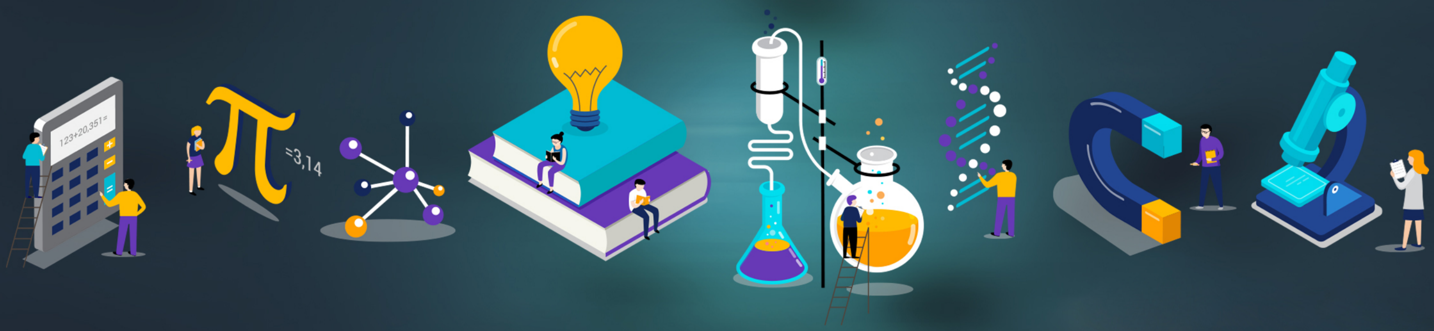


1. Power Series Solution and Special Functions

1.1 Series Solution Method

Series expansion is a method of obtaining one solution of the linear, second-order, homogeneous ODE. The method, will always work, provided the point of expansion is no worse than a regular singular point. In physics this very gentle condition is almost always satisfied. A linear, second-order, homogeneous ODE can be written in the form

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0 \tag{1.1}$$

The most general solution of the equation 1.1 may be written as,

$$y(x)=c_1y_1(x)+c_2y_2(x) \tag{1.2}$$

But a physical problem may lead to a nonhomogeneous, linear, second-order ODE

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=F(x) \tag{1.3}$$

Hence the most general solution to the equation will be of the form,

$$y(x)=c_1y_1(x)+c_2y_2(x)+y_p(x) \tag{1.4}$$

The constants c_1 and c_2 will eventually be fixed by boundary conditions.

There are two series solution method for differential equation,

1. **Simple series expansion method**
2. **Frobenious Method**

1.1.1 Simple Power Series Expansion Method

The simple series expansion method works for differential equations whose solutions are well-behaved at the expansion point $x=0$. We illustrate the method of series solution by solving the following simple equation (which you can easily solve by elementary methods also!).

$$y'=2xy \tag{1.5}$$

We assume a solution of this differential equation in the form of a power series, namely

$$\begin{aligned} y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \\ &= \sum_{n=0}^{\infty} a_n x^n \end{aligned} \tag{1.6}$$

where the a 's are to be found. Differentiating (1.2) term by term, we get

$$\begin{aligned} y' &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} \end{aligned} \tag{1.7}$$

We substitute (1.6) and (1.24) into the differential equation (1.5); we then have two power series equal to each other. Now the original differential equation is to be satisfied for all values of x, that is, y' and $2xy$ are to be the same function of x. Since a given function has only one series expansion in powers of x (see Chapter 1 , Section 11), the two series must be identical, that is, the coefficients of corresponding powers of x must be equal. We get the following set of equations for the a 's:

$$a_1 = 0, \quad a_2 = a_0, \quad a_3 = \frac{2}{3} a_1 = 0, \quad a_4 = \frac{1}{2} a_0 \tag{1.8}$$

or in general:

$$n a_n = 2 a_{n-2}, \quad a_n = \begin{cases} 0, & \text{odd } n, \\ \frac{2}{n} a_{n-2}, & \text{even } n. \end{cases} \tag{1.9}$$

Putting $n = 2m$ (since only even terms appear in this series), we get

$$a_{2m} = \frac{2}{2m} a_{2m-2} = \frac{1}{m} a_{2m-2} = \frac{1}{m} \frac{1}{m-1} a_{2m-4} = \cdots = \frac{1}{m!} a_0 \tag{1.10}$$

Substituting these values of the coefficients into the assumed solution (1.6) gives the solution

$$y = a_0 + a_0 x^2 + \frac{1}{2!} a_0 x^4 + \cdots + \frac{1}{m!} a_0 x^{2m} + \cdots = a_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{m!} \tag{1.11}$$

1.1.2 Classical Linear Oscillator

$$\frac{d^2 y}{dx^2} + \omega^2 y = 0 \tag{1.12}$$

$$\text{with known solutions } y = \sin \omega x, \cos \omega x \tag{1.13}$$

$$\text{We try } y(x) = x^k \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \right) \tag{1.14}$$

$$= \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}, \quad a_0 \neq 0 \tag{1.15}$$

with the exponent k and all the coefficients a_λ still undetermined. Note that k need not be an integer. By differentiating twice, we obtain

$$\frac{dy}{dx} = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1} \tag{1.16}$$

$$\frac{d^2 y}{dx^2} = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} \tag{1.17}$$

By substituting into equation.1.12, we have

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda-2} + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0 \tag{1.18}$$

The coefficients of each power of x on the left-hand side of equation.1.18 must vanish individually.The lowest power of x appearing in equation.1.18 is x^{k-2}, for $\lambda = 0$ in the first summation. The requirement that the coefficient vanish yields,

$$a_0 k(k-1) = 0 \tag{1.19}$$

We had chosen a_0 as the coefficient of the lowest nonvanishing terms of the series 1.15, hence, by definition, $a_0 \neq 0$. Therefore we have,

$$k(k-1) = 0 \tag{1.20}$$

This equation, coming from the coefficient of the lowest power of x, we call the indicial equation. The indicial equation and its roots are of critical importance to our analysis.
From equation.1.20, $k = 0$ or $k = 1$
The only way a power series can be zero is, it's coefficients must be equal to zero. But here the power of x in the equation do not match up. The Coefficent of x in the first term is,$k+\lambda-2$ and for the second term it is,$k+\lambda$, to make them equal, we can replace λ by $\lambda+2$ in the first term. Then we get,

$$\sum_{\lambda=2}^{\infty} a_{\lambda+2}(k+\lambda+2)(k+\lambda+1)x^{k+\lambda} + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0 \tag{1.21}$$

$$\sum_{\lambda=2}^{\infty} a_{\lambda+2}(k+\lambda+2)(k+\lambda+1) + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda = 0 \tag{1.22}$$

Here the coefficients are independent summations and λ is a dummy index. Then we get,

$$a_{\lambda+2}(k+\lambda+2)(k+\lambda+1) + \omega^2 a_\lambda = 0 \tag{1.23}$$

$$a_{\lambda+2} = -a_\lambda \frac{\omega^2}{(k+\lambda+2)(k+\lambda+1)} \tag{1.24}$$

For this example, if we start with a_0, Equation.1.24 leads to the even coefficients a_2, a_4, and so on, and ignores a_1, a_3, a_5, and so on. Since a_1 is arbitrary if $k = 0$ and necessarily zero if $k = 1$,

$$a_3 = a_5 = a_7 = \cdots = 0$$

and all the odd-numbered coefficients vanish. The odd powers of x will actually reappear when the second root of the indicial equation is used.

$$a_{\lambda+2} = -a_\lambda \frac{\omega^2}{(\lambda+2)(\lambda+1)} \tag{1.25}$$

which leads to

$$a_2 = -a_0 \frac{\omega^2}{1 \cdot 2} = -\frac{\omega^2}{2!} a_0 \tag{1.26}$$

$$a_4 = -a_2 \frac{\omega^2}{3 \cdot 4} = +\frac{\omega^4}{4!} a_0 \tag{1.27}$$

$$a_6 = -a_4 \frac{\omega^2}{5 \cdot 6} = -\frac{\omega^6}{6!} a_0, \quad \text{and so on.} \tag{1.28}$$

