Gaussian beam parameter transformation at a refracting surface

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The Fourier transform method is used to calculate the electric field of the astigmatic gaussian beam which appears when an incident stigmatic gaussian beam is refracted at the interface between media with different refractive indices. A beam parameter transformation law is derived which shows that the beam parameters of the refracted astigmatic beam can be calculated from the beam parameters of the incident stigmatic beam and the angles of incidence and refraction. An heuristic derivation of the transformation law based on geometrical optics is also given. The application of the law is illustrated by determining the plane parallel plate astigmatism.

Numerous optical devices contain optical components which change the direction of propagation of a beam of light by refraction. Such devices are, for example, electro-optic light deflectors [1] and optical resonators, which for Q-switching applications are provided with internal Brewster plates. In addition such resonators often consist of laser rods and modulators cut at Brewster's angle.[2]

The beam parameters of a gaussian beam which traverses a refracting surface undergo a transformation. Assuming an incident gaussian beam which is stigmatic, this transformation can be derived as a function of the angles of incidence and refraction. The incident gaussian beam is expanded into plane waves to which the laws of refraction and transmission are applied.[3] The summation of the refracted partial plane waves leads approximately to a refracted gaussian beam with transformed beam parameters. A similar calculation was performed for normal incidence in reference 4. An heuristic derivation of the beam parameter transformation law based on geometrical optics is also given.

As an application of the transformation law, the known astigmatism of a plane parallel plate [5,6] is determined. The assumed approximations do not hold in the vicinity of the total reflection angle. The reflection of a gaussian beam for this case has been examined in reference 7. The more general case of an incident gaussian beam with general astigmatism [8] can be treated along the lines described in this paper. This problem is omitted so that the basic ideas can be followed more easily.

Plane wave expansion of a stigmatic gaussian beam

The following considerations assume an incident gaussian beam which is stigmatic, propagates in the z direction, and has an electric field normal to the plane of incidence (Fig.1). A treatment of the electric field component parallel to the plane of incidence is not necessary. All results for the normal component can be applied analogously to the parallel component. The electric field of a paraxial gaussian beam is approximately normal to the direction of propagation. Under these assumptions the incident beam in medium 1 has an electric field given by:[9]

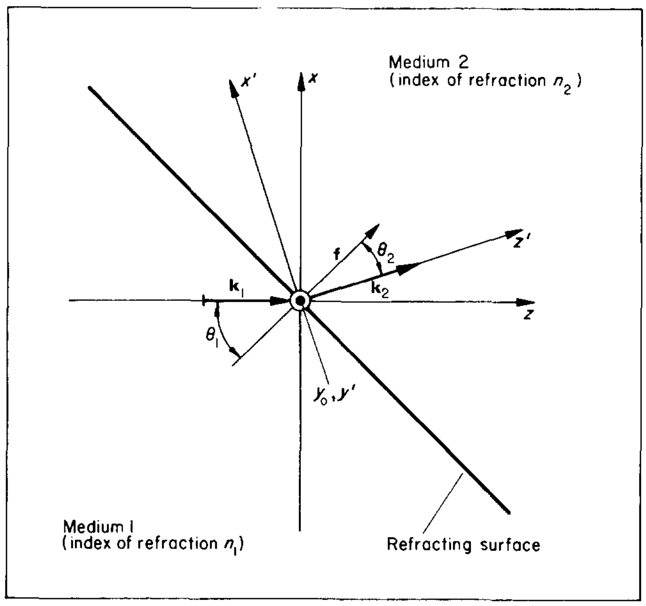

Fig.1 Refraction of plane waves

$$\mathbf{E}_{1,\perp}(\mathbf{x}) = \frac{\hat{\mathbf{E}}_{1,\perp}}{q_1} \exp\left[\mathrm{j}\frac{\pi}{\lambda q_1}(x^2 + y^2) - \mathrm{j}k_1 z\right] \tag{1}$$

where a point in space is denoted by $\mathbf{x} = (x, y, z)$ and $\hat{\mathbf{E}}_{1,\perp}$ is a constant complex electric field vector amplitude which is normal to the direction of propagation, ie to the z direction, and to the plane of incidence. λ is the wavelength of light in free space and $k_1 = 2\pi n_1/\lambda$ is the propagation constant in medium 1, which has the index of refraction

