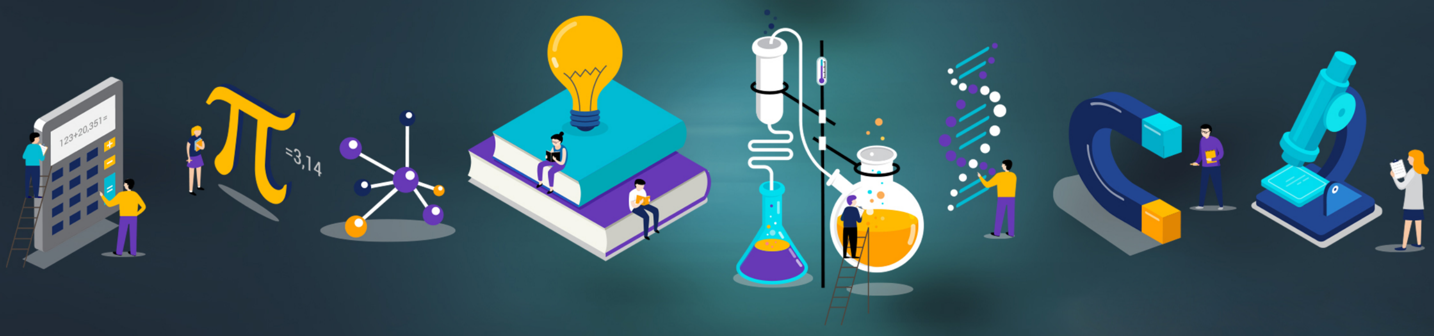


1. VECTOR CALCULUS

Definition 1.0.1 A vector is defined as a physical quantity having magnitude and a direction associated with it.

■ **Example 1.1** Displacement,Velocity, Acceleration, Force ,Torque,Angular momentum etc. ■

1.1 Vector representation

Geometrically a vector is represented by a directed line segment, with length proportional to the magnitude.The direction of the arrow gives the direction of the vector.
We will refer to the start of the arrow as the tail and the end as the tip or head. The vector between two points P and Q will be denoted as, $\overrightarrow{PQ}$(or by a boldface **PQ**). And the magnitude as $|PQ|$. Magnitude will also be called length or norm.

Analytically a three dimensional vector can be specified by an ordered set of three numbers, called its components.The magnitude of the components depend on the coordinate system used. (A vector can be extended to n dimensions). A vector $\vec{A}$ is represented by (A_x, A_y, A_z) in cartesian (rectangular) coordinate system .
Magnitude of vector $\vec{A}$ is given by, $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

1.1.1 Position vector:

Definition 1.1.1 Vectors that start at the origin and terminate at any arbitrary point are called position vectors. These are used to determine the position of a point with reference to the origin.

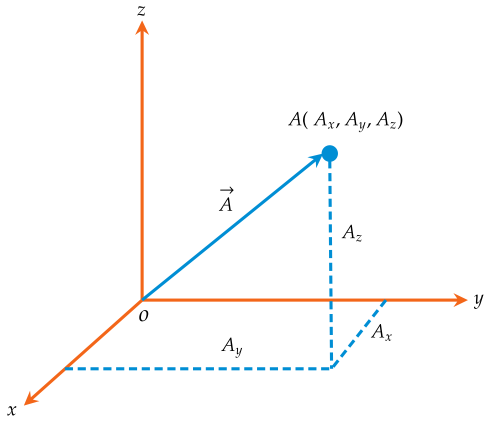


Figure 1.1: Representation of position vector $\vec{A}$

Any vector $\vec{\mathrm{A}}$ in the $3-\mathrm{D}$ right handed rectangular cartesian coordinate system can be represented as,

$$\vec{\mathrm{A}} = \mathrm{A}_x\hat{i} + \mathrm{A}_y\hat{j} + \mathrm{A}\hat{k} \tag{1.1}$$

Where, $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors in direction of x, y and z axis respectively. $\mathrm{A}_x, \mathrm{A}_y$ and A_z are the cartesian components or projections of vector $\vec{A}$ along x, y, z axis.

Exercise 1.1 If A and B are (3,4,5) and $(6,8,9)$, find $\vec{\mathrm{AB}}$. ■

Solution:

$$\begin{aligned}
\overrightarrow{\mathrm{AB}} &= \text{Position vector of B} - \text{Position vector of A} \\
&= (6\hat{i} + 8\hat{j} + 9\hat{k}) - (3\hat{i} + 4\hat{j} + 5\hat{k}) \\
&= 3\hat{i} + 4\hat{j} + 4\hat{k}
\end{aligned}$$

1.1.2 Unit vector:

Definition 1.1.2 A vector quantity having unit magnitude is called unit vector.A unit vector along $\vec{A}$ is defined as,
$\hat{\mathrm{A}} = \frac{\vec{\mathrm{A}}}{|\vec{\mathrm{A}}|} = \frac{(\mathrm{A}_x\hat{i} + \mathrm{A}_y\hat{j} + \mathrm{A}_z\hat{k})}{\sqrt{\mathrm{A}_x^2 + \mathrm{A}_y^2 + \mathrm{A}_z^2}}$

Exercise 1.2 Find unit vector in the direction of vector $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$ ■

Solution:

$$\begin{aligned}
\text{Magnitude of } \vec{a} &= \sqrt{2^2 + 3^2 + 1^2} \\
|\vec{a}| &= \sqrt{4 + 9 + 1} = \sqrt{14} \\
\text{Unit vector in direction of } \vec{a} &= \frac{\vec{a}}{|\vec{a}|} \\
\hat{a} &= \frac{1}{\sqrt{14}}[2\hat{i} + 3\hat{j} + 1\hat{k}] \\
\hat{a} &= \frac{2}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{i} + \frac{1}{\sqrt{14}}\hat{k}
\end{aligned}$$

1.1.3 Direction cosines

In analytical geometry the direction cosines are the angles made by the vector with the three coordinate axes.

Direction cosines of vector $\vec{A}$:

If $\vec{A}$ makes angles α, β, γ with x, y and z axes respectively, then direction cosines of $\vec{A}$ are defined as,

$$l = \cos\alpha = \frac{A_x}{A} \quad ; \quad m = \cos\beta = \frac{A_y}{A} \quad ; \quad n = \cos\gamma = \frac{A_z}{A}$$
$$l^2 + m^2 + n^2 = 1$$

Then the unit vector along $\vec{A}$ can be written as,
$\hat{A} = l\hat{i} + m\hat{j} + n\hat{k}$

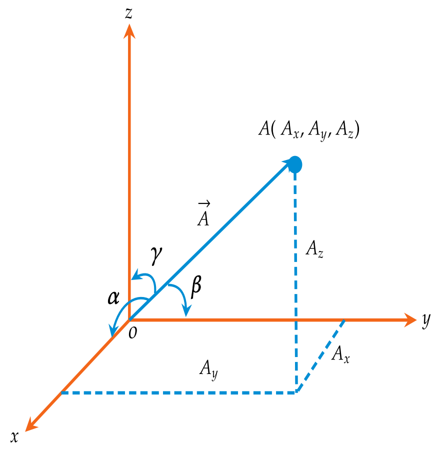


Figure 1.2: Direction cosine.

1.2 Types of vectors

- **Equal vectors** : Vectors having same magnitude and same direction.
- **Null Vectors** : Vectors having coincident initial and terminal point i.e its magnitude is zero and it has any arbitrary direction.
- **Reciprocal Vector :** Vector having same direction as $\vec{A}$ but magnitude reciprocal to that of $\vec{A}$, is known as the reciprocal vector of A Reciprocal vector of $\vec{A}$ is $\vec{A} - \frac{1}{A}\hat{A}$.
- **Negative Vector** : Vectors having same magnitude as $\vec{a}$ but direction opposite to that of $\vec{A}$, is known as the negative vector of $\vec{a}$. Negative vector as $\vec{A}$ is $-\vec{A} = -|A|\hat{A}$.

1.3 Vector operations

1.3.1 Vector addition

- **Triangular law of vector addition:**
  - Place the tail of $\vec{A}$ at the head of $\vec{B}$.
  - The resultant vector $\vec{A} + \vec{B}$ is formed by connecting the tail of the first vector to the head of the last vector.

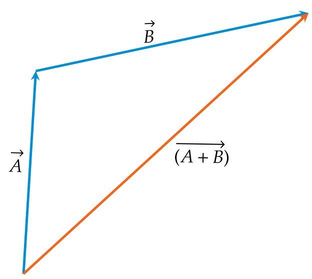


Figure 1.3: Triangular law of vector addition

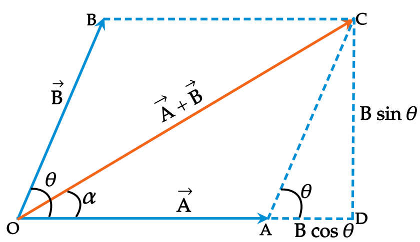


Figure 1.4: Parellelogram law of vector addition

- **Parellelogram law of vector addition:**

When two vectors act at a point, their resultant is found by the law of parallelogram of vectors.(We got to use it often in Electricity and magnetism.)

