

INVARIANTS OF TANGLES WITH FLAT CONNECTIONS IN THEIR COMPLEMENTS

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ABSTRACT. Let G be a simple complex algebraic group. By using a notion of a G -category we define invariants of tangles with flat G -connections in their complements. We also show that quantized universal enveloping algebras at roots of unity provide examples of G -categories.

CONTENTS

Introduction	2
1. Tangles with flat G -connections in their complements	2
1.1. Tangles and diagrams	2
1.2. Flat G -connections in the complement of a tangle and G -tangles	4
1.3. The category of G -tangles	6
2. The category of G -colored diagrams	7
2.1. Factorizable groups and Lie groups	7
2.2. G -colorings of diagrams	7
2.3. The category of G -colored diagrams	8
2.4. The equivalence of categories	9
2.5. A geometric version of the functor F	10
3. Braided G -categories	11
3.1. Braided G -categories	11
3.2. The category of G -colored diagrams is a ribbon G -category	15
3.3. Elementary diagrams	17
3.4. The category of \mathcal{C} -diagrams	18
4. Invariants of framed G -tangles	19
4.1. The functor $\Phi: \mathcal{D}(\mathcal{C}) \rightarrow \mathcal{C}$	19
4.2. Invariants of framed G -tangles	20
5. Quantized universal enveloping algebra of gl_2	20
5.1. The algebra $U_h(gl_2)$	20
5.2. The inner automorphism \mathcal{R}	21
6. The algebra \mathcal{U}	22

Date: July, 2002.

7. The algebra \mathcal{U}_ε	23
8. Conclusion	25
References	25

INTRODUCTION

A breakthrough in the theory of invariants of knots came with the discovery of the Jones polynomial [J] and its generalizations (HOMFLY and Kauffman). Then it was shown in [RT] that such invariants as well as invariants of tangled graphs can be obtained from quantized universal enveloping algebras of simple Lie algebras.

In this paper we extend the construction of invariants of links from [RT] (for details see [T1]). We construct invariants of tangles in $\mathbb{R}^2 \times [0, 1]$ with a flat connection in a principal G -bundle (where G is a simple complex algebraic group) over the complement of a tangle. The invariant is a functor from the category of G -tangles to a given G -category (see sections 1–3 for definitions). The invariant of a tangle with a gauge class of flat connections in the complement is defined as the image of the corresponding G -tangle.

In section 1 of this paper first we describe the moduli space of flat connections in the complement to a tangle in $\mathbb{R}^2 \times [0, 1]$. Then we define the category of tangles with gauge classes of flat connections in their complements. The construction is essentially the same as π -tangles introduced and studied by Turaev in [T2]. In section 2 we define the category of G -colored tangle diagrams and show that it is naturally equivalent to the category of tangles with flat G -connections in their complements. In section 3 we introduce the notion of a G -category, which is very similar but different from that of [T2]. In section 4 we construct invariants of G -tangles. In sections 5–7 we show that representations of \mathcal{U}_ε provide examples of GL_2^* -categories and therefore provide invariants of tangles with GL_2 -connections in their complements.

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1. TANGLES WITH FLAT G -CONNECTIONS IN THEIR COMPLEMENTS

1.1. Tangles and diagrams. Let $I \equiv [0, 1]$ be the closed unit interval. A *geometric tangle* $t \subset \mathbb{R}^2 \times I$ is the image of a smooth embedding $\underbrace{I \sqcup \dots \sqcup I}_k \sqcup \underbrace{S^1 \sqcup \dots \sqcup S^1}_l \rightarrow \mathbb{R}^2 \times I$ with oriented components such that $\partial t \subset \mathbb{R}^2 \times \partial I$, and t intersects the boundary $\mathbb{R}^2 \times \partial I$ perpendicularly. We have $\partial t = \partial_+ t \sqcup \partial_- t$ where $\partial_+ t = (\mathbb{R}^2 \times \{1\}) \cap \partial t$ and $\partial_- t = (\mathbb{R}^2 \times \{0\}) \cap \partial t$. The geometric tangle t is called *standard* if $\partial_+ t = \{(0, 1, 1), (0, 2, 1) \dots (0, n, 1)\}$ and $\partial_- t = \{(0, 1, 0), (0, 2, 0) \dots (0, m, 0)\}$ for some $n, m \in \mathbb{Z}_{>0}$. We will say that such tangle has type (m, n) .

Denote by D_t the image of t under the projection

$$(1) \quad p: \mathbb{R}^2 \times I \rightarrow \mathbb{R} \times I, \quad (x, y, z) \mapsto (y, z)$$

together with the additional information at each self-crossing point about underpassing and overpassing segments. The projection D_t is called *regular* if its only singularities are double points at which the images of the corresponding components of t intersect transversally. The ambient isotopy class of a regular projection D_t is called a *diagram* of the tangle t . When it is not confusing we will use the same notation D_t for diagrams and regular projections.

The diagram of a tangle can be regarded as the isotopy class of a smooth embedding of a graph with four-valent vertices of two types and one-valent vertices. These are the double points and the boundary points, respectively. The edges of the graph are the segments running between the vertices. A vertex of D_t is called *positive* if it is a double point and the angle between upper and lower components is positive. If the angle is negative, the vertex is called *negative*, see Fig. 1. Each such embedding is always a diagram of some tangle. Two diagrams are called *Reidemeister equivalent* if one can be obtained from another by a finite sequence of Reidemeister moves (see Figs 2–4).

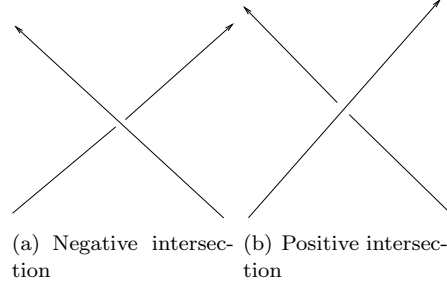


FIGURE 1.

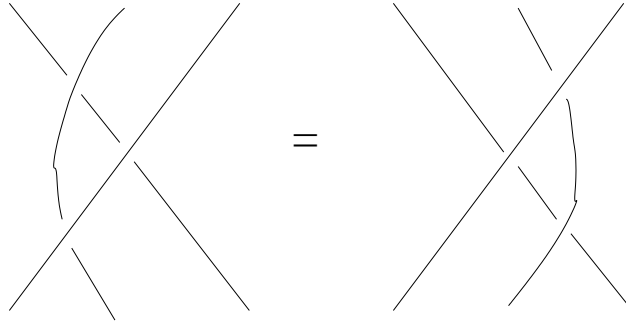


FIGURE 2.

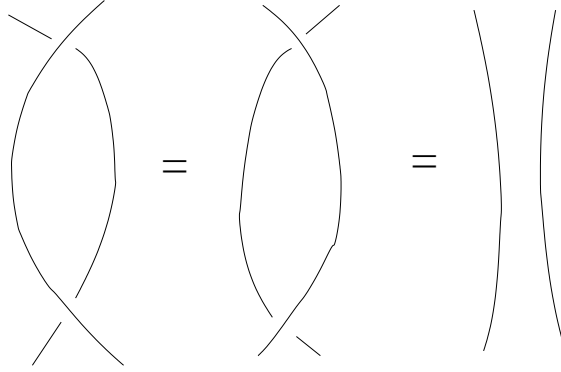


FIGURE 3.

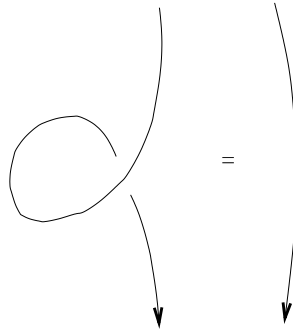


FIGURE 4.

A *tangle* is the isotopy class of a geometric tangle. Tangles are in bijection with Reidemeister classes of diagrams.

A *geometric framing* of a geometric tangle t is a continuous nonsingular section of the normal bundle to t . A framing of a tangle is the isotopy class of a geometric framing. A framing is *standard* if it is parallel to the x -axis at $\partial_{\pm} t$ with positive projection to this axis. It is clear that every framed geometric tangle with such framing at the boundary is isotopically equivalent to a framed geometric tangle with the framing parallel to x -axis. Framed tangle is the isotopy class of a framed geometric tangle with the standard framing.

Two diagrams of tangles are called *framed Reidemeister equivalent* if they are connected by a sequence of moves in Figs 2, 3, 5.

Framed Reidemeister classes of diagrams are in bijection with the framed tangles.

1.2. Flat G -connections in the complement of a tangle and G -tangles. Let E be a trivial principal G -bundle over $\mathbb{R}^2 \times I$ and $A_t \in \Omega^1(\mathbb{R}^2 \times I \setminus t, \mathfrak{g})$ be the 1-form representing a flat connection in E over

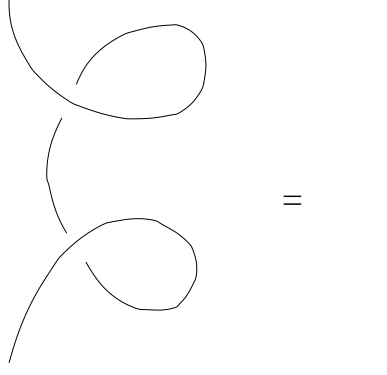


FIGURE 5.

the complement to a standard geometric framed tangle t of type (m, n) . The corresponding parallel transport operators along paths give an equivalent description of the flat connection as a representation of the fundamental groupoid of $(\mathbb{R}^2 \times I) \setminus t$ in G .