By inspection (and mathematical induction),

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!} a_0 \tag{1.29}$$

and our solution is

$$y(x)_{k=0} = a_0 \left[1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} - \frac{(\omega x)^6}{6!} + \cdots\right] \tag{1.30}$$

$$= a_0 \cos \omega x \tag{1.31}$$

If we choose the indicial equation root $k = 1$ Equation.1.24, the recurrence relation becomes

$$a_{j+2} = -a_j \frac{\omega^2}{(j+3)(j+2)} \tag{1.32}$$

Substituting in $j = 0, 2, 4$, successively, we obtain

$$a_2 = -a_0 \frac{\omega^2}{2 \cdot 3} = -\frac{\omega^2}{3!} a_0 \tag{1.33}$$

$$a_4 = -a_2 \frac{\omega^2}{4 \cdot 5} = +\frac{\omega^4}{5!} a_0 \tag{1.34}$$

$$a_6 = -a_4 \frac{\omega^2}{6 \cdot 7} = -\frac{\omega^6}{7!} a_0, \quad \text{and so on.} \tag{1.35}$$

Again, by inspection and mathematical induction,

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n+1)!} a_0 \tag{1.36}$$

For this choice, $k = 1$, we obtain

$$y(x)_{k=1} = a_0 x \left[1 - \frac{(\omega x)^2}{3!} + \frac{(\omega x)^4}{5!} - \frac{(\omega x)^6}{7!} + \cdots\right] \tag{1.37}$$

$$= \frac{a_0}{\omega} \left[(\omega x) - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} - \frac{(\omega x)^7}{7!} + \cdots\right] \tag{1.38}$$

$$= \frac{a_0}{\omega} \sin \omega x \tag{1.39}$$

Power Series Solution (About an Ordinary Point)

Find the power series solution of $\left(1 - x^2\right) y'' - 2xy' + 2y = 0$ about $x = 0$

Since $x = 0$ is an ordinary point of the given differential equation, the solution can be written as

$$y = \sum_{k=0}^{\infty} a_k x^k$$

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} k a_k x^{k-1}$$

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k k(k-1) x^{k-2}$$

Substituting these values in the given equation we get,

$$\left(1 - x^2\right) \sum_k a_k k(k-1) x^{k-2} - 2x \sum_k a_k (k) x^{k-1} + 2 \sum_k a_k x^k = 0$$

$$\sum_{k=2} a_k k(k-1)x^{k-2} - \sum \left(k^2+k-2\right) a_k x^k = 0$$

now equating the coefficient of x^k then

$$\begin{aligned}
(k+2)(k+1)a_{k+2} - \left(k^2+k-2\right) a_k &= 0 \\
a_{k+2} &= \frac{k-1}{(k+1)} a_k \\
\text{For } k = 0 \Rightarrow a_2 &= -a_0 \\
k = 1 \Rightarrow a_3 &= 0 \\
k = 2 \Rightarrow a_4 &= \frac{a_2}{3} = \frac{-a_0}{3} \\
k = 3 \Rightarrow a_5 &= \frac{2}{4} a_3 = 0 \\
\text{Therefore, solution } y &= a_0 + a_1 x + a_2 x^2 + \ldots\ldots \\
&= a_0 \left[1 - x^2 - \frac{x^4}{3} \ldots\ldots\right] + a_1 x
\end{aligned}$$

1.2 Singularity

Consider differential equation,

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \tag{1.40}$$

1. If $P(x)$ and $Q(x)$ remain finite at $x = x_0$ then it is ordinary point
2. If either $P(x)/Q(x)$ or both tends to ∞ at $x = x_0$ then it is singular point.

1.2.1 Kinds of singular points

1. **Regular singularity**

   IF $P(x), Q(x)$ diverges but $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ remain finite at $x \to x_0$ then $x = x_0$ is called regular or non essencial singular points

2. **Irregular singularity**

   $P(x), Q(x)$ diverges and either $(x-x_0)P(x)$ or $(x-x_0)^2Q(x)$ or both remain infinite at $x \to x_0$ then $x = x_0$ is called irregular or essential singular points

Little bit review of Ordinary Differential Equations

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

Using Tailor series expansion method. If a function was continues and differentable you could express it as an infinite series of polynomial terms being added.

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n (x-x_0)^n \\
y'(x) &= \sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1} \\
y''(x) &= \sum_{n=0}^{\infty} n(n-1) a_n (x-x_0)^{n-2}
\end{aligned}$$

Plugging this value into the differential equation we can express as

$$\sum_{n=0}^{\infty} n(a-1)a_n(x-x_0)^{n-2} + \sum_{n=0}^{\infty} na_n(x-x_0)^{n-1}P(x) + Q(x)\sum_{n=0}^{\infty} a_n(x-x_0)^n = R(x)$$

$P(x), Q(x)$ and $R(x)$ are also functions of x. It's also possible to express them as power series

$$P(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n \quad Q(x) = \sum_{n=0}^{\infty} c_n(x-x_0)^n, R(x) = \sum_{n=0}^{\infty} d_n(x-x_0)^n$$

$$\sum_{n=0}^{\infty} n(n-1)a_n(x-x_0)^{n-2} + \sum_{n=0}^{\infty} b_n(x-x_0)^n \sum_{n=0}^{\infty} na_n(x-x_0)^{n-1} + \sum_{n=0}^{\infty} c_n(x-x_0)^n \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

$$= \sum_{n=0}^{\infty} d_n(x-x_0)^n$$

To find a_n, we should properly expand $P(x), Q(x)$ and $R(x)$ abt $x = x_0$ If any one of function is undefined at x_0 then we would not be able to do it.

Example:

$$\frac{d^2y}{dx^2} - 4y = 0,\ y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

It will always continuos and differentiable about $x_0 = 0$, So we will have a valid tylor series expansion about this point and we can proceed with a power series expansion method.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=0}^{\infty} na_n x^{n-1} \qquad y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$

substituting into differential equation

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 4\sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

expanding first two terms of the $\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$ we can express as

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 4\sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

$$\text{combining } \sum_{n=2}^{\infty} [n(n-1)a_n - 4a_{n-2}]x^{n-2} = 0$$

$$\text{since RHS } = 0$$

$$\implies n(n-1)a_n - 4a_{n-2} = 0$$

If relates the n^{th} term in the sequence of coefficient to terms that occured previously in that sequence

$$a_n = \frac{4a_{n-2}}{n(n-1)}$$

n^{th} coefficient is related to two terms back in the sequence
we have two solutions for differential equations, even part and odd part

Case 1 $\rightarrow n =$Even

$$a_2 = \frac{4a_0}{2.1} \quad a_4 = \frac{4a_2}{4.3}$$
$$= \frac{4.4\, a_0}{4.3\, 2.1}$$

A pattern is developing

$$a_4 = \frac{2^4\, a_0}{4!} \quad \text{it follows that}$$
$$a_n = \frac{2^2 a_0}{n!}$$

Since a is even

$$a_{2k} = \frac{2^{2k\, a_0}}{(2k)!}$$

any even number can represent as multiple of 2

here $k = 0, 1, 2, 3$

Case 2 $\rightarrow n =$Odd

$$a_3 = \frac{4\, a_1}{3.2.1}, \; a_5 = \frac{4\, a_3}{5.4} = \frac{4.4.a_1}{5.4.3.2.1}$$
$$a_5 = \frac{2^4\, a_1}{5!} \quad \therefore a_n = \frac{2^{n-1}\, a_1}{n!}$$

It can be expressed as

$$a_{2k+1} = \frac{2^{2k}\, a_1}{(2k+1)!}$$

substituting in $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and expanding,

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots a_{2k} x^{2k} + a_{2k+1} x^{2k+1} + \ldots$$
$$= a_0 + a_1 x + \frac{2^2\, a_0}{2!} x^2 + \frac{2^2\, a_1}{3!} x^3 + \cdots + \frac{2^{2k}\, a_0}{(2k)!} + \frac{2^{2k}\, a_1}{(2k+1)!} + \ldots$$

By seperating even and odd solutions we can express as

$$y(x) = \sum_{k=0}^{\infty} \frac{2^{2k}\, a_0}{(2k)!} x^{2k} + \sum_{k=0}^{\infty} \frac{2^{2k}\, a_1}{(2k+1)!} x^{2k+1}$$

Here we discuss series solution and how to use power series to solve differential equations.Now we can check how to use this methode to solve one of the important problems in maths and physics, The legendre diffrential equation problem.

