1. Complex Analysis

A simple algebraic equation like $x^2 = -1$ may not have a real solution. Introducing complex numbers validates the existence of 'root' for every polynomial with a positive degree . Which then proves the fundamental theorem of algebra. The idea of complex numbers are widely used in Physics and Mathematics.

Definition 1.0.1 A number of the form $x + iy$, where x and y are real numbers and $i = \sqrt{(-1)}$, is called a complex number.

Real Part   : x is called the real part of the complex number, $x + iy$ and is written as, $R(x + iy)$.

Imaginary Part : y is called the imaginary part of the complex number and is written as, $I(x + iy)$.

1.1 Representation of a Complex number

The point whose cartesian coordinates are (x, y) uniquely represents the complex number, $z = x + iy$ on the complex plane z. The diagram in which this representation is carried out is called the Argand's diagram. It's shown in the figure 1.1. Since x is the real part of z we call the x -axis the real axis. Likewise, the y -axis is the imaginary axis.
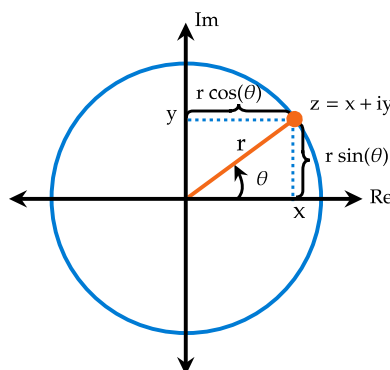

Figure 1.1: Argand Diagram

In terms of the polar coordinates , we have

$$x = r\cos\theta, \quad y = r\sin\theta \tag{1.1}$$

$$z = x + iy = re^{i\theta}$$
$$= r(\cos\theta + i\sin\theta) \qquad (1.2)$$

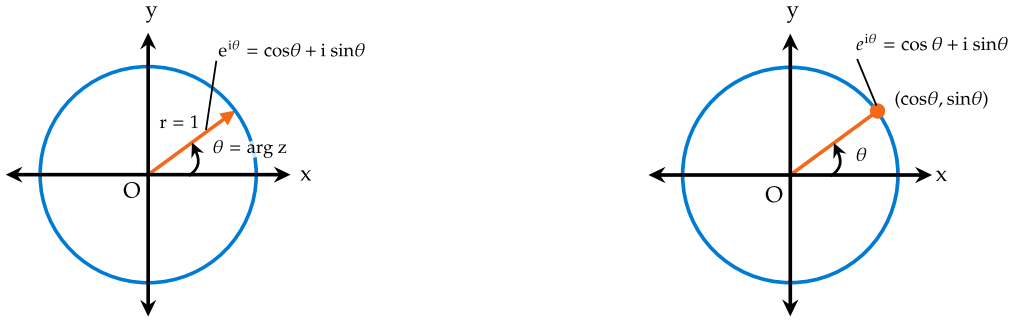Then, the equation 1.2 is known as, Euler's formula

Figure 1.2: Polar representation

1.1.1 Absolute Value

We define the absolute value of a complex number $x + iy$ to be the length r of the vector from the origin to $P(x, y)$.

$$r = |x + iy| = \sqrt{x^2 + y^2}$$

Properties:

- $|z_1 + z_2| \leq |z_1| + |z_2|$

- $|z_1 - z_2| \geq |z_1| - |z_2|$

- $|z_1 z_2| = |z_1||z_2|$

- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

1.1.2 Argument of

The polar angle θ is called the argument of z and it is written as,

$$\theta = \arg z$$

Any integer multiple of 2π may be added to θ to produce another appropriate angle.
From the figure 1.1,

$$\theta = \arg z = \tan^{-1}\left(\frac{y}{x}\right)$$

Properties:

- $\text{Arg}(z_1 z_2 \cdot z_3 \ldots \ldots z_n) = \text{Arg}(z_1) + \text{Arg}(z_2) + \text{Arg}(z_3) + \ldots \ldots + \text{Arg}(z_n)$

- $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$

Exercise 1.1 Find the modulus and principal argument of the complex number $\frac{1+2i}{1-(1-i)^2}$ ∎

Solution:

$$\frac{1+2i}{1-(1-i)^2} = \frac{1+2i}{1-(1-1-2i)} = \frac{1+2i}{1+2i}$$
$$= 1 = 1 + 0i$$

$$\therefore \quad \left| \frac{1+2i}{1-(1-i)^2} \right| = |1+0i| = \sqrt{1^2} = 1$$

$$\text{Principal argument of } \frac{1+2i}{1-(1-i)^2} = \text{Principal argument of} \quad (1+0i)$$

$$\tan^{-1}\frac{0}{1} = \tan^{-1}0$$

$$= 0°$$

1.1.3   Conjugate of a Complex number

The conjugate of a complex number z is represented by,

$$\bar{z} = x - iy$$

Note

$$\frac{z+\bar{z}}{2} = Re\{z\}$$
$$\frac{z-\bar{z}}{2i} = Im\{z\}$$
$$z \cdot \bar{z} = |z|^2$$

1.2   Algebra of Complex numbers

For two Complex numbers, $a+ib$ and $c+id$

Equality:

$$a+ib = c+id$$

Two complex numbers (a,b) and (c,d) are equal if and only $a=c$ and $b=d$.

Addition:

$$(a+ib)+(c+id) \quad = (a+c)+i(b+d)$$

Multiplication:

$$(a+ib)(c+id) = (ac-bd)+i(ad+bc)$$
$$c(a+ib) = ac+i(bc)$$

Polar form:

$$\text{Let,} \quad z_1 = r_1\left(\cos\theta_1 + i\sin\theta_1\right) \quad \text{and} \quad z_2 = r_2\left(\cos\theta_2 + i\sin\theta_2\right)$$
$$z_1.z_2 = r_1 r_2\left(\cos\theta_1 + i\sin\theta_1\right)\left(\cos\theta_2 + i\sin\theta_2\right)$$
$$= r_1 r_2\left[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\left(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\right)\right]$$
$$= r_1 r_1\left[\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right],$$

Division:

$$\frac{c+id}{a+ib} = \frac{(c+id)(a-ib)}{(a+ib)(a-ib)}$$
$$= \frac{(ac+bd)+i(ad-bc)}{a^2+b^2}$$
$$\text{Where,} \quad x = \frac{ac+bd}{a^2+b^2}, \quad \text{and} \quad y = \frac{ad-bc}{a^2+b^2}$$

1.3 Important Identities

1.3.1 Circular functions of Complex numbers

$$\bullet \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \bullet \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\bullet \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \bullet \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

1.3.2 Hyperbolic functions of Complex numbers

$$\bullet \quad \sinh x = \frac{e^x - e^{-x}}{2} \qquad\qquad \bullet \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\bullet \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \bullet \quad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\bullet \quad \operatorname{sech} x = \frac{2}{e^x + e^{-x}} \qquad\qquad \bullet \quad \operatorname{cosech} x = \frac{2}{e^x - e^{-x}}$$

$$\bullet \quad \cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x$$

Note **Relation between Circular and Hyperbolic functions:**

- $\sin ix = i\sinh x$ • $\sinh ix = i\sin x$
- $\cos ix = \cosh x$ • $\cosh ix = \cos x$
- $\tan ix = i\tanh x$ • $\tanh ix = i\tan x$

Theorem 1.3.1 De Moivre's Theorem:

1. For any integer n, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

2. If n is a fraction, then $(\cos n\theta + i\sin n\theta)$ is one of the values .

Exercise 1.2 Express $\frac{(\cos\theta + i\sin\theta)^8}{(\sin\theta + i\cos\theta)^4}$ in the form $(x + iy)$ ∎

Solution:

$$\begin{aligned}
\frac{(\cos\theta + i\sin\theta)^8}{(\sin\theta + i\cos\theta)^4} &= \frac{(\cos\theta + i\sin\theta)^8}{(i)^4\left(\cos\theta + \frac{1}{i}\sin\theta\right)^4} \\
&= \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta - i\sin\theta)^4} = \frac{(\cos\theta + i\sin\theta)^8}{[\cos(-\theta) + i\sin(-\theta)]^4} \\
&= \frac{(\cos\theta + i\sin\theta)^8}{[(\cos\theta + i\sin\theta)^{-1}]^4} = \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta + i\sin\theta)^{-4}} = (\cos\theta + i\sin\theta)^{12} \\
&= \cos 12\theta + i\sin 12\theta
\end{aligned}$$

Note Series expansion of different functions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

1.4 Function of a Complex Variable

1.4.1 Basic Representation

$$W = f(z) = v(x,y) + iv(x,y) \qquad\qquad \text{Real part (u)}, \left(x^2 - y^2\right)^2$$

$$f(z) = z^2 = (x+iy)^2 = \left(x^2 - y^2\right)^2 + i2xy \qquad \text{Imaginary part (v)}, 2xy$$

1.4.2 Existance of $\lim_{z \to z_0} f(z)$:

The limit will exists only if the limiting value is independent of the path along which z approaches z_0

> **Exercise 1.3** Find whether the limit $\lim_{z \to 0} \frac{z}{|z|}$ exist or not.   ∎

Solution:

$$z \to 0 \text{ means } x \to 0 \,\&\, y \to 0$$

For $z = 0$, we have to choose a path passing through a origin.

Therefore, we have choosen a straight line passing through the origin i.e. $y = mx$

$$\lim_{z \to 0} \frac{z}{|z|} = \lim_{\substack{z \to 0 \\ y \to 0}} \frac{x+iy}{\sqrt{x^2+y^2}} = \lim_{x \to 0} \frac{x+imx}{\sqrt{x^2+m^2x^2}} = \frac{1+im}{\sqrt{1+m^2}}$$

Therefore, the limit depends on m i.e. slope of the straight line. Thus, the limiting values is dependent on the path and the limit does not exists.

