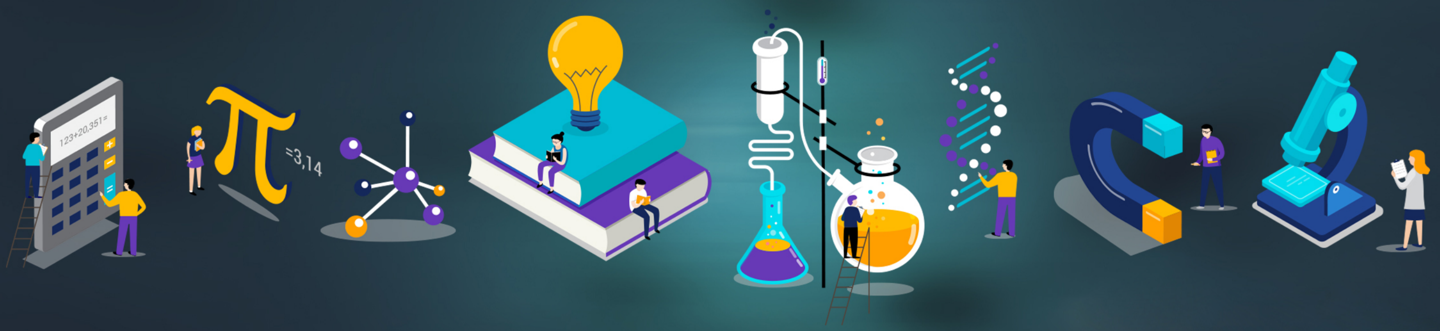

1. Schrodinger Equation - Solutions

Practice Set-1

1. Let v, p and E denote the speed, the magnitude of the momentum, and the energy of a free particle of rest mass m. Then

[NET DEC 2012]

A. $dE/dp = \text{constant}$ **B.** $p = mv$

C. $v = cp/\sqrt{p^2 + m^2c^2}$ **D.** $E = mc^2$

Solution:

$$p = m'v = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow p^2 = \frac{m^2v^2}{1-\frac{v^2}{c^2}} \Rightarrow m^2v^2 = p^2 - \frac{p^2v^2}{c^2}, m \rightarrow \text{ rest mass energy}$$

$$\Rightarrow v^2\left(m^2 + \frac{p^2}{c^2}\right) = p^2 \Rightarrow v^2 = \frac{p^2}{\frac{m^2c^2+p^2}{c^2}} \Rightarrow v = \frac{pc}{\sqrt{p^2+m^2c^2}}$$

The correct option is **(c)**

2. If a particle is represented by the normalized wave function

$$\psi(x) = \begin{cases} \frac{\sqrt{15}\left(a^2-x^2\right)}{4a^{5/2}}, & \text{, or } -a < x < a \\ 0 & \text{, otherwise} \end{cases}$$

the uncertainty Δp in its momentum is

[NET DEC 2012]

A. $2\hbar/5a$ **B.** $5\hbar/2a$

C. $\sqrt{10}\hbar/a$ **D.** $\sqrt{5}\hbar/\sqrt{2}a$

Solution:

$$\Delta p=\sqrt{\langle p^2\rangle-\langle p\rangle^2} \text{ and } \langle p\rangle=\frac{\left\langle\psi\left|-i\hbar\frac{\partial}{\partial x}\right|\psi\right\rangle}{\langle\psi\mid\psi\rangle}$$

$$\Rightarrow\langle p\rangle=\int_{-a}^{a}\frac{\sqrt{15}\left(a^2-x^2\right)}{4a^{5/2}}(-i\hbar)\frac{\sqrt{15}}{4a^{5/2}}\frac{\partial}{\partial x}\left(a^2-x^2\right)dx$$

$$=\int_{-a}^{a}\frac{15}{16a^5}(-i\hbar)\left(a^2-x^2\right)(-2x)dx=+i\hbar\frac{2\times 15}{16\times a^5}\int_{-a}^{a}\left(a^2x-x^3\right)dx=0, \quad (\because \text{ odd function})$$

$$\left\langle p^2\right\rangle=-\hbar^2\times\frac{15}{16a^5}\int_{-a}^{a}\left(a^2-x^2\right)\frac{\partial^2}{\partial x^2}\left(a^2-x^2\right)dx$$

