UE18MA251 Linear Algebra & Its Applications

Assignment

Q1. Equation of the parabola $y = A + Bx + Cx^2$

→ Passes through the points $(1,1)$, $(2,-1)$ & $(3,1)$

→ Using Gaussian Elimination

At $(1,1)$ equation reduces to:

$1 = A + B + C$ — I

At $(2,-1)$ equation reduces to:

$-1 = A + 2B + 4C$ — II

At $(3,1)$ equation reduces to:

$1 = A + 3B + 9C$ — III

$$A + B + C = 1$$
$$A + 2B + 4C = -1$$
$$A + 3B + 9C = 1$$

Writing it in the form $A'x = B'$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}}_{A'} \underbrace{\begin{bmatrix} A \\ B \\ C \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{B'}$$

Solving the system of equations using Gaussian Elimination:

$A':B'$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & 1 \end{array}\right]$$

$R_2' \to R_2 - R_1$, $R_3' \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 8 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 8 & 0 \end{array}\right] \qquad R_3'' \rightarrow R_3' - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 2 & 4 \end{array}\right] \Rightarrow \begin{aligned} A + B + C &= 1 \\ B + 3C &= -2 \\ 2C &= 4 \end{aligned}$$

Using the method of Backward Substitution:

$$2C = 4 \Rightarrow C = 2$$

$$B + 3C = -2 \quad \Rightarrow B = -8$$

$$A + B + C = 1 \quad \Rightarrow A = 7$$

Substituting these values into the equation of the parabola

$$y = A + Bx + Cx^2 \Rightarrow \cancel{y = 7 - 4} \; \boxed{y = 7 - 8x + 2x^2}$$

Q2. LU decomposition of matrix A:

$$A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{bmatrix}$$

$A = LU$

Applying Gaussian Elimination

$R_2' \rightarrow R_2 - 2R_1$, $\quad R_4' \rightarrow R_4 - 5R_1$

$R_3' \rightarrow R_3 + 5R_1$

$$= \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & -4 & 5 & 13 \\ 0 & -4 & 11 & 19 \end{bmatrix} \rightarrow \begin{aligned} R_3'' &\rightarrow R_3' + 2R_2 \\ R_4'' &\rightarrow R_4' + 2R_2 \end{aligned}$$

$$= \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 9 & 11 \end{bmatrix} \longrightarrow R_4''' \rightarrow R_4'' - 3R_3''$$

$$U = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}; \quad E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix} \quad \& \quad U = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

where $A = LU$

Q.3 $T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$

i) $T(1,0,0) = (1,0,1)$

$T(0,1,0) = (2,1,1)$

$T(0,0,1) = (-1,1,-2)$

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

$T(1,0,0) = (1,0,1)$

$= c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$

$\therefore c_1 = 1,\ c_2 = 0,\ c_3 = 1$

$T(0,1,0) = (2,1,1)$

$= c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$

$\therefore c_1 = 2,\ c_2 = 1,\ c_3 = 1$

$T(0,0,1) = (-1,1,-2)$

$= c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$

$\therefore c_1 = -1,\ c_2 = 1,\ c_3 = -2$

$\therefore$ The matrix T relative to the standard basis of R^3 is

$$\cancel{T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}}$$

$$T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

Q3

(ii) $T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow{R_3' = R_3 - R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$

$\xrightarrow{R_3'' \to R_3 + R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Column Space:

$$C(T) = \left\{ c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \;/\; c_1, c_2 \in R \right\}$$

Basis for $C(T) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$

Row space

$C(T^T)$:?