> **Exercise 1.4** Calculate the vahue $\lim_{z \to \infty} \frac{iz^3 + iz - 1}{(2z+3i)(z-i)^2}$ ∎

Solution:

$$\lim_{z \to \infty} \frac{iz^3 + iz - 1}{(2z+3i)(z-i)^2} = \lim_{z \to \infty} \frac{z^3 \left(i + \frac{i}{z^2} - \frac{1}{z^3}\right)}{z\left(2 + \frac{3i}{z}\right) z^2 \left(1 - \frac{i}{z}\right)^2} = \frac{i}{2}$$

1.4.3 Differentiability of Complex Function

$$f'(z) = \operatorname*{Lt}_{\delta z \to 0} \frac{[f(z+\delta z) - f(z)]}{\delta z}$$

The function will be differentiable if limit should exists and it is independent of path along with $\delta z \to 0$.

$$\text{Ex:} \quad f(z) = (4x+y) + i(4y-x) \Rightarrow u = (4x+y) \text{ and } v = (4y-x)$$

$$\Rightarrow f(z+\delta z) = 4(x+\delta x) + (y+\delta y) + i[4(y+\delta y) - (x+\delta x)]$$

$$\Rightarrow f(z+\delta z) - f(z) = 4\delta x + \delta y + i(4\delta y - \delta x)$$

$$\Rightarrow \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{4\delta x + \delta y + i(4\delta y - \delta x)}{\delta z} \Rightarrow \frac{\delta f}{\delta z} = \frac{4\delta x + \delta y - i\delta x + 4i\delta y}{\delta x + i\delta y}$$

$$\text{Along real axis} : \delta x = \delta z, \delta y = 0, \Rightarrow \frac{\delta f}{\delta z} = 4 - i$$

$$\text{Along imaginary axis} : i\delta y = \delta z, \delta x = 0, \Rightarrow \frac{\delta f}{\delta z} = 4 - i$$

$$\text{Along a line} : y = x, \delta y = \delta x, \delta z = (1+i)\delta x, \Rightarrow \frac{\delta f}{\delta z} = \frac{5\delta x + 3i\delta x}{(1+i)\delta \dot{x}} = \frac{5+3i}{1+i} = 4 - i$$

1.5 Complex Analysis Function

A function $f(z)$ is said to be analytic at a point $z = z_0$ if it is single valued and has the derivative at every point in some neighhourhood of z_0. The function $f(z)$ is said to be analytic in a domain D if it is single valued and is differentiable at every point of domain D.

1.5.1 Cauchy Reamann Equations

For a function $f(z) = u + iv$ to be analytic at all points in some region ' R ', the necessary conditions are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\text{Sufficient Condition:} \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \text{ are continuous functions of } x \text{ and } y.$$

$$\text{Derivative of f(z)}: f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Exercise 1.5 Check whether $f(z) = \sin z$ is analytic or not.   ∎

Solution:

$$f(z) = \sin z = \sin(x+iy) = \sin x \cdot \cos(iy) + \cos x \cdot \sin(iy) = \sin x \cdot \cosh y + i \cos x \cdot \sinh y$$

Therefore, $u = \sin x \cdot \cosh y$ and $v = \cos x . sinhy$

$$\frac{\partial u}{\partial x} = \cos x \cdot \cosh y; \frac{\partial u}{\partial y} = \sin x \cdot \sinh y; \frac{\partial v}{\partial x} = -\sin x \cdot \sinh y; \frac{\partial v}{\partial y} = \cos x \cdot \cosh y$$

So, C-R equation is satisfied, given f(z) is analytic.

Exercise 1.6 If the real part of a complex analytic function is $u(x,y) = x + \frac{1}{2}(x^2 - y^2)$, find the corresponding imaginary part.   ∎

Solution:

$$\frac{\partial u}{\partial x} = x + 1 = \frac{\partial v}{\partial y} \Rightarrow v(x,y) = y + xy + f(x)$$

$$\frac{\partial u}{\partial y} = -y = -\frac{\partial v}{\partial x} \Rightarrow v(x,y) = xy + g(y)$$

Therefore, the imaginary part will be $v(x,y) = y + xy + C$ (C = numerical constant)

Exercise 1.7 Example-7: If $f(x,y) = (1+x+y)(1+x-y) + a(x^2-y^2) - 1 + 2iy(1-x-ax)$ is a complex analytic function then find the value of ' a '. ∎

Solution:

$$u(\mathrm{x},\mathrm{y}) = (1+x+y)(1+x-y) + a(x^2-y^2) - 1 \quad \Rightarrow \frac{\partial u}{\partial x} = 2x + 2 + 2ax$$

$$v(x,y) = 2y(1-x-ax) \quad \Rightarrow \frac{\partial v}{\partial y} = 2(1-x-ax)$$

According to Cauchy Reamann equation, $\dfrac{\partial \mathrm{u}}{\partial \mathrm{x}} = \dfrac{\partial \mathrm{v}}{\partial \mathrm{y}} \Rightarrow 4\mathrm{x} = -4a\mathrm{x} \Rightarrow \mathrm{a} = -1$

Exercise 1.8 The harmonic conjugate function of $u(x,y) = 2x(1-y)$ corresponding to a complex analytic function $\omega = u(x,y) + iv(x,y)$ is given $v(x,y) = \alpha x^2 + \beta y + \gamma y^2$ (Taking the integration constant to be zero). Which of the following statement is true ?

a. $\alpha - \gamma = \beta$

b. $\alpha + \gamma + \beta = 0$

c. $\alpha + \gamma = \beta$

d. $\alpha\gamma\beta = 1$.

∎

Solution:

$$u(x,y) = 2x(1-y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2(1-y) = \frac{\partial v}{\partial y} \Rightarrow v = 2y - y^2 + f_1(x)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -2x = -\frac{\partial v}{\partial x} \Rightarrow v = x^2 + f_2(y)$$

Therefore, the imaginary part of the complex function $v = x^2 - y^2 + 2y$

Comparing with the question, $\alpha = 1, \beta = 2, \gamma = -1 \quad \Rightarrow \alpha - \gamma = \beta$

1.5.2 Method for Finding Conjugate Function

Case 1: $f(z) = u + iv$, and u is known.

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \Rightarrow v = -\int \frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy$$

Case 2: $f(z) = u + iv$, and v is known

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy \Rightarrow u = \int \frac{\partial v}{\partial y}dx - \int \frac{\partial v}{\partial x}dy$$

Exercise 1.9 Find the imaginary part of the complex analytic function whose real part is $u(x,y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ ∎

Solution:

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x = \frac{\partial v}{\partial y} \Rightarrow v = 3x^2 y - y^3 + 6xy + f_1(x)$$

$$\frac{\partial u}{\partial y} = -6xy - 6y = -\frac{\partial v}{\partial x} \Rightarrow v = 3x^2 y + 6xy + f_2(y)$$

$$v(x,y) = 3x^2 y - y^3 + 6xy + C$$

1.5.3 Milne-Thomson Method : (To find Analytic function if either ' u ' or ' v ' is given)

Case 1: When 'u' is given,

(1) Find $\dfrac{\partial u}{\partial x} = \phi_1(x,y)$ and $\dfrac{\partial u}{\partial y} = \phi_2(x,y)$

(2) Replace x by z and y by 0 in $\phi_1(x,y)$ and $\phi_2(x,y)$ to get $\phi_1(z,0)$ and $\phi_2(z,0)$.

(3) Find $f(z) = \displaystyle\int \{\phi_1(z,0) - i\phi_2(z,0)\}\,dz + c$

Case 2: When ' v ' is given,

(1) Find $\dfrac{\partial v}{\partial x} = \psi_2(x,y)$ and $\dfrac{\partial v}{\partial y} = \psi_1(x,y)$

(2) Replace x by z and y by 0 in $\psi_1(x,y)$ and $\psi_2(x,y)$ to get $\psi_1(z,0)$ and $\psi_1(z,0)$.

(3) Find $f(z) = \displaystyle\int \{\psi_1(z,0) + i\psi_2(z,0)\}\,dz + c$

Exercise 1.10 Find the analytical function whose imaginary part is $v(x,y) = e^x(x\cos y - y\sin y)$ ∎

Solution:

$$\frac{\partial v}{\partial x} = e^x(x\cos y - y\sin y) + e^x \cos y = \Psi_2(x,y) \quad \Rightarrow \frac{\partial v}{\partial y} = -e^x x\sin y - e^x(\sin y + y\cos y) = \Psi_1(x,y)$$

$$\Psi_1(z,0) = 0 \text{ and } \Psi_1(z,0) = e^z z + e^z \quad \Rightarrow f(z) = \int 0 + i\left[e^z z + e^z\right] dz = ize^z + C$$