The magnitude of Resultant vector $\vec{A}+\vec{B}$ From right angled $\triangle$ OCD.

$$\begin{aligned} OC^2 &= OD^2 + CD^2 \\ &= (OA+AD)^2 + CD^2 \\ &= OA^2 + AD^2 + 2AD \cdot OA + CD^2 \end{aligned}$$

Here, $\quad OA = A \; ; \; AD = B\cos\theta \; ; \; CD = B\sin\theta$

Then,
$$\begin{aligned} OC^2 &= |\vec{A}+\vec{B}|^2 \\ &= A^2 + (B\cos\theta)^2 + (B\sin\theta)^2 + 2AB\cos\theta \\ &= A^2 + B^2 + AB\cos\theta \\ |\vec{A}+\vec{B}| &= \sqrt{A^2+B^2+2AB\cos\theta} \end{aligned}$$

The direction of Resultant vector $\vec{A}+\vec{B}$ with the vector $\vec{A}$

$$\tan\alpha = \frac{B\sin\theta}{A+B\cos\theta}$$

$$\alpha = \tan^{-1}\left(\frac{B\sin\theta}{A+B\cos\theta}\right)$$

If the two vectors are parellel i.e., $\theta = 0$ Then,

$$|\vec{A}+\vec{B}| = \sqrt{A^2+B^2+2AB} = \sqrt{(A+B)^2} = (A+B)$$

If the two vectors are anti-parellel i.e., $\theta = 180$ Then,

$$|\vec{A}+\vec{B}| = \sqrt{A^2+B^2-2AB} = \sqrt{(A-B)^2} = (A-B)$$

Note

If $|\vec{A}| = |\vec{B}| = A$ Then resultant of these two vectors will be,

1. $\theta = 0 \quad \rightarrow \sqrt{A^2+A^2+2AA\cos 0} = \sqrt{A^2+A^2+2A^2} = 2A$
2. $\theta = 60^\circ \quad \rightarrow \sqrt{A^2+A^2+2AA\cos 60} = \sqrt{A^2+A^2+A^2} = \sqrt{3}A$
3. $\theta = 90^\circ \quad \rightarrow \sqrt{A^2+A^2+2AA\cos 90} = \sqrt{A^2+A^2} = \sqrt{2}A$
4. $\theta = 180^\circ \quad \rightarrow \sqrt{A^2+A^2+2AA\cos 180} = \sqrt{A^2+A^2-2A^2} = 0$

Properties of vector addition

- Commutation property: $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$.
- Associative property:$(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$.
- Additive identity:$\mathbf{A}+\mathbf{0} = \mathbf{A}$.
- Additive inverse:$\mathbf{A}+(-\mathbf{A}) = \mathbf{0}$.

1.3.2 Vector multiplication

1.Scaling of vector(Multiplication by scalar)

Scaling a vector means changing it's length by a scale factor. Multiplication of a vector by a positive scalar $'c'$, multiplies the magnitude but leaves the direction unchanged. If $'c'$ is negative, the direction is reversed.

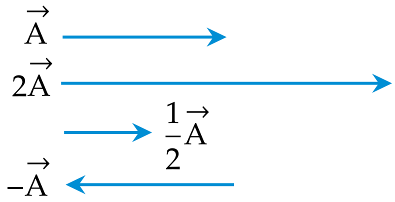


Figure 1.5: Scaling of vector

Properties of scalar multiplication

- Distributive property: $a(\vec{A}+\vec{B}) = a\vec{A}+a\vec{B}$.

2.Dot product or scalar product of two vectors

The dot product of two vectors is defined as

$$\vec{A}\cdot\vec{B} = |A||B|\cos\theta \tag{1.2}$$

Where θ is the angle they form when placed tail to tail. $\vec{A}\cdot\vec{B}$ is itself a scalar.Geometrically $\vec{A}\cdot\vec{B}$ is the product of A times the projection of $\vec{B}$ along $\vec{A}$.

In general, If $\vec{A} = A_z\hat{i}+A_y\hat{j}+A_z\hat{k}$ and $\vec{B} = B_x\hat{i}+B_y\hat{j}+B_z\hat{k}$,

Then we can construct the scalar product of $\vec{A}$ and $\vec{B}$ as,

$$\vec{A}\cdot\vec{B} = A_xB_x+A_yB_y+A_zB_z \tag{1.3}$$

■ **Example 1.2** **Workdone:** If a constant force F acting on a particle displaces it from the point A to B then,

$$\begin{aligned} \text{Work done} &= (\text{component of F along A B }).\text{ Displacement} \\ &= \mathrm{F}\cos\theta . AB \\ &= \vec{F}\cdot\overrightarrow{AB} \\ \text{Work done } &= \text{Force. Displacement} \end{aligned}$$

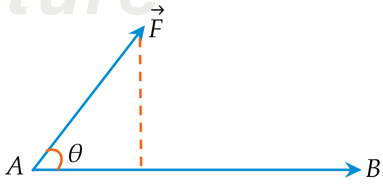


Figure 1.6: Workdone

■

Projection:

Projection of a vector A on B is the component of vector A in the direction of vector B.

$$\text{Projection of } \quad \vec{A} \quad \text{along} \quad \vec{B} = \vec{A}\cos\theta = \vec{A}\cdot\hat{B}$$

.

Properties of Dot product

- Commutative property:$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$.
- Asociative property:$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$.
- For two mutually perpendicular vectors $\vec{A}$ and B, $A\cdot\vec{B} = 0$.
- If the two vectors are parallel, $\vec{A}\cdot\vec{B} = AB$.(since, $\cos 0 = 1$)

- $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot i = 0$
  $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

Exercise 1.3 For the two vectors $\vec{A} = 6\hat{i} + 4\hat{j} + 3\hat{k}$ and $\vec{B} = 2\hat{i} - 3\hat{j} - 3\hat{k}$

$$\begin{aligned}\vec{\mathrm{A}}\cdot\vec{\mathrm{B}} &= \mathrm{A}_x\mathrm{B}_x + \mathrm{A}_y\mathrm{B}_y + \mathrm{A}_z\mathrm{B}_z\\ &= 12 - 12 - 9\\ &= -9\end{aligned}$$

3.Vector product or Cross product

Cross product of two vectors $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$ is defined as a vector that is perpendicular (orthogonal) to both $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$, with a magnitude equal to the area of the parallelogram that the vectors span(This suggest that area may be treated as a vector quantity). Since there are two opposite directions which are so perpendicular to $\vec{A}$ and $\vec{B}$ This does not uniquely determine $\vec{A}\times\vec{B}$. The direction of $\vec{A}\times\vec{B}$ is fixed by a convention, called the Right Hand Rule.

Right Hand Rule :
Stretch out the fingers of the right hand so that the thumb becomes perpendicular to both the index (fore finger) and the middle finger. If the index points in the direction of $\vec{A}$ and the middle finger in the direction of $\vec{B}$ then, $\vec{A}\times\vec{B}$ points in the direction of the thumb.

The vector product of $\vec{A}$ and $\vec{B}$ is defined as,

$$\vec{\mathrm{A}}\times\vec{\mathrm{B}} = |\vec{\mathrm{A}}||\vec{\mathrm{B}}|\sin\theta\hat{n} \tag{1.4}$$

Where $\hat{n}$ is unit vector normal to the plane containing $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$.
Using decomposition of vector into their cartesian components,we can find $\vec{\mathrm{A}}\times\vec{\mathrm{B}}$ as,
If $\vec{\mathrm{A}} = \mathrm{A}_z\hat{i} + \mathrm{A}_y\hat{j} + \mathrm{A}_z\hat{k}$ and $\vec{\mathrm{B}} = \mathrm{B}_x\hat{i} + \mathrm{B}_y\hat{j} + \mathrm{B}_z\hat{k}$, then

$$\vec{\mathrm{A}}\times\vec{\mathrm{B}} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \mathrm{A}_x & \mathrm{A}_y & \mathrm{A}_z\\ \mathrm{B}_x & \mathrm{B}_y & B_z\end{vmatrix}$$

Example 1.3 Angular momentum
The angular momentum of a particle, about a reference point , is defined as the vector product of the potion relative to the reference point, and momentum of the particle

$$L = r\times p$$

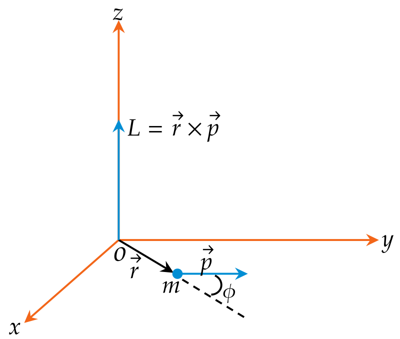


Figure 1.7: Angular momentum

Properties of Cross product

- Distributive property : $\vec{A}\times(\vec{B}+\vec{C}) = (\vec{A}\times\vec{B})+(\vec{A}\times\vec{C})$
- Commutative property : $\vec{A}\times\vec{B} = -(\vec{B}\times\vec{A})$.
- For two collinear vectors (parallel or anti-parallel vectors) $\vec{A}\times\vec{B} = 0$.
- $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

  $\hat{i}\times\hat{j} = \hat{k}\ \ \hat{j}\times\hat{k} = \hat{i}\ \ \hat{k}\times\hat{i} = \hat{j}$.

