions and the Maxwell's equations gives

$$(i)\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+(\omega/c)^2-k^2\right]E_z = 0$$

and

$$(ii)\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+(\omega/c)^2-k^2\right]B_z = 0$$

If $E_z = 0$, we call these TE (transverse electric") waves; if $B_z = 0$, they are called TM ("transverse magnetic") waves; if both $E_z = 0$ and $B_z = 0$, we call them TEM waves. You can prove that TEM waves can't occur in hollow rectangular wave guide.

$$\text{TE, mode} \Rightarrow F_z = 0 \quad B_z = \text{exist}$$
$$\text{TM mode} \Rightarrow E_z = \text{exist}\ B_z = 0$$
$$\text{TEM mode} \Rightarrow E_z = B_z = 0$$

1.7.1 TE Waves in a Rectangular Wave Guide

We need the solution of

$$\text{(ii)}\quad \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+(\omega/c)^2-k^2\right]B_z=0$$

Take

$$B_z(x,y)=X(x)Y(y)$$

and do the separation of variable. You get the solution

$$B_z=B_0\cos(m\pi x/a)\cos(n\pi y/b)$$

with the dispersion relation

$$k=\sqrt{(\omega/c)^2-\pi^2\left[(m/a)^2+(n/b)^2\right]}$$

Define

$$\omega_{mn}=c\pi\sqrt{(m/a)^2+(n/b)^2}$$

Now you see if

$$\omega<\omega_{mn}$$

the wave number is imaginary, and instead of a traveling wave we have exponentially attenuated wave which will rapidly absorbed in the medium. ω_{mn} is called the cut off frequency for a particular mode TE_{mn}. You should remember that ω is angular frequency, not frequency.
In terms of frequency

$$\nu_{mn}=\frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$$

The lowest mode is TE $_{10}$ (usually among a and b the larger is taken to be a.

$$\omega_{10}=c\pi/a$$

Group and phase velocity
Phase velocity is greater than C.

$$v_p=\frac{\omega}{k}=\frac{c}{\sqrt{1-(\omega_{mn}/\omega)^2}}$$

The group velocity

$$v_g=\frac{1}{dk/d\omega}=c\sqrt{1-(\omega_{mn}/\omega)^2}<c$$

1.7.2 TM Waves in a Rectangular Wave Guide

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+(\omega/c)^2-k^2\right]E_z=0$$

Has the soltion of the form.

$$E_z=E_0\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

if either of the m and n is zero the solution itself is zero. So the lowest frequency mode is for TM wave in rectangular wave guide is TE_{11}.

1.8 Resonant cavity

Resonant cavity produced by closing off the two ends of a rectangular wave guide, at $z = 0$ and at $z = d$, making a perfectly conducting empty box.

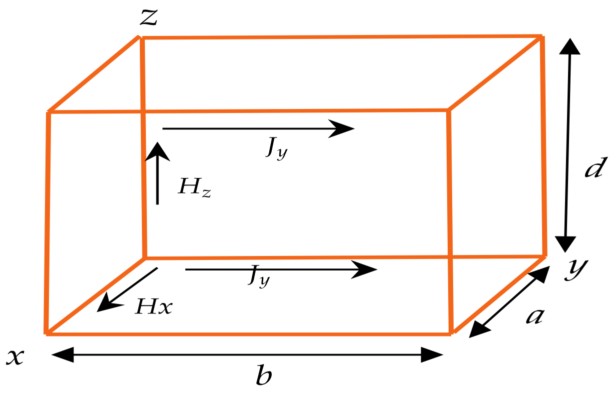


Figure 1.8

We can prove that the cut off frequencies for both TE and TM modes in a resonance cavity are given by

$$\omega_{lmn} = c\pi\sqrt{(l/d)^2 + (m/a)^2 + (n/b)^2}$$
$$\nu_{lmn} = \frac{c}{2}\sqrt{(l/d)^2 + (m/a)^2 + (n/b)^2}$$

Exercise 1.3 What should be the 3rd dimension of a cavity of cross section 1 cm × 1 cm which operates at TE_{103} mode at 24GHz?? ■

Solution: Solution: $l = 1, m = 0, n = 3$. use

$$\nu_{103} = 24\times 10^9; \quad a = 0.01, b = 0.01, c = ?$$

use the formula of ν_{lmn} to get

$$\nu_{lmn} = \frac{c}{2}\sqrt{(l/d)^2 + (m/a)^2 + (n/b)^2}$$
$$\left(\frac{24\times 2\times 10^9}{3\times 10^8}\right)^2 = \left(\frac{1}{0.01}\right)^2 + 0 + \left(\frac{3}{c}\right)^2$$
$$c = 0.02$$

1.9 Dielectric inserted into waveguide

When there is a dielectric / magnetic medium inside of a dielectric instead of vacuum, just replace c by $\frac{1}{\sqrt{\varepsilon\mu}}$

$$\nu_{mn} = \frac{1}{2\sqrt{\varepsilon\mu}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

For most of the material $\mu \approx \mu_0$, so then

$$\nu_{mn} = \frac{1}{2\sqrt{\varepsilon_r\varepsilon_0\mu_0}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$
$$= \frac{c}{2\sqrt{\varepsilon_r}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Exercise 1.4 For a air filled wave guide. the cut off frequency $TE_0 = 1.8756$GHz. What will be the cut off frequency if one dielectric of relative permeability $\varepsilon = 9\varepsilon_0$ is inserted in the dielectric? ■

Solution: you have already derived the formula for cut off frequency for the dielectric inserted wave guide. The ans is

$$\nu_{mn}^{\text{dielectric}} = \nu_{mn}^{\text{vacuum}} / \sqrt{\varepsilon_r} = 1.8756/3\text{GHz}$$

Practice set 1

1. A plane electromagnetic wave is propagating in a lossless dielectric. The electric field is given by

$$\vec{E}(x,y,z,t) = E_0(\hat{x}+A\hat{z})\exp\left[ik_0\{-ct+(x+\sqrt{3}z)\}\right]$$

where c is the speed of light in vacuum, E_0, A and k_0 are constant and $\hat{x}$ and $\hat{z}$ are unit vectors along the x-and z-axes. The relative dielectric constant of the medium ε_r and the constant A are

[NET JUNE 2011]

A. $\varepsilon_r = 4$ and $A = -\frac{1}{\sqrt{3}}$

B. $\varepsilon_r = 4$ and $A = +\frac{1}{\sqrt{3}}$

C. $\varepsilon_r = 4$ and $A = \sqrt{3}$

D. $\varepsilon_r = 4$ and $A = -\sqrt{3}$

2. The magnetic field of the TE_{11} mode of a rectangular waveguide of dimensions $a \times b$ as shown in the figure is given by $H_z = H_0\cos(0.3\pi x)\cos(0.4\pi y)$, where x and y are in cm

[NET JUNE 2011]

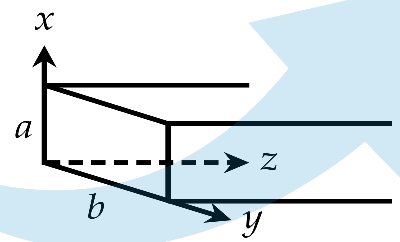


Figure 1.9

A. The dimensions of the waveguide are

A. $a = 3.33$ cm, $b = 2.50$ cm

B. $a = 0.40$ cm, $b = 0.30$ cm

C. $a = 0.80$ cm, $b = 0.60$ cm

D. $a = 1.66$ cm, $b = 1.25$ cm

3. Consider a solenoid of radius R with n turns per unit length, in which a time dependent current $I = I_0\sin\omega t$ (where $\omega R/c << 1$) flows. The magnitude of the electric field at a perpendicular distance $r < R$ from the axis of symmetry of the solenoid, is

[NET DEC 2011]

A. 0

B. $\frac{1}{2r}\omega\mu_0 nI_0R^2\cos\omega t$

C. $\frac{1}{2}\omega\mu_0 nI_0 r\sin\omega t$

D. $\frac{1}{2}\omega\mu_0 nI_0 r\cos\omega t$

4. An electromagnetic wave is incident on a water-air interface. The phase of the perpendicular component of the electric field, $E_\perp$, of the reflected wave into the water is found to remain the same for all angles of incidence. The phase of the magnetic field H

[NET JUNE 2012]

A. does not change

B. changes by $3\pi/2$

C. changes by $\pi/2$

D. changes by π

5. When a charged particle emits electromagnetic radiation, the electric field $\vec{E}$ and the Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ at a larger distance r from emitter vary as $\frac{1}{r^n}$ and $\frac{1}{r^m}$ respectively. Which of the following choices for n and m are correct?