Exercise 1.11 If the real part of a complex analytic function f(z) is given as, $u(x,y) = e^{-2xy}\sin\left(x^2 - y^2\right)$, then f(z) can be written as

a. $ie^{i^2} + C$ 　　　　　　　　　　　　　　　　**b.** $-ie^{ix^2} + C$

c. $-ie^{-iz^2} + C$ 　　　　　　　　　　　　　　**d.** $ie^{-ii^2} + C$

∎

Solution:

$$u(x,y) = e^{-2xy}\sin\left(x^{-2} - y^2\right)$$

$$\Rightarrow \frac{\partial u}{\partial x} = e^{-2xy}(-2y)\sin\left(x^2 - y^2\right) + e^{-2xy}\cos\left(x^2 - y^2\right) 2x = \phi_1(x,y)$$

$$\Rightarrow \frac{\partial u}{\partial y} = e^{-2xy}(-2x)\sin\left(x^2 - y^2\right) + e^{-2xy}\cos\left(x^2 - y^2\right)(-2y) = \phi_2(x,y)$$

$$\therefore \quad \phi_1(z,0) = \cos z^2 \cdot 2z,\ \phi_2(z,0) = \sin z^2(-2z)$$

$$\therefore \quad f(z) = \int \left(\cos z^2 \cdot 2z - i \sin z^2 \cdot (-2z) \right) dz + c = 2 \int \left(\cos z^2 + i \sin z^2 \right) \cdot z \, dz + c$$

$$= 2 \int e^{iz^2} \cdot z \, dz + c = -i e^{iz^2} + c$$

So the correct answer is **option (b)**

1.5.4 Harmonic Function

Any function which satisfies the Laplace's equation, is known as harmonic function. If $u + iv$ is an analytic function, then u, v are conjugate harmonic functions i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Exercise 1.12 Find the values of m, n such that $f(x,y) = x^2 + mxy + ny^2$ is harmonic in nature. ∎

Solution:

$$\text{Since, } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{dy^2} = 0 \Rightarrow 2n + 2 = 0 \Rightarrow n = -1; \, 'm' \text{ can take any value.}$$

1.6 Cauchy's Integral Theorem

If a funciion $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve ' C ', then $\oint_C f(z) dz = 0$

1.6.1 Cauchy's Integral Formula

If $f(z)$ is analytic within or on a closed curve C and if ' a ' is any point within C, where $\frac{f(z)}{z-a}$ is not analytic at $z = a$ then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz \quad \text{and } f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} dz$$

$$\text{Similarly,} \quad f''(a) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^3} dz \quad \text{and } f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$$

Exercise 1.13 Evaluate the integral $\frac{1}{2\pi i} \oint \frac{ze^z}{(z-a)^3}$ ∎

Solution:

$$\frac{1}{2\pi i} \oint \frac{ze^z}{(z-a)^3} = \frac{1}{2\pi i} 2\pi i \frac{1}{2!} \frac{d^2}{dz^2} \left(ze^z \right) \bigg|_{z=a}$$

$$= \frac{ae^a + 2e^a}{2}$$

$$= \frac{1}{2}(a+2)e^a$$

Power Series Expansion of Complex Function

Every analytic function which is analytic at $z = z_0$ can be expanded into power series about $z = z_0$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1 (z-z_0) + a_2 (z-z_0)^2 + \ldots \ldots$$

where, z_0 is the centre of power series.

Radius of Convergence

Imagine a circle of centre z_0 and radius r, then $|z - z_0| = R$, The power series is convergent in the region $|z - z_0| < R$ (i.e. within the circle) and divergent $|z - z_0| > R$
(outside the circle). Therefore, R is known as the radius of convergence of power series and defined as

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Exercise 1.14 Find the radius of convergence of the series

$$\frac{z}{2} + \frac{1.3}{2.5}z^2 + \frac{1.3.5}{2.5.8}z^3 + \ldots\ldots$$

Solution: The coefficient of z^n of the given power series is given by

$$a_n = \frac{1.3.5\ldots(2n-1)}{2.5.8\ldots(3n-1)}$$

$$a_{n+1} = \frac{1.3.5\ldots(2n-1)(2n+1)}{25.8\ldots(3n-1)(3n+2)}$$

$$\text{So } \frac{a_{n+1}}{a_n} = \frac{2n+1}{3n+2} = \frac{2}{3} \cdot \frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{2}{3n}\right)}$$

$$\text{Therefore, } \frac{1}{R} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{2}{3} \cdot \frac{(1+0)}{(1+0)} = \frac{2}{3} \Rightarrow R = \frac{3}{2}$$

Taylor Series Expansion

If a function f(z) is analytic at all points inside and on a circle C, with its center at the point ' a ' and radius ' r ', then at each point z inside C, the function $f(z)$ can be expanded as

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots + \frac{f^n(a)}{n!}(z-a)^n + \ldots$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z-z_0)^n$$

Exercise 1.15 Expand the function $\ln(\cosh x)$ about the point $x = 0$.

Solution:

$$f(x) = f(0) + (x-0)f'(0) + \frac{1}{2!}(x-0)^2 f''(0) + \ldots\ldots$$

$$f'(x) = \frac{1}{\cosh x} \cdot \sinh x = \tanh x, f''(x) = \sec h^2 x$$

$$f'''(x) = 2\operatorname{sech} x(-\operatorname{sech} x \tanh x) = -2\sec h^2 x \tanh x$$

$$f'''(x) = 4\operatorname{sech} h^2 x \cdot \tanh^2 x - 2\sec h^4 x$$

$$\text{Therefore, } f(x) = 0 + 0 + \frac{x^2}{2} + 0 - \frac{x^4}{12} + \ldots\ldots = \frac{x^2}{2} - \frac{x^4}{12} + \ldots..$$

1.7 Laurent Series

Let C_1 and C_2 be two circles with center at z_0. Let $f(z)$ be analytic in the region R between the circles. Then $f(z)$ can be expanded in a series of the form

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

convergent in R. Such a series is called a Laurent series. The "b" series in (4.1) is called the principal part of the Laurent series.

Example 1. Consider the Laurent series (4.2)

$$f(z) = 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots + \left(\frac{z}{2}\right)^n + \cdots$$
$$+ \frac{2}{z} + 4\left(\frac{1}{z^2} - \frac{1}{z^3} + \cdots + \frac{(-1)^n}{z^n} + \cdots\right)$$

Let us see where this series converges. First consider the series of positive powers; by the ratio test (see Chapters 1 and 2), this series converges for $|z/2| < 1$, that is, for $|z| < 2$. Similarly, the series of negative powers converges for $|1|z| < 1$, that is, $|z| > 1$. Then both series converge (and so the Laurent series converges) for $|z|$ between 1 and 2, that is, in a ring between two circles of radii 1 and 2.

We expect this result in general. The " a" series is a power series, and a power series converges inside some circle (say C_2 in Figure 4.1). The " b" series is a series of inverse powers of z, and so converges for $|1/z| <$ some constant; thus the " "b" series converges outside some circle. Then a Laurent series converges between two circles (if it converges at all). (Note that the inner circle may be a point and the outer circle may have infinite radius).
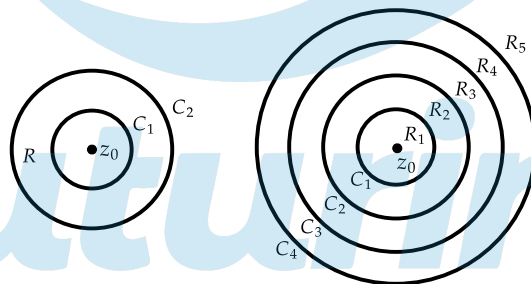
The formulas for the coefficients are

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z - z_0)^{n+1}}, \quad b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z - z_0)^{-n+1}},$$

where C is any simple closed curve surrounding z_0 and lying in R. However, this is not usually the easiest way to find a Laurent series. Like power series about a point, the Laurent series (about z_0) for a function in a given annular ring (about z_0) where the function is analytic, is unique, and we can find it by any method we choose. (See examples below.) If $f(z)$ has several isolated singularities, there are several annular rings, $R_1, R_2, \cdots$, in which $f(z)$ is analytic; then there are several different Laurent series for $f(z)$, one for each ring. The Laurent series which we usually want is the one that converges near z_0. If you have any doubt about the ring of convergence of a Laurent series, you can find out by testing the " a " series and the "b" series separately.

Singularity of Complex Function

Singular Point of an Analytic Function

A point at which the function ceases to be analytic.

Example: $f(z) = \frac{1}{(z-2)}$ has a singularity at z = 2.

Different kinds of singularities exist. These are:

Isolated Singularity

A point z = z_0 is said to be isolated singularity of $f(z)$ if

(a) $f(z)$ is not analytic at z = z_0

(b) $f(z)$ is analytic in the neighbourhood of z = z_0 i.e. there exists a neighbourhood of z = z_0, containing no other singularity.

Exercise 1.16 (i) Function $f(z) = \frac{1}{z}$ is analytic everywhere except at z = 0, therefore z = 0 is an isolated singularity.

(ii) The function $f(z) = \frac{z+2}{(z-1)(z-2)(z-3)}$ has three isolated singularities at $z = 1, 2$ and 3 . ∎

Non-isolated Singularity

$f(z) = \dfrac{1}{\left[\sin \frac{\pi}{z}\right]}$ is not analytic when $\sin \dfrac{\pi}{z} = 0 \Rightarrow \dfrac{\pi}{z} = n\pi \Rightarrow z = \dfrac{1}{n} (n = 0, 1, 2, 3 \ldots \ldots \ldots)$

Thus, $z = 0$ is an non-isolated singularity of $f(z)$, surrounded by a infinite number of other singularity $z = \dfrac{1}{n}$

$f(z) = \dfrac{1}{\sin \pi/z}$ has non-isolated singularity at z = 0

Types of Isolated Singularity

If f(z) is an isolated singular point at z = a, the we can expand $f(z)$ about z = a into laurent series as:

$$\text{f}(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} = \left[a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\right] + \left[\frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \ldots\right]$$

Therefore, three types of singularity are as follows.

(1) Removable Singularity : If the principal part of the Laurent series expansion of $f(z)$ about $z = a$ contains no term i.e. if $b_n = 0$ for all ' n ', then f(z) has a removable singularity at z = a. In this case, Laurent series expansion is $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$

Exercise 1.17 Suppose $f(z) = \frac{\sin z}{z}$, then $\lim_{z\to 0}\left(\frac{\sin z}{z}\right) = 1$, therefore, $z = 0$ is a removable singularity of $f(z)$.