1.3 Legendre Function

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + k(k+1)y = 0$$

to expand it in $x_0 = 0$

$$\frac{d^2y}{dx^2} + \frac{P(x)dy}{dx} + Q(x)y = R(x)(\text{General form})$$

To compare our equation with General form we can express as

$$\frac{d^2y}{dx^2} - \frac{2x}{(1-x^2)}\frac{dy}{dx} + \frac{k(k+1)}{1-x^2}y = 0$$

$P(x), Q(x)$ and $R(x)$ are all defined at $x=0$, So it is possible for regular series solution at $x_0=0$

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y'(x) &= \sum_{n=0}^{\infty} n a_n x^{n-1} \\
y''(x) &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \\
(1-x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=0}^{\infty} n a_n x^{n-1} &+ k(k+1)\sum_{n=0}^{\infty} a_n x^n = 0 \\
\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^n &- \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} k(k+1)a_n x^n = 0
\end{aligned}
$$

To combain the terms, we have to make power of x and limits same, after necessary steps we can express as

$$
\begin{aligned}
&\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} k(k+1)a_n x^n = 0 \\
&\sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) - a_n n(n-1) - 2n a_n + k(k+1)a_n\right] x^n = 0 \\
&a_{n+2}(n+2)(n+1) - a_n n(n-1) - 2n a_n + k(k+1)a_n = 0 \\
&a_{n+2} = \frac{a_n[(n-k)(n+k)+(n-k)]}{(n+2)(n+1)} = \frac{a_n(n-k)(n+k+1)}{(n+2)(n+1)}
\end{aligned}
$$

Case 1 $\quad \rightarrow n =$Even

$$
\begin{aligned}
a_2 &= \frac{a_0 \cdot (-k)(x+1)}{2\cdot 1} \\
a_4 &= \frac{a_2(2-k)(3+k)}{4\cdot 3} = \frac{(k-2)k(k+1)(k+3)a_0}{4!}
\end{aligned}
$$

Case 2 $\quad \rightarrow n =$Odd

$$
\begin{aligned}
a_3 &= \frac{a_1(1-k)(2+k)}{3\cdot 2} \\
a_5 &= \frac{9_3(3-k)(4+k)}{5.4} = \frac{a_1(k-3)(k-1)(k+2)(k+4)}{5!}
\end{aligned}
$$

1. What if $k=1$

$$a_3 = 0$$

all coefficients come later on $a_3=0$ since they depend on a_3

so solution, $y_{odd} = a_1 x = a_1 P_1(x)$

$P_1(x)$ is just a function of x, even series doesnot terminate

2. What if $k=2$

$$
\begin{aligned}
a_2 &= a_0\frac{(-2)(3)}{2} = -3a_0,\ a_4 = 0 \\
y_{\text{even}} &= a_0 - 3a_0x^2 = a_0\left(1-3x^2\right) \\
&= \cdot 5\left(3x^2-1\right) = P_2(x) \text{ for } a_0 = \frac{-1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\text{for } k &= 2, \text{ odd series doesnot terminate}\\
k &= 3\\
y_{odd} &= -\frac{5}{3}a_1x^3 + a_1x\\
&= \frac{1}{2}\left(5x^3 - 3x\right) = P_3(x) \text{ for } a_1 = -3/2
\end{aligned}
$$

$$\text{even series doesnot terminate for odd value of } k$$

we can repeat this for different value of k, for even k, even series terminates and for odd k, odd series terminates P_1, P_2, P_3 they result from series terminating are not like just other polynomial they are special polynomial. P_1, P_2, P_3 are called legendre polynomials, they are solutions to legendre diffrential equations they come up in electromagnetism and quantum mechanics. Bunch of techniques already developed to calculate legendre polynomials more easily and one of these techniques is rodrigues formula $P_k(x) = \frac{1}{2^k k!}\frac{d^k}{dx^k}\left[\left(x^2-1\right)^k\right]$

1.4 Frobenius Method

For power series solution, $y'' + P(x)y' + Q(x)y = R(x)$ only have valid series solution about $x = x_0$ if $P(x), Q(x)$ and $R(x)$ have valid tylor series at x_0. We want them to be continues and differentiable at that point. What if $P(x), Q(x), R(x)$ did not have valid tylor expansions about x_0 but still wanted series solution about x_0? We could do it by using a slightly modified version. All is to do include r to the regular series solution. r is a real number. This is called Frobenius method. Using Frobenius method still can have solution to be power series solution at singular point. But point has to be regular singular point.

$$
\begin{aligned}
&2(x-1)^2y'' - (x-1)y' + y = 0\\
&2y'' - \frac{1}{(x-1)}y' + \frac{1}{(x-1)^2}y = 0
\end{aligned}
$$

x=1 makes this undefined, but point 1 is a regular singular point. So we can express using frobenius method as

$$
\begin{aligned}
&y = \sum_{k=0}^{\infty} a_k(x-1)^{k+\lambda}\\
&y' = \sum_{k=0}^{\infty}(n+\lambda)a_k(x-1)^{k+\lambda-1} \qquad y'' = \sum_{k=0}^{\infty}(k+\lambda-1)(k+\lambda)a_k(x-1)^{k+\lambda-2}\\
&2\sum_{k=0}^{\infty}(k+\lambda-1)(k+\lambda)a_k(x-1)^{k+\lambda-2} - \frac{1}{x-1}\sum_{k=0}^{\infty}(k+\lambda)a_k(x-1)^{k+\lambda-1} + \frac{1}{(x-1)^2}\sum_{k=0}^{\infty}a_k(x-1)^{k+\lambda} = 0\\
&2\sum_{k=0}^{\infty}(k+\lambda-1)(k+\lambda)a_k(x-1)^{k+\lambda-2} - \sum_{k=0}^{\infty}(k+\lambda)a_k(x-1)^{k+\lambda-2} + \sum_{k=0}^{\infty}a_k(x-1)^{k+\lambda-2} = 0\\
&\sum_{k=0}^{\infty}\left[2(k+\lambda)(k+\lambda-1)a_k - (k+\lambda)a_k + a_k\right](x-1)^{k+\lambda-2} = 0
\end{aligned}
$$

expanding few terms

$$[2\lambda(\lambda-1) - \lambda + 1]\,a_0(x-1)^{\lambda-2} + [2\lambda(\lambda+1) - (\lambda+1) + 1]a_1(x-1)^{\lambda-1} + \ldots + = 0$$

equating coefficients of $(x-1)^{\lambda-2}$

$$2\lambda^2 - 3\lambda + 1 = 0$$

This is a quadratic equation and we call it as indicial equation.

$$\lambda_1 = 1 \qquad \lambda_2 = 0.5$$

$\therefore$ For equality to hold $a_1, a_2 .....a_k, .. = 0$, what does this mean for final answer?
$y = \sum_{k=0}^{\infty} a_k (x-1)^{k+\lambda} = a_0 (x-1)^{\lambda}$only first term correspond to $k = 0$ is in the final expression, because of two vaues of r.
$y_1 = a_0(x-1), y_2 = a_0(x-1)^{.5}$.
General solution is : $y = c_1(x-1) + c_2\sqrt{x-1}$

1.5 Bessel Function

Bessel's equation in the usual standard form is,

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right) y = 0 \tag{1.41}$$

Where n is a constant (not necessarily an integer) called the order of the Bessel function $'y'$ which is the solution of equation.1.41. $x = 0$ is a regular singular point of the Bessel differential equation.