[NET DEC 2012]

A. $n = 1$ and $m = 1$

B. $n = 2$ and $m = 2$

C. $n = 1$ and $m = 2$

D. $n = 2$ and $m = 4$

6. A current I is created by a narrow beam of protons moving in vacuum with constant velocity $\vec{u}$. The direction and magnitude, respectively of the Poynting vector $\vec{S}$ outside the beam at a radial distance r (much larger than the width of the beam) from the axis, are

[NET JUNE 2013]

A. $\vec{S} \perp \vec{u}$ and $|\vec{S}| = \frac{I^2}{4\pi^2\varepsilon_0|\vec{u}|r^2}$

B. $\vec{S}||(-\vec{u})$ and $|\vec{S}| = \frac{I^2}{4\pi^2\varepsilon_0|\vec{u}|r^4}$

C. $\vec{S}||\vec{u}$ and $|\vec{S}| = \frac{I^2}{4\pi^2\varepsilon_0|\vec{u}|r^2}$

D. $\vec{S}||\vec{u}$ and $|\vec{S}| = \frac{I^2}{4\pi^2\varepsilon_0|\vec{u}|r^4}$

7. The electric field of an electromagnetic wave is given by

$$\vec{E} = E_0\cos[\pi(0.3x + 0.4y - 1000t)]\hat{k}.$$

The associated magnetic field $\vec{B}$ is

[NET DEC 2013]

A. $10^{-3}E_0\cos[\pi(0.3x + 0.4y - 1000t)]\hat{k}$

B. $10^{-4}E_0\cos^{\pi(0.3x+0.4y-1000t)}\left(4\hat{i} - 3\hat{j}\right)$

C. $E_0\cos^{\pi(0.3x+0.4y-1000t)}\left(0.3\hat{i} + 0.4\hat{j}\right)$

D. $10^2E_0\cos^{\pi(0.3x+0.4y-1000t)}\left(3\hat{i} + 4\hat{j}\right)$

8. A beam of light of frequency ω is reflected from a dielectric-metal interface at normal incidence. The refractive index of the dielectric medium is n and that of the metal is $n_2 = n(1 + i\rho)$. If the beam is polarised parallel to the interface, then the phase change experienced by the light upon reflection is

[NET JUNE 2014]

A. $\tan(2/\rho)$

B. $\tan^{-1}(1/\rho)$

C. $\tan^{-1}(2/\rho)$

D. $\tan^{-1}(2\rho)$

9. An electromagnetically-shielded room is designed so that at a frequency $\omega = 10^7$rad/s the intensity of the external radiation that penetrates the room is 1% of the incident radiation. If $\sigma = \frac{1}{2\pi} \times 10^6(\Omega m)^{-1}$ is the conductivity of the shielding material, its minimum thickness should be (given that $\ln 10 = 2.3$)

[NET JUNE 2014]

A. 4.60 mm

B. 2.30 mm

C. 0.23 mm

D. 0.46 mm

10. A plane electromagnetic wave incident normally on the surface of a material is partially reflected. Measurements on the standing wave in the region in front of the interface such that the ratio of the electric field amplitude at the maxima and the minima is 5 . The ratio of the reflected intensity to the incident intensity is

[NET JUNE 2014]

A. $4/9$

B. $2/3$

C. $2/5$

D. $1/5$

11. A Plane electromagnetic wave is travelling along the positive z-direction. The maximum electric field along the x - direction is 10 V/m. The approximate maximum values of the power per unit area and the magnetic induction B, respectively, are

[NET JUNE 2015]

A. 3.3×10^{-7} watts /m^2 and 10 tesla

B. 3.3×10^{-7} watts /m^2 and 3.3×10^{-8} tesla

C. 0.265 watts / m ^{2} and 10 tesla

D. 0.265 watts $/m^2$ and 3.3×10^{-8} tesla

12. Consider a rectangular wave guide with transverse dimensions $2m \times 1m$ driven with an angular frequency $\omega = 10^9$rad/s. Which transverse electric (TE) modes will propagate in this wave guide?

[NET JUNE 2015]

A. TE_{10}, TE_{01} and TE_{20}

B. TE_{01}, TE_{11} and TE_{20}

C. TE_{01}, TE_{10} and TE_{11}

D. TE_{01}, TE_{10} and TE_{22}

13. The electric and magnetic fields in the charge free region $z>0$ are given by

$$\vec{E}(\vec{r},t)=E_0 e^{-k_1 z}\cos\left(k_2 x-\omega t\right)\hat{j}$$
$$\vec{B}(\vec{r},t)=\frac{E_0}{\omega}e^{-k_1 z}\left[k_1\sin\left(k_2 x-\omega t\right)\hat{\imath}+k_2\cos\left(k_2 x-\omega t\right)\hat{k}\right]$$

where ω, k_1 and k_2 are positive constants. The average energy flow in the x-direction is

[NET JUNE 2015]

A. $\frac{E_0^2 k_2}{2\mu_0\omega}e^{-2k_1 z}$

B. $\frac{E_0^2 k_2}{\mu_0\omega}e^{-2k_1 z}$

C. $\frac{E_0^2 k_1}{2\mu_0\omega}e^{-2k_1 z}$

D. $\frac{1}{2}c\in_0 E_0^2 e^{-2k_1 z}$

14. The x - and z-components of a static magnetic field in a region are $B_x=B_0\left(x^2-y^2\right)$ and $B_z=0$, respectively. Which of the following solutions for its y-component is consistent with the Maxwell equations?

[NET JUNE 2016]

A. $B_y=B_0 xy$

B. $B_y=-2B_0 xy$

C. $B_y=-B_0\left(x^2-y^2\right)$

D. $B_y=B_0\left(\frac{1}{3}x^3-xy^2\right)$

15. The value of the electric and magnetic fields in a particular reference frame (in Gaussian units) are $E=3\hat{x}+4\hat{y}$ and $B=3\hat{z}$ respectively. An inertial observer moving with respect to this frame measures the magnitude of the electric field to be $|E'|=4$. The magnitude of the magnetic field $|B'|$ measured by him is

[IIT JAM 201]

A. 5

B. 9

C. 0

D. 1

16. A waveguide has a square cross-section of side $2a$. For the TM modes of wave vector k, the transverse electromagnetic modes are obtained in terms of a function $\psi(x,y)$ which obeys the equation

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega^2}{c^2}-k^2\right)\right]\psi(x,y)=0$$

with the boundary condition $\psi(\pm a,y)=\psi(x,\pm a)=0$. The frequency ω of the lowest mode is given by

[NET JUNE 2016]

