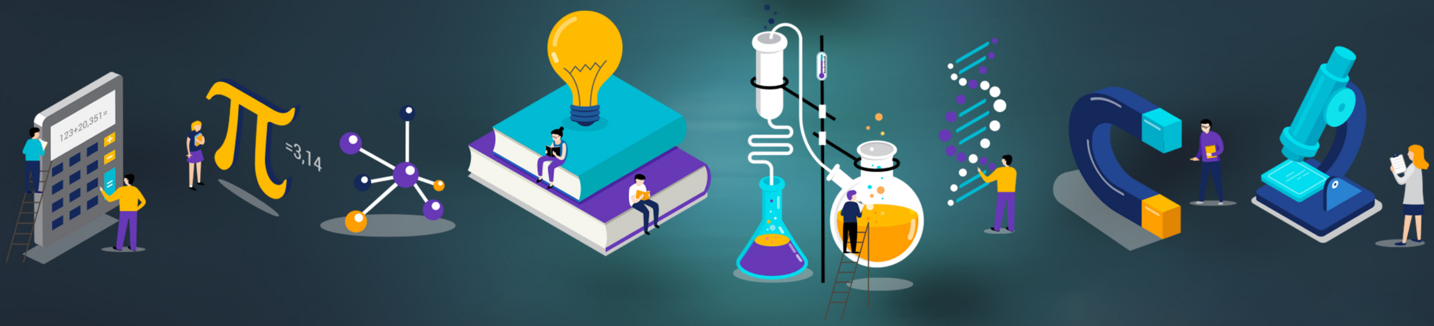


1. Small Oscillations-solutions

Practice set 1

1. A particle of unit mass moves in a potential $V(x)=ax^2+\frac{b}{x^2}$, where a and b are positive constants. The angular frequency of small oscillations about the minimum of the potential is

[NET JUNE 2011]

A. $\sqrt{8b}$ **B.** $\sqrt{8a}$

C. $\sqrt{8a/b}$ **D.** $\sqrt{8b/a}$

Solution:

$$V(x)=ax^2+\frac{b}{x^2}\Rightarrow\frac{\partial V}{\partial x}=0\Rightarrow 2ax-\frac{2b}{x^3}=0\Rightarrow ax^4-b=0\Rightarrow x_0=\left(\frac{b}{a}\right)^{\frac{1}{4}}$$

Since $\omega=\sqrt{\frac{k}{m}}, m=1$

$$k=\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}\quad \text{where } x_0 \text{ is stable equilibrium point.}$$

$$\text{Hence } k=\frac{\partial^2 V}{\partial x^2}=2a+\frac{6b}{x_0^4}=2a+\frac{6b}{b/}=8a \text{ at } x=x_0=\left(\frac{b}{a}\right)^{\frac{1}{4}}$$

Thus, $\omega=\sqrt{8a}$.

The correct option is **(b)**

2. Consider the motion of a classical particle in a one dimensional double-well potential $V(x)=\frac{1}{4}\left(x^2-2\right)^2$. If the particle is displaced infinitesimally from the minimum on the x-axis (and friction is neglected), then

[NET JUNE 2012]

A. the particle will execute simple harmonic motion in the right well with an angular frequency $\omega=\sqrt{2}$

B. the particle will execute simple harmonic motion in the right well with an angular frequency $\omega=2$

C. the particle will switch between the right and left wells

D. the particle will approach the bottom of the right well and settle there

Solution:

$$V(x) = \frac{1}{4}\left(x^2-2\right)^2 \Rightarrow \frac{\partial V}{\partial x} = \frac{2}{4}\left(x^2-2\right) \times 2x = 0 \Rightarrow x = 0, x = \pm\sqrt{2}$$

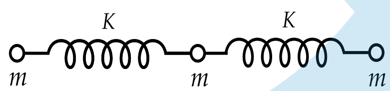
$$\frac{\partial^2 V}{\partial x^2} = 3x^2 - 2$$

At $x = 0, \dfrac{\partial^2 V}{\partial x^2} < 0$ so V is maximum. Thus it is unstable point

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=\pm\sqrt{2}} = 4 \text{ and it is stable equilibrium point with } \omega = \sqrt{\frac{\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}}{\mu}} = 2 \quad \because \mu = 1.$$

