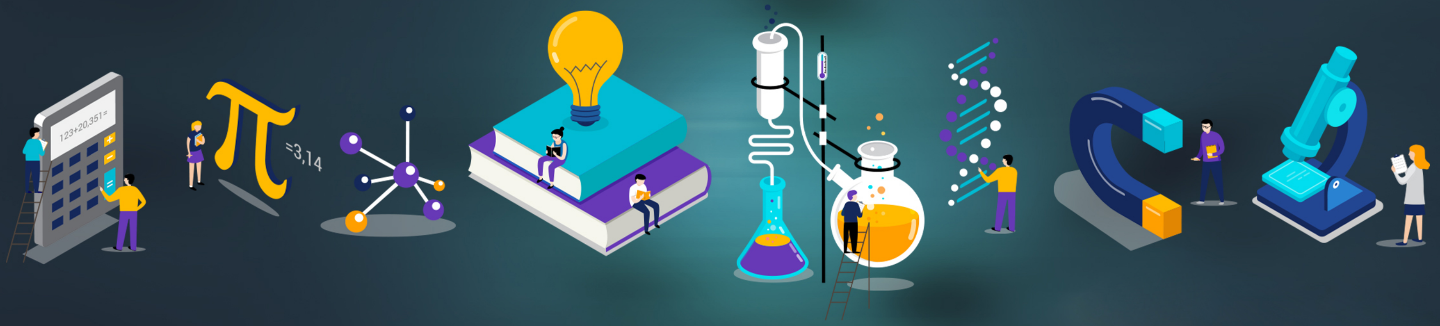

1. Differential Equations

In physics we encounter differential equations all the time. In fact the whole programme of classical mechanics is to develop a second order differential equation using Newton's laws of motion and then solving it. Sometimes these are ordinary differential equations in one variable (abbreviated ODEs). More often the equations are partial differential equations (PDEs) in two or more variables. Simply we can say , differential equations is a relation between a function and its derivatives.

Definition 1.0.1 A differential equation is an equation which involves independent and dependent variables and their derivatives or differentials.

■ **Example 1.1**

- $\frac{dy}{dx} = 4x - 2$
- $\frac{d^2y}{dx^2} = 5\frac{dy}{dx} + 10$
- $(1 + \frac{dy}{dx})^3 = k\frac{dy}{dx}$
- $\frac{dy}{dx} + xy = x^3y^3$
- $\frac{\partial^2 y}{\partial^2 x} = \frac{1}{c^2}\frac{\partial^2 y}{\partial^2 x}$
- $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$

■

1.1 Types of differential equation

There are mainly two types of differential equations,

- **Ordinary differential equations.**

  A differentnial equation involving derivatives with respect to a single variable is called an ordinary differential equation.

- **Partial differential equations.**

  A differential equation involving partial derivatives with respect to more than one independent variable is called a partial differential equation.

1.2 Order and Degree of a differential equation

Order:
The order of a differential equation is the highest differential in the equation.

Degree:
The degree of a differential equation is the power of the highest differential in the equation.

■ **Example 1.2**

- $\left(\frac{\partial^2 y}{\partial x^2}\right)^2 + \left(\frac{\partial y}{\partial x}\right) - \left(\frac{\partial^3 y}{\partial x^3}\right) = xy$ Order=3 ,Degree=2
- $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E\sin\omega t$ Order=2 ,Degree=1
- $\frac{dy}{dx} + xy = x^3y^3$ Order=1 ,Degree=1
- $\left(\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2}\right)^3 = \left[1 + \left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^4\right]^5$ Order=2 ,Degree =3
- $\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^{\frac{1}{2}} = 0$ Order=3 ,Degree=2

■

Exercise 1.1 Find the order and degree of the given differential equations,

1. $\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^{\frac{1}{2}} = 0$
2. $\left[1 + \frac{d^2y}{dx^2}\right]^{\frac{3}{2}} = a\frac{d^2y}{dx^2}$

■

Solution:

1. Here we need to eliminate the radical sign. For this write the equation as

$$\frac{d^3y}{dx^3} = \left(\frac{dy}{dx}\right)^{\frac{1}{2}}$$

Squaring both sides, we get

$$\left(\frac{d^3y}{dx^3}\right)^2 = \frac{dy}{dx}$$

$$\therefore \quad \text{Order} = 3, \text{degree} = 2$$

2. Here we eliminate the radical sign. For this write the equation as

$$\left[1 + \frac{d^2y}{dx^2}\right]^{\frac{3}{2}} = a\frac{d^2y}{dx^2}$$

Squaring both sides, we get

$$\left[1+\frac{d^2y}{dx^2}\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$$

$$\therefore \quad \text{Order} = 2, \text{degree} = 3$$

Note The direction of a curve at a particular point is given by the tangent line at that point and the slope of the tangent is given by $\frac{dy}{dx}$ at that point.