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n_1. The complex beam parameter:

$$q_1 = j \frac{\pi}{\lambda} w_1^2 + \frac{z - z_1}{n_1} \tag{2}$$

is given by the spot size w_1 at the beam waist and the location z_1 of the beam waist on the z axis. The plane wave expansion of the electric field $\mathbf{E}_{1,\perp}(\mathbf{x})$ of the gaussian beam is obtained by using the Fourier transform in three dimensions:

$$\mathbf{F}_{1,\perp}(\mathbf{k}_1) = \frac{1}{(2\pi)^3} \int_R \mathbf{E}_{1,\perp}(\mathbf{x}) \exp(j\mathbf{k}_1 \cdot \mathbf{x})\, d\mathbf{x} \tag{3}$$

where $\mathbf{F}_{1,\perp}(\mathbf{k}_1)$ denotes the complex amplitude of a plane partial wave which propagates in medium 1 in the direction of the wave vector $\mathbf{k}_1 = (k_{1,x}, k_{1,y}, k_{1,z})$. The length of the wave vector in medium 1 is $|\mathbf{k}_1| = k_1 = 2\pi\lambda/n_1$. $d\mathbf{x}$ is the volume element in $\mathbf{x}$ space R. The scalar product of $\mathbf{k}_1$ and $\mathbf{x}$ in the argument of the exponential function is denoted by a dot. Insertion of (1) in (3) results in:

$$\mathbf{F}_{1,\perp}(\mathbf{k}_1) = j \frac{\lambda}{4\pi^2} \hat{\mathbf{E}}_{1,\perp} \exp\left[j \frac{\lambda}{4\pi} q_{1,0} (k_{1,x}^2 + k_{1,y}^2) \right]$$

$$\times\, \delta \left[k_{1,z} - k_1 + \frac{1}{2k_1} (k_{1,x}^2 + k_{1,y}^2) \right] \tag{4}$$

$\delta(z)$ is Dirac's delta function and $q_{1,0}$ is the beam parameter at the location $z = 0$. $q_{1,0}$ is given from (2) by:

$$q_{1,0} = j \frac{\pi}{\lambda} w_1^2 - \frac{z_1}{n_1} \tag{5}$$

Refracted beam

The gaussian beam (1) travels along the z axis. It is composed of the partial plane waves given by (4) refracted at the interface between mediums 1 and 2. A partial plane wave which propagates in medium 1 with amplitude $\mathbf{F}_{1,\perp}$ in the direction of the wave vector $\mathbf{k}_1$ propagates, after crossing the refracting surface, with amplitude $\mathbf{F}_{2,\perp}$ in the direction of the wave vector $\mathbf{k}_2$ in medium 2 (Fig.1). $\mathbf{F}_{2,\perp}$ and $\mathbf{F}_{1,\perp}$ are normal to the plane of incidence. The amplitudes $\mathbf{F}_{2,\perp}$ and $\mathbf{F}_{1,\perp}$ are associated by:

$$\mathbf{F}_{2,\perp}(\mathbf{k}_2) = T_\perp(\mathbf{k}_1)\, \mathbf{F}_{1,\perp}(\mathbf{k}_1) \tag{6}$$

where $T_\perp(\mathbf{k}_1)$ is the amplitude transmission coefficient of an electric field component normal to the plane of incidence. $T_\perp(\mathbf{k}_1)$ and the wave vector $\mathbf{k}_2$ depend on the wave vector $\mathbf{k}_1$. The function $\mathbf{k}_2(\mathbf{k}_1)$ can be obtained from Snell's law of refraction and is:

$$\mathbf{k}_2(\mathbf{k}_1) = \mathbf{k}_1 - (\mathbf{f} \cdot \mathbf{k}_1)\, \mathbf{f}$$

$$+ \left\{ \left[\left(\frac{n_2}{n_1} \right)^2 - 1 \right] (\mathbf{k}_1 \cdot \mathbf{k}_1) + (\mathbf{f} \cdot \mathbf{k}_1)^2 \right\}^{1/2} \mathbf{f} \tag{7}$$

The vector $\mathbf{f} = (\sin\theta_1, 0, \cos\theta_1)$ is a unit vector normal to refracting surface between mediums 1 and 2. The summation of the refracted partial plane waves yields the electric field $\mathbf{E}_{2,\perp}(\mathbf{x})$ of the refracted beam:

$$\mathbf{E}_{2,\perp}(\mathbf{x}) = \int_K \mathbf{F}_{2,\perp}(\mathbf{k}_2) \exp(-j\, \mathbf{k}_2 \cdot \mathbf{x})\, d\mathbf{k}_1 \tag{8}$$