1.5.1 The solution of Bessel's function

$$x^2 y'' + xy' + \left(x^2 - n^2\right) y = 0$$

$x_0 = 0$ is a regular singular point, so we can expand about $x_0 = 0$ using Frobenius method

$$y = \sum_{k=0}^{\infty} a_k x^{k+\lambda} \quad y' = \sum_{k=0}^{\infty} (k+\lambda) a_k x^{k+\lambda-1} \quad y'' = \sum_{k=0}^{\infty} (k+\lambda)(k+\lambda-1) a_k x^{k+\lambda-2}$$

$$\sum_{k=0}^{\infty} (k+\lambda)(k+\lambda-1) a_k x^{k+\lambda-2} + \frac{1}{x}\sum_{k=0}^{\infty} (k+\lambda) a_k x^{k+\lambda-1} + \left(1 - \frac{n^2}{x^2}\right)\sum_{k=0}^{\infty} a_k x^{k+\lambda} = 0$$

$$\sum_{k=0}^{\infty} (k+\lambda)(k+\lambda-1) a_k x^{k+\lambda-2} + \sum_{k=0}^{\infty} (k+\lambda) a_k x^{k+\lambda-2} + \sum_{n=0}^{\infty} a_k x^{k+\lambda} - n^2 \sum_{k=0}^{\infty} a_k x^{k+\lambda-2} = 0$$

$$\sum_{k=0}^{\infty} (k+\lambda)(k+\lambda-1) a_k x^{k+\lambda-2} + \sum_{k=0}^{\infty} (k+\lambda) a_k x^{k+\lambda-2} + \sum_{k=2}^{\infty} a_{k-2} x^{k+\lambda-2} - n^2 \sum_{k=0}^{\infty} a_k x^{k+\lambda-2} = 0$$

$$\sum_{k=2}^{\infty} (k+\lambda)(k+\lambda-1) a_n x^{k+\lambda-2} + \sum_{k=2}^{\infty} (k+\lambda) a_k x^{k+\lambda-2} + \sum_{k=2}^{\infty} a_{k-2} x^{k+\lambda-2} - n^2 \sum_{k=2}^{\infty} a_k x^{k+\lambda-2}$$
$$+\lambda(\lambda-1)a_0 x^{\lambda-2} + \lambda(\lambda+1)a_1 x^{\lambda-1} + \lambda a_0 x^{\lambda-2} + (\lambda+1)a_1 x^{\lambda-1} - n^2 a_0 x^{\lambda-2} - n^2 a_1 x^{\lambda-1} = 0$$

right hand side is 0, so each term of left hand side should be equal to 0 and we can express like this

$$\text{for} x^{\lambda-2}, \lambda(\lambda-1)a_0 + \lambda a_0 - n^2 a_0 = 0, \text{we want } a_0 \neq 0, \text{ so we can write as,}$$
$$\lambda(\lambda-1) + \lambda - n^2 = 0 \quad \lambda^2 - n^2 = 0 \quad \lambda = \pm n, \text{this is our indicial equations}$$

Plugging the values of λ in $x^{\lambda-1}$We can show $a_1 = 0$, and what left with the equation is

$$\sum_{k=2}^{\infty} (k+\lambda)(k+\lambda-1) a_k x^{k+\lambda-2} + \sum_{n=2}^{\infty} (k+\lambda) a_k x^{k+\lambda-2} + \sum_{k=2}^{\infty} a_{k-2} x^{k+\lambda-2} - n^2 \sum_{k=2}^{\infty} a_k x^{k+\lambda-2} = 0$$

$$\sum_{k=2}^{\infty} \left(((k+\lambda)(k+\lambda-1) + (k+\lambda)n^2)a_k + a_{k-2}\right) x^{k+\lambda-2} = 0$$

$$\left[(k+\lambda)(k+\lambda-1) + (k+\lambda) - n^2\right] a_k + a_{k-2} = 0$$

$$a_k = \frac{-a_{k-2}}{[(k+\lambda)^2 - n^2]}$$

substitute $\lambda_1 = n$

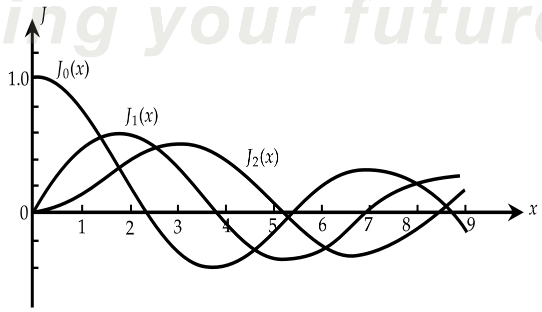
$$a_k = \frac{-a_{k-2}}{k(k+2n)} \implies \text{recursion}$$
$$\text{since } a_1 = 0,\, a_3,\ a_5,\ a_7, ... = 0$$

For even intiger coefficients, $k = 2r$

$$a_{2r} = -\frac{a_{2r-2}}{4r(r+n)}$$
$$a_2 = \frac{-a_0}{4 \cdot 1(1+n)}$$
$$a_4 = -\frac{-a_2}{4 \cdot 2(2+n)} = \frac{(-1)^2 a_0}{4^2(2 \cdot 1)(2+n)(1+n)}$$
$$\therefore a_{2r} = \frac{(-1)^r}{4r} \frac{a_0}{r!(n+r)\cdots(n+1)}$$
$$y_1 = x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{4r} \frac{a_0}{r!(n+r)..(n+1)} x^{2r}$$

For particular value of a_0 this solution is given a special name, $J_n(x)$, Bessel function of I kind and n is order of Bessel function. The solution of Bessel's function, $J_n(x)$ is defined to be a Bessel function of the first kind, of integral order r.

$$\mathrm{J_n(x)} = y = a_0 \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!} \left(\frac{x}{2}\right)^{n+2r} \tag{1.42}$$
$$\text{Where } \mathrm{a_0} = \frac{1}{2^n \Gamma(n+1)}$$
$$\mathrm{n} = 0 \quad \Rightarrow \quad \mathrm{J_0(x)} = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\frac{x}{2}\right)^{2r} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots.. \tag{1.43}$$
$$\mathrm{n} = 1 \quad \Rightarrow \quad \mathrm{J_1}(x) = \frac{x}{2} - \frac{x^3}{2^2 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \ldots. \tag{1.44}$$



The second solution of Bessel's function

We have found just one of the two solutions of Bessel's equation, that is, the one when $n > 0$; we must next find the solution when $n < 0$. We can just replace n by $-n$ in equation.1.42. The solution in $-n$ is usually written as $J_{-n}(x)$

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(r-n)!} \left(\frac{x}{2}\right)^{2r-n}. \tag{1.45}$$

And,

$$J_{-n}(x) = (-1)^n J_n(x) \quad \text{for integral } n \tag{1.46}$$

Properties of Bessel's function.

1. Bessel functions are oscillatory function with varying period and decreasing amplitude.
2. $J_n(x)$ is an even function when n is even and $J_n(x)$ is an odd function when n is odd.

3. $J_0(x)$ has nodes at $x = 2.408, 5.5201$......and $J_1(x)$ has nodes at $x = 3.8317, 7.015$.......

Note The Bessel's functions for $n = \pm\frac{1}{2}$,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$
$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

1.5.2 Generating function of Bessel's function:

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} z^n J_n(x)$$

The coefficient of z^n in the expansion of the generating tünction is the Bessel function of order 1.

