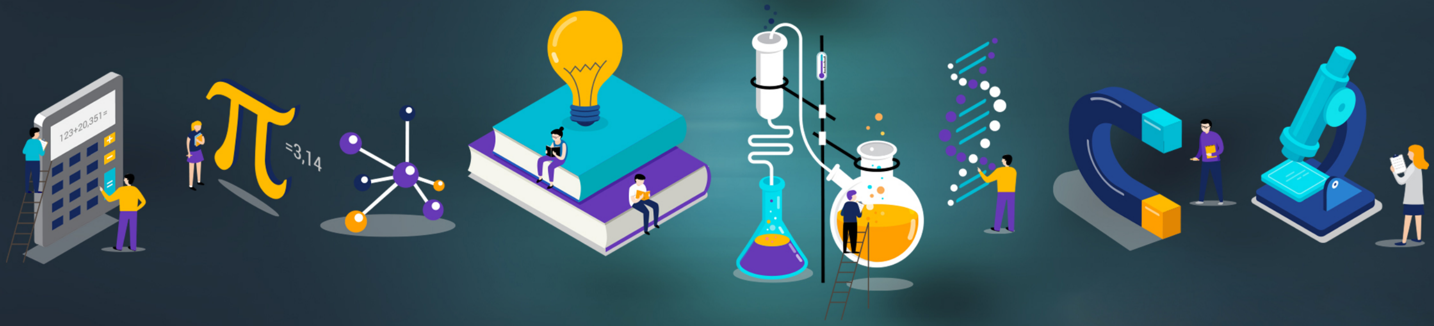


1. Dirac Delta Function

In mathematical models of physical systems we often come across functions that have finite or infinite discontinuities (Potential barriers, Impulse functions). Even though they dont belongs to the generalm definition of functions we can represent them as generalised function or distributions. The most common among them are the step function and the dirac delta function.

1.1 The Step Function

Let's start with the definition of the unit step function, $\theta(x)$:

$$\theta(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}$$

We do not define $\theta(x)$ at $x = 0$. Rather, at $x = 0$ we think of it as in transition between 0 and 1 .The function is called the unit step function because it takes a unit step at $x = 0$. It is sometimes called the **Heaviside function**. The graph of $\theta(x)$ is simple. It is obvious that $\theta(x)$ has a finite jump at $x = 0$. It is sometimes convenient to define $\theta(0)$ to be the average value $\frac{1}{2}$, but this is not always necessary.

$$\begin{aligned} \text{The sum } \; & \theta(x) + \theta(-x) = 1 \\ \text{The difference } \; & \theta(x) - \theta(-x) = \varepsilon(x) \end{aligned}$$

Where, $\varepsilon(x)$ is the signum function.

$$\varepsilon(x) = \begin{cases} +1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \end{cases} \tag{1.1}$$

The function $\varepsilon(x)$ looks like the limit of a tanh (or hyperbolic tangent) function as the 'kink' in the function becomes more and more steep, i.e., as the slope at the origin tends to infinity, as shown in Figure.1.1 In fact, we could define $\varepsilon(x)$ as the limit of a continous sequence of functions $\tanh\left(\frac{x}{\varepsilon(x)}\right)$.

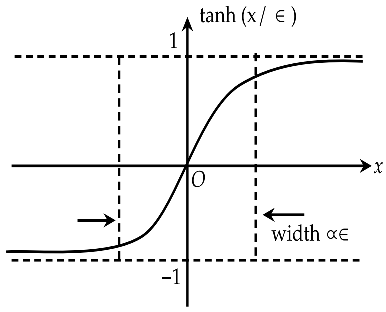


Figure 1.1: The function of $\tanh(x/\varepsilon)$

1.2 Dirac Delta Function

1.2.1 Kronecker delta δ

let us consider a sequence $(a_1, a_2, \ldots) = \{a_j \mid j = 1, 2, \ldots\}$. How do we select a particular member a_i from the sequence? We do so by summing over all members of the sequence with a selector called the **Kronecker delta**, denoted by δ_{ij} and defined as $\delta_{ij} \overset{\text{def.}}{=} \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ It follows immediately that

$$\sum_j \delta_{ij} a_j = a_i$$

$$\sum_j \delta_{ij} = a_i \qquad \text{For each value of } \; i$$

$$\delta_{ij} = \delta_{ji} \qquad \text{Symmetry property.}$$

Now if we have a continuos function, we must replace the summation over j by an integration over x. The role of the specified index i is played by the specified point a. The analog of the Kronecker delta is written like a function, retaining the same symbol δ for it. So we seek a 'function' $\delta(x-a)$ such that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a) \tag{1.2}$$