This integral is the inverse Fourier transform and $d\mathbf{k}_1 = dk_{1x}\, dk_{1y}\, dk_{1z}$ is the volume element in Fourier space K. Under assumptions which are now discussed the integral (8) can be solved approximately. Insertion of (6) in (8) results in:

$$\mathbf{E}_{2,\perp}(\mathbf{x}) = T_\perp(\mathbf{k}_{1,0}) \int_K \mathbf{F}_{1,\perp}(\mathbf{k}_1)$$

$$\times \exp[-j\, \mathbf{k}_2(\mathbf{k}_1) \cdot \mathbf{x}]\, d\mathbf{k}_1 \tag{9}$$

The gaussian beam (1) travels along the z axis, ie in the direction of the wave vector $\mathbf{k}_1 = \mathbf{k}_{1,0} = (0, 0, k_1)$. The amplitudes $\mathbf{F}_{1,\perp}(\mathbf{k}_1)$ of (4) have significant magnitudes only in the vicinity of $\mathbf{k}_1 = \mathbf{k}_{1,0}$. Furthermore the transmission coefficient $T_\perp(\mathbf{k}_1)$ is only a weak function of $\mathbf{k}_1$. Therefore the transmission coefficient $T_\perp(\mathbf{k}_1)$ has been placed, with value $T_\perp(\mathbf{k}_{1,0})$, before the integral (9). After insertion of (4), this integral can be solved in the Fresnel approximation. The δ function in (4) makes it possible to integrate (9) over the $k_{1,z}$ coordinate. Thus the argument:

$$\mathbf{k}_2(\mathbf{k}_1) \cdot \mathbf{x} = \mathbf{k}_2\, (k_{1,x}, k_{1,y}) \cdot \mathbf{x} \tag{10}$$

of the exponential function in (9) is only a function of the coordinates $k_{1,x}$ and $k_{1,y}$. This function is expanded up to the second-order terms in $k_{1,x}$ and $k_{1,y}$ around $k_{1,x} = 0$ and $k_{1,y} = 0$. The position vector $\mathbf{x}$ is measured in the coordinate system x', y', z' (Fig.1); the positive z' axis has the direction of $\mathbf{k}_2(\mathbf{k}_{1,0})$ with $\mathbf{k}_{1,0} = (0, 0, k_1)$ in the x, y, z system. A plane wave travelling in medium 1 in the positive z direction propagates, after crossing the refracting surface, in the positive z' direction in medium 2. The y' axis is normal to the plane of incidence and coincides with the y axis. The expansion of (10) is:

$$\mathbf{k}_2 \cdot \mathbf{x} = \mathbf{k}_{2,0} \cdot \mathbf{x} + a_x\, k_{1,x} + a_y\, k_{1,y}$$

$$+ b_x\, k_{1,x}^2 + 2b_{xy}\, k_{1,x}\, k_{1,y} + b_y\, k_{1,y}^2 \tag{11}$$

The vector $\mathbf{k}_{2,0}$ is parallel to the z' axis, whence:

$$\mathbf{k}_{2,0} \cdot \mathbf{x} = k_2 z' \tag{12}$$

where $|\mathbf{k}_{2,0}| = k_2 = 2\pi n_2/\lambda$ and n_2 is the index of refraction of medium 2. The function (10) can be obtained from (7) and the condition $k_{1,z} = k_1 - (k_{1,x}^2 + k_{1,y}^2)/2\, k_1$, which was imposed in (4), by integrating over $k_{1,z}$ in (9).

Differentiation of (10) with respect to $k_{1,x}$ and $k_{1,y}$ up to second-order terms yields the coefficients:

$$a_x = \frac{\cos\theta_1}{\cos\theta_2} x'$$

$$a_y = y'$$

$$b_x = -\left(\frac{\cos\theta_1}{\cos\theta_2}\right)^2 \frac{z'}{2k_2} + \frac{k_2^2 - k_1^2}{k_1 k_2} \frac{\sin\theta_1}{\cos^3\theta_2} \frac{x'}{2k_2}$$

$$b_{xy} = 0$$

$$b_y = -\frac{z'}{2k_2}$$

$$+ \frac{k_2^2\cos^2\theta_2 - k_1^2\cos^2\theta_1}{k_1 k_2} \frac{\sin\theta_1}{\cos\theta_2} \frac{x'}{2k_2} \tag{13}$$

where $k_{1,x} = 0$ and $k_{1,y} = 0$ are used. The refracted partial plane waves in (9) can now be integrated taking account the expansion (11) and the coefficients in (13) to yield the electric field:

$$\mathbf{E}_{2,\perp}(\mathbf{x}) = \frac{\dfrac{\cos\theta_2}{\cos\theta_1} T_\perp(\mathbf{k}_{1,0})\,\hat{\mathbf{E}}_{1,\perp}}{\left[\left(\dfrac{\cos^2\theta_2}{\cos^2\theta_1} q_{1,0} + \dfrac{z'}{n_2}\right)\left(q_{1,0} + \dfrac{z'}{n_2}\right)\right]^{1/2}}$$