A. $\omega^2=c^2\left(k^2+\frac{4\pi^2}{a^2}\right)$

B. $\omega^2=c^2\left(k^2+\frac{\pi^2}{a^2}\right)$

C. $\omega^2=c^2\left(k^2+\frac{\pi^2}{2a^2}\right)$

D. $\omega^2=c^2\left(k^2+\frac{\pi^2}{4a^2}\right)$

17. An electromagnetic wave (of wavelength λ_0 in free space) travels through an absorbing medium with dielectric permittivity given by $\varepsilon=\varepsilon_R+i\varepsilon_I$ where $\frac{\varepsilon_I}{\varepsilon_R}=\sqrt{3}$. If the skin depth is $\frac{\lambda_0}{4\pi}$, the ratio of the amplitude of electric field E to that of the magnetic field B, in the medium (in ohms) is

[NET JUNE 2017]

A. 120π

B. 377

C. $30\sqrt{2}\pi$

D. 30π

18. An electromagnetic wave is travelling in free space (of permittivity ε_0) with electric field

$$\vec{E}=\hat{k}E_0\cos q(x-ct)$$

The average power (per unit area) crossing planes parallel to $4x+3y=0$ will be

[NET DEC 2017]

A. $\frac{4}{5}\varepsilon_0 c E_0^2$ **B.** $\varepsilon_0 c E_0^2$

C. $\frac{1}{2}\varepsilon_0 c E_0^2$ **D.** $\frac{16}{25}\varepsilon_0 c E_0^2$

19. A circular current carrying loop of radius a carries a steady current. A constant electric charge is kept at the centre of the loop. The electric and magnetic fields, $\vec{E}$ and $\vec{B}$ respectively, at a distance d vertically above the centre of the loop satisfy

[NET DEC 2017]

A. $\vec{E} \perp \vec{B}$ **B.** $\vec{E} = 0$

C. $\vec{\nabla}(\vec{E}\cdot\vec{B}) = 0$ **D.** $\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = 0$

20. A plane electromagnetic wave from within a dielectric medium (with $\varepsilon = 4\varepsilon_0$ and $\mu = \mu_0$) is incident on its boundary with air, at $z = 0$. The magnetic field in the medium is $\vec{H} = \hat{j}H_0\cos\left(\omega t - kx - k\sqrt{3}z\right)$, where ω and k are positive constants. The angles of reflection and refraction are, respectively,

[NET DEC 2017]

A. 45° and 60° **B.** 30° and 90°

C. 30° and 60° **D.** 60° and 90°

21. The electric field $\vec{E}$ and the magnetic field $\vec{B}$ corresponding to the scalar and vector potentials, $V(x,y,z,t) = 0$ and $\vec{A}(x,y,z,t) = \frac{1}{2}\hat{k}\mu_0 A_0(ct - x)$, where A_0 is a constant, are

[NET JUNE 2018]

A. $\vec{E} = 0$ and $\vec{B} = \frac{1}{2}\hat{j}\mu_0 A_0$ **B.** $\vec{E} = -\frac{1}{2}\hat{k}\mu_0 A_0 c$ and $\vec{B} = \frac{1}{2}\hat{j}\mu_0 A_0$

C. $\vec{E} = 0$ and $\vec{B} = -\frac{1}{2}\hat{i}\mu_0 A_0$ **D.** $\vec{E} = \frac{1}{2}\hat{k}\mu_0 A_0 c$ and $\vec{B} = -\frac{1}{2}\hat{i}\mu_0 A_0$

22. The electric field of a plane wave in a conducting medium is given by

$$\vec{E}(z,t) = \hat{i}E_0 e^{-z/3a}\cos\left(\frac{z}{\sqrt{3}a} - \omega t\right)$$

where ω is the angular frequency and $a > 0$ is a constant. The phase difference between the magnetic field $\vec{B}$ and the electric field $\vec{E}$ is

[NET JUNE 2018]

A. 30° and $\vec{B}$ lags behind $\vec{B}$ **B.** 30° and $\vec{B}$ lags behind $\vec{E}$

C. 60° and $\vec{E}$ lags behind $\vec{B}$ **D.** 60° and $\vec{B}$ lags behind $\vec{E}$

23. A hollow waveguide supports transverse electric (TE) modes with the dispersion relation $k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$, where ω_{mn} is the mode frequency. The speed of flow of electromagnetic energy at the mode frequency is

[NET JUNE 2018]

A. c **B.** ω_{mn}/k

C. 0 **D.** ∞

24. In the region far from a source, the time dependent electric field at a point (r,θ,ϕ) is

$$\vec{E}(r,\theta,\phi) = \hat{\phi}E_0\omega^2\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - \frac{r}{c}\right)\right]$$

where ω is angular frequency of the source. The total power radiated (averaged over a cycle) is

[NET JUNE 2018]

A. $\frac{2\pi}{3}\frac{E_0^2\omega^4}{\mu_0 c}$ **B.** $\frac{4\pi}{3}\frac{E_0^2\omega^4}{\mu_0 c}$

C. $\frac{4}{3\pi}\frac{E_0^2\omega^4}{\mu_0 c}$ **D.** $\frac{2}{3}\frac{E_0^2\omega^4}{\mu_0 c}$

25. An electromagnetic wave propagates in a nonmagnetic medium with relative permittivity $\varepsilon = 4$. The magnetic field for this wave is

$$\vec{H}(x,y) = \hat{k}H_0 \cos\left(\omega t - \alpha x - \alpha\sqrt{3}y\right)$$

where H_0 is a constant. The corresponding electric field $\vec{E}(x,y)$ is

[NET DEC 2018]

A. $\frac{1}{4}\mu_0 H_0 c(-\sqrt{3}\hat{i}+\hat{j})\cos\left(\omega t - \alpha x - \alpha\sqrt{3}y\right)$ **B.** $\frac{1}{4}\mu_0 H_0 c(\sqrt{3}\hat{i}+\hat{j})\cos\left(\omega t - \alpha x - \alpha\sqrt{3}y\right)$

C. $\frac{1}{4}\mu_0 H_0 c(\sqrt{3}\hat{i}-\hat{j})\cos\left(\omega t - \alpha x - \alpha\sqrt{3}y\right)$ **D.** $\frac{1}{4}\mu_0 H_0 c(-\sqrt{3}\hat{i}-\hat{j})\cos\left(\omega t - \alpha x - \alpha\sqrt{3}y\right)$

26. Electromagnetic wave of angular frequency ω is propagating in a medium in which, over a band of frequencies the refractive index is $n(\omega) \approx 1 - \left(\frac{\omega}{\omega_0}\right)^2$, where ω_0 is a constant. The ratio $\frac{v_g}{v_p}$ of the group velocity to the phase velocity at $\omega = \frac{\omega_0}{2}$ is

[NET DEC 2018]

A. 3 **B.** $\frac{1}{4}$

C. $\frac{2}{3}$ **D.** 2

Answer key

Q.No.	Answer	Q.No.	Answer
1	**a**	2	**a**
3	**d**	4	**d**
5	**c**	6	**c**
7	**b**	8	**c**
9	**b**	10	**a**
11	**d**	12	**a**
13	**a**	14	**b**
15	**c**	16	**c**
17	**d**	18	**d**
19	**d**	20	**b**
21	**b**	22	**b**
23	**c**	24	**b**
25	**a**	26	**a**

Practice set 2

1. For a plane wave of angular frequency ω and propagation vector $\vec{k}$ propagating in the medium Maxwell's equations reduce to

[GATE 2010]

A. $\vec{k}\cdot\vec{E}=0;\vec{k}\cdot\vec{H}=0;\vec{k}\times\vec{E}=\omega\varepsilon\vec{H};\vec{k}\times\vec{H}=-\omega\mu\vec{E}$