1.5.3 Orthogonal condition of Bessel Function:

$$\int_0^1 x_n \cdot (\alpha x) J_n(\beta x) dx = \frac{\delta_{\alpha\beta}}{2} \cdot [J_{n+1}(\alpha)]^2$$

1.5.4 Recurrencé relations for $J_n(x)$:

1. $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$
2. $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$
3. $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$
4. $J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$
5. $J_n'(x) = -\frac{n}{x} J_n(x) + J_{n-1}(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$

Exercise 1.1 Show that $\sqrt{\frac{\pi x}{2}} J_{3/2}(x) = \frac{\sin x}{x} - \cos x$ ■

Solution:

Using recurrence relation, $2nJ_n = x[J_{n+1} + J_{n-1}]$

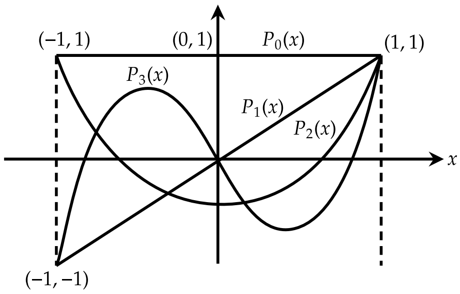
$$\text{puttingn} = 1/2, \text{ we get, } J_{1/2} = x\left[J_{3/2} + J_{-1/2}\right]$$
$$xJ_{3/2} = J_{1/2} - xJ_{-1/2}$$
$$\sqrt{\frac{\pi x}{2}} J_{3/2} = \frac{\sin x}{x} - \cos x \text{ (using the expressions of } J_{1/2} \& J_{-1/2}\text{)}$$

1.6 Legendre Differential Equation

1.6.1 Rodrigue's Formula for Legendre's Polynomial $P_n(x)$:

$$\text{Legendre polynomial of order '}n\text{' : } P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n$$

$$P_0(x)=1;\quad P_1(x)=x;\quad P_2(x)=\frac{1}{2}\left(3x^2-1\right);\quad P_3(x)=\frac{1}{2}\left(5x^3-3x\right)$$



1.6.2 Properties of Legendre's Polynomial

1. $P_n(1)=1$
2. $P_n(-1)=(-1)^n$
3. $\begin{aligned} P_n(x) &= \text{even function, if n} = \text{ even} \\ &= \text{odd function, if n} = \text{ odd} \end{aligned}$
4. $P_n(-x)=(-1)^n P_n(x)$

1.6.3 Generating function of Legendre's polynomial:

$\left(1-2xz+z^2\right)^{-1/2}$ is the generating function for legendre's polynomials i.e.

$$\left(1-2xz+z^2\right)^{-1/2}=\sum_{n=0}^{\infty} z^n P_n(x)$$

The co-efficient of z^n in the exapansion of the generating function is the Legendre's polynomial of order n.

1.6.4 Orthogonal properties of Legendre's polynomial:

$$\int_{-1}^{+1} P_m(x)P_n(x)dx=\frac{2}{2n+1}\delta_{mn} \text{ where } \delta_{mn} = \text{ Kronecker delta } \begin{aligned} &=1 \quad \text{if m} = \text{n} \\ &=0 \quad \text{if } m \neq n \end{aligned}$$

1.6.5 Recurrence relations for $P_n(x)$

1. $nP_n=(2n-1)xP_{n-1}-(n-1)P_{n-2}$
2. $nP_n=xP_n'-P_{n-1}'$
3. $(2n+1)P_n=P_{n+1}'-P_{n-1}'$
4. $(n+1)P_n=P_{n+1}'-xP_n'$
5. $\left(1-x^2\right)P_n'=n\left(P_{n-1}-xP_n\right)$
6. $\left(1-x^2\right)P_n'=(n+1)\left(xP_n-P_{n+1}\right)$

Exercise 1.2 Express $4x^3+6x^2+7x+2$ in terms of Legendre Polynomials. ■

Solution:

$$\begin{aligned} \text{Let, } 4x^3+6x^2+7x+2 &= a_3P_3(x)+a_2P_2(x)+a_1P_1(x)+a_0P_0(x) \\ &= \frac{a_3}{2}\left(5x^3-3x\right)+\frac{a_2}{2}\left(3x^2-1\right)+a_1x+a_0 \end{aligned}$$

$$\text{Comparing the co-efficient of } x^3: \frac{5a_3}{2} = 4 \Rightarrow a_3 = \frac{8}{5}$$
$$\text{Comparing the co-efficient of } x^2: \frac{3a_2}{2} = 6 \Rightarrow a_2 = 4$$
$$\text{Comparing the co-efficient of } x^1: -\frac{3a_3}{2} + a_1 = 7 \Rightarrow a_1 = \frac{47}{5}$$
$$\text{Comparing the co-efficient of } x^0: -\frac{a_2}{2} + a_0 = 2 \Rightarrow a_0 = 4$$
$$\text{Therefore, } 4x^3 + 6x^2 + 7x + 2 = \frac{8}{5}P_3(x) + 4P_2(x) + \frac{47}{5}P_1(x) + 4P_0(x)$$

Exercise 1.3 Show that, $\int_{-1}^{1} x^2 P_{n+1}P_{n-1}dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$ ■

Solution:

$$\text{Using the relation: } (2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$$
$$\text{Putting } (n+1) \text{ in place of n} \Rightarrow (2n+3)xP_{n+1} = (n+2)P_{n+2} + (n+1)P_n$$
$$\text{Putting } (n-1) \text{ in place of n} \Rightarrow (2n-1)xP_{n-1} = nP_n + (n-1)P_{n-2}$$
$$\int_{-1}^{1}(2n+3)(2n-1)x^2P_{n+1}P_{n-1}dx = n(n+2)\int_{-1}^{1}P_{n+2}P_ndx + n(n+1)\int_{-1}^{1}P_nP_ndx +$$
$$(n-1)(n+2)\int_{-1}^{1}P_{n-2}P_{n+2}dx + \int_{-1}^{1}P_nP_{n-2}dx$$
$$\int_{-1}^{1}x^2P_{n-1}P_{n+1}dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

Exercise 1.4 Show that $\sum_n P_n(x) = \frac{1}{\sqrt{2-2x}}$ ■

Solution:

$$\left(1 - 2xz + z^2\right)^{-1/2} = \sum_n z^n P_n(x)$$
$$\text{Putting } z = 1 \text{ in equation, } (1 - 2x + 1)^{-1/2} = \sum_n P_n(x)$$
$$\sum_n P_n(x) = \frac{1}{\sqrt{2-2x}}$$

Exercise 1.5 Prove that $P'_{n+1}(x) + P'_n(x) = P_0 + 3P_1 + 5P_2 + \ldots.. + (2n+1)P_n$ ■

Solution:

$$\text{Using the relation } (2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(\dot{x})$$
$$\text{putting n} = 1 \quad \Rightarrow 3P_1(x) = P'_2(x) - P'_0(x)$$
$$\text{putting n} = 2 \quad \Rightarrow 5P_2(x) = P'_3(x) - P'_1(x)$$
$$\text{putting n} = 3 \quad \Rightarrow 7P_3(x) = P'_4(x) - P'_2(x) \text{ and so on.}$$
Adding all the equations we get,
$$3P_1(x) + 5P_2(x) + 7P_3(x) + \ldots\ldots(2n+1)P_n(x) = -P'_0(x) - P'_1(x) + P'_n(x) + P'_{n+1}(x)$$

$$= 0 - P_0(x) + P_n'(x) + P_{n+1}'(x)$$

$$P_{n+1}'(x) + P_n'(x) = P_0 + 3P_1 + 5P_2 + \ldots.. + (2n+1)P_n$$

1.7 Hermite Differentiation Equation

The differential equation for Hermite functions is,

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0, (\mathrm{n} = 0, 1 \ldots..)$$

$x = 0$ is an ordinary point of the Hermite differential equation

The solution of the Hermite differential equation is a terminating series, so it is called Hermite polynomial of order n, i.e. $\mathrm{H_n(x)}$

$$H_n(x) = \sum_{r=0}^{P} \frac{(-1)^r n!(2x)^{n-2r}}{r!(n-2r)!} \quad \Rightarrow \left[\begin{array}{lll} P & = \frac{n}{2} & \text{if } n = \text{ even} \\ & = \frac{n-1}{2} & \text{if n} = \text{ odd} \end{array} \right]$$

1.7.1 Rodrigue formula for $H_n(x)$:

1. $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$
2. $H_0(x) = 1$
3. $H_1(x) = 2x$
4. $H_2(x) = 4x^2 - 2$
5. $H_3(x) = 8x^3 - 12x$
6. $H_4(x) = 16x^4 - 48x^2 + 12$

$\mathrm{H_n}(x)$ will be even function if n is even and will be odd function if n is odd. The Hermite polynomial of different order differ from the legendre polynomials with respect to the coefficient. So, the nature of the graphs of $\mathrm{H_n(x)}$ will be same as Legendre polynomial $P_\mathrm{n}(x)$.