$$\times \exp\left(\frac{\mathrm{j}\dfrac{\lambda}{\pi}x'^2}{\dfrac{\cos^2\theta_2}{\cos^2\theta_1} q_{1,0} + \dfrac{z'}{n_2}} + \frac{\mathrm{j}\dfrac{\lambda}{\pi}y'^2}{q_{1,0} + \dfrac{z'}{n_2}} - \mathrm{j}\,k_2 z'\right) \tag{14}$$

of the refracted beam, which is an astigmatic gaussian beam. The result (14) is obtained by neglecting the second terms in the sums for b_x and b_y in (13). This approximation is justified for paraxial beams, ie for small x', and for a sufficiently long distance z' from the refracting surface. The approximation is better the smaller the angle of incidence θ_1 because the function $\sin\theta_1$ appears as a coefficient in the second term of the sums for b_x and b_y. For total reflection the angle of refraction is $\theta_2 = \pi/2$ and the expansion (11) breaks down because, since $\cos\theta_2 = \cos\pi/2 = 0$, the coefficients a_x, b_x and b_y of (13) are no longer defined. This means that the result (14) for the refracted beam fails to be valid in the vicinity of the total reflection angle.

Beam parameter transformation law

The stigmatic gaussian beam (1) generates the refracted beam (14) at the interface between mediums 1 and 2 (Fig.1). This generated beam has a simple astigmatism, ie the ellipses of constant intensity and the ellipses of constant phase have identical orientation. This orientation remains unchanged along the z axis. One of the main axes of these ellipses coincides with the y' axis and is thus normal to the plane of incidence, whereas the second main axis coincides with the x' axis and is thus parallel to the plane of incidence.

The beam parameters of the refracted beam :

$$q_{x'} = \left(\frac{\cos\theta_2}{\cos\theta_1}\right)^2 q_{1,0} + \frac{z'}{n_2} \tag{15}$$

for the $x'z'$ plane, and:

$$q_{y'} = q_{1,0} + \frac{z'}{n_2} \tag{16}$$

for the $y'z'$ plane are given by (14). In (15) and (16) $q_{1,0}$ is the beam parameter (5) of the incident stigmatic gaussian beam in medium 1 immediately before the refracting surface. The beam parameters $q_{x',0}$ and $q_{y',0}$ of the refracted astigmatic beam immediately behind the refracting surface result from (15) and (16) on inserting $z' = 0$

$$q_{x',0} = \left(\frac{\cos\theta_2}{\cos\theta_1}\right)^2 q_{1,0} \tag{17a}$$

$$q_{y',0} = q_{1,0} \tag{17b}$$

The equations (17) constitute the law of transformation for the beam parameters of an incident stigmatic gaussian beam crossing a refracting surface. The beam parameter in the plane of incidence is transformed through (17a), whereas the beam parameter in the $y'z'$ plane is, as can be seen from (17b), unchanged.

Heuristic derivation of the beam parameter transformation law

In geometric optics a beam refracted at the interface between mediums 1 and 2 (Fig.2) has a diameter d_1 in medium 1 and, parallel to the plane of incidence, a diameter d_2 in medium 2. Fig.2 shows that:

$$d_2 = \frac{\cos\theta_2}{\cos\theta_1} d_1 \tag{18}$$

The beam diameter normal to the plane of incidence is not changed. The imaginary part of the beam parameter $q_{1,0}$ is proportional to the square of the spot size w_1 as can be seen from (5).

The imaginary part of the beam parameter $q_{x',0}$ is similarly proportional to the square of the spot size $w_{x'}$. Thus the following heuristic substitutions can be performed:

$$d_1^2 \to w_1^2 \to q_{1,0}$$

$$d_2^2 \to w_{x'}^2 \to q_{x',0} \tag{19}$$

Insertion of (19) in (18) yields the transformation law:

$$q_{x',0} = \left(\frac{\cos\theta_2}{\cos\theta_1}\right)^2 q_{1,0}$$

which is the same as (17a). The beam diameter normal to the plane of incidence does not change. Therefore the beam parameter $q_{1,0}$ in this direction is preserved, leading to (17b).

