1. Integral Transforms

1.1 Integral Transforms

Frequently in mathematical physics we encounter pairs of functions related by an expression of the form

$$g(x) = \int_a^b f(t)K(x,t)dt \tag{1.1}$$

where it is understood that a, b, and $K(x,t)$ (called the kernel) will be the same for all function pairs f and g. We can write the relationship expressed in Eq. (1.1) in the more symbolic form

$$g(x) = \mathcal{L}f(t) \tag{1.2}$$

thereby emphasizing the fact that Eq. (1.1) can be interpreted as an operator equation. The function $g(x)$ is called the integral transform of $f(t)$ by the operator $\mathcal{L}$, with the specific transform determined by the choice of a, b, and $K(x,t)$. The operator defined by Eq. (1.1) will be linear:

$$\int_a^b [f_1(t) + f_2(t)]K(x,t)dt = \int_a^b f_1(t)K(x,t)dt + \int_a^b f_2(t)K(t)dt \tag{1.3}$$

$$\int_a^b cf(t)K(\alpha,t)dt = c\int_a^b f(t)K(\alpha,t)dt \tag{1.4}$$

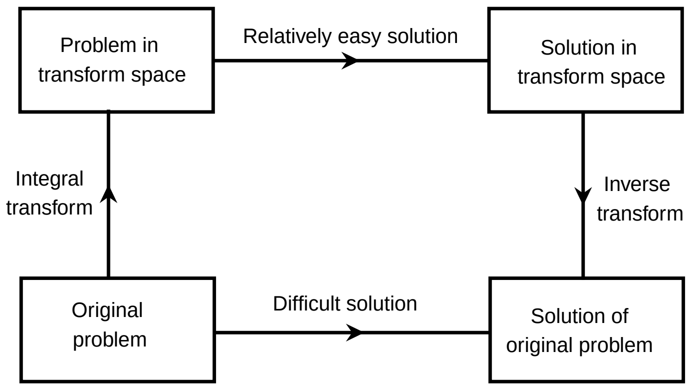
In order for transforms to be useful, we will shortly see that we need to be able to "undo" their effect. From a practical viewpoint, this means that not only must there exist an operator $\mathcal{L}^{-1}$, but also that we have a reasonably convenient and powerful method of evaluating

$$\mathcal{L}^{-1}g(x) = f(t) \tag{1.5}$$

for an acceptably broad range of $g(x)$. The procedure for inverting a transform takes a wide variety of forms that depend on the specific properties of $K(x,t)$, so we cannot write a formula that is as general as that for $\mathcal{L}$ in Eq. (1.1).

Not all superficially reasonable choices for the kernel $K(x,t)$ will lead to operators $\mathcal{L}$ that have inverses, and even for strategically chosen kernels it may be the case that $\mathcal{L}$ and $\mathcal{L}^{-1}$ will only exist for substantially restricted classes of functions. Thus, the entire development of the present chapter is restricted (for any given integral transform) to functions for which the indicated operations can be carried out.

One of the frequent uses of integral transforms is to use one, together with its inverse, to form an **integral representation** of a function that we originally had in an explicit form. This move (which appears to be in the direction of generating greater complexity) has value that arises from the relatively simple behavior of the transforms of differentiation and integration operators. Procedures involving integral representations are also presented in later sections of this chapter.



Fourier Transform

The integral transform that has seen the widest use is the Fourier transform, defined as

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \tag{1.6}$$

The notation for this transform is not entirely universal; some writers omit the prefactor $1/\sqrt{2\pi}$; we keep it because it causes the transform and its inverse to have formulas that are more symmetrical. In applications involving periodic systems, one occasionally encounters a definition with kernel $\exp(2\pi i\omega t/a_0)$, where a_0 is a lattice constant. These differences in notation do not change the mathematics, but cause formulas to differ by powers of 2π or a_0. Caution is therefore advised when combining material from different sources.