1.7.2 Generating function for Hermite polynomial:

e^{2zx-z^2} is the generating function of hemite polynomial i.e.

$$e^{2zx-z^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n$$

The coefficient of z^n in the expansion of e^{2zx-z^2} is $\dfrac{H_n(x)}{n!}$

1.7.3 Orthogonal property of $H_n(x)$:

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 2^n n! \sqrt{\pi} \delta_{mn}$$

1.7.4 Recurrence relation for $\mathrm{H_n}(\mathbf{x})$:

1. $H_n'(x) = 2nH_{n-1}(x)$
2. $2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$
3. $H_n'(x) = 2xH_n(x) - H_{n+1}(x)$
4. $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$
5. $H_n(-x) = (-1)^n H_n(x)$

1.8 Laguerre Differential Equation

$$x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+ny=0$$

$x=0$ is a regular singular point of the Laguerre differential equation.

The solution of Laguerre differentialequation is known as Laguerre polynomial of order n ,i.e. $L_n(x)$

$$L_n(x)=\sum\frac{(-1)^r n! x^r}{(r!)^2(n-r)!}$$

1.8.1 Rodrigue formula for $L_n(x)$:

1. $L_n(x)=\frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right)$
2. $L_0(x)=1$
3. $L_1(x)=1-x$
4. $L_2(x)=\frac{1}{2}\left(x^2-4x+2\right)$
5. $L_3(x)=\frac{1}{6}\left(-x^3+9x^2-18x+6\right)$

Laguerre polynomials are neither even nor odd.

1.8.2 Generating function for Laguerre polynomial:

$$\frac{e^{-xz/(1-z)}}{(1-z)}=\sum_{n=0}^{\infty} z^n L_n(x)$$

1.8.3 Orthogonal property of Laguerre's polynomial:

$$\int_0^{\infty} e^{-x}L_n(x)L_m(x)dx=\delta_{mn}$$

Problem Set -1

1. Let $p_n(x)$ (where $n = 0, 1, 2, \ldots\ldots$) be a polynomial of degree n with real coefficients, defined in the interval $2 \le n \le 4$. If $\int_2^4 p_n(x) p_m(x) dx = \delta_{nm}$, then

[NET/JRF(JUNE-2011)]

A. $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{\frac{3}{2}}(-3-x)$

B. $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{3}(3+x)$

C. $p_0(x) = \frac{1}{2}$ and $p_1(x) = \sqrt{\frac{3}{2}}(3-x)$

D. $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{\frac{3}{2}}(3-x)$

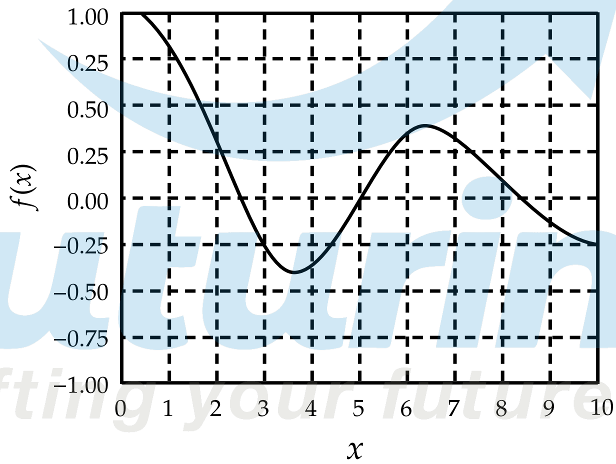
2. The generating function $F(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n$ for the Legendre polynomials $P_n(x)$ is $F(x,t) = \left(1 - 2xt + t^2\right)^{-1/2}$. The value of $P_3(-1)$ is

[NET/JRF(DEC-2011)]

A. $5/2$ **B.** $3/2$ **C.** $+1$ **D.** -1

3. The graph of the function $f(x)$ shown below is best described by

[NET/JRF(DEC-2012)]



A. The Bessel function $J_0(x)$

B. $\cos x$

C. $e^{-x} \cos x$

D. $\frac{1}{x} \cos x$

4. Given that $\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{-t^2+2tx}$ the value of $H_4(0)$ is

[NET/JRF(JUNE-2013)]

A. 12 **B.** 6 **C.** 24 **D.** -6

5. Given $\sum_{n=0}^{\infty} P_n(x) t^n = \left(1 - 2xt + t^2\right)^{-1/2}$, for $|t| < 1$, the value of $P_5(-1)$ is

[NET/JRF(JUNE-2014)]

A. 0.26 **B.** 1 **C.** 0.5 **D.** -1

6. The function $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1}$, satisfies the differential equation

[NET/JRF(DEC-2014)]

A. $x^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} + \left(x^2 + 1\right) f = 0$

B. $x^2 \frac{d^2 f}{dx^2} + 2x \frac{df}{dx} + \left(x^2 - 1\right) f = 0$

C. $x^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} + \left(x^2 - 1\right) f = 0$

D. $x^2 \frac{d^2 f}{dx^2} - x \frac{df}{dx} + \left(x^2 - 1\right) f = 0$

7. The Hermite polynomial $H_n(x)$, satisfies the differential equation

$$\frac{d^2H_n}{dx^2} - 2x\frac{dH_n}{dx} + 2nH_n(x) = 0$$

The corresponding generating function $G(t,x) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n$, satisfies the equation

[NET/JRF(DEC-2015)]

A. $\frac{\partial^2 G}{\partial x^2} - 2x\frac{\partial G}{\partial x} + 2t\frac{\partial G}{\partial t} = 0$

B. $\frac{\partial^2 G}{\partial x^2} - 2x\frac{\partial G}{\partial x} - 2t^2\frac{\partial G}{\partial t} = 0$

C. $\frac{\partial^2 G}{\partial x^2} - 2x\frac{\partial G}{\partial x} + 2\frac{\partial G}{\partial t} = 0$

D. $\frac{\partial^2 G}{\partial x^2} - 2x\frac{\partial G}{\partial x} + 2\frac{\partial^2 G}{\partial x \partial t} = 0$

8. A stable asymptotic solution of the equation $x_{n+1} = 1 + \frac{3}{1+x_n}$ is $x = 2$. If we take $x_n = 2 + \varepsilon_n$ and $x_{n+1} = 2 + \varepsilon_{n+1}$, where ε_n and ε_{n+1} are both small, the ratio $\frac{\varepsilon_{n+1}}{\varepsilon_n}$ is approximately

[NET/JRF(DEC-2016)]

A. $-\frac{1}{2}$ **B.** $-\frac{1}{4}$ **C.** $-\frac{1}{3}$ **D.** $-\frac{2}{3}$

9. The generating function $G(t,x)$ for the Legendre polynomials $P_n(t)$ is

$$G(t,x) = \frac{1}{\sqrt{1-2xt+x^2}} = \sum_{n=0}^{\infty} x^n P_n(t), \text{ for } |x| < 1$$