B. $\vec{k}\cdot\vec{E}=0;\vec{k}\cdot\vec{H}=0;\vec{k}\times\vec{E}=-\omega\varepsilon\vec{H};\vec{k}\times\vec{H}=\omega\mu\vec{E}$

C. $\vec{k}\cdot\vec{E}=0;\vec{k}\cdot\vec{H}=0;\vec{k}\times\vec{E}=-\omega\mu\vec{H};\vec{k}\times\vec{H}=\omega\varepsilon\vec{E}$

D. $\vec{k}\cdot\vec{E}=0;\vec{k}\cdot\vec{H}=0;\vec{k}\times\vec{E}=\omega\mu\vec{H};\vec{k}\times\vec{H}=-\omega\varepsilon\vec{E}$

2. If ε and μ assume negative values in a certain frequency range, then the directions of the propagation vector $\vec{k}$ and the Poynting vector $\vec{S}$ in that frequency range are related as

[GATE 2010]

A. $\vec{k}$ and $\vec{S}$ are parallel

B. $\vec{k}$ and $\vec{S}$ are anti-parallel

C. $\vec{k}$ and $\vec{S}$ are perpendicular to each other

D. $\vec{k}$ and $\vec{S}$ makes an angle that depends on the magnitude of $|\varepsilon|$ and $|\mu|$

3. A plane electromagnetic wave has the magnetic field given by

$$\vec{B}(x,y,z,t)=B_0\sin\left[(x+y)\frac{k}{\sqrt{2}}+\omega t\right]\hat{k}$$

where k is the wave number and $\hat{\imath},\hat{\jmath}$ and $\hat{k}$ are the Cartesian unit vectors in x, y and z directions respectively.
(a) The electric field $\vec{E}(x,y,z,t)$ corresponding to the above wave is given by
(b) The average Poynting vector is given by

[GATE 2011]

4. The space-time dependence of the electric field of a linearly polarized light in free space is given by $\hat{x}E_0\cos(\omega t-kz)$ where E_0,ω and k are the amplitude, the angular frequency and the wavevector, respectively. The time average energy density associated with the electric field is

[GATE 2012]

A. $\frac{1}{4}\varepsilon_0E_0^2$ **B.** $\frac{1}{2}\varepsilon_0E_0^2$

C. $\varepsilon_0E_0^2$ **D.** $2\varepsilon_0E_0^2$

5. A plane electromagnetic wave traveling in free space is incident normally on a glass plate of refractive index $3/2$. If there is no absorption by the glass, its reflectivity is

[GATE 2012]

A. (a) 4% **B.** 16%

C. 20% **D.** 50%

6. A plane polarized electromagnetic wave in free space at time $t=0$ is given by $\vec{E}(x,z)=10\hat{\jmath}\exp[i(6x+8z)]$. The magnetic field $\vec{B}(x,z,t)$ is given by

[GATE 2012]

A. $\vec{B}(x,z,t) = \frac{1}{c}(6\hat{k} - 8\hat{i})\exp[i(6x+8z-10ct)]$

C. $\vec{B}(x,z,t) = \frac{1}{c}(6\hat{k} - 8\hat{i})\exp[i(6x+8z-ct)]$

D. $\vec{B}(x,z,t) = \frac{1}{c}(6\hat{k} + 8\hat{i})\exp[i(6x+8z+ct)]$

B. $\vec{B}(x,z,t) = \frac{1}{c}(6\hat{k} + 8\hat{i})\exp[i(6x+8z-10ct)]$

7. A monochromatic plane wave at oblique incidence undergoes reflection at a dielectric interface. If $\hat{k}_i, \hat{k}_r$ and $\hat{n}$ are the unit vectors in the directions of incident wave, reflected wave and the normal to the surface respectively, which one of the following expressions is correct?

[GATE 2013]

A. $\left(\hat{k}_i - \hat{k}_r\right) \times \hat{n} \neq 0$

B. $\left(\hat{k}_i - \hat{k}_r\right) \cdot \hat{n} = 0$

C. $\left(\hat{k}_i \times \hat{n}\right) \cdot \hat{k}_r = 0$

D. $\left(\hat{k}_i \times \hat{n}\right) \cdot \hat{k}_r \neq 0$

8. The electric field of a uniform plane wave propagating in a dielectric non-conducting medium is given by $\vec{E} = \hat{x}10\cos\left(6\pi \times 10^7 t - 0.4\pi z\right)\mathrm{V}/m$. The phase velocity of the wave is 10^8 m/s

[GATE 2014]

9. The intensity of a laser in free space is 150 m $\mathrm{W/m^2}$. The corresponding amplitude of the electric field of the laser is $\cdots \frac{V}{m}$ $\quad \left(\varepsilon_0 = 8.854 \times 10^{-12} C^2/N.m^2\right)$

[GATE 2014]

10. A long solenoid is embedded in a conducting medium and is insulated from the medium. If the current through the solenoid is increased at a constant rate, the induced current in the medium as a function of the radial distance r from the axis of the solenoid is proportional to

[GATE 2015]

A. r^2 inside the solenoid and $\frac{1}{r}$ outside

B. r inside the solenoid and $\frac{1}{r^2}$ outside

C. r^2 inside the solenoid and $\frac{1}{r^2}$ outside

D. r inside the solenoid and $\frac{1}{r}$ outside

11. The electric field component of a plane electromagnetic wave travelling in vacuum is given by $\vec{E}(z,t) = E_0\cos(kz - \omega t)\hat{i}$. The Poynting vector for the wave is

[GATE 2016]

A. $\left(\frac{c\varepsilon_0}{2}\right)E_0^2\cos^2(kz-\omega t)\hat{j}$

B. $\left(\frac{c\varepsilon_0}{2}\right)E_0^2\cos^2(kz-\omega t)\hat{k}$

C. $c\varepsilon_0 E_0^2\cos^2(kz-\omega t)\hat{j}$

D. $c\varepsilon_0 E_0^2\cos^2(kz-\omega t)\hat{k}$

12. Consider a metal with free electron density of 6×10^{22} cm^{-3}. The lowest frequency of electromagnetic radiation to which this metal is transparent, is 1.38×10^{16} Hz. If this metal had a free electron density of 1.8×10^{23} cm^{-3} instead, the lowest frequency electromagnetic radiation to which it would be transparent is .... $\times 10^{16}$ Hz (up to two decimal places).

[GATE 2017]

13. An infinitely long straight wire is carrying a steady current I. The ratio of magnetic energy density at distance r_1 to that at $r_2\,(=2r_1)$ from the wire is

[GATE 2018]

14. Consider an infinitely long solenoid with N turns per unit length, radius R and carrying a current $I(t) = \alpha\cos\omega t$, where α is a constant and ω is the angular frequency. The magnitude of electric field at the surface of the solenoid is

[GATE 2018]

A. $\frac{1}{2}\mu_0 NR\omega\alpha \sin\omega t$ **B.** $\frac{1}{2}\mu_0 \omega NR \cos\omega t$

C. $\mu_0 NR\omega\alpha \sin\omega t$ **D.** $\mu_0 \omega NR \cos\omega t$

15. A long straight wire, having radius a and resistance per unit length r, carries a current I. The magnitude and direction of the Poynting vector on the surface of the wire is

[GATE 2018]