1.2 Infinite Fourier Transform

The fourier transform of $f(x)[-\infty < x < \infty]$:

$$f(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

Inverse fourier transform of $f(s)[-\infty < x < \infty]$:

$$f(x) = F^{-1}\{f(s)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{-isx} ds.$$

1.3 Fourier Sine Transform

$$f_s(s) = F_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx dx$$

$$f(x) = F_s^{-1}\{f_s(s)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(s) \sin sx ds$$

1.4 Fourier Cosine Transform

$$f_c(s) = F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx dx$$

$$f(x) = F_c^{-1}\{f_c(s)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(s) \cos sx ds$$

Important Formulas:

1. $\int_0^{\infty} \frac{\sin ax}{x} dx = \frac{\pi}{2}$
2. $\int_0^{\infty} e^{-x^2} dx = \sqrt{\frac{\pi}{2}}$
3. $\int_0^{\infty} e^{-ax} \sin bx dx = \frac{b}{a^2+b^2}$
4. $\int_0^{\infty} e^{-ax} \cos bx dx = \frac{a}{a^2+b^2}$
5. $\int_0^{\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$
6. $\int_0^{\infty} e^{-ax^2} \cos bx dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} e^{-b^2/4a}$
7. $\int_{-\infty}^{\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} e^{b^2/4a}$

Exercise 1.1 Find the fourier transform of Gaussian distribution function $f(x) = Ne^{-ax^2}$ where N and 'a' are constant. ■

Solution:

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ikx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty} Ne^{-ax^2}e^{ikx}dx = \frac{N}{\sqrt{2\pi}}e^{-k^2/4a}$$

Exercise 1.2 Find $F\{\delta(x)\}$ ■

Solution:

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ikx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x)e^{ikx}dx = \frac{1}{\sqrt{2\pi}}\left[e^{ikx}\right]_{x=0} = \frac{1}{\sqrt{2\pi}}$$

Exercise 1.3 Find the fourier transform of $f(x) = e^{-a|x|} \quad (-\infty < x < \infty) \quad (a > 0)$ ■

Solution:

$$F\left[e^{-a|x|}\right] = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{ax}e^{isx}dx + \int_{0}^{\infty} e^{-ax}e^{isx}dx\right] = \frac{1}{\sqrt{2\pi}}\left[\left(\frac{e^{(a+is)x}}{(a+is)}\right)_{-\infty}^{0} + \left(\frac{e^{-(a-is)x}}{(is-a)}\right)_{0}^{\infty}\right]$$
$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+is} + \frac{1}{a-is}\right] = \frac{2a}{\sqrt{2\pi}\,(s^2+a^2)} \because$$

1.5 Properties of Fourier Transform

(i) **Linearity theorem:**

$$\text{If } F(x) = a_1 f_1(x) + a_2 f_2(x) + \ldots \text{ then,}$$
$$f(s) = F[f(x)] = a_1 f_1(s) + a_2 f_2(s) + \ldots$$

(ii) **Change of scale property:**

$$\text{If } F[f(x)] = f(s), \text{ then } F\{f(ax)\} = \frac{1}{a}f(\,s/a)$$

(iii) If $F\{f(x)\} = f(s)$, then $F\{f^*(x)\} = f^*(-s)$

(iv) **Shifting property:**

$$\text{If } f(s) \text{ is the Fourier transform of f(x), then } F\{f(x \pm a)\} = e^{\mp isa}f(s)$$

(v) **Modulation theorem:**

If $F\{f(x)\} = f(s)$, then $F\{f(x)\cos ax\} = \frac{1}{2}f(s-a) + \frac{1}{2}f(s+a)$

1.6 Parseval Identity for Fourier Transform

If the Fourier transform of $f(x)$ and $g(x)$ be $f(s)\&g(s)$ respectively, then

(i) $\int_{-\infty}^{\infty} f(s)g^*(s)ds = \int_{-\infty}^{\infty} f(x)g^*(x)dx$

where $g^*(s)$ is complex conjugate of $g(s)$ and $g^*(x)$ is the complex conjugate of $g(x)$.