Exercise 1.4 Find the area of a parallelogram whose adjacent sides are $\hat{i}-2\hat{j}+3\hat{k}$ and $2\hat{i}+\hat{j}-4\hat{k}$. ■

Solution:

$$\text{Vector area of parellelogram} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 1 & -4 \end{vmatrix}$$

$$= (8-3)\hat{i} - (-4-6)\hat{j} + (1+4)\hat{k} = 5\hat{i}+10\hat{j}+5\hat{k}$$

$$\text{Area of parallelogram} = \sqrt{(5)^2+(10)^2+(5)^2} = 5\sqrt{6}$$

4.Triple product

- **Scalar Triple product**
  The scalar triple product of three vectors $\vec{A},\vec{B}$, and $\vec{C}$ is $(\vec{A}\times\vec{B})\cdot\vec{C}$. It is a scalar product because, just like the dot product, it evaluates to a single number. The absolute value of $|(\vec{A}\times\vec{B})\cdot\vec{C}|$ is the volume of the parallelepiped spanned by $\vec{A},\vec{B}$, and $\vec{C}$ (i.e., the parallelepiped whose adjacent sides are the vectors $\vec{A},\vec{B}$, and $\vec{C}$).

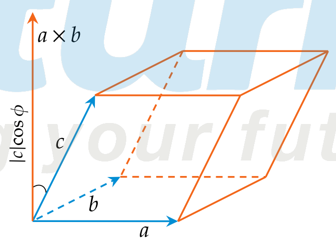


Figure 1.8: Scalar triple product

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

$$\text{In component form } \vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

- **Vector Triple product**
  A vector triple product $\vec{A}\times(\vec{B}\times\vec{C})$ of 3 vectors ,$\vec{A}$, $\vec{B}$ and $\vec{C}$ is simply a vector lying in the plane containing $\vec{A}$, $\vec{B}$ and $\vec{C}$.

  The vector triple product can be simplified by the so-called B A C − C A B rule.The equation is linear in A,B and C.

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

Note

Suppose we have two vectors $\vec{a}$ and $\vec{b}$ as shown in the figure. 1.9 We can write $\vec{b}$ as

$$\vec{b} = \vec{b}_{\parallel} + \vec{b}_{\perp}$$

Where $b_{\parallel}$ is the projection of $\vec{b}$ along $\vec{a}$ and$\vec{b}_{\perp}$ is the projection of $\vec{b}$ perpendicular to $\vec{a}$

$$\begin{aligned}
\vec{b} &= (\vec{b}\cdot\hat{a})\cdot\hat{a} + \vec{b}_{\perp} \\
\vec{b} &= \frac{(\vec{b}\cdot\vec{a})\cdot\vec{a}}{a^2} + \vec{b}_{\perp} \\
\vec{b} &= \frac{(\vec{b}\cdot\vec{a})\cdot\vec{a}}{a^2} + \vec{b} - \frac{(\vec{b}\cdot\vec{a})\cdot\vec{a}}{a^2} \\
\vec{b} &= \frac{(\vec{b}\cdot\vec{a})\cdot\vec{a}}{a^2} + \frac{\vec{b}(\vec{a}\cdot\vec{a}) - (\vec{b}\cdot\vec{a})\vec{a}}{a^2} \\
\vec{b} &= \frac{(\vec{b}\cdot\vec{a})\cdot\vec{a}}{a^2} + \frac{\vec{a}\times(\vec{b}\times\vec{a})}{a^2} \\
\vec{b} &= \vec{b}_{\parallel}(\text{parellell component}) + \vec{b}_{\perp}(\text{perpendicular component})
\end{aligned}$$

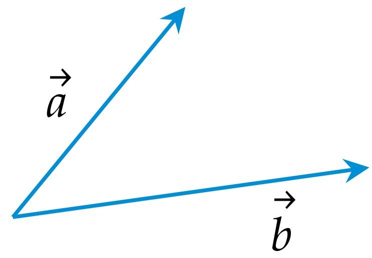


Figure 1.9: vector

1.4 General curvilinear coordinate system

Not all Physical problems are well adapted to a solution in cartesian coordinate system.We have to develop a general system that may be apt for any particular system of intersect. The coordinates in general curvilinear coordinate system be described by three coordinates, q_1, q_2 and q_3
then a position vector $\vec{r}$ in the system can be represented as,

$$\vec{r} = \vec{r}(q_1, q_2, q_3)$$

$$\text{Then,}\quad dr = \frac{\partial r}{\partial q_1}dq_1 + \frac{\partial r}{\partial q_2}dq_2 + \frac{\partial r}{\partial q_3}dq_3$$

Where, $\frac{\partial r}{\partial q_1}, \frac{\partial r}{\partial q_2}, \frac{\partial r}{\partial q_3}$ are the tangent vectors along q_1, q_2 and q_3.

The unit vectors along q_1, q_2 and q_3 are defined as,

$$\hat{e}_1 = \frac{\frac{\partial r}{\partial q_1}}{|\frac{\partial r}{\partial q_1}|} \quad ; \quad \hat{e}_2 = \frac{\frac{\partial r}{\partial q_2}}{|\frac{\partial r}{\partial q_2}|} \quad ; \quad \hat{e}_3 = \frac{\frac{\partial r}{\partial q_3}}{|\frac{\partial r}{\partial q_3}|}$$

Scaling factor:

The factor $|\frac{\partial r}{\partial q_1}|$ is known as the scaling factor. It is denoted as h_1.

Similiarly, $h_2 = |\frac{\partial r}{\partial q_2}|$ and $h_3 = |\frac{\partial r}{\partial q_3}|$

Then the position vector can be written as,

$$\vec{dr} = h_1\hat{e}_1 dq_1 + h_2\hat{e}_2 q_2 + h_3\hat{e}_3 q_3 \tag{1.5}$$

Cartesian coordinate system

$$\begin{aligned}
&\text{Coordinates} &&: q_1 = x \;\; q_2 = y \;\; q_3 = z \\
&\text{Scaling factors} &&: h_1 = 1 \;\; h_2 = 1 \;\; h_3 = 1 \\
&\text{Unit vectors} &&: \hat{e}_1 = \hat{e}_x \;\; \hat{e}_2 = \hat{e}_y \;\; \hat{e}_3 = \hat{e}_z \\
&\text{Position vector} &&: \vec{dr} = \hat{e}_x dx + \hat{e}_y dy + \hat{e}_z dz
\end{aligned}$$

1.5 Differential Operations on Vectors

Gradient (∇) : A derivative on a scalar that gives a vector.
Curl$(\nabla\times)$: A derivative on a vector that gives another vector.
Divergence $(\nabla\cdot)$: A derivative on a vector that gives scalar.

1.5.1 Gradient

The gradient is the multidimensional rate of change of a particular function. Gradient of a continuously differentiable scalar function $\phi(q_1,q_2,q_3)$ is mathematically defined as:

$$\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial q_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial q_2}d\hat{e}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}\hat{e}_3$$

Physical interpretation

Gradient tells you how much something changes as you move from one point to another (such as the pressure in a stream). If a surface $\phi(x,y,z)=c$ passes through a point P. The value of the function at each point on the surface is the same as at P. Then such a surface is called a level surface through P. At each point of the level surafce, the value of scalar function f will be same. Equipotential surface on which value of electrostatic potential is same at all points.

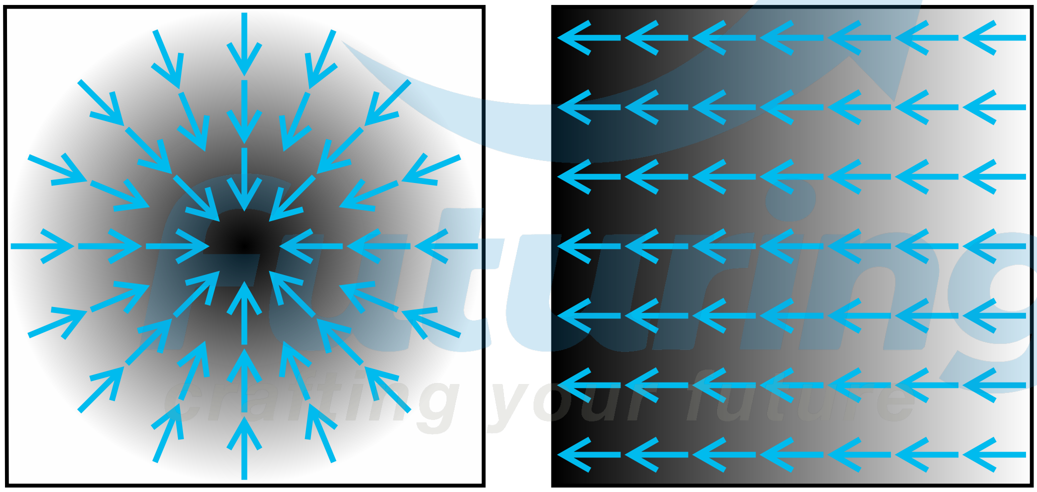


Figure 1.10: The gradient, represented by the blue arrows, denote the direction of greatest change of a scalar function. The values of the function are represented in greyscale and increase in value from white (low) to dark (high).

Product rule

- $\vec{\nabla}(\phi\psi) = \phi\vec{\nabla}\psi + \psi\vec{\nabla}\phi$
- $\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}$

Normal and Directional derivative

Normal:
If $\phi(x,y,z)=c$, represents a family of surfaces for different values of the constant c. On differentiating ϕ,

$$\begin{aligned} \text{We get,} \quad & d\phi = 0 \\ \text{But,} \quad & d\phi = \nabla\phi\cdot d\vec{r} \\ \text{So,} \quad & \nabla\phi\cdot dr = 0 \end{aligned}$$

The scalar product of two vectors $\nabla\phi$ and $d\vec{r}$ being zero, $\nabla\phi$ and $d\vec{r}$ are perpendicular to each other. Then, $d\vec{r}$ is in the direction of tangent to the given surface.