A. $I^2r/2\pi a$, perpendicular to axis of the wire and pointing inwards

B. $I^2r/2\pi a$, perpendicular to axis of the wire and pointing outwards

C. $I^2r/\pi a$, perpendicular to axis of the wire and pointing inwards

D. $I^2r/\pi a$, perpendicular to axis of the wire and pointing outwards

16. An electromagnetic plane wave is propagating with an intensity $I = 1.0 \times 10^5 \mathrm{Wm}^{-2}$ in a medium with $\in = 3 \in_0$ and $\mu = \mu_0$. The amplitude of the electric field inside the medium is $\times 10^3 \mathrm{Vm}^{-1}$ (up to one decimal place). $\left(\in_0 = 8.85 \times 10^{-12} C^2 N^{-1} m^{-2}, \mu_0 = 4\pi \times 10^{-7} NA^{-2}, c = 3 \times 10^8 ms^{-1}\right)$

[GATE 2018]

17. The electric field of an electromagnetic wave in vacuum is given by

$$\vec{E} = E_0 \cos\left(3y + 4z - 1.5 \times 10^9 t\right) \hat{x}$$

The wave is reflected from the $z = 0$ surface. If the pressure exerted on the surface is $\alpha \in E_0^2$, the value of α (rounded off to one decimal place) is

[GATE 2019]

Answer key

Q.No.	Answer	Q.No.	Answer
1	**d**	2	**a**
3		4	**a**
5	**a**	6	**a**
7	**c**	8	**1.5**
9	**10.6**	10	**d**
11	**d**	12	**2.39**
13	**4**	14	**a**
15	**a**	16	**6.6**
17	**0.8**		

2. Potential formulation

2.1 Scalar and vector potentaial

Maxwell's equations are
(i) $\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho$,
(iii) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$,
(ii) $\nabla \cdot \mathbf{B} = 0$,
(iv) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$.
If $\rho(r,t)$ and $J(r,t)$ are bnown electric and magnetic field can be find out using Gauss's law and Bio-savart law.It is difficult to find E ans B if they are time dependant.To solve this problem first we are going to represents the fields in terms of potentials electric potential V and magnetic potential B.
In the static case $\nabla \times E = 0$ So electric field cane written as a negative gradient of some scalar quandity called electric potential V

$$E = -\nabla V$$

(not possible in elctrodynamics)
But $\nabla \cdot B = 0$ always.Which gives

$$B = \nabla \times A$$

Where A is the magnetic vector potential.Putting this value in

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

We will get

$$\nabla \times E = \frac{-\partial}{\partial t}(\nabla \times A)$$

$$\nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0$$

Again the curl of something become zero. so it can be written as negative gradient of potential

$$E + \frac{\partial A}{\partial t} = -\nabla V$$

$$E = -\nabla V - \frac{\partial A}{\partial t}$$

$$\boxed{\begin{array}{l} B = \nabla \times A \\ E = -\nabla V - \frac{\partial A}{\partial t} \end{array}}$$

If A and V are known we can find electric and magnetic field with these two equations.

Putting this equation

$$E = -\nabla V - \frac{\partial A}{\partial t}$$

in Gauss's law we will get,

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot A) = \frac{-\rho}{\varepsilon_0}$$

Putting $B = \nabla \times A$ equation in Ampere/Maxwell's law and rearranging we will get,

$$\left(\nabla^2 A - \mu_0 \varepsilon_0 \frac{\partial^2 A}{\partial t^2}\right) - \nabla \left(\nabla \cdot A + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 J$$

These two equations contain all information in the Maxwell's equations.
However we have succeeded in reducing six problems to find E and B ,down to four(V one component,A three component.),these equations are lengthy and difficult to find solution we have to abandon this potential formulation altogether.

Gauge transformation

To avoid this problem we are transforming the potential equations by adding one extra term to A and V.This is called gauge tranformation.Consider the transformation occuring in the same fields E and B.Then

$$A' = A + \alpha$$

and

$$V' = V + \beta$$

Taking curl on each side of $A' = A + \alpha$ we will get

$$\nabla \times A' = \nabla \times A + \nabla \times \alpha$$

Since B's are same we can written as($\nabla \times B = 0, A' = A$)

$$\nabla \times \alpha = 0$$

Again curl of α is zero.then

$$\alpha = \nabla \lambda$$

The two potentials also gives the same E.so,

$$\nabla \beta + \frac{\partial \alpha}{\partial t} = 0$$

$$\nabla \left(\beta + \frac{\partial \lambda}{\partial t}\right) = 0$$

$$\beta = -\frac{\partial \lambda}{\partial t}$$

Now transformations become,

$$A' = A + \nabla\lambda$$
$$V' = V - \frac{\partial\lambda}{\partial t}$$

2.1.1 Coulomb Gauge and Lorentz gauge

Coluomb gauge

$$\nabla \cdot A = 0$$

is called Coulomb gauge.

importance

$$\nabla \cdot A = 0$$

Then the equation

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot A) = \frac{-\rho}{\varepsilon_0}$$

become,

$$\nabla^2 V = \frac{-\rho}{\varepsilon_0}$$

This is poisson's equation.From this equation V can be findout by using the formula

$$V(r,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r',t)}{r} d\tau'$$

V is easy to get but to find E we need A also($E = -\nabla V - \frac{\partial A}{\partial t}$) which is difficult.

Advantage of the coulomb gauge is that the scalar potential is simply to calculate .The disadvatage is that A is particularly difficult to calculate.

After applying coulomb gauge to the equation

$$\left(\nabla^2 A - \mu_0\varepsilon_0\frac{\partial^2 A}{\partial t^2}\right) - \nabla\left(\nabla \cdot A + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0 J$$

we get

$$\left(\nabla^2 A - \mu_0\varepsilon_0\frac{\partial^2 A}{\partial t^2}\right) = -\mu_0 J + \mu_0\varepsilon_0\nabla\left(\frac{\partial V}{\partial t}\right)$$

Lorentz gauge

$$\nabla \cdot A = -\mu_0\varepsilon_0\frac{\partial}{\partial t}V$$

is called the lorentz gauge.
This designes to eliminate the middle term of the equation

$$\left(\nabla^2 A - \mu_0\varepsilon_0\frac{\partial^2 A}{\partial t^2}\right) - \nabla\left(\nabla \cdot A + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0 J$$

With lorentz gauge this equation become

$$\nabla^2 A - \mu_0\varepsilon_0\frac{\partial^2 A}{\partial t^2} = -\mu_0 J$$

With lorentz gauge this equation $\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot A) = \frac{-\rho}{\varepsilon_0}$ becomes

$$\nabla^2 A - \mu_0 \varepsilon_0 \frac{\partial^2 A}{\partial t^2} = \frac{-\rho}{\varepsilon_0}$$

The virtue of the lorentz gauge is that it treats V and A on the same differential operator

$$\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \equiv \mathscr{D}^2$$

is called d'Alembertian
Then both equation become

$$\mathscr{D}^2 V = \frac{-\rho}{\varepsilon_0}$$

$$\mathscr{D}^2 A = -\mu_0 J$$

Previous year solutions

1. The electric and the magnetic field $\vec{E}(z,t)$ and $\vec{B}(z,t)$, respectively corresponding to the scalar potential $\phi(z,t)=0$ and vector potential $\vec{A}(z,t)=\hat{i}tz$ are

[GATE 2012]

A. $\vec{E}=\hat{i}z$ and $\vec{B}=-\hat{j}t$

B. $\vec{E}=\hat{i}z$ and $\vec{B}=\hat{j}t$

C. $\vec{E}=-\hat{i}z$ and $\vec{B}=-\hat{j}t$

D. $\vec{E}=-\hat{i}z$ and $\vec{B}=-\hat{j}$t

Solution: $\vec{E}=-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}=-\frac{\partial\vec{A}}{\partial t}=-\hat{i}z, \vec{B}=\vec{\nabla}\times\vec{A}=+\hat{j}t$
The correct option is **(d)**

2. If the vector potential $\vec{A}=\alpha x\hat{x}+2y\hat{y}-3z\hat{z}$, satisfies the Coulomb gauge, the value of the constant α is

[GATE 2015]

Solution:

$$\text{Coulomb gauge condition } \vec{\nabla}\cdot\vec{A}=0\Rightarrow\alpha+2-3=0\Rightarrow\alpha=1$$

3. Consider magnetic vector potential $\tilde{A}$ and scalar potential Φ which define the magnetic field $\vec{B}$ and electric field $\vec{E}$. If one adds $\vec{\nabla}\lambda$ to $\vec{A}$ for a well-defined λ, then what should be added to Φ so that $\vec{E}$ remains unchanged up to an arbitrary function of time, $f(t)$?