(ii) $\int_{-\infty}^{\infty} |f(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$

Exercise 1.4 Using parseval's identity show that $\int_b^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \pi/2$ ■

Solution:

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{ixx}dx = \sqrt{\frac{2}{\pi}}\frac{\sin as}{s}$$

$$\text{Using parseval identity, } \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |f(s)|^2 ds$$

$$\int_{-a}^{\infty} l^2 dx = \int_{-\infty}^{\infty} \frac{2}{\pi}\frac{\sin^2 as}{s^2} ds$$

$$2a = \frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds$$

$$\text{Putting } as = x \Rightarrow ads = dx$$

$$\Rightarrow \pi = \int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx$$

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$$

Exercise 1.5 If the fourier transform of $f(x)$ is $f(s)$, then find the fourier transform of $\frac{df}{dx}$ ■

Solution:

$$F\left[\frac{df}{dx}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df}{dx} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[e^{isx} f(x)\right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ise^{isx} f(x)dx$$

Since $f(x)$ should be a well behave function for the fourier tranform to exist, therefore, the first term of the above equation should be equal to zero.

$$\text{Then } F\left[\frac{df}{dx}\right] = -is\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)dx = -isf(s)$$

Exercise 1.6 Find Fourier transform of $f(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a \end{cases}$ ■

Solution:

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} x^2 dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(\frac{e^{isx}}{is}\cdot x^2\right)_{x=-a}^{a} - \frac{2}{is}\int_{-a}^{a} xe^{isx}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a^2}{is}\left(e^{isa}-e^{-isa}\right) - \frac{2}{is}\left[\left(\frac{xe^{isx}}{is}\right)_{x=-a}^{a} - \frac{1}{is}\int_{-a}^{a} e^{isx}dx\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2a^2}{s}\sin(sa) + \frac{2a}{s^2}\cos(sa) - \frac{4}{s^3}\sin(sa)\right]$$

Exercise 1.7 Find the complex fourier transform $e^{-|x|}$ ■

Solution:

$$F\left\{e^{-|x|}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{-|x|}e^{isx}dx + \int_{0}^{\infty} e^{-|x|}\cdot e^{isx}dx\right]$$

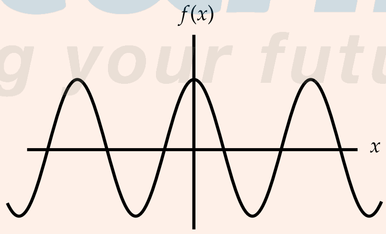
$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{x}\cdot e^{isx}dx + \int_{0}^{\infty} e^{-x(1-is)}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left[\frac{e^{x(1+is)}}{1+is}\right]_{x=-\infty}^{0} + \left[\frac{e^{-x(1-is)}}{-(1-is)}\right]_{x=0}^{\infty}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{1-is} + \frac{1}{1+is}\right]$$

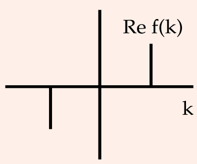
$$= \frac{1}{\sqrt{2\pi}}\left(\frac{2}{1+s^2}\right)$$

Exercise 1.8 The graph of a real periodic function $f(x)$ for the range $[-\infty,\infty]$ is shown below

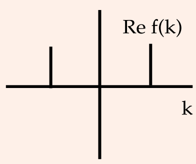


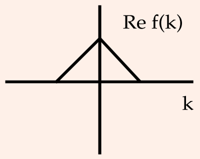
Which of the following graphs represents the real part of its Fourier transform?

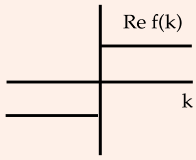
a.



b.



c. 

d. 