Again, $\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \ldots\ldots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} \ldots\ldots$

Since, there is no negative term in the laurent series expansion of $f(z)$ about $z = 0$, hence $z = 0$ is a removable singularity of $f(z)$. ∎

(2) Non-essential singularity or Pole: If the principal part of the Laurent series expansion of $f(z)$ about $z = $ a contains a finite number of terms, saym, i.e. $b_n = 0$ for all $n > m$, then $f(z)$ has a non-essential singularity or a pole of order ' m ' at $z = a$. A polc of order one is also known as simple pole.

Thus if z = a is pole of order mof function f(z), then f(z) will have the Laurent series expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{m} b_n (z - z_0)^{-n}$$

Example : $f(z) = \frac{z}{(z-1)(z+2)^2}$ has a simple pole at $z = 1$ and a pole of order 2 at $z = -2$.

(3) Essential singularity: If the principal part of the Laurrent series expansion of $f(z)$ about $z = a$, contains infinite number of terms i.e. $b_n \neq 0$ for infinitely many values of n, then $f(z)$ has an essential singularity at $z = a$.

Example : $f(z) = e^{1/2}$ has an essential singularity at $z = 0$, since the expansion $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{z^2}2! + \ldots$ is an infinite series of-ve powers of z.

Exercise 1.18 Examine the nature of singularity of the functions: (a) $\sin\left(\frac{1}{1-z}\right)$, (b) $(z-3)\sin\left(\frac{1}{z+2}\right)$. ∎

Solution:

(a) $\sin\left(\dfrac{1}{1-z}\right) = \dfrac{1}{1-z} - \dfrac{1}{(1-z)^3 \cdot 3!} + \dfrac{1}{(1-z)^5 \cdot 5!} - $. so, $z = 1$ is an isolated essential singular point.

(b) $(z-3)\sin\left(\dfrac{1}{z+2}\right) = (z-3)\left[\dfrac{1}{z+2} - \dfrac{1}{(z+2)^3\cdot 3!} + \dfrac{1}{(z+2)^5\cdot 5!} - \ldots\ldots\right]$ so, $z = -2$ is an isolated

essential point

Zero of an Analytic Function

A zero of an analytic function $f(z)$ is a value of z such that $f(z) = 0$

An analytic function f(z) is said to have a zero of order ' m ' at $z = z_0$ if f(z) is expressible as

$$f(z)\,(z-z_0)^m\,\phi(z)$$

where $\phi(z)$ is analytic and $\phi(z_0) \neq 0$. For $m = 1$, $f(z)$ is said to have a simple zero at $z = z_0$

Residue of Complex Fuction
Definition of residue at a pole:
Let, $z = a$ be a pole of order ' m ' of f(z) and C_1 is a circle of radius ' r ' with center at $z = a$ which does not contain singularities except $z = a$, then f(z) is analytic within the annular region $r < |z-a| < R$ can be expanded into Laurrent series within the annulur region as:

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$$

Co-efficient b_1 is known as residue of f(z) at $z = a$ i.e. Res. $f(z = a) = b_1 = \frac{1}{2\pi i}\oint f(z)dz$ **Methods of Finding Residues**

Method 1: Res. $f(z = a) = \underset{z\to a}{Lt}\,(z-a)f(z)$

Method 2: If $f(z) = \frac{\phi(z)}{\psi(z)}$ where $\psi(a) = 0$ but $\phi(a) \neq 0$, then Res. $f(z = a) = \frac{\phi(a)}{\psi'(a)}$
(b) Residue at a pole of order ' n ':
Method 1: Res. $f(z = a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a}$
Method 2: First put $z + a = t$ and expand it into series, then Res. $f(z = a) = $ co-efficient of $1/t$.
(c) Residue at $z = \infty$: Res. $f(z = \infty)\,\underset{z\to\infty}{Lt}\,[-zf(z)]$

Exercise 1.19 Find the singular points of the following function and the corresponding residues:
(a) $f(z) = \frac{1-2z}{z(z-1)(z-2)}$ (b) $f(z) = \frac{z^2}{z^2+a^2}$ (c) $f(z) = z^2 e^{1/z}$ ∎

Solution:

$$(a\,)f(z) = \frac{1-2z}{z(z-1)(z-2)} \Rightarrow \text{Poles} : z = 0, z = 1, z = 2$$

$$\text{Res. } f(z=0) = Lt_{z\to 0}(z-0)f(z) = Lt_{z\to 0}\frac{1-2z}{(z-1)(z-2)} = \frac{1}{2}.$$

$$\text{Res. } f(z=1) = L_{z\to 1}(z-1)f(z) = L_{z\to 1}\frac{1-2z}{z(z-2)} = 1$$

$$\text{Res. } f(z=2) = \underset{z\to 2}{Lt}(z-2)f(z) = \underset{z\to 2}{Lt}\frac{1-2z}{z(z-1)} = -\frac{3}{2}$$

$$(b)\,f(z) = \frac{z^2}{z^2+a^2} \Rightarrow \text{Poles} : z = ia, z = -ia$$

$$\text{Res. } f(z=ia) = \left(\frac{z^2}{2z}\right)_{z=ia} = \frac{1}{2}ia; \text{Res.}\,f(z=-ia) = \left(\frac{z^2}{2z}\right)_{z=-ia} = -\frac{1}{2}ia$$

(c) $f(z) = z^2 e^{1/z} = z^2 \left[1 + \dfrac{1}{z} + \dfrac{1}{z^2 \cdot 2!} + \dfrac{1}{z^3 \cdot 3!} + \ldots \ldots \right] \Rightarrow$ Poles : $z = 0$

Res. $f(z=0) = $ Coefficient of $\dfrac{1}{z} = \dfrac{1}{3!} = \dfrac{1}{6}$

Cauchy's Residue Theorem : If f(z) in single-valued and analytic in a closed curve ' C ', except at a finite number of poles within ' C, then $\oint_C f(z)dz = 2\pi i$ (sum of the residues at poles within ' C')

Exercise 1.20 Evaluate the integral: $\oint_C \frac{4-3z}{z(z-1)(z-3)} dz$ where $|z| = \frac{3}{2}$ ∎

Solution:

$$\hat{f}(z) = \frac{4-3z}{z(z-1)(z-3)} \Rightarrow \text{Poles : } z = 0, z = 1, z = 3$$

But, the given contour is circle centered at the origin and radius $3/2$ units.

Therefore, only $z = 0$ and $z = 1$ within the contour.

$$I = 2\pi i[\text{Re} s.f(z=0) + \text{Re} s.f(z=1)] = 2\pi i \left[\frac{4}{3} - \frac{1}{2} \right] = \frac{5\pi i}{3}$$

Exercise 1.21 Evaluate the integral: $\oint_C \frac{e^{2z}+z^2}{(z-1)^5} dz$ where $|z| = 2$ ∎

Solution:

$$f(z) = \frac{e^{2z}+z^2}{(z-1)^5} \Rightarrow \text{Poles } : z = 1(\text{ order } 5)$$

$$I = 2\pi i \times \text{Re} s.f(z=1) = 2\pi i \times \frac{1}{4!} \frac{d^4}{dz^4} \left[e^{2z} + z^2 \right]_{z=1} = 2\pi i \times \frac{2e^2}{3} = \frac{4\pi i e^2}{3}.$$

Definite Integrals of Trigonometric Functions of $\cos\theta$ **and** $\sin\theta$ **: (Integration round the unit circle)** :
Method: Consider the contour to be a circle centered at the origin and having radius one unit i.e. $|z| = 1$

Assume, $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \Rightarrow d\theta = \dfrac{dz}{iz}$

Therefore, $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} = \dfrac{1}{2}\left(z + \dfrac{1}{z} \right)$ and $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} = \dfrac{1}{2i}\left(z - \dfrac{1}{z} \right)$

and the limit will be changed from $0 \to 2\pi$ to $\oint_C$

The replacements regarding $\cos\theta$ and $\sin\theta$ is to be done only in the denominator of the given integral.

Exercise 1.22 Evaluate the integral: $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}; \quad a > b > 0$ ∎

Solution:

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{dz/iz}{a+b\left(\frac{z^2+1}{2z}\right)} = \int_0^{2\pi} \frac{2dz}{i\left(bz^2+2az+b\right)}$$

The singular points are at $\Rightarrow z = \alpha = \dfrac{-a+\sqrt{a^2-b^2}}{b} \quad \& \quad z = \beta = \dfrac{-a-\sqrt{a^2-b^2}}{b}$

The singular point $z = \beta$ will lie outside the unit circle as $a > b > 0$ while the singular point $z = \alpha$ will lie inside the unit circle which is a simple pole.

$$\text{Res. } f(z = \alpha) = \lim_{z \to \alpha}(z - \alpha)f(z) = \lim_{z \to \alpha}\frac{2}{ib}\frac{(z - \alpha)}{(z - \alpha)(z - \beta)} = \frac{2}{ib(\alpha - \beta)} = \frac{2}{ib} \times \frac{b}{2\sqrt{a^2 - b^2}} = \frac{1}{i\sqrt{a^2 - b^2}}$$

Therefore, by cauchy Residue theorem.

$$I = 2\pi i \times \text{ Residue } = 2\pi i \times \frac{1}{i\sqrt{a^2 - b^2}} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Evaluation of improper integrals between the limit $-\infty$ to $+\infty$:
Theorem I:

If $f(x)$ contain only polynomial terms
then $f(x) = f(x)$
→ find singular points
→ check point lie in upper half

A.   If singular point lie on imaginary axis

$$\int_{-\infty}^{\infty} f(z)dz = \int_{-\infty}^{\infty} f(x)dx = 2\pi i[\Sigma \text{ Res}] - \left(z \overset{\lim}{\to} \infty \quad zf(x)\right)x\pi i$$

B.   If singular point lie on real axis

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(z)dz = \pi i[\Sigma \text{ Res}] - \pi i\left[z \overset{\lim}{\to} \infty \quad zf(z)\right]$$

Theorem II:

If $f(x)$ contains sine and cosine function along with polynomial function rule is same, except second term which is 0 in this case.