$$=-\hbar^2\times\frac{15}{16a^5}\times(-2)\int_{-a}^{a}\left(a^2-x^2\right)dx=\hbar^2\times\frac{15}{16a^5}\times 2\left\{a^2\cdot x-\frac{x^3}{3}\right\}_{-a}^{a}$$

$$=\hbar^2\times\frac{15}{16a^5}\times 2\left[2a^3-\frac{2a^3}{3}\right]=\hbar^2\times\frac{15}{16}\times\frac{2}{a^5}\times 2a^3\left[1-\frac{1}{3}\right]=\frac{15\hbar^2}{4a^2}\times\frac{2}{3}=\frac{5\hbar^2}{2a^2}$$

Now, $\Delta p=\sqrt{\langle p^2\rangle-\langle p\rangle^2}=\sqrt{\frac{5\hbar^2}{2a^2}-0}=\frac{\sqrt{5}\hbar}{\sqrt{2}a}$

The correct option is **(d)**

3. The energies in the ground state and first excited state of a particle of mass $m=\frac{1}{2}$ in a potential $V(x)$ are -4 and -1, respectively, (in units in which $\hbar=1$). If the corresponding wavefunctions are related by $\psi_1(x)=\psi_0(x)\sinh x$, then the ground state eigenfunction is

[NET DEC 2012]

A. $\psi_0(x)=\sqrt{\sec hx}$ **B.** $\psi_0(x)=\sec hx$

C. $\psi_0(x)=\operatorname{sech}^2 x$ **D.** $\psi_0(x)=\sec h^3 x$

Solution: Given that ground state energy $E_0=-4$, first excited state energy $E_1=-1$ and ψ_0,ψ_1 are corresponding wave functions.

Solving Schrödinger equation (use $m=\frac{1}{2}$ and $\hbar=1$)

Solving Schrödinger equation (use $m=\frac{1}{2}$ and $\hbar=1$)

$\frac{-\hbar^2}{2m}\frac{\partial^2\psi_0}{\partial x^2}+V\psi_0=E_0\psi_0 \quad \Rightarrow -\frac{\partial^2\psi_0}{\partial x^2}+V\psi_0=-4\psi_0\ldots\ldots(1)$

$\frac{-\hbar^2}{2m}\frac{\partial^2\psi_1}{\partial x^2}+V\psi_1=E_1\psi_1 \quad \Rightarrow -\frac{\partial^2\psi_1}{\partial x^2}+V\psi_1=-1\psi_1\ldots\ldots\ldots(2)$

Put $\psi_1=\psi_0\sinh x$ in equation (2) one will get

$-\left[\frac{\partial^2\psi_0}{\partial x^2}\cdot\sinh x+2\frac{\partial\psi_0}{\partial x}\cosh x+\psi_0\sinh x\right]+V\psi_0\sinh x=-\psi_0\sinh x$

$-\left[\frac{\partial^2\psi_0}{\partial x^2}+2\frac{\partial\psi_0}{\partial x}\coth x+\psi_0\right]+V\psi_0=-\psi_0$

$\left[-\frac{\partial^2\psi_0}{\partial x^2}+V\psi_0\right]-2\frac{\partial\psi_0}{\partial x}\coth x-\psi_0=-\psi_0$

using relation $-\frac{\partial^2\psi_0}{\partial x^2}+V\psi_0=-4\psi_0$

$-4\psi_0-2\frac{\partial\psi_0}{\partial x}\coth x-\psi_0=-\psi_0 \quad \Rightarrow \frac{d\psi_0}{\psi_0}=-2\tanh x dx\Rightarrow\psi_0=\sec h^2 x$

The correct option is **(c)**

4. If $\psi(x)=A\exp\left(-x^4\right)$ is the eigenfunction of a one dimensional Hamiltonian with eigen value $E=0$, the potential $V(x)$ (in units where $\hbar=2m=1$) is

[NET DEC 2013]