If the function $f(x)$ is defined by the integral equation $\int_0^x f(x')\,dx' = xG(1,x)$, it can be expressed as

[NET/JRF(DEC-2017)]

A. $\sum_{n,m=0}^{\infty} x^{n+m} P_n(1) P_m\left(\frac{1}{2}\right)$

B. $\sum_{n,m=0}^{\infty} x^{n+m} P_n(1) P_m(1)$

C. $\sum_{n,m=0}^{\infty} x^{n-m} P_n(1) P_m(1)$

D. $\sum_{n,m=0}^{\infty} x^{n-m} P_n(0) P_m(1)$

10. In the function $P_n(x)e^{-x^2}$ of a real variable $x, P_n(x)$ is polynomial of degree n. The maximum number of extrema that this function can have is

[NET/JRF(JUNE-2018)]

A. $n+2$ **B.** $n-1$ **C.** $n+1$ **D.** n

11. The polynomial $f(x) = 1 + 5x + 3x^2$ is written as linear combination of the Legendre polynomials $\left(P_0(x) = 1, P_1(x), P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)\right)$ as $f(x) = \sum_n c_n P_n(x)$. The value of c_0 is

[NET/JRF(DEC-2018)]

A. $\frac{1}{4}$ **B.** $\frac{1}{2}$ **C.** 2 **D.** 4

Answer key

Q.No.	Answer	Q.No.	Answer
1	**D**	2	**D**
3	**A**	4	**A**
5	**D**	6	**C**
7	**A**	8	**C**
9	**B**	10	**C**
11	**C**		

Problem Set -2

1. What is the maximum number of extrema of the function $f(x)=P_k(x)e^{-\left(\frac{x^4}{4}+\frac{x^2}{2}\right)}$, where $x\in(-\infty,\infty)$ and $P_k(x)$ is an arbitrary polynomial of degree k ?

[JEST 2015]

A. $k+2$ **B.** $k+6$ **C.** $k+3$ **D.** k

2. The Bernoulli polynominals $B_n(s)$ are defined by, $\frac{xe^{xs}}{e^x-1}=\sum B_n(s)\frac{x^n}{n!}$. Which one of the following relations is true?

[JEST 2015]

A. $\frac{xe^{x(1-s)}}{e^x-1}=\sum B_n(s)\frac{x^n}{(n+1)!}$ **B.** $\frac{xe^{x(1-s)}}{e^x-1}=\sum B_n(s)(-1)^n\frac{x^n}{(n+1)!}$

C. $\frac{xe^{x(1-s)}}{e^x-1}=\sum B_n(-s)(-1)^n\frac{x^n}{n!}$ **D.** $\frac{xe^{x(1-s)}}{e^x-1}=\sum B_n(s)(-1)^n\frac{x^n}{n!}$

3. For which of the following conditions does the integral $\int_0^1 P_m(x)P_n(x)dx$ vanish for $m\neq n$, where $P_m(x)$ and $P_n(x)$ are the Legendre polynomials of order m and n respectively?

[JEST 2018]

A. all $m, m\neq n$ **B.** $m-n$ is an odd integer

C. $m-n$ is a nonzero even integer **D.** $n=m\pm1$

4. The Euler polynomials are defined by $\frac{2e^{xs}}{e^x+1}=\sum_{n=0}^{\infty}E_n(s)\frac{x^n}{n!}$What is the value of $E_5(2)+E_5(3)$?

[JEST 2019]

5. If $F(x,y)=x^2+y^2+xy$, its Legendre transformed function $G(u,v)$, upto a multiplicative constant, is

[JEST 2018]

A. u^2+v^2+uv **B.** u^2+v^2-uv **C.** u^2+v^2 **D.** $(u+v)^2$

6. Consider a function $f(x)=P_k(x)e^{-\left(x^4+2x^2\right)}$ in the domain $x\in(-\infty,\infty)$, where P_k is any polynomial of degree k. What is the maximum possible number of extrema of the function?

[JEST 2019]

A. $k+3$ **B.** $k-3$ **C.** $k+2$ **D.** $k+1$

Answer key			
Q.No.	Answer	Q.No.	Answer
1	**C**	2	**D**
3	**A**	4	**64**
5	**B**	6	**D**

Problem Set -3

1. For real n the cylindrical Bessel function is $J_n(x)$ of order n then $J_n(x)$ will satisfied differential equation

   a. $\frac{d^2J_n}{dx^2}+\frac{1}{x}\left(\frac{dJ_n}{dx}\right)+\left(1+\frac{n^2}{x^2}\right)J_n=0$

   b. $\frac{d^2J_n}{dx^2}+\frac{1}{x}\left(\frac{dJ_n}{dx}\right)+\left(1-\frac{n^2}{x^2}\right)J_n=0$

   c. $\frac{d^2J_n}{dx^2}+x\left(\frac{dJ_n}{dx}\right)+\left(1+\frac{n^2}{x^2}\right)J_n=0$

   d. $\frac{d^2J_n}{dx^2}+x\left(\frac{dJ_n}{dx}\right)+\left(1-\frac{n^2}{x^2}\right)J_n=0$

Solution:

$$\text{The Bessel function is given by } \frac{d^2J_n}{dx^2}+\frac{1}{x}\left(\frac{dJ_n}{dx}\right)+\left(1-\frac{n^2}{x^2}\right)J_n=0$$

So the correct answer is **Option (b)**

2. For real n the cylindrical Bessel function is $J_n(x)$ of order n then value of $\frac{dJ_0}{dx}$ is equivalent to

   a. J_1 **b.** $-J_1$ **c.** $2J_1$ **d.** $-2J_1$

Solution:

$$J_{n+1}(x)=-J_n'(x)+\frac{n}{x}J_n.\text{ for } n=0, J_1=-J_0'$$

So the correct answer is **Option (b)**

3. The differential equation $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+\left[x^2-\lambda\right]y(x)=0$ is spherical Bessel's differential equation of order n then value of λ is given by

   a. n **b.** $n(n+1)$ **c.** $n(n-1)$ **d.** n^2

Solution:

$$\text{Spherical Bessel's differential equation } x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+\left[x^2-n(n+1)\right]y(x)=0$$

So the correct answer is **Option (b)**

4. If $J_n(x)$ is spherical Bessel function of order n then $j_0'(x)$ is equivalent to

   a. $j_1(x)$ **b.** $-j_1(x)$ **c.** $\frac{j_1(x)}{2}$ **d.** $-\frac{j_1(x)}{2}$

Solution:

$$\frac{d}{dx}(j_0(x))=\frac{d}{dx}\left(\frac{\sin x}{x}\right)=\frac{\cos x}{x}-\frac{\sin x}{x^2}=-J_1(x)$$

$$\text{Where } j_1(x)=-\frac{\cos x}{x}+\frac{\sin x}{x^2}$$

So the correct answer is **Option (b)**

5. The solution of the differential equation $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+x^2y(x)=0$ subjected to the condition is given by $y(0)=1$.

a. $\frac{\sin x}{x}$ **b.** $\frac{\cos x}{x}$ **c.** $\frac{\exp(-ix)}{x}$ **d.** $\frac{\exp ix}{x}$

Solution:

$$\text{Spherical Bessel's differential equation } x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+\left[x^2-n(n+1)\right]y(x)=0$$

$$\text{then } x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+x^2y(x)=0 \text{ is spherical Bessel's differential equation for order}$$

$$n=0$$

$$\text{then solution is } J_0(x)=\frac{\sin x}{x} \text{ with boundary condition } y(0)=1.$$