A.   If singular point lie on imaginary axis

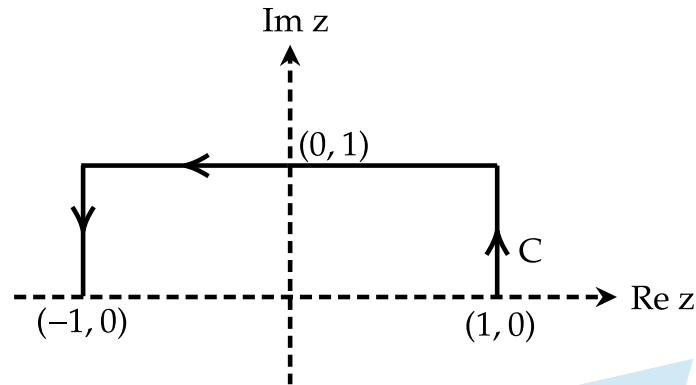$$\int_{-\infty}^{\infty} f(z)dz = \int_{-\infty}^{\infty} f(x)dx = 2\pi i[\Sigma \text{ Res}]$$

B.   If singular point lie on real axis

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(z)dz = \pi i[\Sigma \text{ Res}]$$

Practise Set-1

1. The value of the integral $\int_C dz z^2 e^z$, where C is an open contour in the complex z-plane as shown in the figure below, is:

A. $\frac{5}{e} + e$ **B.** $e - \frac{5}{e}$ **C.** $\frac{5}{e} - e$ **D.** $-\frac{5}{e} - e$

Solution:

If we complete the contour, then by Cauchy integral theorem

$$\int_{-1}^{1} dz z^2 e^z + \int_C dz z^2 e^z = 0 \Rightarrow \int_C dz z^2 e^z = -\int_{-1}^{1} dz z^2 e^z$$

$$= -\left[z^2 e^z - 2z e^z + 2e^z \right]_{-1}^{1} = \frac{5}{e} - e$$

So the correct answer is **Option (C)**

2. Which of the following is an analytic function of the complex variable $z = x + iy$ in the domain $|z| < 2$?

A. $(3 + x - iy)^7$ **B.** $(1 + x + iy)^4 (7 - x - iy)^3$

C. $(1 - x - iy)^4 (7 - x + iy)^3$ **D.** $(x + iy - 1)^{1/2}$

Solution: Put $z = x + iy$. If $\bar{z} = x - iy$ appears in any of the expressions then that expression is non-analytic. For option (D) we have a branch point singularity as the power is $\frac{1}{2}$ which is fractional. Hence only option (B) is analytic.

So the correct answer is **Option (B)**

3. The first few terms in the Laurent series for $\frac{1}{(z-1)(z-2)}$ in the region $1 \le |z| \le 2$ and around $z = 1$ is

A. $\frac{1}{2} \left[1 + z + z^2 + \ldots \right] \left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots \right]$

B. $\frac{1}{1-z} - z - (1-z)^2 + (1-z)^3 + \ldots$

C. $\frac{1}{z^2} \left[1 + \frac{1}{z} + \frac{1}{z^2} + \ldots \right] \left[1 + \frac{2}{z} + \frac{4}{z^2} + \ldots \right]$

D. $2(z - 1) + 5(z - 1)^2 + 7(z - 1)^3 + \ldots$

Solution:

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{1-z} + \frac{1}{(z-1)-1}$$

$$= \frac{1}{1-z} - (1+(1-z))^{-1}$$

$$= \frac{1}{1-z} - \left[1 + (1-z) + \frac{(-1)(-2)}{2!}(1-z)^2 + \frac{(-1)(-2)(-3)}{3!}(1-z)^3 \cdots\right]$$

$$= \frac{1}{1-z} - \left[z + (1-z)^2 - (1-z)^3 + \ldots\right]$$

So the correct answer is **Option (B)**

4. Let $u(x,y) = x + \frac{1}{2}(x^2 - y^2)$ be the real part of analytic function $f(z)$ of the complex variable $z = x + iy$. The imaginary part of $f(z)$ is

A. $y + xy$ **B.** xy **C.** y **D.** $y^2 - x^2$

Solution:

$$u(x,y) = x + \frac{1}{2}(x^2 - y^2), v(x,y) = ?$$

$$\text{Check } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial y} = 1 + x,$$

$$v = y + xy + f(x)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial v}{\partial x} = +y,$$

$$v = yx + f(y)$$

$$y + xy + f(x) = yx + f(y)$$

$$\text{If } f(x) = 0 \qquad f(y) = y$$

$$v = xy + y$$

So the correct answer is **Option (A)**

5. The value of the integral $\int_C \frac{z^3 dz}{(z^2 - 5z + 6)}$, where C is a closed contour defined by the equation $2|z| - 5 = 0$, traversed in the anti-clockwise direction, is

A. $-16\pi i$ **B.** $16\pi i$ **C.** $8\pi i$ **D.** $2\pi i$

Solution:

$$z^2 - 5z + 6 = 0 \Rightarrow z^2 - 2z - 3z + 6$$

$$= 0 \Rightarrow z(z-2) - 3(z-2) = 0 \Rightarrow z = 3, 2$$

$$2|z| = 5 \Rightarrow |z| = 2.5, \text{ only 2 will be inside.}$$

$$\text{Residue} = (z-2)\frac{z^3}{(z-3)(z-2)}\Big|_{z=2} = \frac{8}{2-3}$$

$$= -8 \Rightarrow \int \frac{z^3 dz}{z^2 - 5z + 6} = 2\pi i(-8) = -16\pi i$$

So the correct answer is **Option (A)**

6. With $z = x + iy$, which of the following functions $f(x,y)$ is NOT a (complex) analytic function of z ?

   [NET/JRF(JUNE-2013)]

   A. $f(x,y) = (x + iy - 8)^3 \left(4 + x^2 - y^2 + 2ixy\right)^7$
   B. $f(x,y) = (x + iy)^7 (1 - x - iy)^3$
   C. $f(x,y) = \left(x^2 - y^2 + 2ixy - 3\right)^5$
   D. $f(x,y) = (1 - x + iy)^4 (2 + x + iy)^6$

   Solution:

   $$f(x,y) = (1 - x + iy)^4 (2 + x + iy)^6$$
   $$= \{1 - (x - iy)\}^4 (2 + x + iy)^6$$
   Due to present of $\bar{z} = (x - iy)$

   So the correct answer is **Option (D)**

7. Which of the following functions cannot be the real part of a complex analytic function of $z = x + iy$?

   [NET/JRF(DEC-2013)]

   A. $x^2 y$ **B.** $x^2 - y^2$ **C.** $x^3 - 3xy^2$ **D.** $3x^2 y - y - y^3$

   Solution:

   Let $x^2 y$ be real part of a complex function. Use Milne Thomson's method to write analytic complex function. The real part of that function should be (1) but that is not the case. So this cannot be real part of an analytic function. Also,

   $$z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy, \text{ Real part option (2)}$$
   $$z^3 = (x + iy)^3 = x^3 - iy^3 + 3ixy(x + iy)$$
   $$= x^3 - iy^3 + 3ix^2 y - 3xy^2, \text{ Real part option (3)}$$

   So the correct answer is **Option (A)**

8. Given that the integral $\int_0^\infty \frac{dx}{y^2 + x^2} = \frac{\pi}{2y}$, the value of $\int_0^\infty \frac{dx}{(y^2 + x^2)^2}$ is

   [NET/JRF(DEC-2013)]

   A. $\frac{\pi}{y^3}$ **B.** $\frac{\pi}{4y^3}$ **C.** $\frac{\pi}{8y^3}$ **D.** $\frac{\pi}{2y^3}$

   Solution:

   $$\int_0^\infty \frac{dx}{(y^2 + x^2)^2} = \frac{1}{2} \int_{-\infty}^\infty \frac{dx}{(y^2 + x^2)^2}, \text{ pole is of 2}^{\text{nd}} \text{ order at } x = iy, \text{ residue } = 1/\left(4iy^3\right)$$
   $$\text{Integral } = \left(\frac{1}{2}\right)(2\pi i)\frac{1}{4iy^3} = \frac{\pi}{(4y^3)}$$

9. If C is the contour defined by $|z| = \frac{1}{2}$, the value of the integral

   $$\oint_C \frac{dz}{\sin^2 z}$$

   is

   [NET/JRF(JUNE-2014)]

A. ∞　　　　　　　**B.** $2\pi i$　　　　　　　**C.** 0　　　　　　　**D.** πi

Solution:

$$f(z) = \frac{1}{\sin^2 z} \quad \left(|z| = \frac{1}{2}\right)$$

$$\sin z = z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} \ldots \Rightarrow \frac{1}{\sin^2 z} = \frac{1}{\left(z - \frac{z^3}{3} + \frac{z^5}{5} \cdots\right)^2}$$

$$\Rightarrow \frac{1}{\sin^2 z} = \frac{1}{z^2}\left[1 - \frac{z^2}{\lfloor 3} + \frac{z^4}{\lfloor 5} \ldots\right]^{-2} \Rightarrow \oint_C \frac{dz}{\sin^2 z} = 0$$