1.3 First order differential equation

An equation of the general form

$$\frac{dy}{dx} = -\frac{f(x,y)}{g(x,y)}$$

Is said to be a first order differential equation.The equation contains first and no higher derivatives. The only derivative here $\frac{dy}{dx}$ is a total or ordinarry derivative not a partial one.

1.4 Geometrical meaning of First order First degree differential equations

The solution of every first order first degree differential equations represent a family of curves.

Let, $f\left(x,y,\frac{dy}{dx}\right) = 0$ represents a differential equation of first order and first degree.

Taking $A(x_0,y_0)$ as an initial point,we can find $\frac{dy}{dx}$ at $A(x_0,y_0)$. And with the help of that we can draw the tangent at the point A.

On the tangent line take a neighbouring point $B(x_1,y_1)$. Find $\frac{dy}{dx}$ at the point $B(x_1,y_1)$and draw the tangent at B. And in this way draw another tangent at the point C on the tangent line B . Similarly draw, some more tangents by taking the neighbouring points on them. Again we take another starting point $A'(x_0',y_0')$. We can draw another curve starting from A'. In this way we can draw a number of curves. They form a smooth curve.
That is the given difrential equation represents a family of curves.

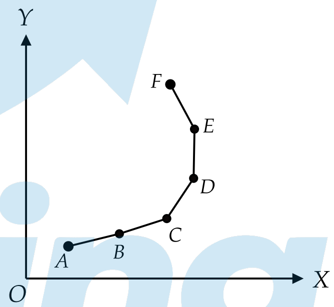

Figure 1.1: Geometrical meaning of Differential equation

1.5 Solution of a differential equation

A solution of a differential equation is any relation between variables which is free of derivatives and which satisfies the differential equation.

General solution:
A general solution is the solution in which the number of arbitrary constants and the order of the differential equation are same.

Particular solution:
A particular solution is the solution which can be obtained by giving particular values to arbitrary constants of general solution.

1.6 Solution of First order differential equations

The solutions of first order differential equations are obtained by various methods,

1.7 Method of seperation of variables

If all functions of x and dx can be arranged on one side and y and dy on the other side, then the variables are separable. The solution of this equation is found by integrating the functions of x and y.

$$f(x)dx = g(y)dy \Longrightarrow \int f(x)dx = \int g(y)dy + C$$

Method of solving:

1. Separate the variables as $f(x)dx = g(y)dy$.
2. Integrate both sides as $\int f(x)dx = \int g(y)dy$.
3. Add an arbitrary constant C on R.H.S.

Exercise 1.2 Solve $\cos(x+y)dy = dx$

Solution:

$$\begin{aligned}
\cos(x+y)dy &= dx \\
\frac{dy}{dx} &= \sec(x+y) \\
\text{Let, } x+y &= z \\
\text{Then, } 1+\frac{dy}{dx} = \frac{dz}{dx} \quad &\Rightarrow \quad \frac{dy}{dx} = \frac{dz}{dx} - 1 \\
\frac{dz}{dx} - 1 = \sec z \quad &\Rightarrow \quad \frac{dz}{dx} = 1 + \sec z \\
\text{Separating the variables, we get,} \\
\frac{dz}{1+\sec z} &= dx \\
\text{On integrating,} \\
\int \frac{\cos z}{\cos z + 1} dz &= \int dx \\
\int \left[1 - \frac{1}{\cos z + 1}\right] dz &= x + C \\
\int \left[1 - \frac{1}{2\cos^2\frac{z}{2} - 1 + 1}\right] dz &= x + C \\
\int \left(1 - \frac{1}{2}\sec^2\frac{z}{2}\right) dz &= x + C \\
\int \left(1 - \frac{1}{2}\sec^2\frac{z}{2}\right) dz &= x + C \\
z - \tan\frac{z}{2} &= x + C \\
x + y - \tan\frac{x+y}{2} &= x + C \\
y - \tan\frac{x+y}{2} &= C
\end{aligned}$$

Exercise 1.3 Solve $e^{dy/dx} = (x+1)$; given $y = 3$ at $x = 0$

Solution: Taking log of both sides we get,

$$\frac{dy}{dx} = \ln(x+1) \quad \Rightarrow \quad dy = \ln(x+1)dx$$

on integration ,

$$\int dy = \int 1 \cdot \ln(x+1)dx \quad \Rightarrow \quad y = x\ln(x+1) - \int \frac{x}{(x+1)}dx + C$$

$$y = x\ln(x+1) - \int \frac{(x+1)-1}{(x+1)}dx + C$$

$$y = x\ln(x+1) - \int \frac{(x+1)}{(x+1)}dx + \int \frac{1}{(x+1)}dx + C$$

$$= x\ln(x+1) - x + \ln(x+1) + C$$

$$y = (x+1)\ln(x+1) - x + C$$

Given: at, $x = 0 \quad y = 3 \Rightarrow \quad C = 3$

Therefore, $y = (x+1)\ln(x+1) - x + 3$

1.8 Solution of Homogeneous differential equation

Homogeneous equations are of the form,

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

Where $f(x,y)$ and $g(x,y)$ are homogeneous functions of the same degree in x and y. Homogeneous functions are those in which all the terms are of n^{th} degree.