Solution:

The given graph in the question is of $f(x) = \cos\omega x$

$$F(\cos\omega x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\cos\omega x e^{ikx}dx = \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{+\infty}\frac{e^{i(k+\omega)x}}{2}dx + \int_{-\infty}^{+\infty}\frac{e^{i(k-\omega)x}}{2}dx\right)$$

$$\Rightarrow F(\cos\omega x)\frac{1}{\sqrt{2\pi}}\left(\frac{1}{2}\delta(k+\omega)+\frac{1}{2}\delta(k-\omega)\right)$$

Therefore, the fourier transform of $f(x) = \cos x\omega x$ consist of two delta functions of same height at $k = \omega$ and $k = -\omega$.

So the correct answer is **option (b)**

Laplace Transforms

The Laplace transform $f(s)$ of a function $F(t)$ is defined by

$$f(s) = \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t) dt. \tag{1.7}$$

A few comments on the existence of the integral are in order. The infinite integral of $F(t)$,

$$\int_0^\infty F(t) dt,$$

need not exist. For instance, $F(t)$ may diverge exponentially for large t. However, if there are some constants s_0, M, and $t_0 \geq 0$ such that for all $t > t_0$

$$\left| e^{-s_0 t} F(t) \right| \leq M, \tag{1.8}$$

the Laplace transform will exist for $s > s_0$. $F(t)$ is then said to be of exponential order. As a counter example, $F(t) = e^{t^2}$ does not satisfy the condition given by Eq. (1.8) and is not of exponential order. Thus, $\mathcal{L}\left\{e^{t^2}\right\}$ does not exist. The Laplace transform may also fail to exist because of a sufficiently strong singularity in the function $F(t)$ as $t \to 0$. For example,

$$\int_0^\infty e^{-st} t^n dt$$

diverges at the origin for $n \leq -1$. The Laplace transform $\mathcal{L}\{t^n\}$ does not exist for $n \leq -1$.

1.7 Important Laplace Transforms:

1. $\mathcal{L}(1) = \frac{1}{s}$

Exercise 1.9

$$\mathcal{L}(1) = \frac{1}{s}$$

■

Solution:

$$\begin{aligned}
\mathcal{L}(1) &= \int_0^\infty 1 \cdot e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty \\
&= -\frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^\infty = -\frac{1}{s}[0-1] \\
&= \frac{1}{s}
\end{aligned}$$

Hence $\mathcal{L}(1) = \dfrac{1}{s}$

2. $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}} (n = 0, 1, 2, \ldots\ldots..)$

Exercise 1.10 $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$ where n and s are positive. ■

Solution:

$$\mathcal{L}(t^n) = \int_0^\infty e^{-st} t^n dt$$

Putting $st = x$ or $t = \dfrac{x}{s}$ or $dt = \dfrac{dx}{s}$

$$\text{Thus we have} \mathcal{L}\left(t^n\right) = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s_r} \quad \Rightarrow \quad \mathcal{L}\left(t^n\right) = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} \cdot x^n dx$$

$$\Rightarrow \mathcal{L}\left(t^n\right) = \frac{\sqrt{n+1}}{s^{n+1}} \Rightarrow \mathcal{L}\left(t^n\right) = \frac{n!}{s^{n+1}} \qquad \left[\begin{array}{cc} [n+1 = \int_0^\infty e^{-x} \cdot x^n dx] \\ \text{and} \quad | \overline{n+1} = n! \end{array}\right]$$

3. $\mathcal{L}\left(e^{at}\right) = \frac{1}{s-a}(s > a)$

Exercise 1.11 $\mathcal{L}\left(e^{at}\right) = \frac{1}{s-a}$, where $s > a$ ■

Solution:

$$\begin{aligned}
\mathcal{L}\left(e^{at}\right) &= \int_0^\infty e^{-st} \cdot e^{at} dt = \int_0^\infty e^{-st+at} \cdot dt \\
&= \int_0^\infty e^{(-s+a)t} \cdot dt = \int_0^\infty e^{-(s-a)t} \cdot dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = -\frac{1}{s-a}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty \\
&= \frac{-1}{(s-a)}(0-1) = \frac{1}{s-a}
\end{aligned}$$