- Normal vector to the level surface : $\vec{\nabla}\phi$
- Unit normal vector to the level surface : $\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$

Exercise 1.5 Find the unit normal to the surface:$x^2 + y^2 = z$ at a point (1,2,5) ■

Solution:

$$\begin{aligned}
\text{Let } \phi &= x^2 + y^2 - z \\
\nabla\phi &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2 + y^2 - z\right) = 2x\hat{i} + 2y\hat{j} - \hat{k} \\
(\nabla\phi)_{1,2,5} &= 2\hat{i} + 4\hat{j} - \hat{k} \\
\text{Unit normal vector } &= \frac{\Delta\phi}{|\Delta\phi|} \\
&= \frac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{4 + 16 + 1}} \\
&= \frac{2}{\sqrt{21}}\hat{i} + \frac{4}{\sqrt{21}}\hat{j} - \frac{\hat{k}}{\sqrt{21}}
\end{aligned}$$

Directional derivative

Directional derivative of ϕ in the direction of $\vec{A}$ is defined as rate of change of ϕ with distance along the direction of $\vec{A}$. It is mathematically defined as the component of $\vec{\nabla}\phi$ in the direction of vector $\vec{A}$ i.e.

$$\vec{\nabla}\phi \cdot \hat{A} = \vec{\nabla}\phi . \frac{\vec{A}}{|\vec{A}|}$$

Exercise 1.6 Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at (1,-2,1) in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$. ■

Solution:

$$\begin{aligned}
\phi(x,y,z) &= x^2yz + 4xz^2 \\
\nabla\phi &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2yz + 4xz^2\right) \\
&= \left(2xyz + 4z^2\right)\hat{i} + \left(x^2z\right)\hat{j} + \left(x^2y + 8xz\right)\hat{k} \\
\nabla\phi \text{ at } (1,-2,1) &= \left\{2(1)(-2)(1) + 4(1)^2\right\}\hat{i} + (1\times 1)\hat{j} + \{1(-2) + 8(1)(1)\}\hat{k} \\
&= (-4+4)\hat{i} + \hat{j} + (-2+8)\hat{k} = \hat{j} + 6\hat{k} \\
\hat{a} = \text{ unit vector } &= \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})
\end{aligned}$$

So, the directional derivative at (1,-2,1)

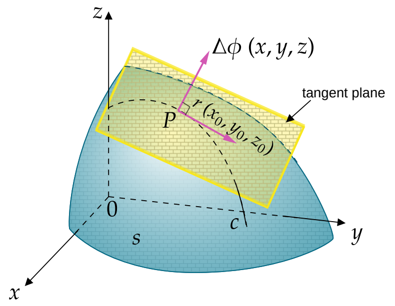
$$\begin{aligned}
&= \nabla\phi \cdot \hat{a} \\
&= (\hat{j} + 6\hat{k}) \cdot \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})
\end{aligned}$$

$$= \frac{1}{3}(-1-12) = \frac{-13}{3}$$

Tangent planes

Consider $\phi(x,y,z) = c$ be the equation of a level surface, and $\vec{r} = x_0 i + y_0\hat{j} + z_0\hat{k}$ be the position vector of any point P(x,y,z) on this surface.

Since, $\vec{\nabla}\phi$ is a vector normal to the surface, it is perpendicular to the tangent plane at P.



Let, $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point on the tangent plane at P to the surface.

Then, $\vec{R} - \vec{r} = (x-x_0)\hat{i} + (y-y_0)\hat{i} + (z-z_0)\hat{k}$ lies in the tangent plane at P and it will be perpendicular to $\vec{\nabla}\phi$
Then the tangent plane at the point P :

$$(\vec{R} - \vec{r}) \cdot \vec{\nabla}\phi = 0$$

$$(x-x_0)\frac{\partial\phi}{\partial x} + (y-y_0)\frac{\partial\phi}{\partial y} + (z-z_0)\frac{\partial\phi}{\partial z} = 0$$

1.5.2 Divergence ($\nabla\cdot$)

The divergence of a vector field measures how much the flow is expanding at a given point. It does not indicate in which direction the expansion is occuring. Hence the divergence is a scalar. Divergence of a continuous differentiable vector point function A specified in a vector field is given by,

$$\nabla \cdot \vec{f} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 f_1) + \frac{\partial}{\partial q_2}(h_3 h_1 f_2) + \frac{\partial}{\partial q_3}(h_1 h_2 f_3)\right]$$

In Cartesian coordinate system,

$$\nabla . f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

You can't have the divergence of a scalar: that's meaningless.

Physical interpretation

$\vec{\nabla} \cdot \vec{A}$ is a measure of how much the vector $\vec{A}$ spreads out (diverges) from a point in space.

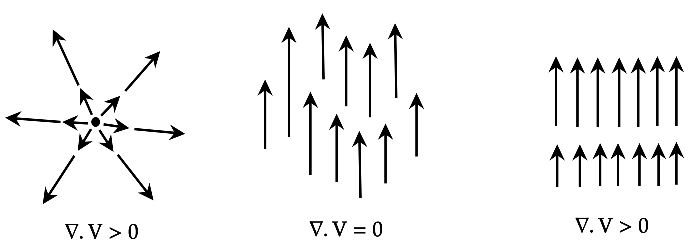


Figure 1.11: Physical intepretation of divergence.

Note
- If $\vec{\nabla} \cdot \vec{A} = 0$, then $\vec{A}$ is known as solenoidal vector field.
- If $\vec{\nabla} \cdot \vec{A} =$ negative, then $\vec{A}$ is known as sink field i.e. vector lines are going inward.
- If $\vec{\nabla} \cdot \vec{A} =$ positive, then $\vec{A}$ is known as source field i.e. vector lines are the going outward.

Product rules

- $\vec{\nabla} \cdot (f\overrightarrow{A}) = f(\vec{\nabla} \cdot \overrightarrow{A}) + \overrightarrow{A} \cdot (\vec{\nabla} f)$
- $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

Exercise 1.7 Calculate $\nabla \cdot \vec{r}$ ■

Solution:

$$
\begin{aligned}
\vec{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\
\nabla \cdot \vec{r} &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) \\
&= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \\
&= 1 + 1 + 1 \\
&= 3
\end{aligned}
$$

1.5.3 Curl

The curl is the vector valued derivative of a vector function. Its operation can be geometrically interpreted as the rotation of a field about a point in space.
From the definition of $\vec{\nabla}$ we construct the curl of a vector $\vec{f} = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$ as

$$
\nabla \times \vec{f} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \partial/\partial q_1 & \partial/\partial q_2 & \partial/\partial q_3 \\ h_1f_1 & h_2f_2 & h_3f_3 \end{vmatrix}
$$

In cartesian coordinate system,

$$
\nabla \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}
$$

Physical interpretation

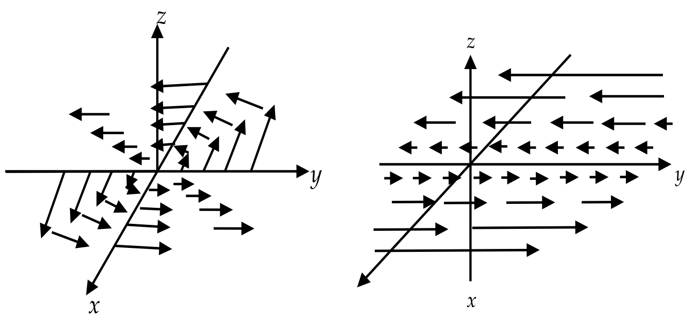


Figure 1.12: Physical interpretation of Curl

The curl of a vector field measures the tendency for the vector field to swirl around. Imagine that the vector field represents the velocity vectors of water in a lake. If the vector field swirls around, then when we stick a paddle wheel into the water, it will tend to spin. The amount of the spin will depend on how we orient the paddle. Thus, we should expect the curl to be vector valued.

- If $\vec{\nabla} \times \vec{V} = 0$, then $\vec{V}$ is known as an irrotational vector and we can write $\vec{V} = \vec{\nabla}\phi$
- If $\vec{\nabla} \times \vec{V} \neq 0$, then $\vec{V}$ is known as rotational vector.

Product rules

- $\vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla} \times \vec{A}) - \vec{A} \times (\vec{\nabla} f)$
- $\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$

Exercise 1.8 Prove that $(y^2 - z^2 + 3yz - 2x)\,\hat{i} + (3xz + 2xy)\,\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is irrotational. ■

Solution: For irrotational, we have to prove Curl $\bar{F} = 0$.

$$\begin{aligned}
\operatorname{Curl}\vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix} \\
&= (3x - 3x)\hat{i} - (-2z + 3y - 3y + 2z)\hat{j} + (3z + 2y - 2y - 3z)\hat{k} \\
&= 0\hat{i} + 0\hat{j} + 0\hat{k} = 0
\end{aligned}$$

Thus, $\vec{F}$ is irrotational.