[JEST 2017]

A. $\frac{\partial\lambda}{\partial t}$

B. $-\frac{\partial\lambda}{\partial t}$

C. $\frac{1}{2}\frac{\partial\lambda}{\partial t}$

D. $-\frac{1}{2}\frac{\partial\lambda}{\partial t}$

Solution: Consider Gauge Transformation

$$\vec{A}'=\vec{A}-\vec{\nabla}\lambda=\vec{A}+\vec{V}(-\lambda) \quad \text{and} \quad \Phi'=\Phi-\frac{\partial(-\lambda)}{\partial t}=\Phi+\frac{\partial\lambda}{\partial t}$$

The correct option is **(a)**

2.2 Lagrangian and hamiltonian of a charged particle in a Electromagnetic field

The force experienced by a particle of charge q at rest in an electric field of intensity E is given by

$$F_1 = qE$$

The force experienced by a moving charge q in a magnetic field B is given by

$$F_2 = q(v \times B)$$

Where V is the velocity of the particle.The direction of F_2 is perpendicular to both v and B.
Total force on a uniformly moving charged particle of charge q is the sum of F_1 and F_2

$$F = F_1 + F_2 = qE + q(v \times B)$$

The above equation is known as lorentz formula.
Maxwell's equation in empty space is given by
$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$
$\nabla \cdot B = 0$
$\nabla \times E = -\frac{\partial B}{\partial t}$
$\nabla \times B = \mu_0 (J + \varepsilon_0 \frac{\partial E}{\partial t})$
We know that if $\nabla \cdot B = 0$,B can be expressed as a curl of some vector A

$$B = \nabla \times A$$

Substitute this equation in $\nabla \times E = -\frac{\partial B}{\partial t}$
$\nabla \times E = -\frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times A) = -\nabla \times \frac{\partial A}{\partial t}$
$\therefore E = -\frac{\partial A}{\partial t} - \nabla s$,since curl of gradient is always zero and s is scalar function
The lorentz force in terms of scalar potential s and vector potential A is given by

$$F = q[E + v \times B]$$

$$F = q\left[-\frac{\partial A}{\partial t} - \nabla s + (v \times \nabla \times A)\right]$$

Let us consider the last part $v \times (\nabla \times A)$

$$v \times (\nabla \times A) = \nabla(A \cdot v) - (\nabla \cdot v)A$$

Also $A = A(x, y, z, t)$
Therfore total time derivative of A is given by
$\frac{dA}{dt} = \frac{\partial A}{\partial x}\dot{x} + \frac{\partial A}{\partial y}\dot{y} + \frac{\partial A}{\partial z}\dot{z} + \frac{\partial A}{\partial t} = \frac{\partial A}{\partial x}v_x + \frac{\partial A}{\partial y}v_y + \frac{\partial A}{\partial z}v_z + \frac{\partial A}{\partial t}$
$= (\hat{i}v_x + \hat{j}v_y + \hat{k}v_z) \cdot \left(\hat{i}\frac{\partial A}{\partial x} + \hat{j}\frac{\partial A}{\partial y} + \hat{k}\frac{\partial A}{\partial z}\right) + \frac{\partial A}{\partial t} = v \cdot \nabla \mathbf{A} + \frac{\partial \mathbf{A}}{dt}$

$$v \cdot \nabla \mathbf{A} = \frac{d\mathbf{A}}{dt} - \frac{\partial \mathbf{A}}{\partial t}$$

Substituting this value in

$$v \times (\nabla \times A) = \nabla(A \cdot v) - (\nabla \cdot v)A$$

we will get

$$\mathrm{v} \times (\nabla \times \mathbf{A}) = \nabla(\mathbf{A} \cdot \mathrm{v}) - \frac{d\mathbf{A}}{dt} + \frac{\partial \mathbf{A}}{\partial t}$$

Then F becomes

$$F = q\left[-\frac{\partial A}{\partial t} - \nabla s + (v \times \nabla \times A)\right]$$

$$
\begin{aligned}
\mathbf{F} &= q\left[-\frac{\partial \mathbf{A}}{dt} - \nabla s + \left\{\nabla(\mathbf{A}\cdot \mathrm{v}) - \frac{d\mathbf{A}}{dt} + \frac{\partial \mathbf{A}}{\partial t}\right\}\right] \\
&= q\left[-\nabla s + \left\{\nabla(\mathbf{A}\cdot \mathrm{v}) - \frac{d\mathbf{A}}{dt}\right\}\right] \\
&= q\left[-\nabla(s - \mathbf{A}\cdot \mathrm{v}) - \frac{d\mathbf{A}}{dt}\right]
\end{aligned}
$$

The x-component of the force is given by

$$
\begin{aligned}
F_x &= q\left[\{-\nabla(s - \mathbf{A}\cdot \mathrm{v})\}_x - \frac{d\mathbf{A}_x}{dt}\right] \\
&= q\left[-\frac{\partial}{\partial x}(s - \mathbf{A}\cdot \mathrm{v}) - \frac{d\mathbf{A}_x}{dt}\right] \\
\text{or}\quad F_x &= q\left[-\frac{\partial}{\partial x}(s - \mathbf{A}\cdot \mathrm{v}) - \frac{d}{dt}\left\{\frac{\partial}{\partial \mathrm{v}_x}(\mathbf{A}\cdot \mathrm{v}\}\right]\right.
\end{aligned}
$$

qs is independent of velocity v_x i.e. $\frac{\partial(qs)}{\partial \mathrm{v}_x} = 0$ So we can add that term to the above equation, it will not make any change.
After rewritting

$$
\begin{aligned}
F_x &= -\frac{\partial U}{\partial x} - \frac{d}{dt}\frac{\partial U}{\partial \mathrm{v_x}} \\
U &= qs - q(\mathbf{A}\cdot \mathbf{v})
\end{aligned}
$$

FRom these it is clear that U is a function of x and v ie q_k and $\dot{q}_k$ U is called generalized potential or velocity dependent potential.