4. $\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}$ $(s > 0)$

Exercise 1.12 $L(\cos at) = \frac{s}{s^2+a^2}$ ■

Solution:

$$\begin{aligned}
\mathcal{L}(\cos at) &= \mathcal{L}\left(\frac{e^{tat}+e^{-iat}}{2}\right) \qquad \left[\because \cos at = \frac{e^{iat}+e^{-iat}}{2}\right] \\
&= \frac{1}{2}\left[\mathcal{L}\left(e^{iat}+e^{-iat}\right)\right] = \frac{1}{2}\left[\mathcal{L}\left(e^{iat}\right)+\mathcal{L}\left(e^{iat}\right)\right] = \frac{1}{2}\left[\frac{1}{s-ia}+\frac{1}{s+ia}\right] = \frac{1}{2}\frac{s+ia+s-ia}{s^2+a^2} \\
&= \frac{s}{s^2+a^2}
\end{aligned}$$

5. $\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$ $(s > 0)$

Exercise 1.13 $\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$ ■

Solution:

$$\begin{aligned}
\mathcal{L}(\sin at) &= \mathcal{L}\left[\frac{e^{iat}-e^{-iat}}{2i}\right] \qquad \left[\because \sin at = \frac{e^{iat}-e^{-iat}}{2i}\right] \\
&= \frac{1}{2i}\left[L\left(e^{iat}-e^{-iat}\right)\right] = \frac{1}{2i}\left[L\left(e^{iat}\right)-\mathcal{L}\left(e^{-iat}\right)\right] \\
&= \frac{1}{2i}\left[\frac{1}{s-ia}-\frac{1}{s+ia}\right] = \frac{1}{2i}\frac{s+ia-s+ia}{s^2+a^2} = \frac{1}{2i}\frac{2ia}{s^2+a^2} = \frac{a}{s^2+2^2}
\end{aligned}$$

6. $\mathcal{L}(\cosh at) = \frac{s}{s^2-a^2}$ $\left(s^2 > a^2\right)$

Exercise 1.14 $\mathcal{L}(\cosh at) = \frac{s}{s^2-a^2}$ ■

Solution:

$$\mathcal{L}(\cosh at) = L\left[\frac{e^{at}+e^{-at}}{2}\right] \qquad \left(\because \cosh at = \frac{e^{at}+e^{-at}}{2}\right)$$

$$= \frac{1}{2}\mathcal{L}\left(e^{at}\right) + \frac{1}{2}L\left(e^{-at}\right) = \frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right] \qquad \left[\mathcal{L}\left(e^{at}\right) = \frac{1}{s-a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$

7. $\mathcal{L}(\sinh at) = \frac{a}{s^2-a^2}\ (s^2 > a^2)$

Exercise 1.15 $L(\sinh at) = \frac{a}{s^2-a^2}$

Solution:

$$\mathcal{L}(\sinh at) = \mathcal{L}\left[\frac{1}{2}\left(e^{at}-e^{-at}\right)\right]$$

$$= \frac{1}{2}\left[\mathcal{L}\left(e^{at}\right) - L\left(e^{-at}\right)\right] = \frac{1}{2}\left[\frac{1}{s-a}-\frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{a}{s^2-a^2}$$

Important properties:

1. Linear Property: $\mathcal{L}\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1\mathcal{L}\left[f_1(t)\right] + a_2\mathcal{L}\left[f_2(t)\right]$
2. Shifting Property: $\mathcal{L}\left[e^{at} f(t)\right] = f(s-a)$
3. Scaling Property: $\mathcal{L}[f(at)] = \frac{1}{a} f\left(\frac{s}{a}\right)$
4. $\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n}(f(s))$
5. $\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_{-\infty}^{\infty} f(s)ds$
6. $\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s} f(s)$
7. $\mathcal{L}\left[f'(t)\right] = sL[f(t)] - f(0)$
8. $\mathcal{L}\left[f''(t)\right] = s^2 L[f(t)] - f'(0) - sf(0)$
9. Laplace transform of a periodic function $f(t)$ having period ' T ' is

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt$$

10. Laplace transform of the unit step function $u(t-a) = u_a(t) = \begin{cases} 0 & \text{for} \quad t < a \\ 1 & \text{for} \quad t \geq a \end{cases}$ is $\frac{e^{-as}}{s}$.
11. Laplace transform of the dirac delta function $\delta(t-a)$ is e^{-as}.

Exercise 1.16 Find the laplace transform of $f(t) = (1+\cos 2t)$

Solution:

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st}\left(1+\cos 2t \cdot dt = \left(\frac{e^{-st}}{-s}\right)_0^\infty + \int_0^\infty e^{-st}\cos 2t dt\right.$$
$$= \frac{1}{s} + \frac{s}{4+s} = \frac{2s^2+4}{s\left(s^2+4\right)}$$

Exercise 1.17 Find the laplace transform of $f(t) = 2\sin 2t \cos 4t$

Solution:

$$\mathcal{L}[f(t)] = \mathcal{L}[2\sin 2t \cdot \cos 4t] = \mathcal{L}[\sin 6t - \sin 2t] = \left(\frac{6}{s^2+36} - \frac{2}{s^2+4}\right)$$

Exercise 1.18 Find the laplace transform of $f(t) = e^{-t}(3\sinh 2t - 5\cosh 2t)$

Solution:

$$\mathcal{L}[f(t)] = \mathcal{L}\left(3e^{-t}\sinh 2t\right) - \mathcal{L}\left(5e^{-t}\cosh 2t\right)$$
$$= 3\frac{2}{(s+1)^2-4} - 5\cdot\frac{(s+1)}{(s+1)^2-4} = \frac{(1-5\,\mathrm{s})}{(\mathrm{s}^2+2\,\mathrm{s}-3)} \quad \text{(Using 2nd property)}$$

Exercise 1.19 Find the laplace transform of the fuction
$f(t) = 1$ for $2n \le t \le 2n+1$
$= 0$ for $2n+1 \le t \le 2n+2 \quad (n = 0,1,2,\ldots..)$

Solution:

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}dt$$
$$= \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}dt\right] = \frac{1}{1-e^{-2s}}\left(\frac{1-e^{-s}}{s}\right)$$
$$= \frac{\left(1-e^{-s}\right)}{s\left(1+e^{-s}\right)\left(1-e^{-s}\right)} = \frac{1}{s\left(1+e^{-s}\right)}$$

Exercise 1.20 Find the laplace transform of $f(t) = \left(1+te^{-t}\right)^3$

Solution:

$$f(t) = \left(1+te^{-t}\right)^3 = 1 + t^3e^{-3t} + 3te^{-t} + 3t^2e^{-2t}$$
$$\mathcal{L}[f(t)] = \mathcal{L}(1) + \mathcal{L}\left(t^3e^{-3t}\right) + \mathcal{L}\left(3te^{-t}\right) + \mathcal{L}\left(3t^2e^{-2t}\right)$$
$$= \frac{1}{s} + (-1)^3\frac{d^3}{ds^3}\left(\frac{1}{s+3}\right) + 3(-1)^1\frac{d}{ds}\left(\frac{1}{s+1}\right) + 3(-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right) \quad \text{(Using 4th property)}$$
$$= \frac{1}{s} + \frac{6}{(s+3)^4} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3}.$$

Futuring

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