Let E_m be a trivial principal G -bundle over \mathbb{R}^2 , and α be the \mathfrak{g} -valued 1-form of a flat connection in E' over $\mathbb{R}^2 \setminus \{(0, 1), \dots, (0, m)\}$. Assume that $\alpha(x, y)$ decays sufficiently fast when y goes to $-\infty$ so that the parallel transport operator of α along a path connecting (x_0, y_0) with (x, y) in \mathbb{R}^2 has a limit as $y_0 \rightarrow -\infty$. Let $[\alpha]$ be the gauge class of α with respect to gauge transformations trivial at ∞ . Let γ_i be a path which starts at $(0, -\infty)$ encircles points $(0, 1), \dots, (0, i)$ and then returns to $(0, -\infty)$ (see Fig. 6). We can identify the class $[\alpha]$ with an element of $G^m = G \times \dots \times G$ so that the group element g_i in the collection $(g_1, \dots, g_m) \in G^m$ is the holonomy along the path γ_i .

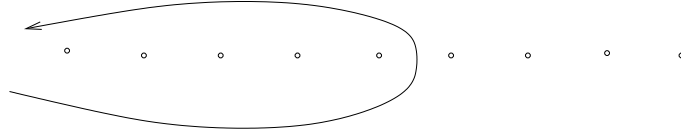


FIGURE 6.

Let α and β be flat connections in E_m and E_n , respectively. Denote by $\mathcal{A}_t(\alpha, \beta)$ the space of flat G -connections over the complement of t which, when restricted to $\mathbb{R}^2 \times \{0\} \setminus \{(0, 1, 0), \dots, (0, m, 0)\}$ and $\mathbb{R}^2 \times \{1\} \setminus \{(0, 1, 1), \dots, (0, n, 1)\}$, coincide with α and β , respectively, and which decay sufficiently fast when $y \rightarrow -\infty$ (so that the parallel along a path connecting (x_0, y_0, z_0) and (x, y, z) has a finite limit when $y_0 \rightarrow -\infty$).

If G_t is the group of gauge transformations trivial at infinity, the quotient space $\mathcal{M}_t([\alpha], [\beta]) = \mathcal{A}_t(\alpha, \beta)/G_t$ is by definition the moduli space of those flat G -connections in the complement of t which, when restricted to two boundary components $\mathbb{R}^2 \times \{0\} \setminus \partial t$ and $\mathbb{R}^2 \times \{1\} \setminus \partial t$, give representatives of the gauge classes $[\alpha]$ and $[\beta]$ respectively.

Following [T2], denote $C_t = (\mathbb{R}^2 \times I) \setminus t$ and consider the fundamental group of C_t with the base point in $\{0\} \times [-\infty, L] \times I$ with $t \subset \mathbb{R} \times [L+1, \infty] \times I$. The set of such base points is contractible and we will write $\pi_1(C_t)$ for the fundamental group, suppressing the base point. We will call such base points left base points. It is clear that an element of $\mathcal{M}_t([\alpha], [\beta])$ defines a representation of $\pi_1(C_t)$. Gauge classes of boundary values $[\alpha]$ and $[\beta]$ become elements of G^m and G^n respectively, defined as holonomies along γ_i (see above). Thus, the space $\mathcal{M}_t([\alpha], [\beta])$ can be identified with a subspace of G -tangles [T2], where a G -tangle is defined as a pair consisting of a framed oriented tangle t and a group homomorphism $\rho: \pi_1(C_t) \rightarrow G$.

Let t and t' be two standard isotopically equivalent geometric tangles. An isotopy bringing t to t' lifts to an isomorphism between G -connections over the complements of t and t' and therefore to an isomorphism between corresponding moduli spaces of flat G -connections. The isomorphism class of the moduli spaces generated by these isomorphisms can be identified with the equivalence class of pairs (t, A_t) with respect to isotopies of tangles, their pull-backs acting on connections and gauge transformations from G_t . We will denote the obtained set of classes by $\mathcal{M}_{[t]}([\alpha], [\beta])$.

From now on we will work with G -tangles. The geometric picture with flat connections will be used only once below when we shall assign a geometrical meaning to G -colorings of diagrams.

Let $v = (x, y, z) \in t \subset \mathbb{R}^2 \times I$ and $\gamma_v \subset C_t$ be a (homotopy) path which starts at a left base point then goes to the point $(x, y + \delta, z)$ "over" the tangle, then returns to the base point "under" the tangle. Here δ is sufficiently small and we assume that t is transversal to (x, y) -plane at (x, y, z) . We will call such path (the homotopy class thereof) *standard* for $v = (x, y, z)$.

1.3. The category of G -tangles. *Objects* of the category $\mathcal{T}(G)$ of G -tangles are finite sequences $\{(\varepsilon_1, g_1), \dots, (\varepsilon_n, g_n)\}$ with $\varepsilon_i = \pm 1$, $g_i \in G$. *Morphisms* from $\{(\varepsilon_1, g_1), \dots, (\varepsilon_m, g_m)\}$ to $\{(\sigma_1, h_1), \dots, (\sigma_n, h_n)\}$ are G -tangles.

Here m is the number of connected components of $(\mathbb{R}^2 \times \{0\}) \cap t$ and n is the number of connected components of $(\mathbb{R}^2 \times \{1\}) \cap t$. The signs ε_i and σ_i show the orientation of the boundary components. If "+", the component is oriented upward and if "-", it is oriented downward. The representation $\rho: \pi_1(C_t) \rightarrow G$ should agree with $\{g_i\}$ and $\{h_i\}$ in the following way. If γ_i^+ is a path from a left base point encircling boundary points $(0, 1, 0), \dots, (0, i, 0)$ in the vicinity of $\mathbb{R}^2 \times \{0\}$ then $\rho(\gamma_i^+) = g_i$. Similarly for a path γ_i^- encircling points $(0, 1, 1), \dots, (0, i, 1)$ in the vicinity of $\mathbb{R}^2 \times \{1\}$ we have $\rho(\gamma_i^-) = h_i$.

The composition of morphisms is defined by gluing tangles. The identity morphism of $\{(\varepsilon_1, g_1), \dots, (\varepsilon_n, g_n)\}$ to itself is the trivial braid with the representation of the fundamental group defined by g_1, \dots, g_n .

Remark 1. *The category $\mathcal{T}(G)$ can also be defined in more geometrical terms of gauge classes of flat connections. Objects are sequences $(\varepsilon_1, \dots, \varepsilon_n; [\alpha])$ where $[\alpha]$ is a gauge class of a flat connection over $\mathbb{R}^2 \setminus \{(0, 1) \dots (0, n)\}$ in the trivial principal G -bundle over \mathbb{R}^2 and $\varepsilon_i = \pm 1$. Morphisms between $(\varepsilon_1, \dots, \varepsilon_m; [\alpha])$*

and $(\sigma_1, \dots, \sigma_n, [\beta])$ are elements of $\mathcal{M}_{[t]}([\alpha], [\beta])$ i.e. equivalence classes of pairs (t, A_t) described in the previous subsection.

2. THE CATEGORY OF G -COLORED DIAGRAMS

2.1. Factorizable groups and Lie groups. We say that group G is *factorizable* into two subgroups $G_{\pm} \subset G$ if any element $g \in G$ can be represented in a unique way as

$$(2) \quad g = g_+ g_-^{-1}$$

where $g_{\pm} \in G_{\pm}$.

If G is a complex algebraic Lie group (later we will focus on this case) we will say that it is factorizable if there exists a Zariski open neighborhood $G' \subset G$ of 1 such that every element of G' has a unique factorization (2). Notice that in this case any $l \in \mathfrak{g} = \text{Lie}(G)$ has a unique decomposition

$$l = l_+ - l_-$$

where $l_{\pm} \in \mathfrak{g}_{\pm} = \text{Lie}(G_{\pm})$.

Remark 2. We can choose $G_+ = \{e\}$ and $G_- = G$. We call it *trivial factorizability*.

Let G be a factorizable group. Define a binary operation

$$g \star h = g_+ h_+ (g_- h_-)^{-1}$$

which obviously defines a group structure on G with the same identity element as for the original group structure. The inverse of g in this group is $i(g) = g_+^{-1} g_-$. This operation corresponds to the multiplication of the group $G_+ \times G_-$ under the mapping $G_+ \times G_- \rightarrow G$, $(g_+, g_-) \mapsto g_+ (g_-)^{-1}$. In what follows, we shall denote this group G^* .

2.2. G -colorings of diagrams. Let t be a standard geometric tangle and D_t , its diagram with the set of edges $E(D_t)$. Assume that G is a factorizable group.