$T^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \xrightarrow[R_3' \to R_3 + R_1]{R_2' \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

$\xrightarrow{R_3'' \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$C(T^T) = \{ c_1 (1, 2, -1) + c_2 (0, 1, 1) \;/\; c_1, c_2 \in R \}$

Basis for $C(T^T) = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

Null space $N(T)$

Echelon form of $T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$Tx = 0$

$$\underset{3\times 3}{\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 2y - z = 0$

$y + z = 0$

z is the free variable x & y are pivot variables

$\therefore$ put $z = 1$ then $y = -1$ $\quad x = 3$

$\therefore$ $N(T) = \left\{ z \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}$

Basis for $N(T) = \left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}$

Left Null space $N(T^T)$

Echelon form of $T^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$Tx = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + z = 0$

$y - z = 0$

z = free variable

x, y = pivot variables

$z = 1 \Rightarrow y = 1 \quad x = -1$

$\therefore$ $N(T^T) = \left\{ z \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$. Basis for $N(T^T) = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$

3 (iii) Eigen values & Eigen Vectors

$$T(x, y, z) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

characteristic equation of T

$$|T - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(1-\lambda)(-2-\lambda)\right] - 2\,[-1] - 1[-1+\lambda] = 0$$

$$(1-\lambda)\left[-3+\lambda+\lambda^2\right] + 2 + 1 - \lambda = 0$$

$$-\cancel{3} + \cancel{\lambda} + \cancel{\lambda^2} + 3\lambda - \cancel{\lambda^2} - \lambda^3 + \cancel{3} - \cancel{\lambda} = 0$$

$$-\lambda^3 + 3\lambda = 0$$

$$\lambda^3 - 3\lambda = 0$$

$$\lambda(\lambda^2 - 3) = 0$$

$$\lambda = 0, \pm\sqrt{3}$$

Sum of eigen values = 0
sum of diagonal elements = 0

Case i) $\lambda = 0$

$$Ax = \lambda x$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [0]$$

Using method of cross multiplication:

$$\frac{x}{\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}}$$

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$$\frac{x}{2+1} = \frac{y}{-1} = \frac{z}{1}$$

$$\frac{x}{3} = \frac{y}{-1} = \frac{z}{1}$$

$\therefore$ Eigen vector for $\lambda = 0$ is $k(3,-1,1)$ $k \neq 0$

$\lambda_2 = \sqrt{3}$

$A - \lambda_2 I =$

$$\begin{pmatrix} -\sqrt{3}+1 & 2 & -1 \\ 0 & -\sqrt{3}+1 & 1 \\ 1 & 1 & -\sqrt{3}-2 \end{pmatrix}$$

$$\Big\downarrow \; R_3' \to R_3 - \frac{R_1}{(1-\sqrt{3})}$$

$A v = \lambda v$

$(A - \lambda I) v = 0$

$$\begin{pmatrix} -\sqrt{3}+1 & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 0 & 2+\sqrt{3} & -\left(\frac{5+3\sqrt{3}}{2}\right) \end{pmatrix}$$

$$\Big\downarrow \; R_3'' \to R_3' - \left[\frac{2+\sqrt{3}}{1-\sqrt{3}}\right] R_2$$

$$\begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$(1-\sqrt{3})y = -z \qquad y = \frac{-z}{1-\sqrt{3}} \qquad y = \left(\frac{1+\sqrt{3}}{2}\right)z$

$\cancel{(++}$ $(1-\sqrt{3})x + 2y = z \Rightarrow x(1-\sqrt{3}) = z + \frac{2z}{1-\sqrt{3}}$ $\therefore x = \left(\frac{3+\sqrt{3}}{2}\right)z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} \frac{\sqrt{3}+3}{2} \\ \frac{\sqrt{3}+1}{2} \\ 1 \end{pmatrix} = V_2 \text{ (eigenvector)}$$

For $\lambda = -\sqrt{3}$

$$\begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{bmatrix} \quad R_3' \to R_3 - \frac{R_2}{(1+\sqrt{3})}$$

$$\begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 0 & 2-\sqrt{3} & (-5+3\sqrt{3})/2 \end{bmatrix} \quad R_3'' \to R_3' - \left(\frac{2-\sqrt{3}}{1+\sqrt{3}}\right) R_2$$

$$\begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow y = \frac{-z}{1+\sqrt{3}} = \cancel{\left(1+\sqrt{\right.}} \left(\frac{1-\sqrt{3}}{2}\right) z$$

$$x = \left(\frac{3-\sqrt{3}}{2}\right) z$$

$\therefore$ Eigen vector $V_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} (3-\sqrt{3})/2 \\ (1-\sqrt{3})/2 \\ 1 \end{pmatrix}$