Lagrangian

$L = T - U = T - \{qs - q(\mathbf{A}\cdot \mathbf{v}).\} = T - qs + q(\mathbf{A}\cdot \mathbf{v})$
T=kinetic energy.$T = \frac{1}{2}mv^2$

$$L = \frac{1}{2}mv^2 - qs + q(\mathbf{A}\cdot \mathbf{v})$$

Momentum $P_K = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial v_k} = mv_k + qA = mv + qA$

Hamiltonian

$$
\begin{aligned}
H &= \sum_k p_k \dot{q}_k - L \\
&= \sum_k p_k \dot{r}_k - L \\
&= \sum_k p_k v_k - L \\
&= \sum_k (mv_k + qA_k)\, v_k - \left[\frac{1}{2}mv^2 - qs + q(\mathbf{v}\cdot \mathbf{A})\right] \\
&= \sum_k \left(mv_k^2 + qA_k v_k\right) - \frac{1}{2}mv^2 + qs - q(\mathbf{v}\cdot \mathbf{A}) \\
&= mv^2 + q(\mathbf{A}\cdot \mathbf{v}) - \frac{1}{2}mv^2 + qs - q(\mathbf{v}\cdot \mathbf{A}) \\
&= \frac{1}{2}mv^2 + qs
\end{aligned}
$$

Futuring

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3. Dipole radiation

Radiation means the generation of electromagnetic waves, which then propagate away from the source to infinity. Classically, electromagnetic waves are created by charged particles accelerating, or, equivalently, currents varying in time. There is no radiation in case of electric and magnetic fields of a point charge moving with constant velocity. The energy in the electromagnetic field is transported along with the particle, but does not propagate away toward infinity. But if the charge velocity is not constant there must be radiation.
As we start the study of radiation, it is interesting to remember that the classical field theory was developed from experiments and observations on macroscopic objects such as charged bodies, insulators, conductors, current-carrying wires, and magnets. The classical theory is therefore at its best when applied to macroscopic electromagnetic phenomena. So, for example, the theory applies to radiation of radio waves by an antenna, or microwaves by a cavity resonator. In this book we concentrate mainly on systems of that kind.
However, much of the radiation around us-in fact much of the radiation in the Universe-comes from microscopic systems and must be described by quantum electrodynamics. For example, sunlight, light from the filament of an incandescent bulb, fluorescent light, or laser light can only be explained by quantum considerations of individual atoms, molecules, or systems of atoms. For these systems the classical theory does not really apply, except perhaps in a qualitative, heuristic way. Only quantum electrodynamics can properly describe radiation by an atom. Despite its limited applicability, the classical theory of radiation is an important part of electromagnetism. Quantum electrodynamics, which has limitations of its own, relies on a foundation of classical electrodynamics.

3.1 Electric dipole radiation

Picture two tiny metal spheres separated by a distance d and connected by a fine wire (figure below); at time t the charge on the upper sphere is $q(t)$, and the charge on the lower sphere is $-q(t)$. Suppose that we drive the charge back and forth through the wire, from one end to the other, at an angular frequency ω :

$$q(t)=q_0\cos(\omega t)$$

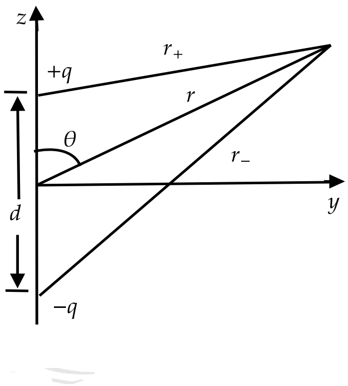

Figure 3.1

The result is an oscillating electric dipole:

$$\mathbf{p}(t) = p_0 \cos(\omega t)\hat{\mathbf{z}}$$

where

$$p_0 = q_0 d$$

is the maximum value of the dipole moment. With the approximation of

$$d \ll \frac{\lambda}{2\pi} \ll r$$

you should not forget that $\frac{\lambda}{2\pi} = \frac{c}{\omega}$ we can calculate the potentials

$$V(r,\theta,t) = -\frac{p_0\omega}{4\pi\varepsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t-r/c)]$$

and

$$\mathbf{A}(r,\theta,t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t-r/c)]\hat{\mathbf{z}}$$

Hence we get the fields

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\hat{\theta}$$

and

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\hat{\phi}$$

The energy radiated by an oscillating electric dipole is determined by the Poynting vector:

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\right\}^2\hat{\mathbf{r}}$$

The intensity is obtained by averaging (in time) over a complete cycle:

$$\langle\mathbf{S}\rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}}$$

The total power radiated is found by integrating $\langle\mathbf{S}\rangle$ over a sphere of radius r :

$$\langle P\rangle = \int \langle\mathbf{S}\rangle \cdot d\mathbf{a} = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\int \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta \, d\phi = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$

3.2 Magnetic dipole radiations

Suppose now that we have a wire loop of radius b (figure below) around which we drive an alternating current:

$$I(t) = I_0 \cos(\omega t)$$

This is a model for an oscillating magnetic dipole,

$$\mathbf{m}(t) = \pi b^2 I(t)\hat{\mathbf{z}} = m_0 \cos(\omega t)\hat{\mathbf{z}}$$

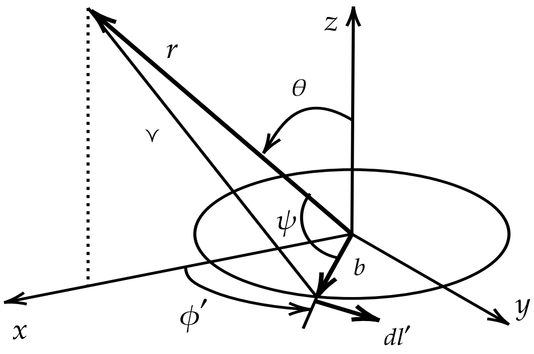

Figure 3.2

where the maximum value of the magnetic dipole moment is.

$$m_0 = \pi b^2 I_0$$

Again with the assumption of

$$b \ll \frac{\lambda}{2\pi} \ll r$$

we get the potentials and fields

$$\mathbf{A}(r,\theta,t) = -\frac{\mu_0 m_0 \omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin[\omega(t-r/c)]\hat{\phi}$$

There is no electric scalar potential as there is no static charge. Hence the fields

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\hat{\phi}$$

and

$$\mathbf{B} = \nabla\times\mathbf{A} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\hat{\theta}$$

The Poynting vector

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{m_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\right\}^2\hat{\mathbf{r}}$$

time average of Poynting vector

$$\langle \mathbf{S}\rangle = \left(\frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\right)\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}}$$

the total radiated power

$$\langle P\rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$$

Note If you compare the formula for power for electric and magnetic dipole radiation you get

$$\frac{P_{\text{magnetic}}}{P_{\text{electric}}} = \left(\frac{m_0}{p_0 c}\right)^2$$

3.3 Radiation from a arbitrary source

Power radiated from arbitrary electric source whose electric dipole moment is changing with respect to time

$$P_{\text{rad}}(t_0) \cong \frac{\mu_0}{6\pi c}\left[\ddot{p}(t_0)\right]^2$$

Power radiated from arbitrary magnetic source whose magnetic dipole moment is changing with respect to time

$$P_{\text{rad}}(t_0) \cong \frac{\mu_0}{6\pi c^3}\left[\ddot{m}(t_0)\right]^2$$

Exercise 3.1 A particle moving straight line with velocity $v = v_0 + \alpha t$. How much energy does it radiate? ■

Solution: The position of the particle

$$x = x_0 + v_0 t + \frac{1}{2}\alpha t^2$$

Now dipole moment

$$\begin{aligned} \mathbf{p} &= qx\hat{i} \\ \Rightarrow p &= q\left(x_0 + v_0 t + \frac{1}{2}\alpha t^2\right) \\ \Rightarrow \ddot{p} &= q\alpha \\ \text{Power } &= \frac{\mu_0}{6\pi c} q^2 \alpha^2 \end{aligned}$$

This is the famous larmor formula.
This formula can be used for a point charge moving with acceleration α

Exercise 3.2 A parallel-plate capacitor C, with plate separation d, is given an initial charge $(\pm)Q_0$. It is then connected to a resistor R, and discharges, $Q(t) = Q_0 e^{-t/RC}$. What fraction of its initial energy $\left(Q_0^2/2C\right)$ does it radiate away? ■

Solution: Power radiated is

$$\frac{\mu_0}{6\pi c}\ddot{p}^2$$

In our case

$$p = Qd, \text{ so } \ddot{p} = \ddot{Q}d = Q_0\left(\frac{1}{RC}\right)^2 e^{-t/RC} d$$