So the correct answer is **Option (C)**

10. The principal value of the integral $\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^3} dx$ is

A. -2π　　　　　　　**B.** $-\pi$　　　　　　　**C.** π　　　　　　　**D.** 2π

Solution:

$$\text{Let } f(z) = \frac{e^{i2z}}{z^3}$$

$$\lim_{2 \to 0}(z-0)^3 f(z) = \lim_{z \to 0}(z-0)^3 \frac{e^{i2z}}{z^3}$$

$$= 1(\text{ finite and } \neq 0) \Rightarrow z = 0 \text{is pole of order 3.}$$

$$\text{Residue } R = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2}\left[(z-0)^3 \frac{e^{i2z}}{z^3}\right] = -2$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)dx = \pi i \Sigma R = \pi i(-2) = -2\pi i \Rightarrow \text{Im. Part}$$

$$= -2\pi \Rightarrow \int_{-\infty}^{\infty} f(x)dx = -2\pi$$

So the correct answer is **Option (A)**

11. The Laurent series expansion of the function $f(z) = e^2 + e^{1/2}$ about $z = 0$ is given by

A. $\sum_{n=-\infty}^{\infty} \frac{z^n}{n!}$ for all $|z| < \infty$　　　　　　　**B.** $\sum_{n=0}^{\infty} \left(z^n + \frac{1}{z^n}\right) \frac{1}{n!}$ only if $0 < |z| < 1$

C. $\sum_{n=0}^{\infty} \left(z^n + \frac{1}{z^n}\right) \frac{1}{n!}$ for all $0 < |z| < \infty$　　　　　　　**D.** $\sum_{n=-\infty}^{\infty} \frac{z^n}{n!}$ only if $|z| < 1$

Solution:

$$e^z = \left(1 + z + \frac{z^2}{2!} + \ldots\right) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \text{ and } e^{1/z}$$

$$= 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \ldots = \sum_{n=0}^{\infty} \frac{1}{z^n n!}$$

$$\Rightarrow f(z) = \left(e^z + e^{1/2}\right) = \sum_{n=0}^{\infty} \left(z^n + \frac{1}{z^n}\right) \frac{1}{n!}, \text{ for all } 0 < |z| < \infty$$

So the correct answer is **Option (C)**

12. Consider the function $f(z) = \frac{1}{z}\ln(1-z)$ of a complex variable $z = re^{i\theta}(r \geq 0, \quad -\infty < \theta < \infty)$. The singularities of $f(z)$ are as follows:

                                                                             [NET/JRF(DEC-2014)]

   A. Branch points at $z = 1$ and $z = \infty$; and a pole at $z = 0$ only for $0 \leq \theta < 2\pi$

   B. Branch points at $z = 1$ and $z = \infty$; and a pole at $z = 0$ for all θ other than $0 \leq \theta < 2\pi$

   C. Branch points at $z = 1$ and $z = \infty$; and a pole at $z = 0$ for all θ

   D. Branch points at $z = 0, z = 1$ and $z = \infty$.

Solution:

$$\text{For } f(z) = \frac{1}{z}\ln(1-z) = \frac{1}{z}\left(-z - \frac{z^2}{2} - \frac{z^3}{3} - \ldots\right)$$

$$= -1 - \frac{z}{2} - \frac{z^2}{3} - \ldots$$

There is no principal part and when $z \to 0, f(z) = -1$. So there is removable singularity at $z = 0$. Also $z = 1$ and $z = \infty$ is Branch point.

None of the above is correct

13. The value of integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$

                                                                                [NET/JRF(JUNE-2015)]

   A. $\frac{\pi}{\sqrt{2}}$ **B.** $\frac{\pi}{2}$ **C.** $\sqrt{2}\pi$ **D.** 2π

Solution:

$$\int_{-\infty}^{\infty} \frac{dz}{1+z^4} \quad \because |z| = R$$

$$\text{Now, pole } z = e^{(2n+1)\frac{\pi}{4}}$$

$$n = 0, \quad \Rightarrow z_0 = e^{\frac{i\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, n$$

$$= 2 \Rightarrow z_2 = \frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

$$n = 1 \Rightarrow z_1 = e^{\frac{i3\pi}{4}} = \frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, n$$

$$= 3 \Rightarrow z_3 = +\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

only z_0 and z_1 lies in contour

$$\text{i.e., residue at } \left(z = e^{\frac{i\pi}{4}}\right) = \frac{1}{4}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$\text{residue at } \left(z = e^{\frac{i3\pi}{4}}\right) = \frac{1}{4}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$\text{now } \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\pi i \Sigma \text{Re} S = \frac{\pi}{\sqrt{2}}$$

So the correct answer is **Option (A)**

14. The function $\frac{z}{\sin \pi z^2}$ of a complex variable z has

A. A simple pole at 0 and poles of order 2 at $\pm\sqrt{n}$ for $n = 1, 2, 3 \ldots$

B. A simple pole at 0 and poles of order 2 at $\pm\sqrt{n}$ and $\pm i\sqrt{n}$ for $n = 1, 2, 3 \ldots$

C. Poles of order 2 at $\pm\sqrt{n}, n = 0, 1, 2, 3 \ldots$

D. Poles of order 2 at $\pm n, n = 0, 1, 2, 3 \ldots$

Solution:

$$f(z) = \frac{z}{\sin \pi z^2} = \frac{z}{\pi z^2 \frac{\sin \pi z^2}{\pi z^2}}$$

at $z = 0$, it is a simple pole since, $\lim_{z \to 0} \frac{\sin \pi z^2}{\pi z^2} = 1$

Also, $\sin \pi z^2 = \sin n\pi \Rightarrow \pi z^2$
$$= \pm n\pi, z = \pm\sqrt{n}, \pm i\sqrt{n}$$
$$\lim_{z \to \sqrt{n}} (z - \sqrt{n})^2 \cdot \frac{z}{\sin \pi z^2}, \text{exists. So its pole of order 2}$$

So the correct answer is **Option (B)**

15. The value of the contour integral $\frac{1}{2\pi i} \oint_C \frac{e^{4z}-1}{\cosh(z)-2\sinh(z)} dz$ around the unit circle C traversed in the anti-clockwise direction, is

A. 0   **B.** 2   **C.** $\frac{-8}{\sqrt{3}}$ **D.** $-\tanh\left(\frac{1}{2}\right)$

Solution:

$$f(z) = \frac{e^{4z}-1}{\cosh z - 2\sinh z} = \frac{e^{4z}-1}{\frac{e^z+e^{-z}}{2} - (e^z - e^{-z})}$$

$$= \frac{e^{42}-1}{-\frac{e^z}{2} + \frac{3}{2}e^{-z}}$$

$$\Rightarrow f(z) = \frac{2e^2 \left(e^{4z}-1\right)}{(3 - e^{2z})} = \frac{2\left(e^{5z} - e^z\right)}{(3 - e^{2z})}$$

For pole at $z = z_0, 3 - e^{2\xi_0} = 0 \Rightarrow e^{2z_0}$

$$= 3 \Rightarrow z_0 = \frac{\ln 3}{2}$$

It has simple pole at z_0

$$\text{Re}(z_0) = \lim_{z \to z_0} (z - z_0) f(z) = \lim_{2 \to z_0} (z - z_0) \frac{2\left(e^{5z} - e^2\right)}{3 - e^{22}}$$

$$= \lim_{z \to z_0} \frac{(z - z_0) \times 2\left(5e^{5z} - e^z\right) + 2\left(e^{5z} - e^z\right) \times 1}{-2e^{2z}}$$

$$= -\left(\frac{e^{5z_0} - e^{z_0}}{e^{2z_0}}\right)$$

$$= -\left(\frac{(\sqrt{3})^5 - \sqrt{3}}{3}\right) = -\left(\frac{9\sqrt{3} - \sqrt{3}}{3}\right) = -\frac{8}{\sqrt{3}}$$

$$\frac{1}{2\pi i}\oint f(z)dz = \frac{1}{2\pi i} \times 2\pi i \sum \text{Residue} = -\frac{8}{\sqrt{3}}$$

So the correct answer is **Option (C)**

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16. Let $u(x,y) = e^{ax}\cos(by)$ be the real part of a function $f(z) = u(x,y)+iv(x,y)$ of the complex variable $z = x+iy$, where a,b are real constants and $a \neq 0$. The function $f(z)$ is complex analytic everywhere in the complex plane if and only if

<div align="right">

[NET/JRF(JUNE-2017)]
</div>

    A. $b=0$ **B.** $b=\pm a$ **C.** $b=\pm 2\pi a$ **D.** $b=a\pm 2\pi$

Solution:

The function $f(z)$ will be analytic everywhere in the complex plane if and only if it satisfies the Cauchy Riemann equation in that region.