Definition 1. The map $E(D_t) \rightarrow G$ which associates to edge e element $x_e \in G$ is called a G -coloring of diagram D_t if at each double point it satisfies the relations

$$x_{b_v} = (x_{a_v})_{\pm}^{-1} x_{c_v} (x_{a_v})_{\pm}, \quad x_{a_v} = (x_{c_v})_{\mp} x_{d_v} (x_{c_v})_{\mp}^{-1},$$

depending on whether the intersection is positive or negative. Here the enumeration of edges is the same as on Fig. 7.

Let $x_L, x_R: G \times G \rightarrow G$ be mappings acting as

$$(3) \quad x_L(x, y) = x_- y x_-^{-1}, \quad x_R(x, y) = x_L(x, y)_+^{-1} x_L(x, y)_+$$

In terms of these maps the definition above means that at positive double points we have $(x_a, x_b) = (x_L(x_c, x_d), x_R(x_c, x_d))$ and at negative double points $(x_c, x_d) = (x_L(x_a, x_b), x_R(x_a, x_b))$.

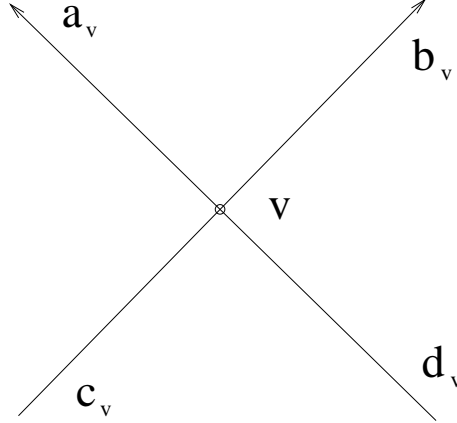


FIGURE 7.

The following proposition is due to Weinstein and Xu [WX] in the context of factorizable Poisson–Lie groups.

Proposition 1. *The map $\mathcal{R}: G \times G \rightarrow G \times G$ acting as $(x, y) \mapsto (x_L(y, x), x_R(y, x))$ satisfies the set-theoretical Yang–Baxter equation:*

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$$

Here all mappings act from $G^{\times 3}$ to itself. The mapping \mathcal{R}_{12} acts as \mathcal{R} in the first two factors and trivially in the last one. The other mappings act in a similar way. The subindices indicate in which factors the mapping acts non-trivially.

It follows from this proposition that if the G -coloring of lower edges of both diagrams on Fig. 2 are x, y, z , then the colorings of upper edges of the diagrams (which are determined by the diagrams and the coloring of lower edges) are the same for both diagrams. In other words, the G -coloring is compatible with the third Reidemeister move. It is easy to see that the G -coloring is also compatible with other framed Reidemeister moves.

Let (D, c) and (D', c') be two G -colored diagrams which are Reidemeister equivalent. Then, since in each Reidemeister move the coloring of a new diagram is uniquely defined by the coloring of the initial diagram, c' is uniquely determined by c . Thus, we have Reidemeister classes of G -colored diagrams.

2.3. The category of G -colored diagrams. Let G be a factorizable group. Define category $\mathcal{D}(G)$ of G -colored diagrams as follows.

Objects of the category are sequences $\{(\varepsilon_1, x_1), \dots, (\varepsilon_n, x_n)\}$ where $\varepsilon_i = \pm$ and $x_i \in G$ and the empty set.

Morphisms between $\{(\varepsilon_1, x_1), \dots, (\varepsilon_n, x_n)\}$ and $\{(\sigma_1, y_1), \dots, (\sigma_m, y_m)\}$ are Reidemeister classes of G -colored diagrams with the orientation of the boundary edges (adjacent to 1-valent vertices) defined by ε_i and σ_j as it is shown on Fig. 8 and with the G -colorings of the boundary edges given by $\varepsilon_i(x_i)$ and $\sigma_j(y_j)$

where $\varepsilon(x)$ is defined as

$$(4) \quad \varepsilon(x) = \begin{cases} x = x_+ x_-^{-1} & \text{for } \varepsilon = +1 \\ i(x) = x_+^{-1} x_- & \text{for } \varepsilon = -1 \end{cases}$$

The operation $x \rightarrow i(x)$ is taking the inverse in G^* . The identity morphism is shown on Fig. 9.

Composition of Reidemeister classes of G -colored diagrams (D_1, c_1) and (D_2, c_2) is the Reidemeister class of the G -colored diagram $(D_1 \circ D_2, c)$ where $D_1 \circ D_2$ is the diagram obtained by gluing D_1 and D_2 and the coloring c is induced by colorings c_1 and c_2 ,

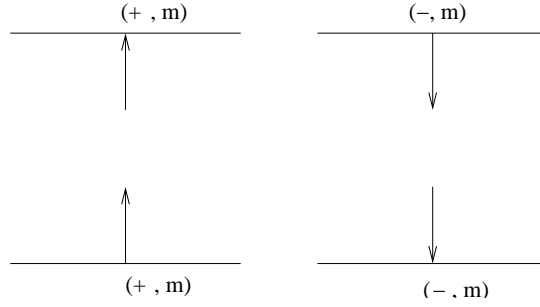


FIGURE 8.

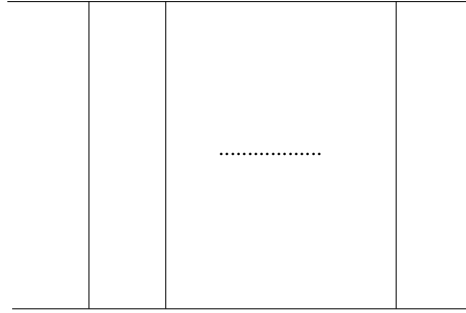


FIGURE 9.

2.4. The equivalence of categories. Here we will prove the equivalence of categories $\mathcal{T}(G) \simeq \mathcal{D}(G)$.

Consider the map $F : \mathcal{T}(G) \rightarrow \mathcal{D}(G)$ acting on objects as

$$F((\varepsilon_1, g_1), \dots, (\varepsilon_n, g_n)) = \{(\varepsilon_1, x_1) \dots (\varepsilon_n, x_n)\}$$

Here $x_i \in G$ are related to $g_i \in G$ via

$$(5) \quad g_i = (x_1)_+^{\varepsilon_1} \dots (x_i)_+^{\varepsilon_i} (x_i)_-^{-\varepsilon_i} (x_1)_-^{-\varepsilon_1}$$

For a G -tangle (t, ρ) define

$$F((t, \rho)) = [(D_t, c)]$$

where D_t is a diagram of the tangle t , c is the coloring of D_t which we define below, and $[(D_t, c)]$ the colored framed Reidemeister class of (D, c) .

Consider a standard path γ_v associated with point $v \in t$. Let e_1, \dots, e_i be the edges of D_t intersected by the projection of γ_v , and $\varepsilon_1, \dots, \varepsilon_i$ the signs of the projections of their orientations to the vertical axis. Then the holonomy $g_v = g_i$ along γ_v associated to ρ is given by formula (5), where x_1, \dots, x_i now are the colors of edges e_1, \dots, e_i . It is easy to see that this correspondence between ρ and the edge colors is one to one, and thus the mapping F is invertible.

It is easy to see that the map F is a functor. To prove this it remains to show that $F(fg) = F(f)F(g)$ for morphisms f and g , which is obvious. It is also clear that this functor is an equivalence of categories.

2.5. A geometric version of the functor F . Now consider a geometric description of the category $\mathcal{T}(G)$ and describe the functor F in this terms.

To define the action of F on objects we consider a special representative $\tilde{\alpha}$ of the gauge class $[\alpha]$. Namely, $\tilde{\alpha}$ is continuous and vanishes outside the strips $\{(x, y) \mid x \in \mathbb{R}, i - \epsilon < y < i + \epsilon\}$ for some $0 < \epsilon < 1/2$. Let γ_i^\pm be paths connecting points $(0, i - 1/2)$ and $(0, i + 1/2)$ which go around $(0, i)$ in the clock-wise direction for γ_i^- and in the counter clock-wise direction for γ_i^+ . These paths are shown as $(-)$ and $(+)$ paths respectively on Fig. 10.

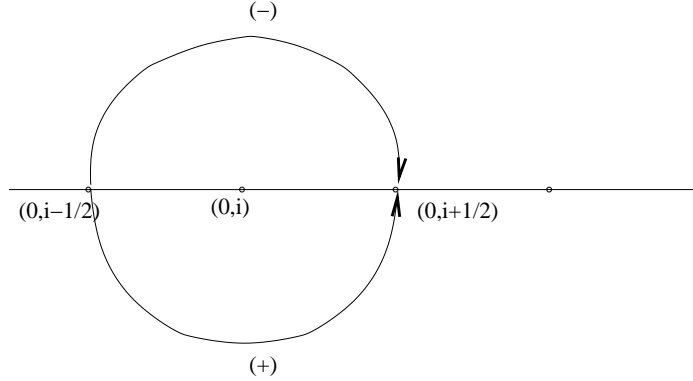


FIGURE 10.