Note

1. The divergence of a curl of a vector field vanishes.
   $\nabla \cdot (\nabla \times u) = 0$
   If $\nabla \cdot v = 0 \Longrightarrow v = \nabla \times u$
2. The curl of gradient of a scalar field vanishes.
   $\nabla \times (\nabla \phi) = 0$
   If $\nabla \times \psi = 0 \Longrightarrow \psi = \nabla u$

1.5.4 Laplacian

The divergence of the gradient of a scalar function is called the Laplacian. In general culvilinear coordinate system laplacian can be written as,

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial_{u_3}}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial_{u_3}}\right)\right]$$

In cartesian coordinate sytem,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

1.5.5 Important identities

1. $\nabla \cdot \nabla \vec{A} = \nabla^2 A = \frac{\partial^2 \vec{A}}{\partial x^2} + \frac{\partial^2 \vec{A}}{\partial y^2} + \frac{\partial^2 \vec{A}}{\partial z^2}$ (The Laplace operator.)
2. $\nabla \times \nabla \vec{A} = 0$
3. $\nabla \cdot \nabla \times \vec{A} = 0$
4. $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) \times \nabla^2 \vec{A}$
5. $\nabla(\nabla \cdot \vec{A}) = \nabla \times (\nabla \times \vec{A}) + \nabla^2 \vec{A}$

	Cylindrical polar(ρ,ϕ,z)	**Spherical polar**(r,θ,ϕ)
Scale factor	$h_1=1$ $h_2=r$ $h_3=1$	$h_1=1$ $h_2=r$ $h_3=r\sin\theta$
Gradient	$\frac{\partial f}{\partial\rho}\hat{\rho}+\frac{1}{\rho}\frac{\partial f}{\partial\phi}\hat{\phi}+\frac{\partial f}{\partial z}\hat{z}$	$\hat{r}\frac{\partial f}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial f}{\partial\theta}+\hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}$
Divergence	$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho)+\frac{1}{\rho}\frac{\partial}{\partial\phi}(F_\phi)+\frac{\partial}{\partial z}(F_z)$	$\frac{1}{\mathrm{r}^2}\frac{\partial}{\partial\mathrm{r}}(r^2F_\mathrm{r})+\frac{1}{\mathrm{r}\sin\theta}\frac{\partial}{\partial\theta}(F_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial \mathrm{F}_\phi}{\partial\phi}$
Curl	$\frac{1}{\rho}\begin{vmatrix}\hat{\rho} & \hat{\phi} & \hat{z}\\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ F_\rho & F_\phi & F_z\end{vmatrix}$	$\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi\\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi\\ F_r & rF_\theta & r\sin\theta F_\phi\end{vmatrix}$
Laplacian	$\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{\partial^2 f}{\partial z^2}$	$\frac{1}{\mathrm{r}^2}\frac{\partial}{\partial\mathrm{r}}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{\mathrm{r}^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$

Table 1.1: Differential operators in general culvilinear coordinate system

6. $\nabla(\vec{A}+\vec{B})=\nabla\cdot\vec{A}+\nabla\cdot\vec{B}$
7. $\nabla\times(\vec{A}+\vec{B})=\nabla\times\vec{A}+\nabla\times\vec{B}$
8. $\nabla\cdot(\vec{A}\times\vec{B})=\vec{B}\cdot(\nabla\times\vec{A})-\vec{A}\cdot(\nabla\times\vec{B})$
9. $\nabla\times(\vec{A}\times\vec{B})=(B\cdot\nabla)A-B(\nabla\cdot A)-(A\nabla)\quad B+A(\nabla B)$

1.6 Integral Calculus

1.6.1 Line integration of vectors

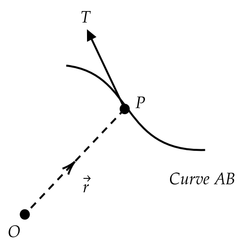


Figure 1.13: Line integration

The integration of a vector function $\vec{F}$ along a curve is known as line integration of vectors.Infact the line integral along the curve is the integral of $\vec{F}$ along the tangent to the curve. Consider a pont P in the curve in figure 1.13 such that the position vector of P is given by $\vec{r}$.
The component of $\vec{F}$ along the tangent at P = $\left(\vec{F}\cdot\frac{d\vec{r}}{ds}\right)$

Then the Line integral of $\vec{F}$ from A to B along the curve C will be,

$$\text{Line integral} = \int_c \left(\vec{F}\cdot\frac{d\vec{r}}{ds}\right) ds = \int_c \vec{F}\cdot d\vec{r}$$

Note

- If $\vec{F}$ represents the variable force acting on a particle along arc AB, then the total work done $W_{AB} = \int_A^B \vec{F}\cdot d\vec{r}$
- If $\vec{V}$ represents the velocity of a liquid then $\oint_c \vec{V}\cdot d\vec{r}$ is called the circulation of $\vec{V}$ round closed curve C
- When the path of integration is a closed curve then notation of integration is $\oint$ in place of $\int$.

Example 1.4 Workdone

Work done by a conservative field $\vec{A}$ in moving a particle from point P to Q will be

$$\int_P^Q \vec{F}\cdot\overrightarrow{dr} = \int_P^Q \vec{\nabla}\phi\cdot\overrightarrow{dr} = \int_P^Q d\phi = \phi_Q - \phi_P = \text{ independent of path.}$$

Ordinarily, the value of a line integral depends critically on the particular path taken from a to b, but there is an important special class of vector functions for which the line integral is independent of the path, and is determined entirely by the end points (A force that has this property is called conservative).
$\vec{A}$ in moving a particle around a closed path C is $\oint_c \vec{F}\cdot\overrightarrow{dr} = 0$ ■

Exercise 1.9 If a force $\vec{F} = 2x^2y\hat{i} + 3xy\hat{j}$ displaces a particle in the xy-plane from (0,0) to (1,4) along a curve $y = 4x^2$. Find the work done. ■

Solution:

$$\begin{aligned}
\text{Work done} &= \int_c \vec{F}\cdot\overrightarrow{dr} \\
&= \int_c (2x^2y\hat{i} + 3xy\hat{j})\cdot(dx\hat{i} + dy\hat{j}) \\
&= \int_c (2x^2ydx + 3xydy)
\end{aligned}$$

$$\begin{bmatrix} \vec{r} = x\hat{i} + y\hat{j} \\ \overrightarrow{dr} = dx\hat{i} + dy\hat{j} \end{bmatrix}$$

Putting the values of y and dy, we get

$$= \int_0^1 \cdot \left[2x^2\left(4x^2\right)dx + 3x\left(4x^2\right)8xdx\right] \quad \begin{bmatrix} y = 4x^2 \\ dy = 8xdx \end{bmatrix}$$

$$= 104\int_0^1 x^4 dx = 104\left(\frac{x^5}{5}\right)_0^1 = \frac{104}{5}$$

1.6.2 Surface integration of vectors

It's the two dimensional analog of line integral. Physically, it can be thought of as flow of a fluid through a surface. It is the integration of a vector on an open or closed surface.
For a function $F(x,y,z)$ the surface integral over a surface S is given as,

$$S = \iint_S (\mathbf{F}\cdot\hat{n})dS = \iint_S \mathbf{F}\cdot d\mathbf{S}$$

where n is the unit normal vector to an element ds and

$$\hat{n} = \frac{\text{grad } f}{|\text{grad } f|} \qquad ds = \frac{dxdy}{(\hat{n}\cdot\hat{k})}$$

Note If $\iint_S (\vec{F}\cdot\hat{n})ds = 0$, then $\vec{F}$ is said to be a solenoidal vector point function.

Example 1.5 Flux

Flux $= \iint_S (\vec{F}\cdot\hat{n})ds$ where, $\bar{F}$ represents the velocity of a liquid.

Exercise 1.10 Evaluate $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot\overrightarrow{ds}$ where S is the surface of the sphere $x^2+y^2+z^2 = a^2$ in the first octant.

Solution: Here, $\phi = x^2+y^2+z^2-a^2$

Vector normal to the surface

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-a^2) = 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a} \qquad [\because x^2+y^2+z^2 = a^2]$$

$$\vec{F} = yz\hat{i}+zx\hat{j}+xy\hat{k}$$

$$\vec{F}\cdot\hat{n} = (yz\hat{i}+zx\hat{j}+xy\hat{k})\cdot\left(\frac{x\hat{i}+\hat{y}+z\hat{k}}{a}\right) = \frac{3xyz}{a}$$

$$\iint_S F\cdot\hat{n}ds = \iint_S (\vec{F}\cdot\hat{n})\frac{dxdy}{|\hat{k}\cdot\hat{n}|}$$

$$= \int_0^a\int_0^{\sqrt{a^2-x^2}} \frac{3xyzdxdy}{a\left(\frac{z}{a}\right)}$$

$$= 3\int_0^a\int_0^{\sqrt{a^2-x^2}} xydydx$$

$$= 3\int_0^a x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{3}{2}\int_0^a x\left(a^2-x^2\right)dx$$

$$= \frac{3}{2}\left(\frac{a^2x^2}{2} - \frac{x^4}{4}\right)_0^a$$

$$= \frac{3}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right)$$

$$= \frac{3a^4}{8}.$$

1.6.3 Volume Integration of Vectors

Volume integral refers to the integral over a 3 dimensional domain. Volume integral of a vector field $\vec{F}$ within the volume V can be written as,

$$\text{Volume integral}=\iiint_V \vec{F}\cdot dV$$

Where, dV is the infinitesimal volume element

$dV = dxdydz$ -In Cartesian cooordinate system

$dV = r^2 sin\theta dr d\theta d\phi$ -In Spherical polar cordinate

$dV = dV = rd\theta drdz$ -In Cylindrical polar coordinate system

Exercise 1.11 If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, evaluate $\iiint_V \vec{F}dv$ where, v is the region bounded by the surfaces $x = 0, y = 0, x = 2, y = 4, \quad z = x^2, \quad z = 2$ ■