Exactly as in the discrete case of the Kronecker delta, we impose the normalization and symmetry properties,

$$\int_{-\infty}^{\infty} \delta(x-a) \, dx = 1 \quad \text{ and } \quad \delta(x-a) = \delta(a-x) \tag{1.3}$$

$\delta(x-a)$ is more like the kernel of an integral

$$\int_{-\infty}^{\infty} \delta(x-a) \, dx = 0 \quad \text{ if } \quad x \neq a \quad \text{Or the limit of integration excludes the point `}a\text{'} \tag{1.4}$$

1.3 Various representations of Delta function

The delta function may be approximated by the sequences of functions,

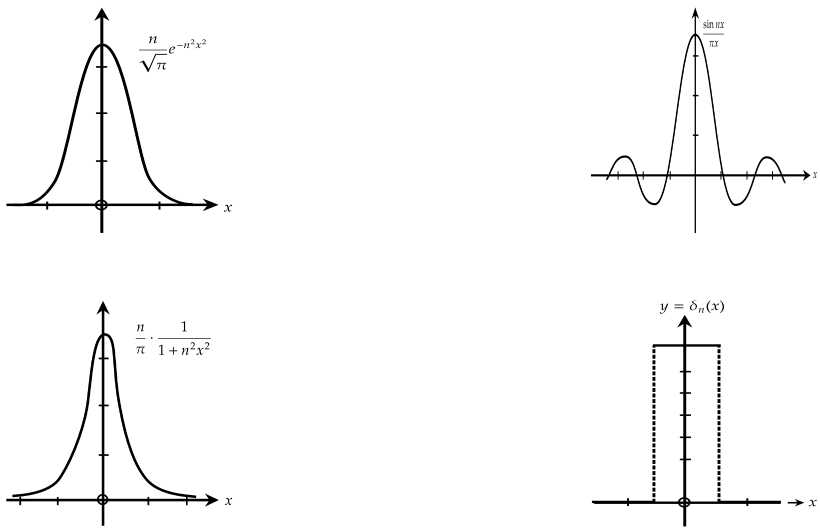
$$\delta_n(x) = \begin{cases} 0, & x < -\frac{1}{2n} \\ n, & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0, & x > \frac{1}{2n} \end{cases}$$

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} \exp\left(-n^2 x^2\right)$$

$$\delta_n(x) = \frac{n}{\pi} \cdot \frac{1}{1+n^2 x^2}$$

$$\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi} \int_{-n}^{n} e^{ixt} dt$$

These approximations have varying degrees of usefulness.



1.3.1 Rectangular Function

Let us consider a rectangular fuunction defined as,

$$R_\varepsilon = \begin{cases} \frac{1}{2\varepsilon} & a-\varepsilon < x < a+\varepsilon \\ 0 & \text{Otherwise} \end{cases} \tag{1.5}$$

We can consider this as a rectangular window of width 2ε and height $\frac{1}{2\varepsilon}$ so that the area is unity. Let us plot the function as shown in the figure below,

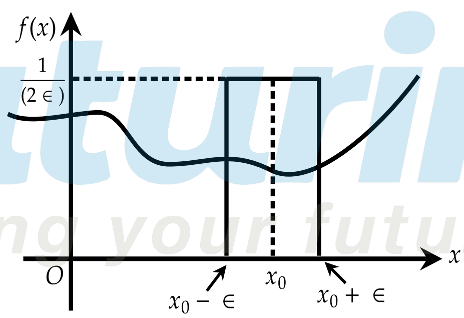


Figure 1.2: Rectangular function

When the window is centred at the choosen point x_0, the integral of $f(x)$ multiplied by this window function is,

$$\frac{1}{2\varepsilon}\int_{a-\varepsilon}^{a+\varepsilon} dx f(x)$$

if we take the limit,$\varepsilon \to 0$ The window becomes vanishingly small and it's height become arbitrarily large such that the area under the curve remains the same.