Define the action of F on objects as:

$$F : (\varepsilon_1, \dots, \varepsilon_n; [\alpha]) \mapsto \{(\varepsilon_1, x_1), \dots, (\varepsilon_n, x_n)\}$$

where $x_i = \text{hol}_{\gamma_i}(\tilde{\alpha})$ and $\gamma_i = \gamma_i^+ \circ (\gamma_i^-)^{-1}$.

To define the action of F on morphisms, first, let us look at the geometry of the projection p . The preimages of edges of D_t form a system of “walls” $p^{-1}(e) \subset \mathbb{R}^2 \times I$, $e \in E(D_t)$, intersecting at lines which are preimages of the vertices of D_t . Let A_t be a flat connection over the complement of t . If the tangle is not a link all chambers bounded by walls including the “outer” chambers are simply connected so the flat

connection A_t can be trivialized inside these chambers, except a thin neighborhood of walls. If the tangle is a link, the outer chamber is not simply connected, but we still can trivialize the flat connection inside this chamber since the link can be placed into a ball and the outside of a ball is a simply connected region in $\mathbb{R}^2 \times I$. Thus, we can trivialize the flat connection in each chamber outside of a thin neighborhood of walls. Assume that this thin neighborhood is such that at the boundary it is inside of strips $x \in \mathbb{R}$, $i - \epsilon < y < i + \epsilon$. The tangle t separates each wall into two semi-infinite parts. Let e be an edge of D_t , denote by $p^{-1}(e)_-$ the part of the wall $p^{-1}(e)$ which is semi-infinite in the negative x -direction and by $p^{-1}(e)_+$ the other part of this wall. Since the connection is flat and since now it is trivial inside the chambers (except of a thin neighborhood of walls), the holonomy through the wall $p^{-1}(e)$ along any path that is based on a pair of points separated by this wall and which are outside of a thin neighborhood of the wall, depends only on the homotopy class of the path. Let us call the path positive if its orientation together with the orientation of the edge e and with the direction of the projection p form positively oriented triple of vectors in \mathbb{R}^3 with the standard orientation. This produces two elements $x_{\pm}(e)$ of G which we can assign to the edge e corresponding to holonomies along positive paths through $p^{-1}(e)_{\pm}$. The product $x(e) = x_+(e)x_-(e)^{-1}$ is the holonomy along a closed path that crosses first the wall $p^{-1}(e)_+$ in the positive direction and then $p^{-1}(e)_-$ in the negative direction. If G is factorizable there exists unique pair $x(e)_{\pm}$ which factorizes $x(e)$ as above with $x(e)_{\pm} \in G_{\pm}$, and one can choose the connection A_t so that $x_{\pm}(e) = x(e)_{\pm}$. This gives us an assignment $e \mapsto x(e)$ of group elements to edges of the diagram. Notice that this assignment does not depend on base points inside chambers and depends only on the gauge class of the flat connection.

The holonomy along a path connecting two points based inside chambers is the product of holonomies through the walls intersected by this path. Given a tangle t and its diagram D_t these holonomies can be computed by the *wall crossing rule* (see Fig. 15). Horizontal edges on Fig. 15 represent parts of the path. Vertical edges represent parts of the tangle (walls). Under-crossings and over-crossings show whether the path went through $p^{-1}(e)_-$ or $p^{-1}(e)_+$ respectively where e is the corresponding edge of the diagram. The holonomy gained at the crossing is given in terms of the G coloring of e .

To show that the map assigning group elements to edges constructed above gives G -colorings, one should consider pairs of paths from Figs 11–14. Isotopy equivalence of these paths implies the equality of the corresponding holonomies. Computation of them according to the “wall-crossing rules” described above gives the identities in the definition of the G -coloring.

3. BRAIDED G -CATEGORIES

3.1. Braided G -categories. A *braided group* is a pair $(G, \mathcal{R} : G \times G \rightarrow G \times G)$ where G is a group and the map \mathcal{R} satisfies the following requirements

- (1) $m \circ \mathcal{R} = m'$
- (2) $\mathcal{R} \circ (m \times \text{id}) = (m \times \text{id}) \circ \mathcal{R}_{23} \circ \mathcal{R}_{13}$
- (3) $\mathcal{R} \circ (\text{id} \times m) = (\text{id} \times m) \circ \mathcal{R}_{12} \circ \mathcal{R}_{13}$

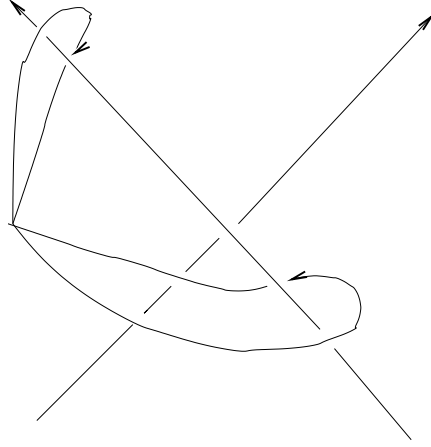


FIGURE 11.

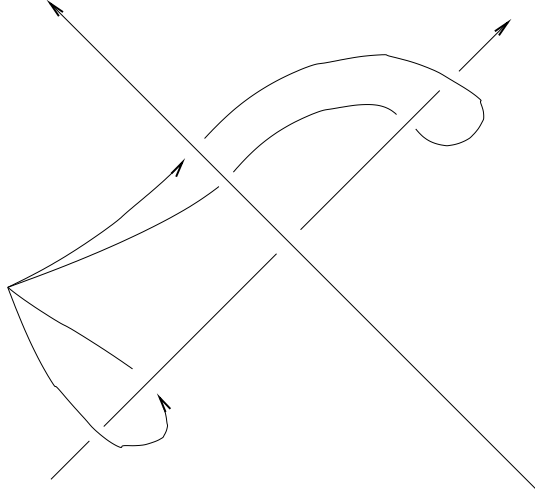


FIGURE 12.

where m is the multiplication and m' is the opposite multiplication in G . In particular, \mathcal{R} satisfies the set-theoretical Yang–Baxter equation

$$\mathcal{R}_{12} \circ \mathcal{R}_{13} \circ \mathcal{R}_{23} = \mathcal{R}_{23} \circ \mathcal{R}_{13} \circ \mathcal{R}_{12}.$$

An example of a braided group is the pair (G^*, \mathcal{R}) where G^* is a factorized group G with multiplication $m(x, y) = x \star y$ and \mathcal{R} is given by (3).

For a set A we will say that category \mathcal{C} is *fibered over* A if

- There is a projection $\pi : \text{Ob}(\mathcal{C}) \rightarrow A$
- $\text{Hom}_{\mathcal{C}}(X, Y) = \emptyset$, if $\pi(X) \neq \pi(Y)$

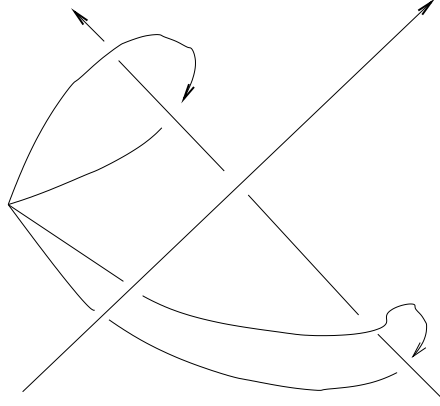


FIGURE 13.

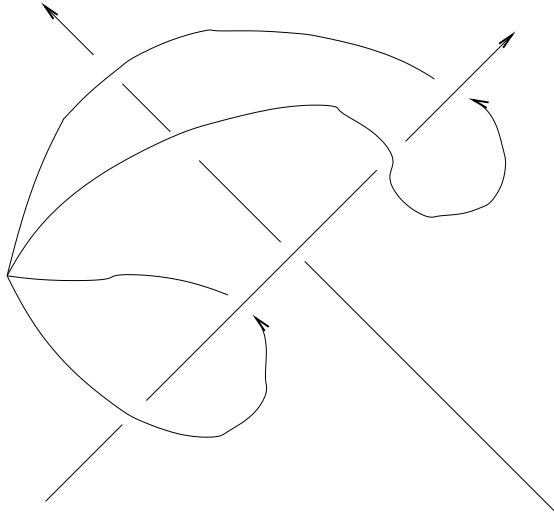


FIGURE 14.

Recall that a monoidal category is a category with a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$. This functor is given together with the natural transformations $a : \otimes \circ \otimes \times id \simeq \otimes \circ id \times \otimes$ which is called an associativity constraint. This natural transformation should satisfy the pentagon identity [ML]. In addition to this, a monoidal category has an identity object $\mathbb{1}$ given together with a system of functorial isomorphisms $l_X : \mathbb{1} \otimes X \simeq X$ and $r_X : X \otimes \mathbb{1} \simeq X$ which satisfy some natural conditions [ML].