3 (iv) $T = QR$

$$T = \begin{array}{c} \begin{matrix} a & b & c \end{matrix} \\ \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \end{array}$$

$$q_1 = \frac{a}{\|a\|} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right) \cdot \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\therefore q_2 = \begin{bmatrix} 0.4082 \\ 0.8165 \\ -0.4082 \end{bmatrix}$$

$$q_3 = c - (q_1^T c) q_1 - (q_2^T c) q_2$$

$$= \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - [-2.1213] \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} - [1.2247] \begin{bmatrix} 0.4082 \\ 0.8165 \\ -0.4082 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$T = QR$

$$T = \begin{bmatrix} 1/\sqrt{2} & 0.4082 & 0 \\ 0 & 0.8165 & 0 \\ 1/\sqrt{2} & -0.4082 & 0 \end{bmatrix} \begin{bmatrix} 1.414 & 2.1213 & -2.1213 \\ 0 & 1.2247 & 1.2247 \\ 0 & 0 & 0 \end{bmatrix}$$

Q4. $y = c + dx$

$$\begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

Assume $x = \hat{x}$

$$\left[\hat{x} = \frac{A^T b}{A^T A}\right]$$

$A^T A \hat{x} = A^T b$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

$4c + 2d = 28$ & $2c + 30d = 34$

$c = 193/29 = 6.65$ & $d = 20/29 = 0.689$

$\therefore y = 6.65 + 0.689x$

$$P = A\hat{x} = \begin{bmatrix} 3.8965 \\ 7.3448 \\ 8.0344 \\ 8.7241 \end{bmatrix}$$

Q5.

$$x_1 + x_2 + 3x_3 + 4x_5 = 0$$

$$N(A) : \begin{pmatrix} -x_2 - 3x_3 - 4x_5 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_4 + \begin{pmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} x_5$$

$$\therefore B = \begin{bmatrix} -1 & -3 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{5\times4}$$

$$P = B(B^T B)^{-1} B^T$$

$$B^T B = \begin{bmatrix} 2 & 3 & 0 & 4 \\ 3 & 10 & 0 & 12 \\ 0 & 0 & 1 & 0 \\ 4 & 12 & 0 & 17 \end{bmatrix}$$

$$(B^T B)^{-1} = \begin{bmatrix} 26/27 & -1/9 & 0 & -4/27 \\ -1/9 & 2/3 & 0 & -4/9 \\ 0 & 0 & 1 & 0 \\ -4/27 & -4/9 & 0 & 11/27 \end{bmatrix}$$

$$B(B^T B)^{-1} = \begin{bmatrix} -1/27 & -1/9 & 0 & -4/27 \\ 26/27 & -1/9 & 0 & -4/27 \\ -1/9 & 2/3 & 0 & -4/9 \\ 0 & 0 & 1 & 0 \\ -4/27 & -4/9 & 0 & 11/27 \end{bmatrix}$$

$$P = B(B^T B)^{-1} B^T = \begin{bmatrix} 26/27 & -1/27 & -1/9 & 0 & -4/27 \\ -1/27 & 26/27 & -1/9 & 0 & -4/27 \\ -1/9 & -1/9 & 2/3 & 0 & -4/9 \\ 0 & 0 & 0 & 1 & 0 \\ -4/27 & -4/27 & -4/9 & 0 & 11/27 \end{bmatrix}$$