A. $12x^2$ **B.** $16x^6$

C. $16x^6+12x^2$ **D.** $16x^6-12x^2$

Solution: Schrodinger equation $-\nabla^2\psi+V\psi=0$ (where $\hbar=2m=1$ and $E=0$)

$$-\frac{\partial^2}{\partial x^2}\left(Ae^{-x^4}\right)+VAe^{-x^4}=0\Rightarrow-\frac{\partial}{\partial x}\left[e^{-x^4}\times-4x^3\right]+Ve^{-x^4}=0$$

$$4\left[\left\{3x^2e^{-x^4}+x^3\left(-4x^3e^{-x^4}\right)\right\}\right]+Ve^{-x^4}=0\Rightarrow 12x^2e^{-x^4}-16x^6e^{-x^4}+Ve^{-x^4}=0$$

$\Rightarrow V=16x^6-12x^2$

The correct option is **(d)**

5. A particle of mass m moves in one dimension under the influence of the potential $V(x)=-\alpha\delta(x)$, where α is a positive constant. The uncertainty in the product $(\Delta x)(\Delta p)$ in its ground state is

[NET JUNE 2016]

A. $2\hbar$ **B.** $\frac{\hbar}{2}$

C. $\frac{\hbar}{\sqrt{2}}$ **D.** $\sqrt{2}\hbar$

Solution: $V(x)=-\alpha\delta(x)$
For this potential wavefunction

$$\psi(x)=\begin{cases}\sqrt{\alpha}e^{\alpha x}, & x<0\\ \sqrt{\alpha}e^{-\alpha x}, & x>0\end{cases}$$

which evenfunction about $x=0$ so $\langle x\rangle=0,\langle p\rangle=0$

$$\left\langle p^2\right\rangle=-\hbar^2\int_{-\infty}^{\infty}\psi^*\frac{d^2}{dx^2}\psi dx=-\hbar^2\int_{-\infty}^{0}\sqrt{\alpha}e^{\alpha x}\frac{d^2}{dx^2}\sqrt{\alpha}e^{\alpha x}dx-\hbar^2\int_{0}^{\infty}\sqrt{\alpha}e^{-\alpha x}\frac{d^2}{dx^2}\sqrt{\alpha}e^{-\alpha x}dx$$

$$=-\hbar^2\alpha^3\int_{-\infty}^{0}e^{2\alpha x}dx-\hbar^2\alpha^3\int_{0}^{\infty}e^{-2\alpha x}dx=-\frac{\hbar^2\alpha^3}{2\alpha}-\frac{\hbar^2\alpha^3}{2\alpha}=-\hbar^2\alpha^2,\text{ which is not possible}$$

so, we will use the formula $\langle p\rangle^2=\hbar^2\int_{-\infty}^{\infty}\left|\frac{d\psi}{dx}\right|^2dx=\hbar^2\alpha^2,\Delta p=\sqrt{\langle p^2\rangle-\langle p\rangle^2}=\hbar\alpha$ now,

$$\Delta x.\Delta p=\frac{1}{\sqrt{2}\alpha}.\hbar\alpha=\frac{\hbar}{\sqrt{2}}$$

The correct option is **(c)**

6. Consider the two lowest normalized energy eigenfunctions $\psi_0(x)$ and $\psi_1(x)$ of a one dimensional system. They satisfy $\psi_0(x)=\psi_0^*(x)$ and $\psi_1(x)=\alpha\frac{d\psi_0}{dx}$, where α is a real constant. The expectation value of the momentum operator in the state ψ_1 is

[NET DEC 2016]

A. $-\frac{\hbar}{\alpha^2}$ **B.** 0

C. $\frac{\hbar}{\alpha^2}$ **D.** $\frac{2\hbar}{\alpha^2}$

Solution: $\psi_1(x)=\alpha\frac{d\psi_0}{dx}$

$$\langle p_x \rangle = \int_{-\infty}^{\infty} \psi_1^* p_x \psi dx = \int_{-\infty}^{\infty} \psi_1^* \left(-i\hbar \frac{\partial \psi_1}{\partial x}\right) dx = \int_{-\infty}^{\infty} \alpha^* \frac{d\psi_0}{dx}(-i\hbar\alpha)\frac{d^2\psi_0}{dx^2}dx$$
$$= -i\hbar|\alpha|^2 \int_{-\infty}^{\infty} \frac{d\psi_0}{dx}\frac{d^2\psi_0}{dx^2}dx$$