A monoidal category fibered over a group G is called a G -category if

- $\pi(X \otimes Y) = \pi(X)\pi(Y)$
- $\pi(\mathbb{1}) = e \in G$

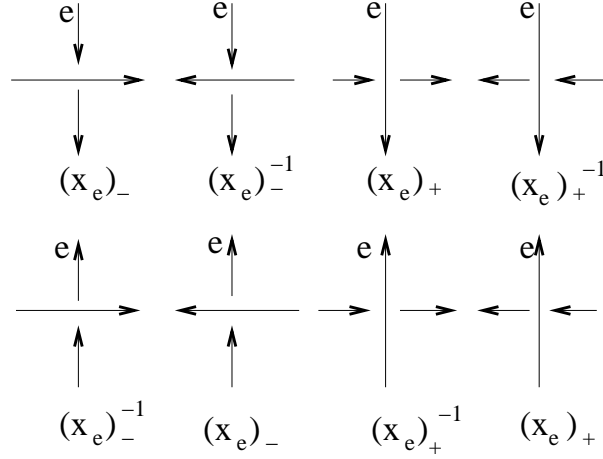


FIGURE 15.

Associativity constraint and functorial morphisms r_X and l_X should act fiber-wise. From now on we will work only with strict monoidal categories: we assume that the associativity constrain is trivial (see [ML] for details on what this exactly means).

Recall that a monoidal category is called rigid if any object X has a left dual object X^* , the injection and evaluation mappings $i_X : \mathbb{1} \rightarrow X \otimes X^*$ and $e_X : X^* \otimes X \rightarrow \mathbb{1}$, and if the triple X^*, i_X, e_X is unique up to an isomorphism. In a rigid monoidal category double dual is not necessary isomorphic to the object itself, and, in particular, each object has left and right duals.

A G -category \mathcal{C} is called *rigid* if it is a rigid monoidal category and in addition to the properties listed above one has

$$\pi(X^*) = \pi(*X) = \pi(X)^{-1}$$

where $*X$ and X^* are left and right duals to X respectively. The evaluation and injection morphisms act fiber-wise.

Now assume that the group G is a braided group. A category \mathcal{C} is called braided rigid G -category if it is a rigid G -category and in addition to this it has the following properties:

- (1) There exists a functor $B : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \xrightarrow{B} & \mathcal{C} \times \mathcal{C} \\ \pi \times \pi \downarrow & & \downarrow \pi \times \pi \\ G \times G & \xrightarrow{\tilde{\mathcal{R}}} & G \times G \end{array}$$

where $\tilde{\mathcal{R}} = \mathcal{R} \circ P$, and $P(x, y) = (y, x)$. We will write $B : (X, Y) \rightarrow (X_L(X, Y), X_R(X, Y))$ for the action of B . This property is the lifting of the property 1. of braided group G to the category \mathcal{C} .

- (2) The functor B satisfies the following identities (for functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C} \times \mathcal{C}$)

$$B \circ (\otimes \times id) = (id \times \otimes) \circ (B \times id) \circ (id \times B)$$

and the same identity for B^{-1} . This properties of B are liftings of properties 2. and 3. of the braided group G to the category \mathcal{C} .

- (3) There exists an isomorphism of functors c which makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \xrightarrow{B} & \mathcal{C} \times \mathcal{C} \\ \otimes \searrow & \xrightarrow{c} & \swarrow \otimes \\ & \mathcal{C} & \end{array}$$

In other words, there exists a system of functorial isomorphisms

$$c^{X,Y} : X \otimes Y \rightarrow X_L(X, Y) \otimes X_R(X, Y).$$

- (4) The commutativity constraint should satisfy the hexagon axioms

$$c^{X \otimes Y, Z} = (c^{X, X_L(Y, Z)} \otimes id)(id \otimes c^{Y, Z})$$

$$c^{X, Y \otimes Z} = (c^{X, Y} \otimes id)(c^{X_R(X, Y), Z} \otimes id)$$

A braided G -category is called a *ribbon category* if it in addition to being a G -category has a system of functorial morphisms $\{\mu_X : X \rightarrow X^{**}\}_{X \in \text{Ob}(\mathcal{C})}$ such that

- $\mu_{X \otimes Y} = \mu_X \otimes \mu_Y$
- $\mu_{X^*} = (\mu_X^*)^{-1}$
- $\mu_{\mathbb{I}} = id$

When it will not be misleading we will shorten the name “ rigid braided ribbon G -category” to “ribbon G -category”. The theorem 1 provides an example of a G -category.

When $G_+ = e$ are $G_- = G$ the notion of the G -category introduced above is equivalent to the one introduced in [T2].

3.2. The category of G -colored diagrams is a ribbon G -category.

Theorem 1. *The category of G -colored framed diagrams is a ribbon G^* -category where $G^* = G_+ \times G_-$.*

Proof. First, let us check that \mathcal{R} satisfies the required properties. The first identity for \mathcal{R} is equivalent to

$$x_L(x, y) \cdot x_R(x, y) = x \cdot y$$

The second identity for \mathcal{R} is equivalent to

$$x_L(x, y \cdot z) = x_L(x, y) \cdot x_L(x_R(x, y), z)$$

and

$$x_R(x, y \cdot z) = x_R(x_R(x, y), z)$$

and similar identities assure the last property of \mathcal{R} . Here the multiplication is taken in G^* . All these identities are easy to check.

Now let us describe the structure of a G^* -category explicitly. Define the G^* structure on $\mathcal{D}(G)$ as

$$\pi((\varepsilon_1, x_1), \dots, (\varepsilon_n, x_n)) = \varepsilon_1(x_1) \cdots \varepsilon_n(x_n)$$

Here the product is taken in G^* , $x \cdot y = x_+ y_+ y_-^{-1} x_-^{-1}$, and $\varepsilon(x)$ is defined in (4).

The monoidal structure is the same as for the category of diagrams. The tensor product of objects is

$$\{(\varepsilon_1, x_1), \dots, (\varepsilon_n, x_n)\} \otimes \{(\sigma_1, y_1), \dots, (\sigma_m, y_m)\} = \{(\varepsilon_1, x_1), \dots, (\varepsilon_n, x_n), (\sigma_1, y_1), \dots, (\sigma_m, y_m)\}$$

The tensor product of morphisms is shown on Fig. 16. The identity object is the empty set.

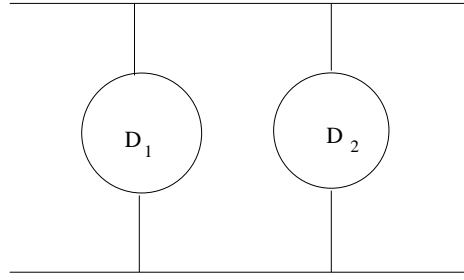


FIGURE 16.

The object dual to $\{(\varepsilon_1, x_1), \dots, (\varepsilon_n, x_n)\}$ is $\{(-\varepsilon_n, i(x_n)), \dots, (-\varepsilon_1, i(x_1))\}$ with the evaluation and the injection morphisms given by diagrams from Fig. 17 and Fig. 18 with the G -colorings induced by objects.

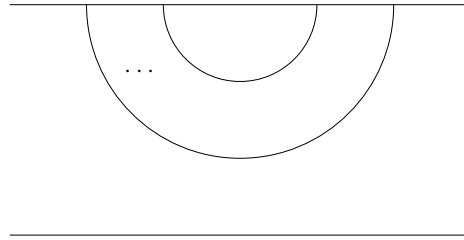


FIGURE 17.

To describe the braiding, let us first define the functor $B : \mathcal{D}(G) \times \mathcal{D}(G) \rightarrow \mathcal{D}(G) \times \mathcal{D}(G)$ as follows. On objects it acts as

$$\begin{aligned} B(\{(\varepsilon_1, x_1), \dots, (\varepsilon_n, x_n)\}, \{(\sigma_1, y_1), \dots, (\sigma_m, y_m)\}) \\ = (\{(\sigma_1, y_1^L), \dots, (\sigma_m, y_m^L)\}, \{(\varepsilon_1, x_1^R), \dots, (\varepsilon_n, x_n^R)\}) \end{aligned}$$

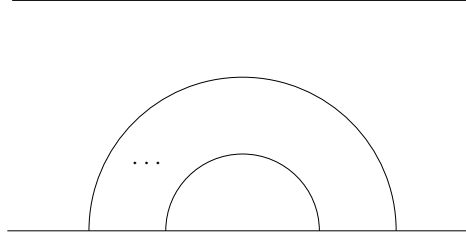


FIGURE 18.

where $(x_1^L, \dots, x_m^L, y_1^R, \dots, y_n^R)$ is the image of $(x_1, \dots, x_n, y_1, \dots, y_m)$ with respect to the map

$$(s_n \cdots s_{n+m-1})(s_{n-1} \cdots s_{n+m-2}) \cdots (s_2 \cdots s_{n+1})(s_1 \cdots s_n) : G^{\times(n+m)} \rightarrow G^{\times(n+m)}$$

where $s_i = \check{\mathcal{R}}_{ii+1}$.

If $[(D_i, c_i)]$ is a morphism $(\varepsilon^{(i)}, x^{(i)}) \rightarrow (\sigma^{(i)}, y^{(i)})$,

$$B((D_1, c_1), (D_2, c_2)) = ((D_2, c'_2), (D_1, c'_1)),$$

Here colorings c'_1 and c'_2 are determined by c_1 and c_2 and by the corresponding objects.