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\text{Hence } ae^{ax}\cos(by) = \frac{\partial v}{\partial y} \qquad (i)$$

$$\text{and } be^{ax}\sin(by) = \frac{\partial v}{\partial x} \qquad (ii)$$

From equation (i)

$$v(x,y) = \frac{ae^{ax}\sin(by)}{b} + c(y) \qquad (iii)$$

Differentiating partially with x gives

$$\frac{\partial v}{\partial x} = \frac{a^2 e^{ax}\sin(by)}{b} \qquad (iv)$$

From equation (iii) and (iv)

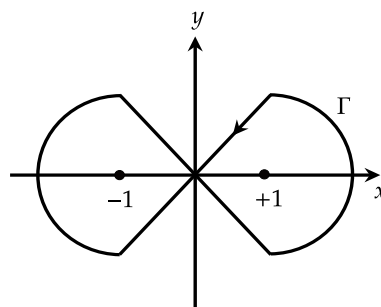$$be^{ax}\sin(by) = \frac{a^2 e^{ax}\sin(by)}{b}$$
$$\Rightarrow b^2 = a^2 \Rightarrow b = \pm a$$

So the correct answer is **Option (B)**

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17. The integral $\oint_\Gamma \frac{ze^{i\pi z/2}}{z^2-1}dz$ along the closed contour Γ shown in the figure is

<div align="right">

[NET/JRF(JUNE-2017)]
</div>

    A. 0 **B.** 2π **C.** -2π **D.** $4\pi i$

Solution:

$$f(z) = \frac{ze^{iz\pi/2}}{(z+1)(z-1)}$$

For $z = +1$ anti-clockwise

$$I = 2\pi i \lim_{z \to 1} \frac{ze^{i\pi z/2}}{(z+1)} = \frac{2\pi i}{2} e^{i\pi/2} = \pi i e^{i\pi/2}$$

For $z = -1$

$$I = -2\pi i \lim_{z \to -1} \frac{ze^{i\pi z/2}}{(z-1)} = -2\pi i \times \frac{(-1)e^{-i\pi/2}}{(-2)} = -\pi i e^{-i\pi/2}$$

$$\text{Integral } = \pi i \frac{\left(e^{i\pi/2} - e^{-i\pi/2}\right)}{2i} \times 2i = 2\pi i^2 \sin\frac{\pi}{2} = -2\pi$$

So the correct answer is **Option (C)**

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18. What is the value of a for which $f(x,y) = 2x + 3\left(x^2 - y^2\right) + 2i(3xy + ay)$ is an analytic function of complex variable $z = x + iy$

    A. 1   **B.** 0   **C.** 3   **D.** 2

Solution:

$$f(x,y) = 2x + 3\left(x^2 - y^2\right) + 2i(3xy + \alpha y)$$
$$u = 2x + 3\left(x^2 - y^2\right), v = 2(3xy + \alpha y)$$

C-R conditions: $u_x = v_y, u_y = -v_x$

$$2 + 3(2x) = 2(3x + \alpha) \Rightarrow \alpha = 1 \Rightarrow -6y = -6y$$

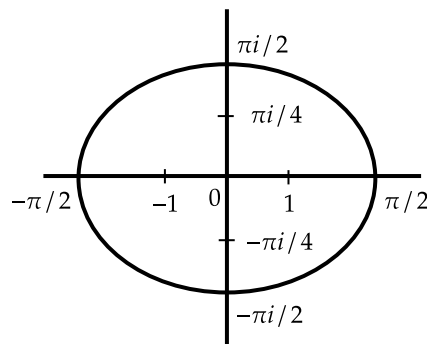So the correct answer is **Option (A)**

---

19. The value of the integral $\oint_C \frac{dz}{z} \frac{\tanh 2z}{\sin \pi z}$, where C is a circle of radius $\frac{\pi}{2}$, traversed counter-clockwise, with centre at $z = 0$, is

    A. 4   **B.** $4i$ **C.** $2i$ **D.** 0

Solution:

$$\oint_C \frac{dz}{z} \frac{\tanh 2z}{\sin \pi z} dz$$

$$z = 0, 1, -1, \frac{\pi i}{4}, \frac{-\pi i}{4}$$

$$f(z) = \frac{2z - \frac{1}{3}(2z)^3 + \frac{2}{15}(2z)^5 \ldots}{z\left(\pi z - \frac{\pi^3 z^3}{3!} + \ldots\right)}$$

$$\frac{2}{\pi z}\left(1 - \frac{1}{2}z^2 + \ldots\right)\left(1 - \frac{\pi^2 z^2}{2!} + \ldots\right)$$

$$b_1 = \frac{2}{\pi}$$

As $\operatorname{Re} z = 1, \dfrac{\tanh^2}{-\pi}$ and $\operatorname{Re} z = -1, \dfrac{\tanh^2}{-\pi}$

$$\operatorname{Re} z = \frac{i\pi}{4} = -\frac{1}{\pi}\left(2\operatorname{cosec} h\frac{\pi^2}{4}\right)$$

$$\operatorname{Re} z = \frac{-i\pi}{4} = -\frac{1}{\pi}\left(2\operatorname{cosec} h\frac{\pi^2}{4}\right)$$

$I = 2\pi i \Sigma R = 4i$ only when 0 lies inside, otherwise wrong question.

So the correct answer is **Option (B)**

20. The integral $I = \int_C e^z dz$ is evaluated from the point $(-1, 0)$ to $(1, 0)$ along the contour C, which is an arc of the parabola $y = x^2 - 1$, as shown in the figure. The value of I is
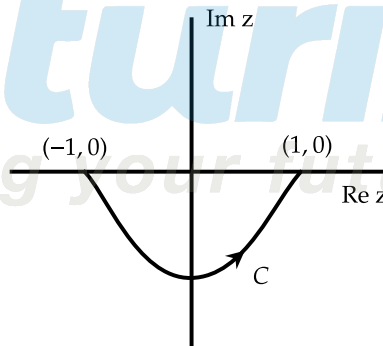
A. 0   **B.** $2\sinh 1$ **C.** $e^{2i}\sinh 1$ **D.** $e + e^{-1}$

Solution:

$$\int_C f(z)dz = 2\pi i \Sigma R$$

$$\int_C f(z)dz + \int_1^{-1} e^x dx = 0$$

$$\int_C f(z)dz = -\int_1^{-1} e^x dx = \int_1^{-1} e^x dx$$

$$= \frac{(e^1 - e^{-1})}{2} \cdot 2 = 2\sinh 1$$

So the correct answer is **Option (B)**

21. The contour C of the following integral

$$\oint_C dz \frac{\sqrt{(z-1)(z-3)}}{(z^2-25)^3}$$

in the complex z plane is shown in the figure below.
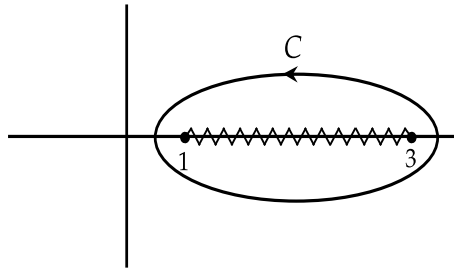
This integral is equivalent to an integral along the contours

A.

B.

C.

D.
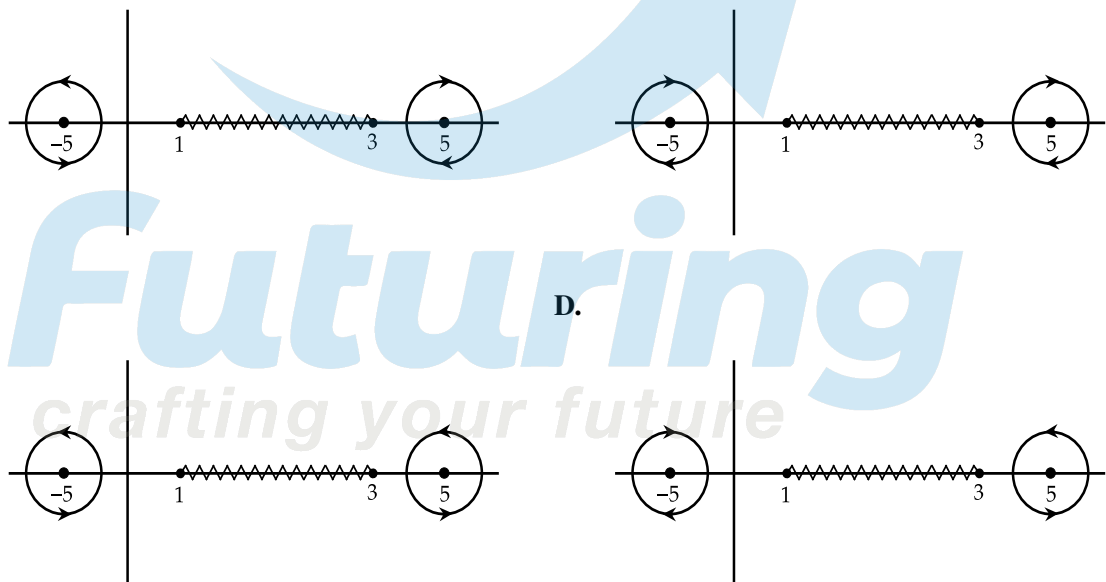
Solution:

$z = 1, 3$ are branch points ∞ is not a branch point 1 branch cut 3

So the correct answer is **Option (C)**

22. Let C be the circle of radius $\frac{\pi}{4}$ centered at $z = \frac{1}{4}$ in the complex z-plane that is traversed counter-clockwise. The value of the contour integral $\oint_C \frac{z^2}{\sin^2 4z} dz$ is

    A. 0   **B.** $\frac{i\pi^2}{4}$ **C.** $\frac{i\pi^2}{16}$ **D.** $\frac{i\pi}{4}$

Solution:
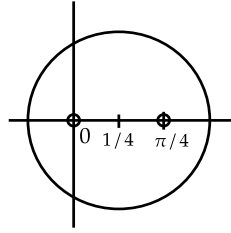
$$f(z) = \left(\frac{\pi}{\sin 4z}\right)^2$$

$$z_0 = 0, \frac{\pi}{4} \text{ are poles}$$

$$4z = n\pi, z = 0, \frac{\pi}{4}$$

Others are outside the contour.

$$\text{Residue at } z = 0 \text{ is } \left[\frac{\pi}{4z - \frac{4^3 z^3}{3!} + \ldots}\right]^2$$

$$= \left[\frac{1}{4 - \frac{4^3 z^2}{3!} + \ldots}\right]^2 \qquad \text{No terms for } \frac{1}{z}, b_1 = 0$$

$$= \left[4 - \frac{4^3 z^2}{3!} + \ldots\right]^{-2}$$

$$\text{Residue for } z = \frac{\pi}{4}$$

$$z - \frac{\pi}{4} = t$$

$\sin(4t + \pi) = -\sin 4t$ \quad (But square so no effect)

$$\left[\frac{t + \frac{\pi}{4}}{\sin 4\left(t + \frac{\pi}{4}\right)}\right]^2$$

$$\left(\frac{t + \frac{\pi}{4}}{\sin 4t}\right)^2 = \frac{t^2 + \frac{\pi^2}{4} + 2t \cdot \frac{\pi}{4}}{\sin^2 4t}$$

$$\frac{\pi}{2} \frac{t}{16t^2[1 - \ldots]^2} = \frac{\pi}{32t}[1 - \ldots]^{-2}\text{(from first term)}$$

$$b_1 = \frac{\pi}{32}$$

$$\oint_C \frac{z^2}{\sin^2 4z} dz = 2\pi i \left[0 + \frac{\pi}{32}\right] = \frac{i\pi^2}{16}$$