Method of solving

1. Put, $y = vx$, then $\frac{dy}{dx} = v + x\frac{dy}{dx}$.
2. Separate v and x and then integrate.

$$\frac{dy}{dx} = f(y/x)$$

$$\Rightarrow \quad y/x = v$$

$$\Rightarrow \quad \frac{dv}{f(v)-v} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{f(v)-v} = \log x + C$$

Exercise 1.4 Solve the differential equation $(x^2 - y^2)\,dx + 2xy\,dy = 0$, given that $y = 1$ when $x = 1$ ■

Solution:

$$(x^2 - y^2)\,dx + 2xydy = 0$$

$$(x^2 - y^2)\,dx = -2xydy$$

$$\frac{dy}{dx} = -\frac{x^2 - y^2}{2xy} = \frac{y^2 - x^2}{2xy}$$

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

We get $v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x \cdot vx}$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2-1}{2v}$$

$$\Rightarrow \quad x \cdot \frac{dv}{dx} = \frac{v^2-1}{2v} - v$$

$$= \frac{v^2-1-2v^2}{2v}$$

$$= -\left[\frac{v^2+1}{2v}\right]$$

$$\Rightarrow \quad \frac{2v}{v^2+1} \cdot dv = -\frac{dx}{x}, \quad x \neq 0$$

$$\Rightarrow \quad \int \frac{2v}{v^2+1} \cdot dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \log\left(v^2+1\right) = -\log|x| + c$$

$$\Rightarrow \quad \log\left(v^2+1\right) + \log|x| = \log c$$

$$\Rightarrow \quad \left(v^2+1\right)|x| = c$$

Now, putting $v = y/x$

$$\left(y^2/x^2+1\right)|x| = c$$

$$\Rightarrow \left(x^2+y^2\right) = c|x|$$

Which is similar to $x = 1$ and $y = 1$, we get, $c = 2$

Putting value of $c = 2$, We get

$$x^2+y^2 = 2x \text{ or } x^2+y^2 = 2(-x)$$

$x = 1$ and $y = 1$

Do not satisfy $x^2+y^2 = 2(-x)$

Hence, $x^2+y^2 = 2x$ is the required solution.

1.8.1 Equations reducible to homogeneuos form

Let a differential equation be,

$$\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$$

Type-1

If, in the above equation, $\frac{a}{A} \neq \frac{b}{B}$

Then we can substitute $x = X + h$, $y = Y + k$, (h, k being constants)

The given differential equation reduces to

$$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{A(X+h)+B(Y+k)+C}$$

$$= \frac{aX+bY+ah+bk+c}{AX+BY+Ah+Bk+C}$$

Choose h, k so that $ah+bk+c = 0$

$$Ah+Kk+C = 0$$

Then the given equation becomes homogeneous

$$\frac{dY}{dX} = \frac{aX+bY}{AX+BY}$$

Type-2

$$\text{If } \frac{a}{A} = \frac{b}{B},$$

Then the value of h,k will not be finite.

$$\frac{a}{A} = \frac{b}{B} = \frac{1}{m}$$

$$A = am \quad , \quad B = b\,\mathrm{m}$$

$$\text{The given equation becomes} \quad \frac{dy}{dx} = \frac{ax+by+c}{m(ax+by)+C}$$

Now put $ax+by = z$ and apply the method of variables separable.

Exercise 1.5 Solve $(x+2y)(dx-dy) = dx+dy$

Solution:

$$(x+2y)(dx-dy) = dx+dy$$

$$\Rightarrow (x+2y-1)dx-(x+2y+1)dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+2y-1}{x+2y+1}$$

$$\text{Let } x+2y = z$$

$$1+2\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dz}{dx} = \frac{3z-1}{z+1} \quad (\text{Since, } \frac{dy}{dx} = \frac{x+2y-1}{x+2y+1})$$

$$\int \frac{z+1}{3z-1}dz = \int dx$$

$$\text{L.H.L} = \int \frac{z+1}{3z-1}dz$$

multiply numerator and denominator by 3, we get,

$$\begin{aligned}
\text{L.H.L} &= \int \frac{1}{3}\frac{(3z+3)}{(3z-1)}dz \\
&= \int \frac{1}{3}\frac{(3z-1+4)}{(3z-1)}dz \\
&= \int \frac{1}{3}\frac{(3z-1)+4}{(3z-1)}dz \\
&= \int \frac{1}{3}\left\{\frac{(3z-1)}{(3z-1)} + \frac{4}{(3z-1)}\right\}dz \\
&= \int \left\{\frac{1}{3} + \frac{1}{3}\frac{4}{(3z-1)}\right\}dz \\
&= \int \left\{\frac{1}{3} + \frac{4}{9}\frac{1}{(3z-1)}\right\}dz \\
&= \frac{1}{3}z + \frac{4}{9}\ln(3z-1)+c
\end{aligned}$$