Then in the limit, $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0} R_\varepsilon = \delta(x-a)$$

An explicit form of the dirac delta function make sense only when it occures in an integral like,

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx \tag{1.6}$$

when it acts on ordinary function like $f(x)$ and an integration over x is carried out . Then the value of 1.6

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = \int_{a-\varepsilon}^{a+\varepsilon} \delta(x-a) f(x) dx \tag{1.7}$$

in the range $a-\varepsilon, a+\varepsilon$ as $\varepsilon \to 0$ the function is almost constant

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a) \lim_{\varepsilon \to 0} \int_{a-\varepsilon}^{a+\varepsilon} R_{\varepsilon} dx \tag{1.8}$$

$$= f(a) \tag{1.9}$$

1.4 Integral Representations for the Delta Function

Integral transforms, such as the Fourier integral

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(i\omega t) dt \tag{1.10}$$

Lead to the corresponding integral representations of Dirac's delta function. For example, take

$$\delta_n(t-x) = \frac{\sin n(t-x)}{\pi(t-x)} = \frac{1}{2\pi} \int_{-n}^{n} \exp(i\omega(t-x)) d\omega \tag{1.11}$$

Using Equation 1.11. We have

$$f(x) = \lim_{n \to \infty} \int_{-\infty}^{\infty} f(t) \delta_n(t-x) dt,$$

where $\delta_n(t-x)$ is the sequence in Eq. (1.192) defining the distribution $\delta(t-x)$. Note that Eq. (1.193a) assumes that $f(t)$ is continuous at $t=x$. If we substitute Eq. (1.192) into Eq. (1.193a) we obtain

$$f(x) = \lim_{n \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \int_{-n}^{n} \exp(i\omega(t-x)) d\omega dt.$$

Interchanging the order of integration and then taking the limit as $n \to \infty$, we have the Fourier integral theorem, Eq. (15.20). With the understanding that it belongs under an integral sign, as in Eq. (1.193a), the identification

1.5 The derivative of the δ function

Consider a bell shaped symmetric function peaked around $x = x_0$ and parametrized by ε as $\varepsilon \to 0$ the function resembles a dirac delta and the derivative starts to oscillate between positive and negative values around $x = x_0$. So the derivative $\delta'(n-n_0)$ is more singular than $8(x-x_0)$

$$\text{Finally} \quad \frac{d^n}{dx^n} \delta(n) \text{ is odd} = n \text{ odd}$$

$$\delta'(-x) = -\delta^0(x) \text{ even} = n \text{ even}$$

$$\text{and} \quad \delta'(x) = \frac{-\delta(x)}{x}$$

$$\text{with} \int_{-\infty}^{+\infty} \delta'(x) f(x) dx = -f'(a)$$

Proof: $\int\limits_{-\infty}^{+\infty} \delta'(x-a) f(x) dx = f(x) \int\limits_{-\infty}^{+\infty} \delta'(x-a) - \int\limits_{-\infty}^{+\infty} (f'(x)\delta'(x-a)dx)dx$

$= 0 - \int f'(x)\delta(x-a) = -f'(a)$,proved

1.6 The occurance of δ-function in physical problems

1. The dirac delta function models the density of a point source. Could be change density mass density etc. eg:$3\delta(x-x_0)\rightarrow$ a charge $3c$ placed at $x=x_0$

2. Suppose $P(x)dx$ represent the probability distribution for age group of students than if we have a part probability say N student has exactly age x_g then $P(x)dx+N\delta(x-x_g)$

3. An impulse function in 3-D

$$\begin{aligned}
\delta^3(r) &= \delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \\
&= \frac{1}{(2\pi)^3}\int_{+\infty}^{-\infty} dk_1 \int_{+\infty}^{-\infty} dk_2 \int_{+\infty}^{-\infty} dk_3 e^{ikx_1+k_2y+k_3z} \\
\delta^3(r) &= \frac{1}{(2\pi)^3}\int d^3k e^{ik\cdot r}
\end{aligned}$$

The dimension of $\delta(x)$will be the inverse x
if x has a dimenssion L
Then $[\delta(x)]\left[L^{-1}\right]$

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