So power radiated

$$\frac{dW_r}{dt} = \frac{\mu_0}{6\pi c}\frac{(Q_0 d)^2}{(RC)^4} e^{-2t/RC}$$

The total energy radiated is

$$\begin{aligned} W_r &= \frac{\mu_0}{6\pi c}\frac{(Q_0 d)^2}{(RC)^4}\int_0^\infty e^{-2t/RC}\,dt \\ &= \frac{\mu_0}{6\pi c}\frac{(Q_0 d)^2}{(RC)^4}\left[-\frac{RC}{2}e^{-2t/RC}\right]\Bigg|_0^\infty \\ &= \frac{\mu_0}{6\pi c}\frac{(Q_0 d)^2}{(RC)^4}\frac{RC}{2} = \frac{\mu_0}{12\pi c}\frac{(Q_0 d)^2}{(RC)^3} \end{aligned}$$

Exercise 3.3 An electron is released from rest and falls under the influence of gravity. In the first centimeter, what fraction of the potential energy lost is radiated away? ■

Solution: Let the y direction is in the downwards direction then
$\mathbf{p} = -ey\hat{\mathbf{y}}, y = \frac{1}{2}gt^2$, so $\mathbf{p} = -\frac{1}{2}get^2\hat{\mathbf{y}}; \ddot{\mathbf{p}} = -ge\hat{\mathbf{y}}$
Hence the power radiated is

$$P = \frac{\mu_0}{6\pi c}(ge)^2$$

Now the time the charge takes to fall a distance h is given by $h = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{2h/g}$, so the energy radiated in falling a distance h is

$$U_{\text{rad}} = Pt = \frac{\mu_0(ge)^2}{6\pi c}\sqrt{2h/g}$$

The potential energy lost is $U_{\text{pot}} = mgh$. So the fraction is

$$f = \frac{U_{\text{rad}}}{U_{\text{pot}}} = \frac{\mu_0 g^2 e^2}{6\pi c}\sqrt{\frac{2h}{g}\frac{1}{mgh}} = \frac{\mu_0 e^2}{6\pi mc}\sqrt{\frac{2g}{h}}$$
$$= \frac{\left(4\pi \times 10^{-7}\right)\left(1.6 \times 10^{-19}\right)^2}{6\pi\left(9.11 \times 10^{-31}\right)\left(3 \times 10^8\right)}\sqrt{\frac{(2)(9.8)}{(0.01)}} = 2.76 \times 10^{-22}$$

Exercise 3.4 In Bohr's theory of hydrogen, the electron in its ground state was supposed to travel in a circle of radius 5×10^{-11} m, held in orbit by the Coulomb attraction of the proton. According to classical electrodynamics, this electron should radiate, and hence spiral in to the nucleus. Show that $v \ll c$ for most of the trip (so you can use the Larmor formula), and calculate the lifespan of Bohr's atom. (Assume each revolution is essentially circular.) ■

Solution:

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r^2} = ma = m\frac{v^2}{r} \Rightarrow v = \sqrt{\frac{1}{4\pi\varepsilon_0}\frac{q^2}{mr}}$$

At the beginning ($r_0 = 0.5$ A)

$$\frac{v}{c} = \left[\frac{\left(1.6 \times 10^{-19}\right)^2}{4\pi\left(8.85 \times 10^{-12}\right)\left(9.11 \times 10^{-31}\right)\left(5 \times 10^{-11}\right)}\right]^{-\frac{1}{2}}\frac{1}{3 \times 10^8}$$
$$= 0.0075$$

and when the radius is one hundredth of this v/c is only 10 times greater (0.075), so for most of the trip the velocity is safely nonrelativistic. From the Larmor formula,

$$P = \frac{\mu_0 q^2}{6\pi c}\left(\frac{v^2}{r}\right)^2 = \frac{\mu_0 q^2}{6\pi\varepsilon_0}\left(\frac{1}{4\pi\varepsilon_0}\frac{q^2}{mr^2}\right)^2$$

since $a = v^2/r$, and $P = -dU/dt$ where U is the (total) energy of the electron

$$\begin{aligned} U = U_{\text{kin}} + U_{\text{pot}} &= \frac{1}{2}mv^2 - \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r} \\ &= \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\frac{q^2}{r}\right) - \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r} \\ &= -\frac{1}{8\pi\varepsilon_0}\frac{q^2}{r} \end{aligned}$$

So

$$-\frac{dU}{dt} = -\frac{1}{8\pi\varepsilon_0}\frac{q^2}{r^2}\frac{dr}{dt} = P = \frac{q^2}{6\pi\varepsilon_0 c^3}\left(\frac{1}{4\pi\varepsilon_0}\frac{q^2}{mr^2}\right)^2$$

and hence

$$\frac{dr}{dt} = -\frac{1}{3c}\left(\frac{q^2}{2\pi\varepsilon_0 mc}\right)^2\frac{1}{r^2}$$

or

$$\begin{aligned} dt = -3c\left(\frac{2\pi\varepsilon_0 mc}{q^2}\right)^2 r^2 dr \Rightarrow t \\ = -3c\left(\frac{2\pi\varepsilon_0 mc}{q^2}\right)^2 \int_{r_0}^{0} r^2 dr \\ = c\left(\frac{2\pi\varepsilon_0 mc}{q^2}\right)^2 r_0^3 \end{aligned}$$

or

$$\begin{aligned} t = &\left(3\times 10^8\right)\times \\ &\left[\frac{2\pi\left(8.85\times 10^{-12}\right)\left(9.11\times 10^{-31}\right)\left(3\times 10^8\right)}{\left(1.6\times 10^{-19}\right)^2}\right]^2\left(5\times 10^{-11}\right)^3 \\ =&1.3\times 10^{-11}\ \text{s} \end{aligned}$$

Previous Year solutions

1. An electron is decelerated at a constant rate starting from an initial velocity u (where $u << c$) to $u/2$ during which it travels a distance s. The amount of energy lost to radiation is

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A. $\frac{\mu_0 e^2 u^2}{3\pi mc^2 s}$

B. $\frac{\mu_0 e^2 u^2}{6\pi mc^2 s}$

C. $\frac{\mu_0 e^2 u}{8\pi mcs}$

D. $\frac{\mu_0 e^2 u}{16\pi mcs}$

Solution: Total power radiated $P = \frac{\mu_0 q^2 a^2}{6\pi c}$

Total energy radiated in time t is $E = P \cdot t = \frac{\mu_0 e^2 a^2}{6\pi c} \cdot t = \frac{\mu_0 e^2 a^2}{6\pi c} \times \frac{u}{2a}$

$$\left[\because v = u - at \Rightarrow \frac{u}{2} = u - at \Rightarrow t = \frac{u}{2a}\right]$$

$$\Rightarrow E = \frac{\mu_0 e^2 au}{12\pi c}$$

Fraction of initial $K.E.$ lost due to radiation $= \frac{E}{\frac{1}{2}mu^2} = \frac{2E}{mu^2}$

$$= \frac{2}{mu^2} \times \frac{\mu_0 e^2 au}{12\pi c} = \frac{\mu_0 e^2 a}{6\pi mcu}$$

$$\left[\therefore s = ut - \frac{1}{2}at^2 = u \times \frac{u}{2a} - \frac{1}{2}a \times \frac{u^2}{4a^2} = \frac{u^2}{2a} - \frac{u^2}{8a} = \frac{3u^2}{8a} \Rightarrow a = \frac{3u^2}{8s}\right]$$

$$= \frac{\mu_0 e^2}{6\pi mcu} \times \frac{3u^2}{8s} = \frac{\mu_0 e^2 u}{16\pi mcs}$$

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