Then,

$$\int \frac{z+1}{3z-1}dz = \int dx \Rightarrow \quad \frac{1}{3}(x+2y)+\frac{4}{9}\ln(3x+6y-1) = 2x+c$$
$$3x-3y+a = 2\ln(3x+6y-1)$$
$$4\ln(3x+6y-1) = (6x-6y)+c$$
$$2\ln(3x+6y-1) = (3x-3y)+c$$

1.9 Linear equation of first order

If a differential equation has its dependent variables and its derivatives occur in the first degree and are not multiplied together, then the equation is said to be linear. The standard equation of a linear equation of first order is given as

$$\frac{dy}{dx}+Py = Q$$

Where P and Q are functions of x.

$$\text{Integrating factor } = (\text{ I.F. }) = e^{\int P\cdot dx}$$
$$\Rightarrow y\cdot e^{\int P\cdot dx} = \int Q\cdot e^{\int P\cdot dx}dx+C$$
$$\Rightarrow \quad y(\text{I.F.}) = \int Q(\text{I.F.})dx+C$$

Exercise 1.6 Solve the differential equation $\frac{dy}{dx}-\frac{y}{x} = 2x^2, x>0$. ■

Solution: we know,

$$\frac{dy}{dx}+\left(\frac{-1}{x}\right)y = 2x^2$$
$$\frac{dy}{dx}+Py = Q, \text{ where } P = -\frac{1}{x} \text{ and } Q = 2x^2$$
$$I.F = e^{\int P\cdot dx} = e^{\int -1/x\cdot dx}$$
$$= e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

Multiplying both sides with $I.F$, we get

$$\frac{1}{x}\cdot\frac{dy}{dx}-\frac{1}{x^2}\cdot y = 2x$$

Integrating both sides w.r.t. x, we get

$$y\cdot\left(\frac{1}{x}\right) = \int 2x\cdot dx+C$$
$$\Rightarrow \quad y\cdot\frac{1}{x} = x^2+C$$
$$\Rightarrow \quad y = x^3+Cx, x>0 \quad \text{is the required solution.}$$

1.9.1 Equations reducible to linear form

The differential equation of the form,

$$\frac{dy}{dx}+p(x)y = f(x)y^n$$

is called the Bernoulli's equation or equation reducible to linear form. It can be done by dividing by y^n and substituting $\frac{1}{y^{n-1}} = z$

$$\begin{aligned} \frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}}P &= Q \\ \text{Put} \quad \frac{1}{y^{n-1}} &= z \\ \frac{(1-n)}{y^n}\frac{dy}{dx} &= \frac{dz}{dx} \\ \Rightarrow \quad \frac{1}{y^n}\frac{dy}{dx} &= \frac{dz}{1-n} \\ \frac{1}{1-n}\frac{dz}{dx} + Pz &= Q \quad \text{or} \\ \frac{dz}{dx} + P(1-n)z = Q(1-n) \end{aligned}$$

Which is a linear equation and can be solved easily.

Exercise 1.7 Solve $\frac{dy}{dx} + xy = x^3y^3$ ■

Solution: We have,

$$\begin{aligned} \frac{dy}{dx} + xy &= x^3y^3 \\ \frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} &= x^3 \\ \text{putting} \quad \frac{1}{y^2} &= z \\ \Rightarrow \quad \frac{-2}{y^3}\frac{dy}{dx} &= \frac{dz}{dx} \quad \Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx} = \frac{-1}{2}\frac{dz}{dx} \\ \therefore \quad -\frac{1}{2}\frac{dz}{dx} + xz &= x^3 \quad \Rightarrow \quad \frac{dz}{dx} - 2xz = -2x^3 \\ \therefore \text{I.F.} = e^{-\int 2xdx} &= e^{-x^2} \\ ze^{-x^2} &= -2\int x^3 e^{-x^2}dx \\ \text{Let} \quad -x^2 &= t \Rightarrow -2xdx = dt \\ ze^{-x^2} &= \int te^t dt = te^t - e^t + c \\ \text{put} \quad z = y^{-2} \text{and} \quad t &= -x^2 \\ \therefore \frac{e^{-x^2}}{y^2} &= -x^2e^{-x^2} - e^{-x^2} + c \\ \frac{1}{y^2} &= -x^2 - 1 + Ce^{x^2} \end{aligned}$$

1.10 Exact differential equations

A differential equation of the form , $Mdx + Ndy = 0$ is said to be exact if it satisfy the following condition,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\frac{\partial M}{\partial y}$ – Differential co-efficient of M with respect to y keeping x constant

$\frac{\partial N}{\partial x}$ – Differential co-efficient of N with respect to x, keeping y constant.