So the correct answer is **Option (a)**

6. $H_n(x)$ is Hermite polynomials of order n then $H_n(x)=(-1)^n f(x)\frac{d^n(W(x))}{dx^n}$, then $f(x)$ and $W(x)$ are respectively

a. $f(x)=\exp\left(x^2\right),W(x)=\exp\left(-x^2\right)$

b. $f(x)=\exp\left(-x^2\right),W=\exp\left(x^2\right)$

c. $f(x)=W(x)=\exp\left(x^2\right)$

d. $f(x)=W(x)=\exp\left(-x^2\right)$

Solution:

$$H_n(x)=(-1)^n\exp\left(x^2\right)\frac{d^n\left(\exp\left(-x^2\right)\right)}{dx^n}$$

$$\text{So after comparing } H_n(x)=(-1)^n f(x)\frac{d^n(W(x))}{dx^n}$$

$$f(x)=\exp\left(x^2\right),W(x)=\exp\left(-x^2\right)$$

So the correct answer is **Option (a)**

7. The solution of differential equation $\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\lambda y(x)=0$ is Hermilte polynomial of order n then value of λ is

a. n **b.** $-n$ **c.** $2n$ **d.** $-2n$

Solution:

$$\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2ny(x)=0 \text{ is Hermite differential equation}$$

So the correct answer is **Option (c)**

8. The Rodrigues formula for Laguerre polunomial is given by

a. $L_n(x)=\frac{e^{-x}}{n!}\left(\frac{d}{dx}\right)^n\left(x^ne^{-x}\right)$ **b.** $L_n(x)=\frac{e^x}{n!}\left(\frac{d}{dx}\right)^n\left(x^ne^x\right)$

c. $L_n(x)=\frac{e^{-x}}{n!}\left(\frac{d}{dx}\right)^n\left(x^ne^x\right)$ **d.** $L_n(x)=\frac{e^x}{n!}\left(\frac{d}{dx}\right)^n\left(x^ne^{-x}\right)$

Solution:

$$L_n(x) = \frac{e^x}{n!}\left(\frac{d}{dx}\right)^n \left(x^n e^{-x}\right)$$

So the correct answer is **Option (d)**

9. It is given that operator $x - \frac{d}{dx} = -\exp\left(\frac{x^2}{2}\right)\frac{d}{dx}\exp\left(-\frac{x^2}{2}\right)$ If then the normalized wave function for harmonic oscillation is $\psi(x) = \left(\pi^{1/2}2^n\lfloor n)^{-1/2}\exp\left(-\frac{x^2}{2}\right)H_n(x)\right.$, then $\psi_n(x)$ is equivalent to

- **a.** $\psi_n(x) = \left(\pi^{1/2}2^n\lfloor n)^{-1/2}\left(x - \frac{d}{dx}\right)^n \exp\left(-\frac{x^2}{2}\right)\right.$
- **b.** $\psi_n(x) = \left(\pi^{1/2}2^n\lfloor n)^{-1/2}\left(x - \frac{d}{dx}\right)^{2n} \exp\left(\frac{x^2}{2}\right)\right.$
- **c.** $\psi_n(x) = \left(\pi^{1/2}2^n\lfloor n)^{-1/2}\left(x - \frac{d}{dx}\right)^n \exp\left(-x^2\right)\right.$
- **d.** $\psi_n(x) = \left(\pi^{k/2}2^n\lfloor n)^{-1/2}\left(x - \frac{d}{dx}\right)^{2n} \mathrm{cxp}\left(-x^2\right)\right.$

Solution:

$$H_n(x) = (-1)^n \exp\left(x^2\right)\frac{d^n\left(\exp\left(-x^2\right)\right)}{dx^n}$$

$$x - \frac{d}{dx} = -\exp\left(\frac{x^2}{2}\right)\frac{d}{dx}\exp\left(-\frac{x^2}{2}\right) \Rightarrow \left(x - \frac{d}{dx}\right)\exp\left(-\frac{x^2}{2}\right)$$

$$= -\exp\left(\frac{x^2}{2}\right)\frac{d}{dx}\exp\left(-\frac{x^2}{2}\right)\Big)\exp\left(-\frac{x^2}{2}\right)$$

$$x\exp\left(-\frac{x^2}{2}\right) - \frac{d\exp\left(-\frac{x^2}{2}\right)}{dx} = -\exp\left(\frac{x^2}{2}\right)\frac{d}{dx}\exp\left(-x^2\right)$$

$$\Rightarrow \left(x - \frac{d}{dx}\right)\exp\left(-\frac{x^2}{2}\right) = \exp\left(\frac{x^2}{2}\right)\left(-2x\exp\left(-x^2\right)\right) = 2x\exp - \frac{x^2}{2} = H_1\left(\exp - \frac{x^2}{2}\right)$$

where $2x = H_1(x)$

Similarly $\left(x - \frac{d}{dx}\right)^n \exp\left(-\frac{x^2}{2}\right) = H_n \exp\left(-\frac{x^2}{2}\right)$

$$\psi_n(x) = \left(\pi^{1/2}2^n\lfloor n)^{-1/2}\left(x - \frac{d}{dx}\right)^n \exp\left(-\frac{x^2}{-2}\right)\right.$$

So the correct answer is **Option (a)**

10. The solution of differential equation $x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + \lambda y(x) = 0$ is Laguerre polynomials of order n then value of λ is

a. n **b.** $-n$ **c.** $2n$ **d.** $-2n$

Solution:

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny(x) = 0 \text{ is Laguerre differential equation.}$$

So the correct answer is **Option (a)**

11. The generating function $F(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n$ for the Legendre polynomials $P_n(x)$ is $F(x,t) = \left(1 - 2xt + t^2\right)^{-1/2}$. The value of $P_2(-1)$ is

a. $5/2$ **b.** $3/2$ **c.** $+1$ **d.** -1

Solution:

The generating function for Legendre polynomial is $F(x,t) = \left(1-2xt+t^2\right)^{-1/2}$. Thus

$$P_2(x) = \frac{1}{2}\left(3x^2-1\right) \Rightarrow P_2(-1) = \frac{1}{2}(3-1) = 1$$

So the correct answer is **Option (c)**

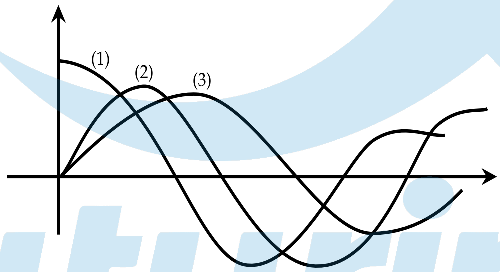
12. If we observe plot of Bessel functions $J_0(x), J_1(x)$, and $J_2(x)$ we find their maxima at x_0, x_1 and x_2 respectively. Then which of the following is true

a. $x_0 < x_1 < x_2$ **b.** $x_0 > x_1 > x_2$

c. $x_0 < x_1 = x_2$ **d.** $x_0 = x_1 < x_2$

Solution: So the correct answer is **Option (a)**

13. Which one of the following is correctly matched?



a. (1) J_0, (2) J_2, (3) J_1 **b.** (1)J_0, (2) J_1, (3)J_2

c. (1) J_2, (2) J_1, (3) J_0 **d.** None of the above

Solution: So the correct answer is **Option (b)**

14. If the generating function of Legendre polynomial is $\frac{1}{\sqrt{1-6t+t^2}}$, then coefficient of t^2 is

a. 11 **b.** -11 **c.** 13 **d.** -13

Solution:

The generating function for the polynomial solutions of the Legendre ODE is given by

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Thus $x = 3$ and $n = 2$.

$$P_2(x) = \frac{1}{2}\left(3x^2-1\right) \Rightarrow P_2(3) = \frac{1}{2}\left(3\times 3^2-1\right) = 13$$

So the correct answer is **Option (c)**

15. Which of the following relation is true for Bessel's differential equation?

a. $J_0'(x) = J_1(x)$

b. $J_0'(x) = -J_2(x)$

c. $J_0'(x) = J_2(x)$

d. $J_0'(x) = -J_1(x)$

Solution: So the correct answer is **Option (d)**