The commutativity morphism is represented by the diagram on Fig. 19 and it is a mapping

$$\begin{aligned} \{(\varepsilon_1, x_1), \dots, (\varepsilon_n, x_n)\} \otimes \{(\sigma_1, y_1), \dots, (\sigma_m, y_m)\} \\ \rightarrow \{(\sigma_1, y_1^L), \dots, (\sigma_m, y_m^L)\} \otimes \{(\varepsilon_1, x_1^R), \dots, (\varepsilon_n, x_n^R)\} \end{aligned}$$

The coloring of the diagram on Fig. 19 is determined by the objects.

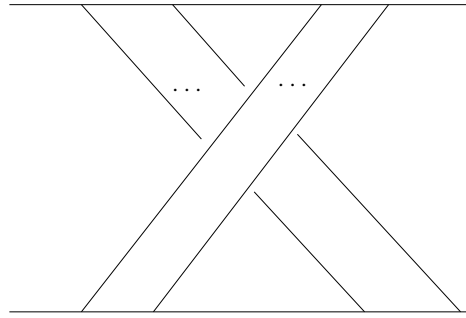


FIGURE 19.

□

3.3. Elementary diagrams. The following fact is a key for construction of invariants of tangles via braided monoidal categories.

Proposition 2. *All morphisms in the category $\mathcal{D}(G)$ are compositions of tensor products of elementary diagrams. Elementary diagrams are given on Fig. 20.*

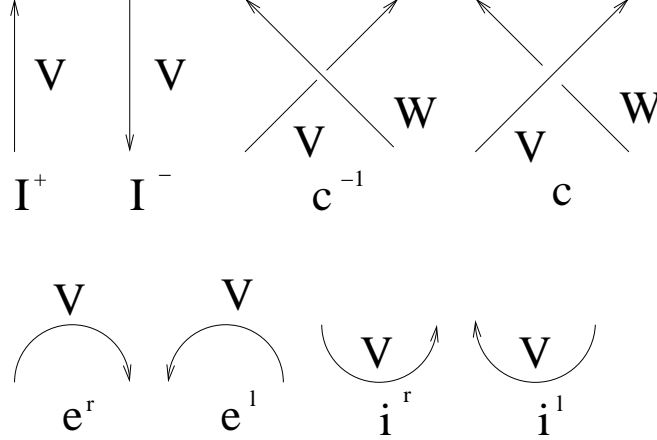


FIGURE 20.

The proof of this proposition and the definition of elementary diagrams are the same as for the category of framed tangles (see [T1]).

3.4. The category of \mathcal{C} -diagrams. Let \mathcal{C} be a braided G -category.

Definition 2. *A pair (D, a) where D is a diagram and $a: E(D) \rightarrow \text{Ob}(\mathcal{C})$ is called a \mathcal{C} -diagram if at each double point the values of the map a on adjacent edges satisfy the following conditions (the edges are enumerated as in Fig. 7):*

- *If a double point is positive, $X_a = X_L(X_c, X_d)$, $X_b = X_R(X_c, X_d)$.*
- *If the double point is negative, $X_c = X_L(X_a, X_b)$, $X_d = X_R(X_a, X_b)$.*

If G is factorizable and \mathcal{R} is given by (3), then it is clear that each \mathcal{C} -diagram defines a G -colored diagram defined by the composition map $\pi \circ a: E(D) \rightarrow G$.

Now let us define the category $\mathcal{D}(\mathcal{C})$ of \mathcal{C} -diagrams.

Objects of $\mathcal{D}(\mathcal{C})$ are finite sequences $\{(\varepsilon_1, X_1), \dots, (\varepsilon_n, X_n)\}$, where $\varepsilon_i = \pm$ and $X_i \in \text{Ob}(\mathcal{C})$.

Morphisms from $\{(\varepsilon_1, X_1), \dots, (\varepsilon_n, X_n)\}$ to $\{(\sigma_1, Y_1), \dots, (\sigma_m, Y_m)\}$ are framed \mathcal{C} -diagrams (D, a) with $a(e_i^+) = Y_i^{\varepsilon_i}$, $a(e_i^-) = X_i^{\sigma_i}$. Here e_i^+ are edges of D adjacent to the upper boundary, enumerated from left to right $i = 1, \dots, m$, and e_i^- are edges adjacent to the lower boundary, with $i = 1, \dots, n$.

The identity morphism of $(\varepsilon_1, X_1), \dots, (\varepsilon_n, X_n)$ is the trivial braid with the orientation of components defined by ε_i and with $X_i \in \text{Ob}(\mathcal{C})$ assigned to i -th strand.

The following theorem is a generalization of Theorem 1 about G -colored diagrams.

Theorem 2. *The category $\mathcal{D}(\mathcal{C})$ is a ribbon G -category.*

The proof is parallel to the one of the theorem 1.

4. INVARIANTS OF FRAMED G -TANGLES

4.1. **The functor Φ :** $\mathcal{D}(\mathcal{C}) \rightarrow \mathcal{C}$. As in the case of framed tangles and tangled framed graphs the first step in the construction of invariants of G -tangles will be the construction of a rigid G -braided monoidal functor from the category of \mathcal{C} -diagrams to the category \mathcal{C} .

Let \mathcal{C} be a ribbon G -category.

Theorem 3. *There exists a unique covariant functor $\Phi : \mathcal{D}(\mathcal{C}) \rightarrow \mathcal{C}$ such that*

- $\Phi(\{(\varepsilon_1, X_1), \dots, (\varepsilon_n, X_n)\}) = X_1^{\varepsilon_1} \otimes \dots \otimes X_n^{\varepsilon_n}$, where $X^+ = X$, $X^- = X^*$.
- Φ is a monoidal functor, i.e.

$$\Phi((D_1, a_1) \otimes (D_2, a_2)) = \Phi((D_1, a_1)) \otimes \Phi((D_2, a_2))$$

where (D_i, a_i) are \mathcal{C} -diagrams.

- Values of Φ on elementary diagrams are:

(1) If $I^\varepsilon : (\varepsilon, X) \rightarrow (\varepsilon, X)$ is the identity morphism then

$$\Phi(I^\varepsilon) = id_{X^\varepsilon}.$$

(2) For morphism $e_X^{r,l} : (\pm, X) \otimes (\mp, X) \rightarrow 1$ we have:

$$\Phi(e_X^r) = e_{X^*} \circ (\mu_X \otimes id) : X \otimes X^* \rightarrow X^{**} \otimes X^* \rightarrow \mathbb{1},$$

$$\Phi(e_X^l) = e_X : X^* \otimes X \rightarrow \mathbb{1}$$

(3) For $i_X^{r,l} : \mathbb{1} \rightarrow (\mp, X) \otimes (\pm, X)$ we have

$$\Phi(i_X^r) = (\mu_X^{-1} \otimes id) \circ i_{X^*} : \mathbb{1} \rightarrow X \otimes X^*$$

$$\Phi(i_X^l) = i_X : \mathbb{1} \rightarrow X^* \otimes X,$$

(4) If $c : (+, X) \otimes (+, Y) \rightarrow (+, X_L(X, Y)) \otimes (+, X_R(X, Y))$ is the braiding morphism,

$$\Phi(c) = c^{X,Y} : X \otimes Y \rightarrow X_L \otimes X_R.$$

(5) For the inverse morphism $c^{-1} : (+, X_L) \otimes (+, X_R) \rightarrow (+, X) \otimes (+, Y)$ we have:

$$\Phi(c^{-1}) = (c^{X,Y})^{-1} : X_L \otimes X_R \rightarrow X \otimes Y.$$

- This functor Φ is rigid monoidal and G -braided.

The proof of this theorem is completely parallel to the corresponding theorem describing invariants of framed tangles.

4.2. Invariants of framed G -tangles. Let \mathcal{C} be a ribbon G -category and $A = \{A_x\}_{x \in G}$ be a family of objects such that $\pi(A_x) = x$. In other words A is a section of $\pi : \mathcal{C} \rightarrow G$. Assume that objects from this family have the following property:

$$B : (A_x, A_y) \rightarrow (A_{x_L(x,y)}, A_{x_R(x,y)})$$

This implies that the braiding morphisms act as:

$$c^{A_x, A_y} : A_x \otimes A_y \rightarrow A_{x_L(x,y)} \otimes A_{x_R(x,y)}$$

It also implies that

$$c^{A_x^*, A_y} : A_x^* \otimes A_y \rightarrow A_{x_R(y, x_1)} \otimes A_{x_1}^*$$

where $x_1 \in G$ is such that $x = x_L(y, x_1)$. This can be derived from the previous formula and from the axioms for the evaluation and injection morphisms. The action of the braiding on other duals can be computed similarly.

Now, let us fix such family of objects in \mathcal{C} to define a \mathcal{C} -coloring of D for a given G -coloring c of the diagram D . We construct invariants of G -tangles as follows:

- given a G -tangle (t, ρ) define the G -colored diagram $[(D, c)]$ as in section 2 using the equivalence of categories of G -tangles and G -diagrams.
- given a G -colored diagram $[(D, c)]$ define a \mathcal{C} -diagram $[(D, a)]$ as above.
- apply the functor Φ to the \mathcal{C} -diagram $[(D, a)]$.