Method of solving:
Step 1: Integrate M w.r.t. x keeping y constant
Step 2: Integrate w.r.t. y, only those terms of N which do not contain x.
Step 3: Result of 1 + Result of 2 = Constant.

Exercise 1.8 Solve $\left(x^2+2xy\right)dx+\left(x^2+y^2\right)dy=0$ ■

Solution:

$$\text{Here,}\quad M=\left(x^2+2xy\right)\quad\text{and}\quad N=\left(x^2+y^2\right)$$
$$\Rightarrow \frac{\partial M}{\partial y}=2x$$
$$\text{and}\quad \frac{\partial N}{\partial x}=2x$$
Hence, the given equation is exact
$$\int\left(x^2+2xy\right)dx+\int y^2dy=c$$
$$\frac{x^3}{3}+x^2y+\frac{y^3}{3}=C$$
The solution is:
$$\int\left(x^2+2xy\right)dx+\int y^2dy=c=\frac{x^3}{43}+x^2y+\frac{y^3}{3}=6$$

1.10.1 Equations reducible to Exact form

A differential equation which is not exact can be reduced to exact form by multiplying it by a constant , here the integrating factor.

Type 1: If $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$ is a function of x alone, say $f(x)$, then I.F $=e^{\int f(x)dx}$

Type 2: If $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$ is a function of y alone, say $f(y)$, then I.F $=e^{\int f(y)dy}$

Exercise 1.9 Solve $(2x\log x-xy)dy+2ydx=0$ ■

Solution:

$$M=2y,\quad N=2x\log x-xy$$
$$\frac{\partial M}{\partial y}=2,\quad \frac{\partial N}{\partial x}=2(1+\log x)-y$$
Here,
$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2-2-2\log x+y}{2x\log x-xy}$$
$$=\frac{-(2\log x-y)}{x(2\log x-y)}=-\frac{1}{x}=f(x)$$
$$\text{I.F.}=e^{\int f(x)dx}=e^{\int -\frac{1}{x}dx}$$
$$=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$$
On multiplying the given differential equation by $\frac{1}{x}$, we get

$$\frac{2y}{x}dx + (2\log x - y)dy = 0$$
$$\Rightarrow \quad \int \frac{2y}{x}dx + \int -ydy = c$$
$$\Rightarrow \quad 2y\log x - \frac{1}{2}y^2 = c$$

1.11 Orthogonal trajectories and Family of curves

Given a one-parameter family of plane curves, it's orthogonal trajectories are another one-parameter family of curves, each one of which is perpendicular to all the curves in the original family. For instance, if the original family consisted of all circles having center at the origin, it's orthogonal trajectories would be all rays (half-lines) starting at the origin.

Definition 1.11.1 Two families of curves are such that every curve of either family cuts each curve of the other family at right angles. They are called orthogonal trajectories of each other.

Orthogonal trajectories arise in different contexts in applications. If the original family represents the lines of force in a gravitational or electrostatic field, its orthogonal trajectories represent the equipotentials, the curves along which the gravitational or electrostatic potential is constant.

■ **Example 1.3**

1. The path of an electric field is perpendicular to equipotential curves.
2. In fluid flow, the stream lines and equipotential lines are orthogonal trajectories.
3. The lines of heat flow is perpendicular to isothermal curves.

■

1.11.1 Finding orthogonal trajectories to the curve

Let a family of curves be given by the equation

$$g(x,y) = C$$

Where C is a constant. For the given family of curves, we can draw the orthogonal trajectories, that is another family of curves $f(x,y) = C$ that cross the given curves at right angles.

Method of solving

Family of curves given ,then to find orthogonal trajectories.

1. By differentiating the equation of curves find the differential equations in the form $f\left(x,y,\frac{dy}{dx}\right) = 0$
2. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$
   ($m_1m_2 = -1$, where,m_1=given family,m_2=orthogonal family)
3. Solve the differential equation of the orthogonal trajectories i.e., $f\left(x,y,-\frac{dx}{dy}\right) = 0$

Orthogonal trajectories given , then to find Family of curves .

1. Solve the differential equation of the orthogonal trajectory $f\left(x,y,\frac{dx}{dy}\right) = 0$ using appropriate method.

Note **Self-orthogonal:**. If the family of orthogonal trajectory is the same as the given family of curves,then it's called self orthogonal.