Orthogonal complement of B is Q

Plane equations are the equations of their normals

$\therefore Q = [1 \quad 1 \quad 3 \quad 0 \quad 4]$

Q 6. For a matrix to be positive definite :-

- All pivots (w/o row exchange) are positive

$\therefore a > 0$

- All submatrices are positive

$$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix} \begin{array}{l} — R_1 \\ — R_2 \\ — R_3 \end{array}$$

$$R_2 = R_2 - \frac{2}{a} R_1$$

$$R_3 = R_3 - \frac{2}{a} R_1$$

$$A = \begin{bmatrix} a & 2 & 2 \\ 0 & (a^2-4)/a & (2a-4)/a \\ 0 & (2a-4)/a & (a^2-4)/a \end{bmatrix}$$

$$R_3 = R_3 - \frac{2}{a+2} R_2$$

$$A = \begin{bmatrix} a & 2 & 2 \\ 0 & (a^2-4)/a & (2a-4)/a \\ 0 & 0 & \left(\frac{a^2-4}{a}\right) - \frac{2}{(a+2)} \times \frac{2a-4}{a} \end{bmatrix}$$

$a > 0$

$$\frac{a^2-4}{a} > 0 \Rightarrow a(a^2-4) > 0$$

$$a > 0 ; a > 2$$

$$\left(\frac{a^2-4}{a}\right) - \left(\frac{2}{a+2}\right) \times \left(\frac{2a-4}{a}\right) > 0 \Rightarrow a > 2$$

Which 3x3 matrix (symmetric) B produces the function $f = x^T A x$?

$$f = 2(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3)$$

$$x^T A x = (x \quad y \quad z) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= (x_1 \quad x_2 \quad x_3) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= a_{11} x_1^2 + a_{22} x_2^2 + a_{33} x_3^2 + 2a_{12} x_1 x_2 + 2a_{13} x_1 x_3 + 2a_{23} x_2 x_3$$

Comparing with f $\quad a_{11} = 2 \quad 2a_{12} = -2 \quad 2a_{23} = -2$

$a_{22} = 2 \quad \therefore a_{12} = -1 \quad \therefore a_{23} = -1$

$a_{33} = 2 \quad 2a_{13} = 0$

$\therefore a_{13} = 0$

$\therefore$ The matrix B is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Q.7. SVD of A $U \Sigma V^T$

$$A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}_{3\times 2}$$

$$\cancel{A^T A} \quad A^T A = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}_{2\times 2}$$

$$[A^T A - \lambda I] = 0$$

Eigen values $\lambda_1 = 90$, $\lambda_2 = 0$

$\lambda = 90$

$$\begin{bmatrix} 81-90 & -27 \\ -27 & 9-90 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -9 & -27 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$-9x - 27y = 0$$

$$x = -3y$$

$x_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ normalising $x_1 = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$

$\lambda = 0$

$$\begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} 81 & -27 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$3x = y$$

$x_2 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$ normalising. $x_2 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$

$$V = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

$$V^T = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 10 & -20 & -20 \\ 20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix}$$

Eigen values (λ) are : 90, 0, 0

Eigen vector for $\lambda_1 = 90$

$$\begin{bmatrix} -80 & -20 & -20 \\ -20 & -50 & 40 \\ -20 & 40 & -50 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\text{Gaussian elimination}} \begin{bmatrix} -80 & -20 & -20 \\ 0 & -45 & 45 \\ 0 & 0 & 0 \end{bmatrix}$$

$$y = z \quad \& \quad x = -\frac{z}{2}$$

$$u_1 = \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

Eigen vector for λ_2 & $\lambda_3 = 0$

$$\begin{bmatrix} -10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \ u_2 = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 10 & -20 & -20 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \quad x = 2y + 2z$$

$$u_3 = \begin{pmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{pmatrix}$$

$$\therefore U = \begin{bmatrix} -1/3 & 2/\sqrt{5} & 2/\sqrt{5} \\ 2/3 & 1/\sqrt{5} & 0 \\ 2/3 & 0 & 1/\sqrt{5} \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$\begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} -1/3 & 2/\sqrt{5} & 2/\sqrt{5} \\ 2/3 & 1/\sqrt{5} & 0 \\ 2/3 & 0 & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 3/\sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

x — x — x