It is clear, from the definition of every step here, that the composition map is an invariant of (t, ρ) with values in morphisms of the category \mathcal{C} .

In the following section we will describe a $GL_2(\mathbb{C})$ -category associated with the quantized universal enveloping algebra of $gl_2(\mathbb{C})$.

5. QUANTIZED UNIVERSAL ENVELOPING ALGEBRA OF gl_2

5.1. The algebra $U_h(gl_2)$. The algebra $U_h(gl_2)$ over the ring $\mathbb{C}[[h]]$ is generated by elements H, G, X , and Y with defining relations

$$[H, G] = 0, [H, X] = 2X, [H, Y] = -2Y,$$

$$[G, X] = 2X, [G, Y] = -2Y,$$

$$[X, Y] = \frac{e^{\frac{hH}{2}} - e^{-\frac{hG}{2}}}{e^{\frac{h}{2}} - e^{-\frac{h}{2}}}$$

The Hopf algebra structure on $U_h(gl_2)$ is defined by the action of the comultiplication on generators:

$$(6) \quad \Delta H = H \otimes 1 + 1 \otimes H, \quad \Delta G = G \otimes 1 + 1 \otimes G,$$

$$(7) \quad \Delta X = X \otimes e^{\frac{hH}{2}} + 1 \otimes X, \quad \Delta Y = Y \otimes 1 + e^{-\frac{hG}{2}} \otimes Y$$

Elements H, G, X and Y "correspond to" elements $2e_{11}, 2e_{22}, e_{12}$ and e_{21} respectively.

The algebra $U_h(gl_2)$ is the Drinfeld double of the quantized universal enveloping algebra $U_h(b) \subset U_h(sl_2)$ where b is a Borel subalgebra in sl_2 . As the double of a Hopf algebra it is quasitriangular with the universal R -matrix

$$R = \exp\left(\frac{h}{4}H \otimes G\right) \prod_{n \geq 0} (1 + e^{\frac{h}{2}}(e^{\frac{h}{2}} - e^{-\frac{h}{2}})^2 X \otimes Y e^{-nh})$$

This is the element of $U_h(gl_2)^{\otimes 2}$ which one should consider as a formal power series in h .

Since R is the universal R -matrix it satisfies the following identities:

$$(8) \quad \begin{aligned} R\Delta(a)R^{-1} &= \sigma \cdot \Delta(a) \\ (\Delta \otimes id)(R) &= R_{13}R_{23} \\ (id \otimes \Delta)(R) &= R_{13}R_{12} \end{aligned}$$

where σ is the permutation operator $\sigma(a \otimes b) = b \otimes a$. In particular, R satisfies the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

5.2. The inner automorphism \mathcal{R} . Define the inner automorphism

$\mathcal{R} : U_h(gl_2)^{\otimes 2}[[h]] \rightarrow U_h(gl_2)^{\otimes 2}[[h]]$ as

$$(9) \quad \mathcal{R}(x \otimes y) = R(x \otimes y)R^{-1}$$

It is easy to compute the action of \mathcal{R} on generators.

Theorem 4. *The following identities hold.*

$$\begin{aligned} \mathcal{R}(1 \otimes e^{\frac{hH}{2}}) &= (1 \otimes e^{\frac{hH}{2}})(1 + e^{\frac{h}{2}}(e^{\frac{h}{2}} - e^{-\frac{h}{2}})^2 e^{-\frac{hH}{2}} X \otimes Y e^{\frac{hG}{2}})^{-1} \\ \mathcal{R}(1 \otimes e^{\frac{hG}{2}}) &= (1 \otimes e^{\frac{hG}{2}})(1 + e^{\frac{h}{2}}(e^{\frac{h}{2}} - e^{-\frac{h}{2}})^2 e^{-\frac{hH}{2}} X \otimes Y e^{\frac{hG}{2}})^{-1} \\ \mathcal{R}(X \otimes 1) &= X \otimes e^{\frac{hG}{2}} \\ \mathcal{R}(1 \otimes Y) &= e^{-\frac{hH}{2}} \otimes Y \end{aligned}$$

The theorem follows immediately from the commutation relations between generators and from the equation

$$f(zq^{-1}; q) = (1 + zq^{-1})f(z; q)$$

for the function

$$f(z; q) = \prod_{n=0}^{\infty} (1 + zq^n).$$

The action of \mathcal{R} on elements $1 \otimes X$, $Y \otimes 1$, $e^{\frac{hH}{2}} \otimes 1$, and $e^{\frac{hG}{2}} \otimes 1$ can be derived from the formulae above and from the identity (8).

The Yang-Baxter equation for R implies the Yang-Baxter equation for \mathcal{R} :

$$\mathcal{R}_{12} \cdot \mathcal{R}_{13} \cdot \mathcal{R}_{23} = \mathcal{R}_{23} \cdot \mathcal{R}_{13} \cdot \mathcal{R}_{12}$$

6. THE ALGEBRA \mathcal{U}

The algebra \mathcal{U} is generated over $\mathbb{C}[t, t^{-1}]$ by elements K, L, E and F with the following defining relations

$$KL = LK, \quad KE = t^2 EK, \quad KF = t^{-2} FK,$$

$$LE = t^2 EL, \quad LF = t^{-2} FL,$$

$$EF - FE = (t - t^{-1})(K - L^{-1})$$

The center of \mathcal{U} is generated freely by Laurent polynomials in KL^{-1} and

$$(10) \quad c = EF + Kt^{-1} + L^{-1}t$$

This is a Hopf algebra with

$$\Delta(K) = K \otimes K, \quad \Delta(L) = L \otimes L,$$

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + L^{-1} \otimes F.$$

The map $\phi : \mathcal{U} \rightarrow U_h(gl_2)$ acting on generators as

$$\phi(K) = \exp\left(\frac{hH}{2}\right), \quad \phi(L) = \exp\left(\frac{hG}{2}\right), \quad \phi(t) = e^{\frac{h}{2}}$$

$$\phi(E) = (e^{\frac{h}{2}} - e^{-\frac{h}{2}})X, \quad \phi(F) = (e^{\frac{h}{2}} - e^{-\frac{h}{2}})Y$$

extends to a homomorphism of Hopf algebras.

The algebra \mathcal{U} is not quasitriangular. Instead, there is an outer automorphism of the division ring $\mathcal{U}^{\otimes 2}$ of $\mathcal{U}^{\otimes 2}$ which we denote by the same letter \mathcal{R} as the automorphism (9) which acts on generators as

$$\mathcal{R}(1 \otimes K) = (1 \otimes K)(1 + tK^{-1}E \otimes FL)^{-1}$$

$$\mathcal{R}(1 \otimes L) = (1 \otimes L)(1 + tK^{-1}E \otimes FL)^{-1}$$

$$\mathcal{R}(E \otimes 1) = E \otimes L$$

$$\mathcal{R}(1 \otimes F) = K^{-1} \otimes F$$

Define its action on generators $K \otimes 1$, $L \otimes 1$, $1 \otimes E$, and $F \otimes 1$ such that

$$\mathcal{R}(\Delta(a)) = \sigma \circ \Delta(a)$$

where a is one of the generators of \mathcal{U} .

It is clear that the homomorphism ϕ brings the outer automorphism \mathcal{R} to (9).

7. THE ALGEBRA \mathcal{U}_ε

Let ε be a primitive root of 1 of an odd degree ℓ . Denote by \mathcal{U}_ε the specialization of \mathcal{U} to $t = \varepsilon$. The following theorem is a version of the corresponding facts for simple Lie algebras proved in [DC-K].

Theorem 5. • Elements E^ℓ , F^ℓ , $K^{\pm\ell}$, and $L^{\pm\ell}$ generate a central subalgebra $Z_0 \subset \mathcal{U}_\varepsilon$.

- Z_0 is a Hopf subalgebra with

$$\begin{aligned}\Delta(K^\ell) &= K^\ell \otimes K^\ell, \quad \Delta(L^\ell) = L^\ell \otimes L^\ell, \\ \Delta(E^\ell) &= E^\ell \otimes K^\ell + 1 \otimes E^\ell, \quad \Delta(F^\ell) = F^\ell \otimes 1 + L^{-\ell} \otimes F^\ell.\end{aligned}$$

- The algebra \mathcal{U}_ε is a free Z_0 -module of dimension ℓ^4 .
- The center $Z(\mathcal{U}_\varepsilon)$ is generated by Z_0 and by the element (10) modulo the relation

$$\prod_{j=0}^{\ell-1} (c - K\varepsilon^{j+1} - L^{-1}\varepsilon^{-j-1}) = E^\ell F^\ell$$

- Let α, β, a and b be coordinates on the group $B_+ \times B_-$ such that for $b_\pm \in B_\pm$ we have:

$$b_+ = \begin{pmatrix} 1 & \beta \\ 0 & \alpha \end{pmatrix}, \quad b_- = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$$

Then the map $F^\ell \rightarrow ba^{-1}$, $E^\ell \rightarrow \beta$, $K^\ell \rightarrow \alpha$, $L^\ell \rightarrow a$ is a homomorphism of Hopf algebras $Z_0 \rightarrow C(B_+ \times B_-)$

- \mathcal{U}_ε is semisimple over a Zariski open subvariety of $\text{Spec}(Z_0) \simeq B_+ \times B_-$.