Exercise 1.10 Find the orthogonal trajectories of the family of straight lines $y = Cx$, where C is a parameter. ■

Solution:

$$
\begin{aligned}
\text{we have,} \quad y &= Cx \\
\text{differentiating the given equation we get} & \\
dy &= cdx \\
dy &= \frac{y}{x}dx \quad (\because c = \frac{y}{x}) \\
\frac{dy}{dx} &= \frac{y}{x} \\
\frac{dy}{dx_{(ortho)}} &= \frac{-x}{y} \\
\text{using variable separable method} & \\
(-xdx &= ydy) \\
\int -xdx &= \int ydy \\
-\frac{x^2}{2} &= \frac{y^2}{2} + C \\
\frac{x^2}{2} + \frac{y^2}{2} &= C \\
x^2 + y^2 &= 2C \Longrightarrow \text{Represents the family of circles}
\end{aligned}
$$

Exercise 1.11 Find the family of curves of the given trajectory, $\frac{dy}{dx} = \frac{y}{x}$ ■

Solution:

$$
\begin{aligned}
\text{given,} \ \frac{dy}{dx} &= \frac{y}{x} \\
\text{using variable seperable method, we get} & \\
\int xdx &= \int ydy \\
\frac{x^2}{2} &= \frac{y^2}{2} + C \\
\frac{x^2}{2} - \frac{y^2}{2} &= C \\
x^2 - y^2 &= 2C \Longrightarrow \text{family of hyperbolas}
\end{aligned}
$$

1.12 Second order differential equations

1.12.1 Linear differential equation

If the degree of the dependent variable and all derivatives is one, such differential equations are called linear differential equations.

■ **Example 1.4**

1. $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = x^2 + x + 1$

2. $\frac{d^2x}{dx^2} - \frac{dy}{dx} - 3y = x$

3. $2\frac{d^2x}{dt^2} - \frac{dx}{dt} - 3x = f(t)$

1.12.2 Non-Linear differential equation

If the degree of the dependent variable and / or its derivatives are of greater than 1 such differential equations are called non-linear differential equations.

Example 1.5

1. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y^2 = \sin x$
2. $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y^2 = e^x$
3. $\left(\frac{d^2x}{dt^2}\right)^2 + \frac{dx}{dt} + x = f(t)$

1.12.3 Homogeneous differential equation

A differential equation of the form $y'' + P(x)y' + Q(x)y = F(x)$, is said to be non-homogeneous if $F(x) = 0$

Example 1.6

1. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$
2. $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y^2 = 0$

1.12.4 Nonhomogeneous differential eqaution

A differential equation of the form $y'' + P(x)y' + Q(x)y = F(x)$, is said to be non-homogeneous if $F(x) \neq 0$

Example 1.7

1. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + y = x^2 + 2$
2. $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y^2 = e^x$

1.13 Linear independance and dependance of solutions

Two solutions of a differential equation, $y_1(x)$ and $y_2(x)$ are said to be linearly independant if

$$Ay_1(x) + By_2(x) \neq 0$$

given, $A \neq 0$ and $B \neq 0$

1.13.1 Wronskian

The Wronskian of two functions $y_1(x)$ and $y_2(x)$ is given by,

$$W(y_1, y_2, x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

1. If $W(y_1, y_2, x) = 0$, then $y_1(x)$ and $y_2(x)$ are linearly dependent.
2. If $W(y_1, y_2, x) \neq 0$, then $y_1(x), y_2(x)$ are linearly independent.

1.14 Linear Second Order Differential Equations With Constant Coefficients

The general form of the linear differential equation of second order is

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

Where P and Q are constants and R is a function of x or constant.

Note **Differential operator:**
A differential operator can be represented as, $D=\frac{d}{dx}$
Then a differential equation can be written in terms of differntial operators as,

$$D^2y+PDy+Qy=R$$
$$\left(D^2+PD+Q\right)y=R$$
$$\text{Where,}\quad Dy=\frac{dy}{dx},\ \text{and } D^2y=\frac{d^2y}{dx^2}$$

$\frac{1}{D}$ stands for the operation of integration.

1.15 Solution of Second order homogeneous differential equation.

1.15.1 Method of solving

1. Let $y=C_1e^{mx}$ be the trial solution

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0 \tag{1.1}$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in 1.1 then,
$C_1e^{mx}\left(m^2+Pm+Q\right)=0\Rightarrow m^2+Pm+Q=0$. It is called Auxiliary equation.

2. Solve the auxiliary equation.

Case 1:

Roots are real and distinct
If m_1 and m_2 are the roots, then the C.F. is

$$y=C_1e^{m_1x}+C_2e^{m_2x}$$

Case 2:

Roots are real and equal
If both the roots are m,m then the C.F. is

$$y=\left(C_1+C_2x\right)e^{mx}$$

Case 3:

Roots are Imaginary
If the roots are $\alpha\pm i\beta$, then the solution will be

$$\begin{aligned}
y&=C_1e^{(\alpha+i\beta)x}+C_2e^{(\alpha-i\beta)x}=e^{\alpha x}\cdot\left[C_1e^{i\beta x}+C_2e^{-i\beta x}\right]\\
&=e^{\alpha x}\left[C_1(\cos\beta x+i\sin\beta x)+C_2(\cos\beta x-i\sin\beta x)\right]\\
&=e^{\alpha x}\left[(C_1+C_2)\cos\beta x+i\left(C_1-C_2\right)\sin\beta x\right]\\
&=e^{\alpha x}[A\cos\beta x+B\sin\beta x]
\end{aligned}$$

Roots	Basis of solution	General solution
Real and equal(repeated root m)	e^{mx} and xe^{mx}	$y = (C_1 + C_2 x) e^{mx}$
Real and distinct(m_1 , m_2)	$e^{m_1 x}$, $e^{m_2 x}$	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$
Imaginary roots ($\alpha \pm i\beta$)	$e^{(\alpha+i\beta)x}, e^{(\alpha-i\beta)x}$	$e^{\alpha x}[A\cos\beta x + B\sin\beta x]$

Table 1.1: Roots of homogeneous second order DE

Exercise 1.12 Solve $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$.