Integrate by parts

$$I = -i\hbar|\alpha|^2 \left(\frac{d\psi_0}{dx}\frac{d\psi_0}{dx}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d^2\psi_0}{dx^2}\frac{d\psi_0}{dx}dx \right) = 0 - (-i\hbar)|\alpha|^2 \int_{-\infty}^{\infty} \frac{d\psi_0}{dx}\frac{d^2\psi_0}{dx^2}dx$$
$$I = 0 - (-i\hbar)|\alpha|^2 \int_{-\infty}^{\infty} \frac{d\psi_0}{dx}\frac{d^2\psi_0}{dx^2}dx$$
$$\frac{d\psi_0}{dx} \to \frac{\psi_0}{\alpha}, \quad \psi_0 = 0, x \to \infty$$
$$I = 0 - I \Rightarrow 2I = 0 \Rightarrow I = 0 \Rightarrow \langle p_x \rangle = 0$$

The correct option is **(b)**

7. A particle in one dimension is in a potential $V(x) = A\delta(x-a)$. Its wavefunction $\psi(x)$ is continuous everywhere. The discontinuity in $\frac{d\psi}{dx}$ at $x = a$ is

[NET DEC 2016]

A. $\frac{2m}{\hbar^2}A\psi(a)$ **B.** $A(\psi(a) - \psi(-a))$

C. $\frac{\hbar^2}{2m}A$ **D.** 0

Solution: $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + A\delta(x-a)\psi(x) = E\psi(x)$
Integrates both side within limit

$$a - \varepsilon \text{ to } a + \varepsilon$$
$$-\frac{\hbar^2}{2m}\int_{a-\varepsilon}^{a+\varepsilon} \frac{d^2\psi}{dx^2}dx + \int_{a-\varepsilon}^{a+\varepsilon} A\delta(x-a)\psi dx = E\int_{a-\varepsilon}^{a+\varepsilon} \psi(x)dx$$
$$-\frac{\hbar^2}{2m}\left(\frac{d\psi_{II}}{dx} - \frac{d\psi_I}{dx}\right) + A\psi(a) = 0$$
$$\frac{d\psi_{II}}{dx} - \frac{d\psi_I}{dx} = \frac{2mA}{\hbar^2}\psi(a)$$
$$\text{so discontinues in } \frac{d\psi}{dx} \text{ at } x = a \text{ is } \frac{2mA}{\hbar^2}\psi(a).$$

The correct option is **(a)**

8. The normalized wavefunction of a particle in three dimensions is given by $\psi(r,\theta,\varphi) = \frac{1}{\sqrt{8\pi a^3}}e^{-r/2a}$ where $a > 0$ is a constant. The ratio of the most probable distance from the origin to the mean distance from the origin, is [You may use $\int_0^{\infty} dx x^n e^{-x} = n!$]

[NET DEC 2017]

A. $\frac{1}{3}$ **B.** $\frac{1}{2}$

C. $\frac{3}{2}$ **D.** $\frac{2}{3}$

Solution: $\psi(r,\theta,\varphi) = \frac{1}{\sqrt{8\pi a^3}}e^{\frac{-r}{2a}}$

$$\langle r \rangle = \iiint r\psi^*\psi r^2 dr \sin\theta d\theta d\phi = \frac{3}{2}(2a) = 3a$$

one can compare the wave function at hydrogen atom with Bohr radius $a_0 = 2a$ most probable distance,

$$\frac{d}{dr} r^2 e^{-r/a} = 0$$
$$r_P = 2a$$
$$\frac{r_p}{\langle r \rangle} = \frac{2a}{3a} = \frac{2}{3}$$

The correct option is **(d)**

9. The normalized wavefunction in the momentum space of a particle in one dimension is $\phi(p) = \frac{\alpha}{p^2+\beta^2}$, where α and β are real constants. The uncertainty Δx in measuring its position is

[NET DEC 2017]

A. $\sqrt{\pi}\frac{\hbar\alpha}{\beta^2}$

B. $\sqrt{\pi}\frac{\hbar\alpha}{\beta^3}$

C. $\frac{\hbar}{\sqrt{2}\beta}$

D. $\sqrt{\frac{\pi}{\beta}\frac{\hbar\alpha}{\beta}}$

Solution: $\phi(p) = \frac{\alpha}{p^2+\beta^2}$

From inverse Fourier transformation Normalize, $\psi(x) = \sqrt{\frac{\beta}{\hbar}} e^{-\frac{\beta|x|}{\hbar}}$ $\langle x \rangle = 0$

$$\langle x^2 \rangle = \frac{\beta}{h} \int_{-\infty}^{\infty} x^2 e^{-2\frac{\beta|x|}{\hbar}} dx = \frac{\hbar^2}{2\beta^2}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{\hbar}{\sqrt{2}\beta}$$