The correct option is **(b)**

3. Three particles of equal mass (m) are connected by two identical massless springs of stiffness constant (K) as shown in the figure



If x_1, x_2 and x_3 denote the horizontal displacement of the masses from their respective equilibrium positions the potential energy of the system is

[NET DEC 2012]

A. $\frac{1}{2}K\left[x_1^2+x_2^2+x_3^2\right]$

B. $\frac{1}{2}K\left[x_1^2+x_2^2+x_3^2-x_2\left(x_1+x_3\right)\right]$

C. $\frac{1}{2}K\left[x_1^2+2x_2^2+x_3^2-2x_2\left(x_1+x_3\right)\right]$

D. $\frac{1}{2}K\left[x_1^2+2x_2^2-2x_2\left(x_1+x_3\right)\right]$

Solution:

$$V = \frac{1}{2}K\left(x_2-x_1\right)^2 + \frac{1}{2}K\left(x_3-x_2\right)^2$$

$$V = \frac{1}{2}K\left(x_2^2+x_1^2-2x_2x_1\right) + \frac{1}{2}K\left(x_3^2+x_2^2-2x_3x_2\right)$$

$$V = \frac{1}{2}K\left[x_1^2+2x_2^2+x_3^2-2x_2\left(x_1+x_3\right)\right]$$

The correct option is **(c)**

4. The time period of a simple pendulum under the influence of the acceleration due to gravity g is T. The bob is subjected to an additional acceleration of magnitude $\sqrt{3}g$ in the horizontal direction. Assuming small oscillations, the mean position and time period of oscillation, respectively, of the bob will be

[NET JUNE 2014]

A. 0° to the vertical and $\sqrt{3}T$

B. 30° to the vertical and $T/2$

C. 60° to the vertical and $T/\sqrt{2}$

D. 0° to the vertical and $T/\sqrt{3}$

Solution:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$g' = \sqrt{3g^2+g^2} = \sqrt{4g^2} = 2g$$

$$T' = 2\pi\sqrt{\frac{l}{2g}} \Rightarrow T' = 2\pi\sqrt{\frac{l}{g}}\cdot\frac{1}{\sqrt{2}} \Rightarrow T' = \frac{T}{\sqrt{2}}$$

$$T\cos\theta = mg, T\sin\theta = \sqrt{3}mg \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = 60^\circ$$

The correct option is **(c)**

5. A particle of mass m is moving in the potential $V(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$ where a, b are positive constants. The frequency of small oscillations about a point of stable equilibrium is

[NET DEC 2014]

A. $\sqrt{a/m}$ **B.** $\sqrt{2a/m}$

C. $\sqrt{3a/m}$ **D.** $\sqrt{6a/m}$

Solution:

$$\because V(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$$

$$\frac{\partial V}{\partial x} = 0 \Rightarrow -ax+bx^3 = 0 \Rightarrow x\left[-a+bx^2\right] = 0 \Rightarrow x = \pm\left(\frac{a}{b}\right)^{\frac{1}{2}}, 0$$

$$\because \frac{\partial^2 V}{\partial x^2} = -a+3bx^2$$

$$\text{At } x=0, \frac{\partial^2 V}{\partial x^2} = -a \text{ (Negative so it is unstable point)}$$

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=\pm\left(\frac{a}{b}\right)^{\frac{1}{2}}} = -a+3b\frac{a}{b} = 2a \text{ (Positive so it is stable point)}$$

$$\Rightarrow \omega = \sqrt{\frac{\frac{\partial^2 V}{\partial x^2}}{m}} = \sqrt{\frac{2a}{m}}$$

The correct option is **(b)**

6. A particle of mass m, kept in potential $V(x) = -\frac{1}{2}kx^2 + \frac{1}{4}\lambda x^4$ (where k and λ are positive constants), undergoes small oscillations about an equilibrium point. The frequency of oscillations is

[NET JUNE 2018]