Solution:

Given equation can be written as,

$$\left(D^2 - 8D + 15\right) y = 0$$

Here auxiliary equation is,

$$m^2 - 8m + 15 = 0$$

$$(m-3)(m-5) = 0 \quad \therefore m = 3, 5$$

Hence, the required solution is,

$$y = C_1 e^{3x} + C_2 e^{5x}$$

Exercise 1.13 Solve the differential equation: $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

Solution: We have

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

$$\Rightarrow \left(D^2 + 6D + 9\right) y = 0$$

Auxiliary equation is $\quad D^2 + 6D + 9 = 0$

$$\Rightarrow \quad (D+3)^2 = 0 \Rightarrow D = -3, -3$$

the solution, y $= (c_1 + c_2 x) e^{-3x}$

Exercise 1.14 Solve $\left(D^3 - 1\right) y = 0$

Solution:

we have, $\left(D^3 - 1\right) y = 0$

The characteristic equation is,

$$m^3 - 1 = 0 \Rightarrow m = 1, \frac{-1 \pm \sqrt{3}i}{2}$$

$$\text{y } = Ae^x + e^{-x/2} \cdot \left[B\cos\frac{\sqrt{3}}{2}x + C\sin\frac{\sqrt{3}}{2}x\right]$$

1.16 Solution of Second order nonhomogeneous differential equation.

The solution of a differential equation of the form,

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

consists of two parts,a complementary function and a particular integral.
Complete Solution = Complementary Function + Particular Integral

$$\boxed{y \;=\; C.F \;+\; P.I}$$

Finding complementary function

Complementary function is the solution obtained by solving the equation replacing R.H.S by 0.Same as that explained in finding solution to homogeneous differential equations.

Finding Particular solution

Particular integral (P.I) depends on the form of $R(x)$
If the differential equation is of the form,

$$\left(D^n+k_1D^{n-1}+k_2D^{n-2}+\cdots+k_n\right)y=R(x)$$

Then the particular integral of the equation is given by,

$$\text{P.I.} = \frac{1}{D^n+k_1D^{n-1}+k_2D^{n-2}+\cdots+k_n}R(x)$$

The following cases arise for particular integrals:

1. When $X=e^{ax}$, then

$$\begin{aligned}\text{P.I.} &= \frac{1}{f(D)}e^{ax}\\ &= \frac{1}{f(a)}e^{ax}, \quad \text{If } f(a)\neq 0\end{aligned}$$

If $f(a)=0$, then

$$\text{P.I.} = \frac{x}{f'(a)}e^{ax}, \quad \text{If } f'(a)\neq 0$$

If $f'(a)=0$, then

$$\text{P.I.} = \frac{x^2}{f''(a)}e^{ax}, \quad \text{If } f''(a)\neq 0$$

2. When $X=\sin ax$, then

$$\text{P.I.} = \tfrac{1}{f(D^2)}\sin ax$$

$$=\tfrac{1}{f(-a^2)}\sin ax, \text{ if } f\left(-a^2\right)\neq 0$$

3. When $X=\cos ax$, then

$$\begin{aligned}\text{P.I.} &= \frac{1}{f(D^2)}\cos ax\\ &= \frac{1}{f(-a^2)}\cos ax, \text{ if } f\left(-a^2\right)\neq 0\end{aligned}$$

4. When $X=x^m$, then

$$\text{P.I.} = \frac{1}{f(D)}x^m=[f(D)]^{-1}x^m$$

Expansion of $[f(D)]^{-1}$ is to be carried up to the term D^m because $(m+1)^{\text{th}}$ and higher derivatives of x^m are zero.