Solution:

$$\begin{aligned}
\iiint_V \vec{F}dv &= \iiint (2z\hat{i} - x\hat{j} + y\hat{k})dxdydz \\
&= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})dz \\
&= \int_0^2 dx \int_0^4 dy \left[z^2\hat{i} - xz\hat{j} + yz\hat{k}\right]_{x^2}^2 \\
&= \int_0^2 dx \int_0^4 dy \left[4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2y\hat{k}\right] \\
&= \int_0^2 dx \left[4y\hat{i} - 2xy\hat{j} + y^2\hat{k} - x^4y\hat{i} + x^3y\hat{j} - \frac{x^2y^2}{2}\hat{k}\right]_0^4 \\
&= \int_0^2 \left(16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k}\right) dx \\
&= \left[16x\hat{i} - 4x^2\hat{j} + 16x\hat{k} - \frac{4x^5}{5}\hat{i} + x^4\hat{j} - \frac{8x^3}{3}\hat{k}\right]_0^2 \\
&= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k} = \frac{32\hat{i}}{5} + \frac{32\hat{k}}{3} \\
&= \frac{32}{15}(3\hat{i} + 5\hat{k})
\end{aligned}$$

1.7 Theorems

1.7.1 Divergence Theorem

Definition 1.7.1 The surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume V enclosed by the surface S. Mathematically

$$\iint_S \vec{F}\cdot\hat{n}ds = \iiint_V div\vec{F}\cdot dV = \iiint_V (\vec{\nabla}\cdot\vec{F})dV$$

Where $\hat{n}$ is the outward normal to ' S ' indicating the positive direction of S.

This theorem is applicable only for closed surfaces and it converts surface integral into volume integral and vice versa.
The divergence theorem is a mathematical statement of the physical fact that, in the absence of the creation or destruction of matter, the density within a region of space can change only by having it flow into or away from the region through its boundary.

Exercise 1.12 Evaluate $\iint_S \vec{F}\cdot\hat{n}ds$ where S is the surface of the sphere $x^2+y^2+z^2=16$ and $\vec{F}=3x\hat{i}+4y\hat{j}+5z\hat{k}$
By Gauss's divergence theorem, ■

Solution:

$$\iint_S \vec{F}\cdot\hat{n}ds = \iint_v\int \nabla\cdot\vec{F}dv$$
$$Here, \vec{F}=3x\hat{i}+4y\hat{j}+5z\hat{k}$$
$$\nabla\cdot\vec{F}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(3x\hat{i}+4y\hat{j}+5z\hat{k})$$
$$\nabla\cdot\vec{F}=3+4+5=14$$

Putting the value of ∇.F, we get

$$\iint_S \vec{F}\cdot\hat{n}ds = \iint_v\int 14\cdot dv$$

Where v is volume of a sphere

$$=14v$$
$$=14\tfrac{4}{3}\pi(4)^3=\frac{3584\pi}{3}$$

1.7.2 Stoke's Theorem

Definition 1.7.2 Surface integral of the component of curl $\vec{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\vec{F}$ taken along the closed curve C.

Mathematically $\oint_C \vec{F}\cdot\overrightarrow{dr}=\iint_S(\vec{\nabla}\times\vec{F})\hat{n}ds=\iint_S(\vec{\nabla}\times\vec{F})\cdot\overrightarrow{ds}$

where $\hat{n}=\cos\alpha\hat{i}+\cos\beta\hat{j}+\cos\gamma\hat{k}$ is a unit external normal to any surface dS

If we apply Stoke's theorem to a closed surface. Since it has no perimeter, The line integral vanishes. So,

$$\iint_S(\vec{\nabla}\times\vec{F})\cdot\overrightarrow{ds}=0\rightarrow \text{For } S\text{, a closed surface}$$

Exercise 1.13 Evaluate by Stokes theorem $\oint_C(yzdx+zxdy+xydz)$ where C is the curve $x^2+y^2=1, z=y^2$ ■

Solution: Here we have

$$\oint yzdx+zxdy+xydz=\int(yz\hat{i}+zx\hat{j}+xy\hat{k})\cdot(\hat{i}dx+\hat{j}dy+kdz)$$
$$=\oint F.dx$$
$$=\int \text{curl}F\cdot nds=0$$
$$\because \text{ Curl }\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz & zx & xy\end{vmatrix}$$
$$=(x-x)\hat{i}+(y-y)\hat{j}+(z-z)\hat{k}=0$$

1.7.3 Green's theorem (In a plane)

Definition 1.7.3 If $\phi(x,y), \psi(x,y), \frac{\partial\phi}{\partial y}$ and $\frac{\partial\psi}{\partial x}$ be continuous functions over a region R bounded by simple closed curve C in $x-y$ plane, then $\oint_C(\phi dx+\psi dy) = \iint_R\left(\frac{\partial\psi}{\partial x}-\frac{\partial\phi}{\partial y}\right)dxdy$.

Green's theorem is mainly used for the integration of line combined with a curved plane .We can write Green's theorem as

$$\int_c \vec{F}\cdot d\vec{r} = \iint_R (\nabla\times\vec{F})\cdot\hat{k}dR$$

Where, $\vec{F}=\phi\hat{i}+\psi\hat{j}, \bar{r}=x\hat{i}+y\hat{j}, \hat{k}$ is a unit vector along z -axis and $dR = dxdy$

Exercise 1.14 A vector field $\vec{F}$ is given by $\vec{F}=\sin y\hat{i}+x(1+\cos y)\hat{j}$ Evaluate the line integral $\int_C \vec{F}\cdot\overrightarrow{dr}$ where C is the circular path given by $x^2+y^2=a^2$. ■

Solution:

$$\vec{F} = \sin y\hat{i}+x(1+\cos y)\hat{j}$$

$$\int_C \vec{F}\cdot\overrightarrow{dr} = \int_C [\sin y\hat{i}+x(1+\cos y)\hat{j}]\cdot(\hat{i}dx+\hat{j}dy)$$

$$= \int_C \sin ydx+x(1+\cos y)dy$$

On applying Green's Theorem, we have

$$\oint_c(\phi dx+\psi dy) = \iint_S\left(\frac{\partial\psi}{\partial x}-\frac{\partial\phi}{\partial y}\right)dxdy$$

$$= \iint_S [(1+\cos y)-\cos y]dxdy$$

where S is the circular plane surface of radius a.

$$= \iint_S dxdy = \text{ Area of circle} = \pi a^2.$$

Practice Set -1

1. Let $\vec{a}$ and $\vec{b}$ be two distinct three dimensional vectors. Then the component of $\vec{b}$ that is perpendicular to $\vec{a}$ is given by

[NET/JRF(JUNE-2011)]

A. $\frac{\vec{a}\times(\vec{b}\times\vec{a})}{a^2}$ **B.** $\frac{\vec{b}\times(\vec{a}\times\vec{b})}{b^2}$ **C.** $\frac{(\vec{a}\cdot\vec{b})b}{b^2}$ **D.** $\frac{(\vec{b}\cdot\vec{a})\vec{a}}{a^2}$

2. The equation of the plane that is tangent to the surface $xyz=8$ at the point $(1,2,4)$ is

[NET/JRF(DEC-2011)]

A. $x+2y+4z=12$ **B.** $4x+2y+z=12$
C. $x+4y+2=0$ **D.** $x+y+z=7$

A vector perpendicular to any vector that lies on the plane defined by $x+y+z=5$, is

[NET/JRF(JUNE-2012)]

A. $\hat{i}+\hat{j}$ **B.** $\hat{j}+\hat{k}$ **C.** $\hat{i}+\hat{j}+\hat{k}$ **D.** $2\hat{i}+3\hat{j}+5\hat{k}$

3. A unit vector $\hat{n}$ on the xy-plane is at an angle of 120° with respect to $\hat{i}$. The angle between the vectors $\vec{u}=a\hat{i}+b\hat{n}$ and $\vec{v}=a\hat{n}+b\hat{i}$ will be 60° if

[NET/JRF(JUNE-2013)]

A. $b=\sqrt{3}a/2$ **B.** $b=2a/\sqrt{3}$ **C.** $b=a/2$ **D.** $b=a$

4. The unit normal vector of the point $\left[\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right]$ on the surface of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$is

[NET/JRF(DEC-2012)]

A. $\frac{bc\hat{i}+ca\hat{j}+ab\hat{k}}{\sqrt{a^2+b^2+c^2}}$ **B.** $\frac{a\hat{i}+b\hat{j}+c\hat{k}}{\sqrt{a^2+b^2+c^2}}$ **C.** $\frac{b\hat{i}+c\hat{j}+a\hat{k}}{\sqrt{a^2+b^2+c^2}}$ **D.** $\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

5. Let $\vec{r}$ denote the position vector of any point in three-dimensional space, and $r=|\vec{r}|$. Then

[NET/JRF(DEC-2014)]

A. $\vec{\nabla}\cdot\vec{r}=0$ and $\vec{\nabla}\times\vec{r}=\vec{r}/r$ **B.** $\vec{\nabla}\cdot\vec{r}=0$ and $\nabla^2 r=0$
C. $\vec{\nabla}\cdot\vec{r}=3$ and $\nabla^2\vec{r}=\vec{r}/r^2$ **D.** $\vec{\nabla}\cdot\vec{r}=3$ and $\vec{\nabla}\times\vec{r}=0$