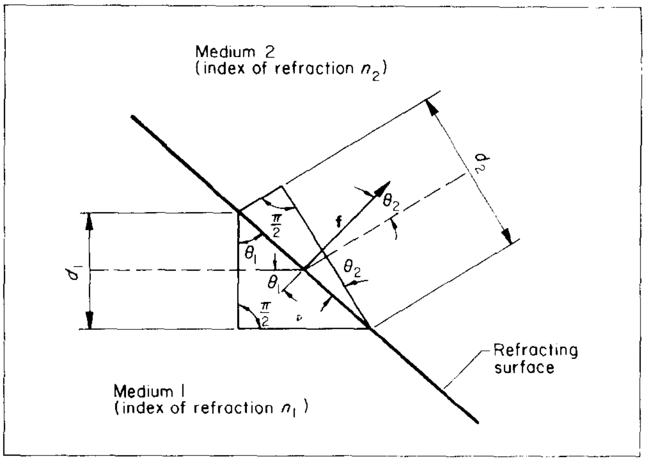

Fig.2 Beam transformation at a refracting surface using geometrical optics

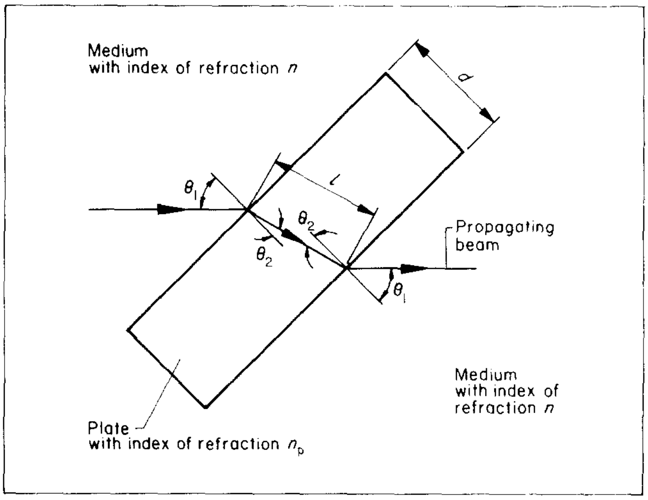

Fig.3 Derivation of the plane-parallel plate astigmatism

Example: astigmatism of a plane-parallel plate

The astigmatism of the plane-parallel plate is determined by application of the transformation law (17a and 17b) and by taking account of the beam parameter transformation due to beam propagation.[9] The plate has a refractive index of n_p and is embedded in a medium with an index of refraction n (Fig.3). A stigmatic gaussian beam incident from the left may be characterized by the beam parameter $q_{1,0}$. After entering the plate, the beam parameters are:

$$q_x = \frac{\cos^2\theta_1}{\cos^2\theta_2} q_{1,0}$$

$$q_y = q_{1,0} \tag{20}$$

which can be obtained from (17a) and (17b), where q_x and q_y are the beam parameters parallel and normal to the plane of incidence respectively. Within the plate the beam travels a distance l, whereupon q_x and q_y become:

$$q_x = \frac{\cos^2\theta_2}{\cos^2\theta_1} q_{1,0} + \frac{l}{n_p}$$

$$q_y = q_{1,0} + \frac{l}{n_p} \tag{21}$$

When the beam leaves the plate the law (17a, 17b) must again be applied, hence:

$$q_x = \left(\frac{\cos^2\theta_2}{\cos^2\theta_1} q_{1,0} + \frac{l}{n_p}\right)\frac{\cos^2\theta_1}{\cos^2\theta_2}$$

$$q_y = q_{1,0} + \frac{l}{n_p} \tag{22}$$

The beam parameters q_x and q_y are now different, signifying that the beam is astigmatic. The astigmatism A introduced by the plate is equal to the distance between the beam waists parallel and normal to the plane of incidence. The astigmatism is given by:

$$A = n \operatorname{Re}(q_y - q_x) \tag{23}$$

where Re(z) denotes the real part of a complex number z. Insertion of (22) in (23) yields an expression for the astigmatism:

$$A = \frac{n\,l}{n_p \cos\theta_2}\left(1 - \frac{\cos^2\theta_1}{\cos^2\theta_2}\right) \tag{24}$$

for a plane-parallel plate, where $l = d/\cos\theta_2$ and d is the thickness of the plate. The result (24) has been derived differently in references 5 and 6.

Conclusions

An incoming stigmatic gaussian beam was decomposed into plane waves. After applying the known laws of refraction and transmission at an interface between two media with different indices of refraction, and after summation of the refracted plane waves, an astigmatic gaussian beam results. Thus the beam parameter transformation law (17a, 17b) is obtained which forms a basis for the calculation of astigmatism in optical arrangements with refracting surfaces, eg electro-optic light deflectors.[1] The transformation is also useful for computing the modes of optical resonators containing optical elements with Brewster plates. The basic concepts presented in this paper can also be used to calculate the beam parameter transformation, at a refracting surface, of a beam with general astigmatism.[8]

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