5. When $X = e^{ax}v(x)$, then

$$\text{P.I.} = \tfrac{1}{f(D)}e^{ax}v(x)$$
$$\text{P.I.} = e^{ax}\tfrac{1}{f(D+a)}v(x)$$

6. When $X = xv(x)$, then

$$\begin{aligned}\text{P.I.} &= \frac{1}{f(D)}xv(x)\\ &= \left[x - \frac{f'(D)}{f(D)}\right]\cdot\frac{1}{f(D)}v(x)\end{aligned}$$

Exercise 1.15 Solve the differential equation:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$$

Solution: We have

$$\begin{aligned}\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y &= 5e^{3x}\\ \left(D^2+6D+9\right)y &= 5e^{3x}\\ \text{Auxiliary equation is}\quad D^2+6D+9 &= 0\\ (D+3)^2 = 0 &\Rightarrow D = -3,-3\\ \text{The solution, y} &= (c_1+c_2x)\,e^{-3x}\\ \text{Particular integral} &= \frac{1}{D^2+6D+9}\cdot 5e^{3x}\\ &= \frac{5e^{3x}}{(3)^2+6(3)+9} = \frac{5e^{3x}}{36}\\ \text{The complete solution is given by} y &= \text{C.F.} + \text{P.I}\\ y &= (c_1+c_2x)\,e^{-3x} + \frac{5e^{3x}}{36}\end{aligned}$$

1.17 Euler - Cauchy Differential equation

A homogeneous differential equation of the form,

$$a_nx^n\frac{d^ny}{dx^n} + a_{(n-1)}x^{(n-1)}\frac{d^{(n-1)}y}{dx^{(n-1)}} + \cdots\cdots + a_2x^2\frac{d^2y}{dx^2} + a_1x\frac{dy}{dx} + a_0y = \mathrm{Q(x)} \tag{1.2}$$

Where, $a_0, a_1, a_2 \cdots a_n$ are constants is called a Euler - Cauchy differential equation.

$$\text{Put, x} = \mathrm{e}^z \Rightarrow \log \mathrm{x} = \mathrm{z} \Rightarrow \frac{1}{x} = \frac{dz}{dx}$$
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dx}\cdot\frac{dz}{dx}\\ &= \frac{1}{x}\frac{dy}{dz}\\ \mathrm{x}\frac{dy}{dx} &= \frac{dy}{dz}\end{aligned}$$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right)\\
&= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)\\
&= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}\\
x^2\frac{dy}{dx^2} &= -\frac{dy}{dz} + \frac{d^2y}{dz^2}
\end{aligned}$$

Then neglecting the higher orders , the given equation becomes,

$$a_2\frac{d^2y}{dz^2} + (a_1 - a_2)\frac{dy}{dz} + a_0 y = \mathrm{Q}$$

Linear second order non-homogeneous differential equation

Exercise 1.16 $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 20y = (x+1)^2$ ■

Solution:

$$\begin{aligned}
x &= e^z\\
\log x = z \Rightarrow \frac{1}{x} &= \frac{dz}{dx}\\
\text{Let, } x\frac{dy}{dx} &= \frac{dy}{dz}\\
\text{and,} x^2\frac{d^2y}{dx^2} &= \frac{d^2y}{dz^2} - \frac{dy}{dz}\\
\text{Then we get,} \frac{d^2y}{dz^2} + \frac{dy}{dz} - 20y &= (e^z+1)^2\\
\text{C.F.} &= c_1e^{4z} + c_2e^{-5z}\\
&= c_1x^4 + c_2x^{-5}\\
\text{P . I .} &= \frac{1}{D^2+D-20}(e^z+1)^2\\
&= \frac{1}{D^2+D-20}\left(e^{2z}+2e^z+1\right)\\
&= -\frac{1}{14}e^{2z} - \frac{1}{9}e^z - \frac{1}{20}\\
\text{Total solution } y &= c_1x^4 + c_2x^{-5} - \frac{1}{14}x^2 - \frac{1}{9}x - \frac{1}{20}
\end{aligned}$$

1.18 Singular Points

The concept of singular point or singularity (as applied to a differential equation) stems from the ts usefulness in classifying Ordinary Differential Equations and investigating the feasibility of a series solution (Series Solution method will be explained in the next section) If we write our second-order homogeneous differential equation (in y) as,

$$y'' + P(x)y' + Q(x)y = 0 \tag{1.3}$$

We are ready to define ordinary and singular points. If the functions $P(x)$ and $Q(x)$ remain finite at $x = x_0$, point $x = x_0$ is an ordinary point. However, if either $P(x)$ or $Q(x)$ (or both) diverges as $x \to x_0$, point x_0 is a singular point. Using equation 1.3 we may distinguish between two kinds of singular points.

1. If either $P(x)$ or $Q(x)$ diverges as $x \to x_0$ but $(x-x_0)\,P(x)$ and $(x-x_0)^2\,Q(x)$ remain finite as $x \to x_0$, then $x = x_0$ is called a **regular**, or **nonessential, singular point.**

2. If $P(x)$ diverges faster than $\frac{1}{(x-x_0)}$ so that $(x-x_0)\,P(x)$ goes to infinity as $x \to x_0$, or $Q(x)$ diverges faster than $\frac{1}{(x-x_0)}^2$ so that $(x-x_0)^2\,Q(x)$ goes to infinity as $x \to x_0$, then point $x = x_0$ is labeled an **irregular**, or **essential, singularity**.

Futuring
crafting your future