So the correct answer is **Option (C)**

23. A function of a complex variable z is defined by the integral $f(z) = \oint_\Gamma \frac{w^2 - 2}{w - z} dw$, where Γ is a circular contour of radius 3, centred at origin, running counter-clockwise in the w - plane. The value of the function at $z = (2 - i)$ is

   A. 0   **B.** $1 - 4i$ **C.** $8\pi + 2\pi i$ **D.** $-\frac{2}{\pi} - \frac{i}{2\pi}$

Solution:
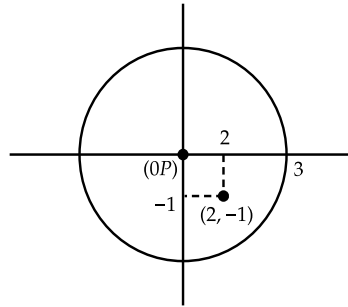
$$f(z) = \oint_\Gamma \frac{w^2 - 2}{w - z} dw$$

$\omega = z$ is a simple pole.

$$\text{Residue } \lim_{\omega \to z} (\omega - z) \frac{(\omega^2 - 2)}{(\omega - z)} = (2 - i)^2 - 2$$

$$= 4 - 1 - 4i - 2 = (1 - 4i)$$

$$f(z) = \oint_\Gamma \frac{w^2 - 2}{w - z} dw = 2\pi i(1 - 4i) = 2\pi i + 8\pi$$

So the correct answer is **Option (C)**

Answer key			
Q.No.	Answer	Q.No.	Answer
1	**C**	2	**B**
3	**B**	4	**A**
5	**A**	6	**D**
7	**A**	8	-
9	**C**	10	**A**
11	**C**	12	-
13	**A**	14	**B**
15	**C**	16	**B**
17	**C**	18	**A**
19	**B**	20	**B**
21	**C**	22	**C**
23	**C**		

Practise Set-2

1. The value of the integral $\oint_C \frac{e^z \sin(z)}{z^2} dz$, where the contour C is the unit circle: $|z-2|=1$, is

    A. $2\pi i$ **B.** $4\pi i$ **C.** πi **D.** 0

2. Which of the following statements is TRUE for the function $f(z) = \frac{z\sin z}{(z-\pi)^2}$?

    A. $f(z)$ is analytic everywhere in the complex plane

    B. $f(z)$ has a zero at $z = \pi$

    C. $f(z)$ has a pole of order 2 at $z = \pi$

    D. $f(z)$ has a simple pole at $z = \pi$

3. For the function $f(z) = \frac{16z}{(z+3)(z-1)^2}$, the residue at the pole $z = 1$ is (your answer should be an integer)———-

4. The value of the integral

$$\oint_C \frac{z^2}{e^z+1} dz$$

    where C is the circle $|z| = 4$, is

    A. $2\pi i$ **B.** $2\pi^2 i$ **C.** $4\pi^3 i$ **D.** $4\pi^2 i$

5. Consider a complex function $f(z) = \frac{1}{z\left(z+\frac{1}{2}\right)\cos(z\pi)}$. Which one of the following statements is correct?

    A. $f(z)$ has simple poles at $z = 0$ and $z = -\frac{1}{2}$

    B. $f(z)$ has second order pole at $z = -\frac{1}{2}$

    C. $f(z)$ has infinite number of second order poles

    D. $f(z)$ has all simple poles

6. Consider $w = f(z) = u(x,y) + iv(x,y)$ to be an analytic function in a domain D. Which one of the following options is NOT correct?

    A. $u(x,y)$ satisfies Laplace equation in D

    B. $v(x,y)$ satisfies Laplace equation in D

    C. $\int_1^{z_2} f(z)dz$ is dependent on the choice of the contour between z_1 and z_2 in D

    D. $f(z)$ can be Taylor expended in D

7. A function $y(z)$ satisfies the ordinary differential equation $y'' + \frac{1}{z}y' - \frac{m^2}{z^2}y = 0$, where $m = 0,1,2,3,\ldots$. Consider the four statements P, Q, R, S as given below.
P : z^m and z^{-m} are linearly independent solutions for all values of m
Q: z^m and z^{-m} are linearly independent solutions for all values of $m > 0$
R : $\ln z$ and 1 are linearly independent solutions for $m = 0$
S: z^m and $\ln z$ are linearly independent solutions for all values of m
The correct option for the combination of valid statements is

     A. P, R and S only   **B.** P and R only   **C.** Q and R only   **D.** R and S only

8. Which of the following is an analytic function of z everywhere in the complex plane?

                                                                                [GATE 2016]

     A. z^2 **B.** $(z^*)^2$ **C.** $|z|^2$ **D.** $\sqrt{z}$

9. The contour integral $\oint \frac{dz}{1+z^2}$ evaluated along a contour going from $-\infty$ to $+\infty$ along the real axis and closed in the lower half-plane circle is equal to................ (up to two decimal places).

                                                                                 [GATE 2017]

10. The imaginary part of an analytic complex function is $v(x,y) = 2xy + 3y$. The real part of the function is zero at the origin. The value of the real part of the function at $1 + i$ is .... (up to two decimal places)

                                                                                 [GATE 2017]

11. The absolute value of the integral

$$\int \frac{5z^3 + 3z^2}{z^2 - 4} dz$$

   over the circle $|z - 1.5| = 1$ in complex plane, is ... (up to two decimal places).

                                                                                 [GATE 2018]

12. The pole of the function $f(z) = \cot z$ at $z = 0$ is

                                                                                 [GATE 2019]

     A. A removable pole   **B.** An essential singularity

     C. A simple pole   **D.** A second order pole

13. The value of the integral $\int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2+a^2} dx$, where $k > 0$ and $a > 0$, is

                                                                                 [GATE 2019]

     A. $\frac{\pi}{a} e^{-ka}$ **B.** $\frac{2\pi}{a} e^{-ka}$ **C.** $\frac{\pi}{2a} e^{-ka}$ **D.** $\frac{3\pi}{2a} e^{-ka}$

14. The value of the integral $\int_0^{\infty} \frac{\ln x}{(x^2+1)^2} dx$ is

                                                                                 [JEST 2012]

     a. 0   **b.** $\frac{-\pi}{4}$

     c. $\frac{-\pi}{2}$ **d.** $\frac{\pi}{2}$

15. Compute $\lim_{z \to 0} \frac{\text{Re}(z^2) + \text{Im}(z^2)}{z^2}$

                                                                               [JEST 2013]

     a. The limit does not exist   **b.** 1

     c. $-i$ **d.** -1

16. The value of limit

$$\lim_{z \to i} \frac{z^{10} + 1}{z^6 + 1}$$

   is equal to

                                                                                [JEST 2014]

     a. 1   **b.** 0

     c. $\frac{-10}{3}$ **d.** $\frac{5}{3}$

17. The value of integral

$$I = \oint \frac{\sin z}{2z - \pi} dz$$

with c a circle $|z| = 2$, is

    a. 0   **b.** $2\pi i$

    c. πi **d.** $-\pi i$

18. Given an analytic function $f(z) = \phi(x,y) + i\psi(x,y)$, where $\phi(x,y) = x^2 + 4x - y^2 + 2y$ If C is a constant, which of the following relations is true?

    a. $\psi(x,y) = x^2 y + 4y + C$ **b.** $\psi(x,y) = 2xy - 2x + C$

    c. $\psi(x,y) = 2xy + 4y - 2x + C$ **d.** $\psi(x,y) = x^2 y - 2x + C$

19. Which one is the image of the complex domain $\{z \mid xy \geq 1, x + y > 0\}$ under the mapping $f(z) = z^2$, if $z = x + iy$?

    a. $\{z \mid xy \geq 1, x + y > 0\}$ **b.** $\{z \mid x \geq 2, x + y > 0\}$

    c. $\{z \mid y \geq 2 \forall x\}$ **d.** $\{z \mid y \geq 1 \forall x\}$

20. The integral $I = \int_1^\infty \frac{\sqrt{x-1}}{(1+x)^2} dx$ is

    a. $\frac{\pi}{\sqrt{2}}$ **b.** $\frac{\pi}{2\sqrt{2}}$

    c. $\frac{\sqrt{\pi}}{2}$ **d.** $\sqrt{\frac{\pi}{2}}$

21. The integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} dx \text{ is}$$

    a. $\frac{\pi}{e}$ **b.** πe^{-2}

    c. π **d.** zero

22. Consider the function $f(x,y) = |x| - i|y|$. In which domain of the complex plane is this function analytic?

    a. First and second quadrants   **b.** Second and third quadrants

    c. Second and fourth quadrants   **d.** Nowhere

Answer key			
Q.No.	Answer	Q.No.	Answer
1	**D**	2	**C**
3	**3(NAT)**	4	**C**
5	**A**	6	**C**
7	**C**	8	**A**
9	**π(NAT)**	10	**3(NAT)**
11	**81.64(NAT)**	12	**C**
13	**A**	14	**B**
15	**A**	16	**D**
17	**C**	18	**C**
19	**-**	20	**B**
21	**A**	22	**C**