The correct option is **(c)**

10. The product $\Delta x \Delta p$ of uncertainties in the position and momentum of a simple harmonic oscillator of mass m and angular frequency ω in the ground state $|0\rangle$, is $\frac{\hbar}{2}$. The value of the product $\Delta x \Delta p$ in the state, $e^{-i\hat{p}\ell/\hbar}|0\rangle$ (where ℓ is a constant and $\hat{p}$ is the momentum operator) is

[NET DEC 2018]

A. $\frac{\hbar}{2}\sqrt{\frac{m\omega\ell^2}{\hbar}}$

B. $\hbar$

C. $\frac{\hbar}{2}$

D. $\frac{\hbar^2}{m\omega\ell^2}$

Solution: The correct option is **(c)**

Practice Set-2

1. Which of the following is an allowed wavefunction for a particle in a bound state? N is a constant and $\alpha, \beta > 0$. **[GATE 2010]**

A. $\psi = N\frac{e^{-\alpha r}}{r^3}$

B. $\psi = N\left(1 - e^{-\alpha r}\right)$

C. $\psi = Ne^{-\alpha x}e^{-\beta\left(x^2+y^2+z^2\right)}$

D. $\psi = \begin{cases} \text{non - zero constant} & \text{if } r < R \\ 0 & \text{if } r > R \end{cases}$

Solution: The correct option is **(c)**

Common data questions for 2 and 3

The wavefunction of particle moving in free space is given by, $\psi = \left(e^{ikx} + 2e^{-ikx}\right)$

2. The energy of the particle is **[GATE 2012]**

A. $\frac{5\hbar^2k^2}{2m}$

B. $\frac{3\hbar^2k^2}{4m}$

C. $\frac{\hbar^2k^2}{2m}$

D. $\frac{\hbar^2k^2}{m}$

Solution: $H\psi = E\psi, H\psi = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = \frac{-\hbar^2}{2m}\left[(ik)(ik)e^{ikx} + 2(-ik)(-ik)e^{-ikx}\right]$
$\Rightarrow H\psi = \frac{\hbar^2k^2}{2m}\left(e^{ikx} + 2e^{-2ikx}\right) = \frac{\hbar^2k^2}{2m}\psi$
The correct option is **(c)**

3. The probability current density for the real part of the wavefunction is

A. 1

B. $\frac{\hbar k}{m}$

C. $\frac{\hbar k}{2m}$

D. 0

Solution: The real part of the wave function $\psi_{\text{real}} = \cos kx + 2\cos kx$
Current density for real part of wave function=0
The correct option is **(d)**

4. Consider the wave function $Ae^{ikr}\left(r_0/r\right)$, where A is the normalization constant. For $r = 2r_0$, the magnitude of probability current density up to two decimal places, in units of $\left(A^2\hbar k/m\right)$ is **[GATE 2013]**

Solution: $\vec{J} = |\psi|^2\frac{\hbar k}{m} = |A|^2\left|\frac{r_0}{r}\right|^2\frac{\hbar k}{m} \Rightarrow J = |A|^2\left|\frac{r_0}{2r_0}\right|^2\frac{\hbar k}{m} \Rightarrow J = |A|^2\frac{\hbar k}{4m} = (0.25)|A|^2\frac{\hbar k}{m}$

5. The recoil momentum of an atom is p_A when it emits an infrared photon of wavelength 1500 nm, and it is p_B when it emits a photon of visible wavelength 500 nm. The ratio $\frac{p_A}{p_B}$ is **[GATE 2014]**

A. $1:1$

B. $1:\sqrt{3}$

C. $1:3$

D. $3:2$

Solution: $p = \frac{h}{\lambda}$, $\frac{p_A}{p_B} = \frac{\lambda_B}{\lambda_A}$, $\frac{\lambda_B}{\lambda_A} = \frac{500}{1500} = 1:3$
The correct option is **(c)**