A. $\frac{1}{2\pi}\sqrt{\frac{2\lambda}{m}}$ **B.** $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$

C. $\frac{1}{2\pi}\sqrt{\frac{2k}{m}}$ **D.** $\frac{1}{2\pi}\sqrt{\frac{\lambda}{m}}$

Solution:

$$V = -\frac{1}{2}kx^2 + \frac{1}{4}\lambda x^4$$

$$\frac{dV}{dx} = 0 \quad -kx+\lambda x^3 = 0$$

$$x = 0, \quad x^2 = \frac{k}{\lambda} \Rightarrow x = x_0 = \sqrt{\frac{k}{\lambda}}$$

$$\frac{d^2V}{dx^2} = -k \quad \text{at } x=0 \quad \text{so } x=0 \text{ is unstable part}$$

$$\frac{d^2V}{dx^2} = 2k \text{ at } x_0 = \sqrt{\frac{k}{\lambda}} \text{ so } x_0 = \sqrt{\frac{k}{x}} \text{ is stable equation point}$$

$$\omega = \sqrt{\frac{\left.\frac{d^2V}{dx^2}\right|_{x=x_0}}{m}} = \sqrt{\frac{2k}{m}} \quad f = \frac{1}{2\pi}\sqrt{\frac{2k}{m}}$$

The correct option is **(c)**

Practice set 2 solutions

1. A particle is placed in a region with the potential $V(x)=\frac{1}{2}kx^2-\frac{\lambda}{3}x^3$, where $k,\lambda>0$. Then,
[GATE 2010]

   A. $x=0$ and $x=\frac{k}{\lambda}$ are points of stable equilibrium

   B. $x=0$ is a point of stable equilibrium and $x=\frac{k}{\lambda}$ is a point of unstable equilibrium

   C. $x=0$ and $x=\frac{k}{\lambda}$ are points of unstable equilibrium

   D. There are no points of stable or unstable equilibrium

Solution:

$$V=\frac{1}{2}kx^2-\frac{\lambda x^3}{3}\Rightarrow\frac{\partial V}{\partial x}=kx-\lambda x^2=0\Rightarrow x=0,x=\frac{k}{\lambda}$$
$$=\frac{\partial^2 V}{\partial x^2}=k-2\lambda x$$
$$At x=0,\frac{\partial^2 V}{\partial x^2}=+ve(\text{ Stable })$$
$$\text{at } x=\frac{k}{\lambda},\frac{\partial^2 V}{\partial x^2}=-ve\text{ (unstable)}$$

The correct option is **(b)**

2. Two bodies of mass m and $2m$ are connected by a spring constant k. The frequency of the normal mode is
[GATE 2011]

   A. $\sqrt{3k/2m}$

   B. $\sqrt{k/m}$

   C. $\sqrt{2k/3m}$

   D. $\sqrt{k/2m}$

Solution:

$$\omega=\sqrt{\frac{k}{\mu}}=\sqrt{\frac{k}{\frac{2m}{3}}}=\sqrt{\frac{3k}{2m}}$$
$$\text{where reduce mass }\mu=\frac{2mm}{2m+m}=\frac{2m}{3}.$$

The correct option is **(a)**

3. A particle of unit mass moves along the x-axis under the influence of a potential, $V(x)=x(x-2)^2$. The particle is found to be in stable equilibrium at the point $x=2$. The time period of oscillation of the particle is
[GATE 2012]

   A. $\frac{\pi}{2}$

   B. π

   C. $\frac{3\pi}{2}$

   D. 2π

Solution:

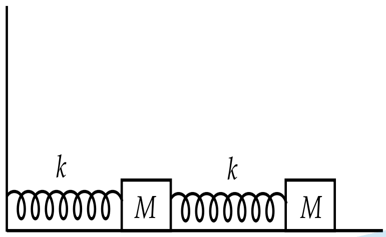
$$V(x)=x(x-2)^2\Rightarrow\frac{\partial V}{\partial x}=(x-2)^2+2x(x-2)=0\Rightarrow x=2,x=\frac{2}{3}$$

$$\frac{\partial^2 V}{\partial x^2} = 2(x-2)+2(x-2)+2x \Rightarrow \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=2} = 2\times 2 = 4$$

$$\omega = \sqrt{\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=2}} \Rightarrow \omega = \frac{2\pi}{T} = 2 \Rightarrow T = \pi$$

The correct option is **(b)**

4. Consider two small blocks, each of mass M, attached to two identical springs. One of the springs is attached to the wall, as shown in the figure. The spring constant of each spring is k. The masses slide along the surface and the friction is negligible. The frequency of one of the normal modes of the system is,
[GATE 2013]