6. Consider the three vectors $\vec{v}_1=2\hat{i}+3\hat{k}, \vec{v}_2=\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{v}_3=5\hat{i}+\hat{j}+a\hat{k}$ where $\hat{i},\hat{j}$ and $\hat{k}$ are the standard unit vectors in a three-dimensional Euclidean space. These vectors will be linearly dependent if the value of a is

[NET/JRF(JUNE-2018)]

A. $\frac{31}{4}$ **B.** $\frac{23}{4}$ **C.** $\frac{27}{4}$ **D.** 0

Note * For the 4^{th} question answer will be $\frac{bc\hat{i}+ca\hat{j}+ab\hat{k}}{\sqrt{b^2c^2+c^2a^2+a^2b^2}}$

Answer key

Q.No.	Answer	Q.No.	Answer
1	**a**	2	**b**
3	**c**	4	**Incorrect option**
5	**d**	6	**a**

Practice Set -2

1. If a force $\vec{F}$ is derivable from a potential function $V(r)$, where r is the distance from the origin of the coordinate system, it follows that

[GATE 2011]

A. $\vec{\nabla} \times \vec{F} = 0$ **B.** $\vec{\nabla} \cdot \vec{F} = 0$ **C.** $\vec{\nabla} V = 0$ **D.** $\nabla^2 V = 0$

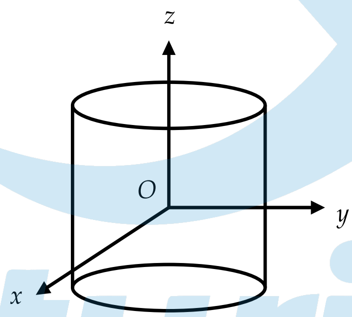
2. The unit vector normal to the surface $x^2 + y^2 - z = 1$ at the point $P(1,1,1)$ is

[GATE 2011]

A. $\frac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}$ **B.** $\frac{2\hat{i}+\hat{j}-\hat{k}}{\sqrt{6}}$ **C.** $\frac{\hat{i}+2\hat{j}-\hat{k}}{\sqrt{6}}$ **D.** $\frac{2\hat{i}+2\hat{j}-\hat{k}}{3}$

3. Consider a cylinder of height h and radius a, closed at both ends, centered at the origin. Let $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ be the position vector and $\hat{n}$ be a unit vector normal to the surface. The surface integral $\int_S \vec{r} \cdot \hat{n} ds$ over the closed surface of the cylinder is

[GATE 2011]



A. $2\pi a^2(a+h)$ **B.** $3\pi a^2 h$ **C.** $2\pi a^2 h$ **D.** Zero

4. Identify the correct statement for the following vectors $\vec{a} = 3\hat{i} + 2\hat{j}$ and $\vec{b} = \hat{i} + 2\hat{j}$

[GATE 2012]

A. The vectors $\vec{a}$ and $\vec{b}$ are linearly independent

B. The vectors $\vec{a}$ and $\vec{b}$ are linearly dependent

C. The vectors $\vec{a}$ and $\vec{b}$ are orthogonal

D. The vectors $\vec{a}$ and $\vec{b}$ are normalized

5. If $\vec{A}$ and $\vec{B}$ are constant vectors, then $\vec{\nabla}(\vec{A} \cdot (\vec{B} \times \vec{r}))$ is

[GATE 2013]

A. $\vec{A} \cdot \vec{B}$ **B.** $\vec{A} \times \vec{B}$ **C.** $\vec{r}$ **D.** Zero

6. The unit vector perpendicular to the surface $x^2 + y^2 + z^2 = 3$ at the point $(1,1,1)$ is

[GATE 2014]

A. $\frac{\hat{x}+\hat{y}-\hat{z}}{\sqrt{3}}$ **B.** $\frac{\hat{x}-\hat{y}-\hat{z}}{\sqrt{3}}$ **C.** $\frac{\hat{x}-\hat{y}+\hat{z}}{\sqrt{3}}$ **D.** $\frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}$

7. The direction of $\vec{\nabla} f$ for a scalar field $f(x,y,z) = \frac{1}{2}x^2 - xy + \frac{1}{2}z^2$ at the point $P(1,1,2)$ is

[GATE 2016]

A. $\frac{(-\hat{j}-2\hat{k})}{\sqrt{5}}$ **B.** $\frac{(-\hat{j}+2\hat{k})}{\sqrt{5}}$ **C.** $\frac{(\hat{j}-2\hat{k})}{\sqrt{5}}$ **D.** $\frac{(\hat{j}+2\hat{k})}{\sqrt{5}}$

8. In spherical polar coordinates (r,θ,ϕ), the unit vector $\hat{\theta}$ at $(10,\pi/4,\pi/2)$ is

[GATE 2018]

A. $\hat{k}$

B. $\frac{1}{\sqrt{2}}(\hat{j}+\hat{k})$

C. $\frac{1}{\sqrt{2}}(-\hat{j}+\hat{k})$

D. $\frac{1}{\sqrt{2}}(\hat{j}-\hat{k})$

9. Given $\vec{V}_1=\hat{i}-\hat{j}$ and $\vec{V}_2=-2\hat{i}+3\hat{j}+2\hat{k}$, which one of the following $\vec{V}_3$ makes $\left(\vec{V}_1,\vec{V}_2,\vec{V}_3\right)$ a complete set for a three dimensional real linear vector space?

[GATE 2018]

A. $\vec{V}_3=\hat{i}+\hat{j}+4\hat{k}$

B. $\vec{V}_3=2\hat{i}-\hat{j}+2\hat{k}$

C. $\vec{V}_3=\hat{i}+2\hat{j}+6\hat{k}$

D. $\vec{V}_3=2\hat{i}+\hat{j}+4\hat{k}$

Answer key

Q.No.	Answer	Q.No.	Answer
1	**a**	2	**d**
3	**b**	4	**a**
5	**b**	6	**d**
7	**b**	8	**d**
9	**d**	10	

Practice Set-3

1. Three unit vectors $\vec{a}, \vec{b}, \vec{c}$ $\left(\vec{b}\right.$ and $\vec{c}$ are not parallel) are such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\sqrt{3}}{2}\vec{c}$. The angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$, respectively are

a. $30^\circ, 90^\circ$ **b.** $150^\circ, 90^\circ$

c. $60^\circ, 90^\circ$ **d.** $90^\circ, 30^\circ$

Solution:

$$(\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\sqrt{3}}{2}\vec{c} \Rightarrow \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}) = \frac{\sqrt{3}}{2}\vec{c}$$

Comparing coefficients of $\vec{b}$ and $\vec{c}$ on both sides.

$$\vec{a} \cdot \vec{c} = 0 \Rightarrow \vec{a} \perp \vec{c}$$
$$\vec{a} \cdot \vec{b} = \frac{\sqrt{3}}{2}$$
$$\Rightarrow ab\cos\theta = -\frac{\sqrt{3}}{2}$$
$$\Rightarrow \cos\theta = -\frac{\sqrt{3}}{2}$$
$$\Rightarrow \theta = 150^\circ$$

The angle which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$ are $150^\circ, 90^\circ$ respectively.Correct option is (b)

2. Find the angle between the two surfaces $5x + y + z = 1$ and $3x + 3y + 3z = 5$.

a. $\cos^{-1}\left(\frac{7}{3}\right)$ **b.** $\cos^{-1}\left(\frac{7}{9}\right)$

c. $\cos^{-1}\left(\frac{7}{27}\right)$ **d.** $\cos^{-1}\left(\frac{21}{9}\right)$

Solution:

$$\text{Given} \quad \phi_1 : 5x + y + z = 1 \quad \text{and} \quad \phi_2 : 3x + 3y + 3z = 5$$

therefore,

$$\cos\theta = \frac{\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2}{\left|\vec{\nabla}\phi_1\right|\left|\vec{\nabla}\phi_2\right|}$$
$$= \frac{(5\hat{i} + \hat{j} + \hat{k}) \cdot (3\hat{i} + 3\hat{j} + 3\hat{k})}{\sqrt{27} \cdot \sqrt{27}}$$
$$= \frac{21}{27}$$
$$\Rightarrow \theta = \cos^{-1}\left(\frac{7}{9}\right)$$

3. The equation of the plane that is tangent to the surface $xyz = 8$ at the point (1,2,4) is

a. $x+2y+4z=12$ **b.** $4x+2y+z=12$

c. $x+4y+2=0$ **d.** $x+y+z=7$

Solution:

To get a normal at the surface let's take the gradient

$$\begin{aligned}\vec{\nabla}(xyz) &= yz\hat{i}+zx\hat{j}+\hat{k}xy \\ &= 8\hat{i}+4\hat{j}+2\hat{k}\end{aligned}$$

We want a plane perpendicular to this so:

$$(\vec{r}-\vec{r}_0)\cdot\frac{(8\hat{i}+4\hat{j}+2\hat{k})}{\sqrt{64+16+4}}=0$$

$$|(x-1)\hat{i}+(y-2)\hat{j}+(z-4)\hat{k}|\cdot[8\hat{i}+4\hat{j}+2\hat{k}]=0$$

$$\Rightarrow 4x+2y+z=12$$

4. If $\vec{A}=\hat{i}yz+\hat{j}xz+\hat{k}xy$, then the integral $\oint_C \vec{A}\cdot d\vec{l}$ (where C is along the perimeter of a rectangular area bounded by $x=0, x=a$ and $y=0, y=b$) is

a. $\frac{1}{2}\left(a^3+b^3\right)$ **b.** $\pi\left(ab^2+a^2b\right)$ **c.** $\pi\left(a^3+b^3\right)$ **d.** 0