6. The state of a system is given by $|\psi\rangle = |\phi_1\rangle + 2|\phi_2\rangle + 3|\phi_3\rangle$, where $|\phi_1\rangle, |\phi_2\rangle$ and $|\phi_3\rangle$ form an orthonormal set. The probability of finding the system in the state $|\phi_2\rangle$ is

[GATE 2016]

Solution: Probability that ψ in state $|\phi_2\rangle = \frac{2^2}{1^2+2^2+3^2} = \frac{4}{1+4+9} = \frac{4}{14} = \frac{2}{7} = 0.28$

7. The Compton wavelength of a proton is fm. (up to two decimal places).

[GATE 2017]

Solution: $\left(m_p = 1.67 \times 10^{-27}\ \text{kg}, h = 6.626 \times 10^{-34} J_S, e = 1.602 \times 10^{-19} C, c = 3 \times 10^8\ \text{ms}^{-1}\right)$

8. Consider a one-dimensional potential well of width 3 nm. Using the uncertainty principle $\left(\Delta x \cdot \Delta p \geq \frac{\hbar}{2}\right)$, an estimate of the minimum depth of the well such that it has at least one bound state for an electron is $\left(m_e = 9.31 \times 10^{-31}\ \text{kg}, h = 6.626 \times 10^{-34} J_s, e = 1.602 \times 10^{-19}\text{C}\right)$

[GATE 2017]

A. 1μeV
B. 1meV
C. 1eV
D. 1MeV

Solution: $E = \frac{p^2}{2m}, \Delta p = \frac{\hbar}{2\Delta x} \Rightarrow \Delta p = \frac{\hbar}{2a}$
So, $E = \frac{\hbar^2}{8ma^2} = \frac{h^2}{32\pi^2 ma^2} = \frac{\left(6.6\times10^{-34}\right)^2}{32\times10\times9.31\times10^{-31}\times9\times10^{-18}} = .001 \times 10^{-19}\ \text{J} \approx 1\text{meV}$
The correct option is **(b)**

9. The Hamiltonian for a quantum harmonic oscillator of mass m in three dimensions is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$$

where ω is the angular frequency. The expectation value of r^2 in the first excited state of the oscillator in units of $\frac{\hbar}{m\omega}$ (rounded off to one decimal place) is

[GATE 2018]

Solution: $\langle r^2\rangle = \langle x^2\rangle + \langle y^2\rangle + \langle z^2\rangle$

$$= \frac{\hbar}{2m\omega}\left[(2n_x+1) + (2n_y+1) + (2n_z+1)\right]$$

For first excited state $n_x = 1, n_y = 0, n_z = 0$ Hence it is triply degenerate one can take

$$n_x = 0, n_y = 1, n_z = 0 \text{ or } n_x = 0, n_y = 0, n_z = 1$$

putting any one combination, expectation value of $r^2 = \frac{5}{2}\frac{\hbar}{m\omega} = 2.5\frac{\hbar}{m\omega}$

10. Consider the motion of a particle along the x - axis in a potential $V(x) = F|x|$. Its ground state energy E_0 is estimated using the uncertainty principle. Then E_0 is proportional to

[GATE 2019]

A. $F^{1/3}$
B. $F^{1/2}$
C. $F^{2/5}$
D. $F^{2/3}$

Solution: $E = \frac{p^2}{2m} + F|x| \quad E = \frac{p^2}{2m} + Fx$ for $x > 0 \quad E = \frac{p^2}{2m} - Fx < 0$ from uncertainty theory

$$\Delta x.\Delta p = \hbar \Rightarrow \Delta p = \frac{\hbar}{\Delta x}$$
$$E = \frac{(\Delta p)^2}{2m} + F(\Delta x) \Rightarrow E = \frac{\hbar^2}{2m(\Delta x)^2} + F\Delta x$$

For minimum energy,

$$\frac{dE}{d\Delta x} = -\frac{\hbar^2}{m(\Delta x)^3} + F = 0 \Rightarrow \Delta x = \left(\frac{\hbar^2}{mF}\right)^{1/3} \frac{\hbar^2}{2m}\left(\frac{mF}{\hbar^2}\right)^{2/3} + F\left(\frac{\hbar^2}{mF}\right)^{1/3} \Rightarrow E \propto F^{2/3}$$

THe correct option is **(d)**