A. $\sqrt{\frac{3+\sqrt{2}}{2}}\sqrt{\frac{k}{M}}$

B. $\sqrt{\frac{3+\sqrt{3}}{2}\sqrt{\frac{k}{M}}}$

C. $\sqrt{\frac{3+\sqrt{5}}{2}}\sqrt{\frac{k}{M}}$

D. $\sqrt{\frac{3+\sqrt{6}}{2}\sqrt{\frac{k}{M}}}$

Solution:

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2-x_1)^2$$

$$= \frac{1}{2}kx_1^2 + \frac{1}{2}k\left(x_2^2+x_1^2-2x_2x_1\right) = \frac{1}{2}k\left(2x_1^2+x_2^2-2x_2x_1\right)$$

$$T = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}; \quad V = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}$$

$$\begin{vmatrix} 2k-\omega^2 m & -k \\ -k & k-\omega^2 m \end{vmatrix} = 0 \Rightarrow \left(2k-\omega^2 m\right)\left(k-\omega^2 m\right) - k^2 = 0 \Rightarrow \omega = \sqrt{\frac{3+\sqrt{5}}{2}}\sqrt{\frac{k}{m}}$$

THe correct option is **(c)**

5. Two masses m and $3m$ are attached to the two ends of a massless spring with force constant K. If $m = 100$ g and $K = 0.3$ N/m, then the natural angular frequency of oscillation is Hz.
[GATE 2014]

Solution:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$$

$$\mu = \frac{m_1\cdot m_2}{m_1+m_2} = \frac{3m\cdot m}{4m} = \frac{3m}{4}$$

$$\omega = \sqrt{\frac{4k}{3m}} = 2 \Rightarrow f = 0.318 \text{ Hz}$$

6. A particle of mass m is in a potential given by

$$V(r) = -\frac{a}{r} + \frac{ar_0^2}{3r^3}$$

where a and r_0 are positive constants. When disturbed slightly from its stable equilibrium position it undergoes a simple harmonic oscillation. The time period of oscillation is

[GATE 2014]

A. $2\pi\sqrt{\frac{mr_0^3}{2a}}$

B. $2\pi\sqrt{\frac{mr_0{}^3}{a}}$

C. $2\pi\sqrt{\frac{2mr_0^3}{a}}$

D. $4\pi\sqrt{\frac{mr_0^3}{a}}$

Solution:

$$V(r) = -\frac{a}{r} + \frac{ar_0^2}{3r^3}$$

For equilibrium

$$\frac{\partial V}{\partial r} = \frac{a}{r^2} - \frac{3ar_0^2}{3r^4} = 0, \quad r = \pm r_0$$

$$\frac{\partial^2 V}{\partial r^2} = -\frac{2a}{r^3} + \frac{4ar_0^2}{r^5}\bigg|_{r_0} = -\frac{2a}{r_0^3} + \frac{4ar_0^2}{r_0^5} = \frac{2a}{r_0^3}$$

$$\omega = \sqrt{\frac{\frac{\partial^2 V}{\partial r^2}\Big|_{r_0}}{m}} \Rightarrow T = 2\pi\sqrt{\frac{mr_0^3}{2a}}$$

The correct option id **(a)**

7. Two identical masses of 10gm each are connected by a massless spring of spring constant 1 N/m. The non-zero angular eigenfrequency of the system is. .rad/s. (up to two decimal places)

[GATE 2017]

Solution:

$$\omega = \sqrt{\frac{k}{\mu}}, \quad \text{where } \mu = \frac{m}{2} = \frac{10}{2 \times 1000} = \frac{1}{200} \text{ and } k = 1N/m, \quad \omega = 14.14$$

8. In the context of small oscillations, which one of the following does NOT apply to the normal coordinates?

[GATE 2018]

A. Each normal coordinate has an eigen-frequency associated with it

B. The normal coordinates are orthogonal to one another

C. The normal coordinates are all independent

D. The potential energy of the system is a sum of squares of the normal coordinates with constant coefficients

Solution: Normal co-ordinate must be independent. It is not necessary that it should orthogonal.
The correct option is **(b)**

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