Solution:

$$\oint_C \vec{A}\cdot d\vec{l}=\int_S(\vec{\nabla}\times\vec{A})d\vec{a}=0$$

$$\because \vec{\nabla}\times\vec{A}=0$$

5. Value of the integral $\oint\left(xydy-y^2dx\right)$, where c is the square cut from the quadrant by the lines $x=1$ and $y=1$ will be (use Green's theorem to change the line integral into double integral)

a. $\frac{1}{2}$ **b.** 1 **c.** $\frac{3}{2}$ **d.** $\frac{5}{3}$

Solution:

We know that Green's theorem is given by

$$\oint_c \phi dx+\psi dy=\iint_R\left(\frac{\partial\psi}{\partial x}-\frac{\partial\phi}{\partial y}\right)dxdy$$

$$\begin{aligned}\text{Here,}I &= \oint\left(xydy-y^2dx\right) \\ &= \oint\left(-y^2\right)dx+(xy)dy\end{aligned}$$

Hence, we can deduce

$$\begin{aligned}\phi &= -y^2 \\ \psi &= xy \\ \frac{\partial\psi}{\partial x} &= y \\ \frac{\partial\phi}{\partial y} &= -2y\end{aligned}$$

Substituting in Green's theorem, we get

$$I=\int_{y=0}^{1}\int_{x=0}^{1}[y-(-2y)]dxdy=\int_{y=0}^{1}\int_{x=0}^{1}3ydxdy$$
$$=\int_{y=0}^{1}[3xy]_{x=0}^{1}dy=\int_{y=0}^{1}3ydy$$
$$=\frac{3}{2}$$

Thus the correct option is (c).

6. Evaluate $\int_C \vec{F}\cdot\overrightarrow{dr}$, where $F=x^2\hat{i}+y^3\hat{j}$ and curve C is the arc of parabola $y=x^2$ in the $x-y$ plane from (0,0) to (1,1)

a. $\frac{7}{12}$ **b.** $\frac{7}{12}$

c. $\frac{7}{12}$ **d.** $\frac{7}{12}$

Solution:

Along the curve C,$y=x^2 \Rightarrow dy=2xdx$;

$$\vec{r}=x\hat{i}+x^2\hat{j}\Rightarrow d\vec{r}=dx\hat{i}+2xd\hat{x}j$$

Therefore, $\displaystyle\int_C \vec{F}\cdot d\vec{r}=\int_{x=0}^{1}\left[x^2dx+x^6(2x)dx\right]$

$$=\int_0^1\left(x^2+2x^7\right)dx=\left[\frac{x^3}{3}+\frac{2x^8}{8}\right]_0^1$$
$$=\frac{7}{12}$$

7. At any point of the curve $x=3\cos t, y=3\sin t, z=4t$, find The unit tangent vector

a. $\frac{1}{\sqrt{10}}(-3\sin t\hat{i}+3\cos t\hat{j}+4\hat{k})$ **b.** $\frac{1}{10}(-3\sin t\hat{i}+3\cos t\hat{j}+4\hat{k})$

c. $\frac{1}{5}(-3\sin t\hat{i}+3\cos t\hat{j}+4\hat{k})$ **d.** $\frac{1}{5(\sin t+\cos t)}(-3\sin t\hat{i}+3\cos t\hat{j}+4\hat{k})$

Solution:

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}\Rightarrow\vec{r}=(3\cos t)\hat{i}+(3\sin t)\hat{j}+(4t)\hat{k}$$
$$\frac{d\vec{r}}{dt}=(-3\sin t)\hat{i}+(3\cos t)\hat{j}+4\hat{k}\quad\text{which is the required tangent vector.}$$

Magnitude of tangent vector

$$=\sqrt{(-3\sin t)^2+(3\cos t)^2+(4)^2}=5$$

Unit tangent vector

$$=\frac{1}{5}(-3\sin t\hat{i}+3\cos t\hat{j}+4\hat{k})$$

8. The unit normal vector of the point $\left[\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right]$ on the surface of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is

a. $\frac{bc\hat{i}+ca\hat{j}+ab\hat{k}}{\sqrt{b^2c^2+c^2a^2+a^2b^2}}$ **b.** $\frac{a\hat{i}+b\hat{j}+c\hat{k}}{\sqrt{a^2+b^2+c^2}}$

c. $\frac{b\hat{i}+c\hat{j}+a\hat{k}}{\sqrt{a^2+b^2+c^2}}$ **d.** $\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

Solution:

$$\text{Here,}\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

$$\text{Unit normal vector is}\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$$

$$\text{So}\vec{\nabla}\phi = \left(i\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$= \frac{2x\hat{i}}{a^2} + \frac{2y\hat{j}}{b^2} + \frac{2z\hat{k}}{c^2}$$

$$\vec{\nabla}\phi\Big|_{\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)} = \frac{2}{a\sqrt{3}}\hat{i} + \frac{2}{b\sqrt{3}}\hat{j} + \frac{2}{c\sqrt{3}}\hat{k}$$

$$|\vec{\nabla}\phi| = \sqrt{\frac{4}{3a^2} + \frac{4}{3b^2} + \frac{4}{3c^2}}$$

$$= \frac{2}{\sqrt{3}}\sqrt{\frac{b^2c^2 + a^2c^2 + a^2c^2}{a^2b^2c^2}}$$

$$\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}\Bigg|_{\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)} = \frac{\frac{2}{a\sqrt{3}}\hat{i} + \frac{2}{b\sqrt{3}}\hat{j} + \frac{2}{c\sqrt{3}}\hat{k}}{\frac{2}{\sqrt{3}}\frac{\sqrt{b^2c^2+c^2a^2+a^2b^2}}{abc}}$$

$$= \frac{bc\hat{i} + ca\hat{j} + ab\hat{k}}{\sqrt{b^2c^2 + c^2a^2 + a^2b^2}}$$

The correct option is (a)

9. For the vector field $\vec{A} = xz^2\hat{i} - yz^2\hat{j} + z\left(x^2 - y^2\right)\hat{k}$, the volume integral of the divergence of $\vec{A}$ out of the region defined by $-a \le x \le a, -b \le y \le b$ and $0 \le z \le c$ is:

a. $\frac{4}{3}abc\left[a^2 - b^2\right]$

b. $\frac{2}{3}abc\left[a^2 - b^2\right]$

c. $\frac{1}{3}abc\left[a^2 - b^2\right]$

d. $abc\left[a^2 - b^2\right]$

Solution:

$$\text{Since,}\vec{A} = xz^2\hat{i} - yz^2\hat{j} + z\left(x^2 - y^2\right)\hat{k} \Rightarrow \vec{\nabla}\cdot\vec{A} = z^2 - z^2 + \left(x^2 - y^2\right) = x^2 - y^2$$

$$\text{Thus,} \int_V (\vec{\nabla}\cdot\vec{A})d\tau$$

$$= \int_{x=-a}^{x=+a}\int_{y=-b}^{y=+b}\int_{z=0}^{z=c}\left(x^2 - y^2\right)dxdydz$$

$$= \int_{y=-b}^{y=+b}\int_{z=0}^{z=c}\left[\frac{x^3}{3} - y^2x\right]_{-a}^{+a}dydz$$

$$= \int_{y=-b}^{y=+b}\int_{z=0}^{z=c}\left[\frac{2}{3}a^3 - 2ay^2\right]dydz$$

$$\Rightarrow \int_V (\vec{\nabla}\cdot\vec{A})d\tau = \int_{z=0}^{z=c}\left[\frac{2}{3}a^3y - 2a\frac{y^3}{3}\right]_{-b}^{+b}dz = \int_{z=0}^{z=c}\left[\frac{4}{3}a^3b - \frac{4}{3}ab^3\right]dz$$

$$= \frac{4}{3}abc\left[a^2 - b^2\right]$$

Correct option is (a)

10. The value of $\oint \vec{F}\cdot d\vec{r}$, where C is the curve bounded by $x^2+y^2 \ge 4; x^2+y^2 \le 16; x \ge 0$ and $\vec{F} = -y\hat{i} + x\hat{j} + z\hat{k}$ is ....

a. 12π **b.** 24π **c.** $\frac{14\pi}{3}$ **d.** $\frac{10\pi}{3}$

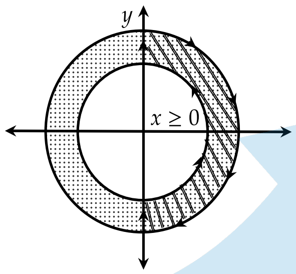
Solution:

Using Stoke's theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(1+1) = 2\hat{k}$$



$$\text{And } d\vec{S} = dxdy\hat{k}$$

$$\therefore \quad \iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = 2\iint dxdy$$

Put, $x = r\cos\theta, y = r\sin\theta$ and $dxdy = rdrd\theta$

$$= 2\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{r=2}^{4} rdrd\theta$$

$$= 2\left(\frac{r^2}{2}\right)_2^4(\pi) = \pi\times 12 = 12\pi$$

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