Let $x \in GL_2^*$ be an irreducible Z_0 -character and $I_x \subset \mathcal{U}_\varepsilon$ be the corresponding ideal. The quotient algebra

$$A_x = \mathcal{U}_\varepsilon / I_x$$

is finite-dimensional of dimension ℓ^4 . There are three natural structures of a left module on A_x . For $a \in \mathcal{U}_\varepsilon$ denote by $[a]$ the class of a in A_x . Then these three actions are:

- $\pi(a)[b] = [ab]$,
- $\phi(a)[b] = [bS(a)]$,
- $\psi(a)[b] = [bS^{-1}(a)]$.

Assume that $x \in GL_2^*$ is generic, i.e. that A_x is semisimple. Fix an isomorphism of algebras $\phi_x : A_x \simeq \oplus_{i=1}^n \text{Mat}(k_i)$. For the algebra \mathcal{U}_ε it is known [DC-K] that $n = \ell^2$ and $k_i = \ell$ for all $i = 1, \dots, n$. Define

$$t : A_x \rightarrow \mathbb{C}, \quad t(a) = \sum_{i=1}^n t_i \text{Tr}(\phi_x^i(a))$$

where Tr is the matrix trace in $\text{Mat}(k_i)$ and $\phi_x^i : A_x \rightarrow \text{Mat}(k_i)$ is the i -th component of ϕ_x . It is clear that $t(a)$ does not depend on a particular choice of ϕ_x . Indeed, any other such isomorphism differs from ϕ_x by an inner automorphism of $\oplus_{i=1}^n \text{Mat}(k_i)$. Since trace is cyclically invariant, the value of $t(a)$ for such isomorphism will be the same as for ϕ_x . Thus, for generic x , we have an invariant bilinear form on A_x :

$$(a, b) = t(ab).$$

It is a scalar product if $t_i \neq 0$ for each $i = 1, \dots, n$.

Fix a scalar product on A_x as above. This gives an isomorphism of vector spaces $A_x^* \simeq A_x$. It is easy to verify that the pairing between \mathcal{U}_ε -modules (A_x, ϕ) and (A_x, π) given by the map

$$(11) \quad e_x : (A_x, \phi) \otimes (A_x, \pi) \rightarrow \mathbb{C}$$

acting as $a \otimes b \mapsto t(ab)$ is \mathcal{U}_ε -invariant with respect to the diagonal action. Indeed, using Sweedler's notation $\Delta(c) = \sum_c c^{(1)} \otimes c^{(2)}$ for the comultiplication of element c , we have:

$$\varepsilon_x \left(\sum_c a S(c^{(1)}) \otimes c^{(2)} b \right) = \sum_c t(a S(c^{(1)}) c^{(2)} b) = \varepsilon(c) t(ab)$$

Similarly

$$e_x \left(\sum_c c^{(1)} a \otimes b S^{-1}(c^{(2)}) \right) = \sum_c t(c^{(1)} a b S^{-1}(c^{(2)})) = \varepsilon(c) t(ab)$$

Therefore the map $e_x : (A_x, \pi) \otimes (A_x, \psi) \rightarrow \mathbb{C}$ defined as in (11) is also \mathcal{U}_ε -invariant.

Let linear mapping $i_x : \mathbb{C} \rightarrow A_x \otimes A_x$ be defined by the formula

$$i_x(1) \mapsto \sum_i e_i \otimes e^i,$$

$\{e_i\}$ is a linear basis of A_x and $\{e^i\}$ the corresponding dual basis. It is easy to see that it is a morphism of \mathcal{U}_ε -modules $\mathbb{C} \rightarrow (A_x, \pi) \otimes (A_x, \phi)$ and $\mathbb{C} \rightarrow (A_x, \psi) \otimes (A_x, \pi)$. Indeed, let $a_i^j = t(a e_i e^j)$ for any $a \in A_x$, then

$$\sum_a \sum_i a^{(1)} e_i \otimes e^i S(a^{(2)}) = \sum_a \sum_{i,j} (a^{(1)})_i^j e_j \otimes e^i (S(a^{(2)})) = \sum_a \sum_j e_j \otimes e^j a^{(1)} S(a^{(2)}) = \varepsilon(a) \sum_i e_i \otimes e^i$$

which implies the first statement and the second statement can be proved similarly.

Thus, for the object (A_x, π) we have the left dual (A_x, ϕ) (and the right dual (A_x, ψ)).

Theorem 6. *The subspace $Z_0 \otimes Z_0 \subset \mathcal{U}_\varepsilon \otimes \mathcal{U}_\varepsilon$ is invariant with respect to the action of the automorphism \mathcal{R} .*

Proof. From the action of \mathcal{R} on generators of \mathcal{U}_ε and from the relations between generators we have:

$$\mathcal{R}(1 \otimes K^\ell) = (1 \otimes K^\ell)(1 + K^{-\ell} E^\ell \otimes F^\ell L^\ell)^{-1}$$

$$\mathcal{R}(1 \otimes L^\ell) = (1 \otimes L^\ell)(1 + K^{-\ell} E^\ell \otimes F^\ell L^\ell)^{-1}$$

$$\mathcal{R}(E^\ell \otimes 1) = E^\ell \otimes L^\ell$$

$$\mathcal{R}(1 \otimes F^\ell) = K^{-\ell} \otimes F^\ell.$$

The comultiplication acts on ℓ -th powers of generators as:

$$\Delta(K^\ell) = K^\ell \otimes K^\ell, \quad \Delta(L^\ell) = L^\ell \otimes L^\ell$$

$$\Delta(E^\ell) = E^\ell \otimes K^\ell + 1 \otimes E^\ell$$

$$\Delta(F^\ell) = F^\ell \otimes 1 + L^{-\ell} \otimes F^\ell.$$

These formulae and the defining property $\mathcal{R}(\Delta(a)) = \sigma \circ \Delta(a)$ describe the action of \mathcal{R} on generators of $Z_0 \otimes Z_0$. In particular, it is clear that the image is in $Z_0 \otimes Z_0$. \square

Comparing the action of \mathcal{R} on generators of $Z_0 \otimes Z_0$ with the identification of Z_0 and $C(GL_2^*)$ we have the following statement.

Theorem 7. *The automorphism \mathcal{R} is the pull-back of the mapping $b : GL_2^* \times GL_2^* \rightarrow GL_2^* \times GL_2^*$ acting as follows. First, identify GL_2^* with the Zariski open subvariety in GL_2 via the factorization mapping. Then the mapping b acts as*

$$(x, y) \mapsto (x_L(x, y), x_R(x, y))$$

where $x_L(x, y) = x_- y x_-^{-1}$ and $x_R(x, y) = (x_L(x, y))_+^{-1} x (x_L(x, y))_+$.

Let I_x be the ideal in \mathcal{U}_ε with the Z_0 character $x \in GL_2^*$. From now on we will consider only generic points x and therefore we can identify GL_2 with GL_2^* . The two theorems have an important corollary. The mapping \mathcal{R} acts as:

$$\mathcal{R}(I_x \otimes I_y) \subset I_{x_R(x, y)} \otimes I_{x_L(x, y)}$$

This implies that the mapping \mathcal{R} induces the isomorphism of algebras:

$$\mathcal{R}(x, y) : A_x \otimes A_y \rightarrow A_{x_R(x, y)} \otimes A_{x_L(x, y)}$$

This mapping is also an isomorphism of the tensor product of left \mathcal{U}_ε -modules (A_x, π) .

Finally, consider the category of \mathcal{U}_ε -modules generated by tensor products of (A_x, π) and their duals. It is clear that this category is a braided rigid monomial G -category with $G = GL_2^*$ and with the braiding given by the composition $\sigma \circ \mathcal{R}$.

8. CONCLUSION

In this paper we have constructed invariants of tangles with flat connections in their complements. An example of such construction is described for $GL_2(\mathbb{C})$. This is rather simple example related to quantum invariant constructed in [Ka]. More interesting examples related to irreducible representations of quantized universal enveloping algebras of simple Lie algebras will be analyzed in a separate paper.

The complement of a tangle is a rather special 3-manifold. The construction of invariants of 3-manifolds with G -flat connections in them for any simple Lie group G is the next step. The case of $G = PSL_2(\mathbb{C})$ was studied in [BB].

An interesting, but somewhat speculative question: how to relate the constructed invariants to a topological quantum field theory defined “phenomenologically” in terms of functional integrals. We expect that this theory will be Chern–Simons theory with complex simple G . The corresponding boundary conformal field theory should be complex Wess–Zumino–Witten theory on a surface with boundary with boundary operators “parametrized” by elements of G .

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