Book Practice

#8 In a vector space V; show that $(a+b)(x+y) = ax + ay + bx + by$ for $x, y \in V$, $a, b \in F$

by closure of F $a+b = c \in F$

$\Rightarrow c(x+y) = cx + cy$ by VS7

$= (a+b)x + (a+b)y = (ax + bx) + (ay + by)$ by VS8

$= ax + bx + ay + by$ by VS2 associativity

#10 Suppose V is a set of all differentiable real valued functions. Show it is a vector space.

Let Let $f, g \in \mathcal{F}(S, F)$ a vector space. WNTST $\partial(f+g) = \partial(g+f) \in \mathcal{F}(S, F)$

VS1) $\partial(f+g) = \partial(g+f)$ by $\mathcal{F}(S, F)$ a vector space

$= f' + g' = g' + f'$ by linearity of derivatives

VS2) Let $f, g, z \in \mathcal{F}(S, F)$ a vector space where $\forall f, g, z \; (f+g)+z = f+(g+z)$

$\lim_{h \to 0} \frac{f(s+h) - f(s) + g(s+h) - g(s)}{h} + \lim_{h \to 0} \frac{z(s+h) - z(s)}{h} = \lim_{h \to 0} \frac{f(s+h) - f(s)}{h} + \lim_{h \to 0} \frac{g(s+h) - g(s) + z(s+h) - z(s)}{h}$

WNTST $(f+g)(s) = f(s) + g(s)$ $\qquad = (f' + g') + z' = f' + (g' + z')$

$f'(a)$ $\lim_{h \to 0} \frac{(f(a)+h) - f(a)}{h}$

$= f(s+h) + g(s+h)$ since $f, g \in \mathcal{F}(S, F)$

$(f+g)(s) \Rightarrow (f' + g')(s) = \lim_{h \to 0} \frac{((f+g)(s+h) - (f+g)(s)}{h} = \lim_{h \to 0} \frac{(f(s)+h) - f(s)}{h} + \lim_{h \to 0} \frac{g(s)+h - g(s)}{h}$

$(f+g)'(s)$

$= f'(s) + g'(s)$

WNTST $(f+g)(s) = f(s)+g(s)$

$(f+g)(s) \in Y$ $\quad (f+g)(s) = f(s)+g(s) \quad \forall s \in X$

by definition by EXAMPLE 3 where $f, g \in \mathcal{F}(S, F)$ as a vector space.

then $z = f+g$ s.t. $\lim_{h \to 0} \frac{z(s+h) - z(s)}{h} - z'(s) = f'(s) + g'(s)$

ii) let $f \in \mathcal{F}(S, F)$ a vector space

$(cf)(s) = cf(s)$

$\Rightarrow \lim_{h \to 0} \frac{cf(s+h) - cf(s)}{h} = c \lim_{h \to 0} \frac{f(s+h) - f(s)}{h} = cf'(s)$

$(cf)'s$

for (VS6) let $a, b \in F$ & $f \in \mathcal{F}(S, F)$ a vector space. $s \in S$

WNTST for $(ab)f(s) = a(bf(s))$

$\Rightarrow \partial(ab)f(s) = \partial a(bf(s))$

$\Rightarrow ab\,\partial f(s) = abf'(s) = a(\partial bf(s)) = a(bf'(s))$ by linearity of limits

$(ab)f'(s) = a(bf'(s))$

$\Rightarrow$ VS6 applies to $\mathcal{F}'(S, F)$ as Vector Space

(#12) Let $f(-t)=f(t)$ be an even function $\forall t \in S$

Let E be the set of all functions s.t. $\forall t \in S$ and $\forall f \in F$ be a field $f(-t)=f(t)$

$= E(S,F) \ni f$. Show E is a vector Space.

we assume that $f(s+t)=f(s)+f(t)$ & $cf(s)=(cf)(s)$

VS1) for $f,g \in E(S,\mathbb{R})$ $(f+g)(s)=(f+g)(-s)=f(-s)+g(-s)$

$(f+g)(s) \qquad = g(-s)+f(-s)$ since $g(-s) \in \mathbb{R}$ field

VS2) for $f,g,z \in E(S,\mathbb{R})$ $(f+g)(s)+z(s)=f(s)+g(s)+z(s)=f(s)+(g+z)(s)$ since $\mathbb{R}$ is a field

VS3) $(f+0)(s)=f(s)+0(s)=f(s)$

$f(s+0)=f(s)$

VS4) $s+-s=0$ s.t. $f(s+-s)=f(s)+f(-s)=2f(s)$

$f(s-s)=f(s)-f(s)=0$

VS5) $f(1s)=1f(s)=f(s) \in E(s,\mathbb{R})$

VS6) $f(abx)=abf(x)=af(bx)$ by Example 3

VS7) $a(f+g)(s)=a(f+g)(-s)=a(f(s)+g(s))=af(s)+g(s)a$ by $\mathbb{R}$ as a Ring

VS8) . . .

Problems 1.3 Subspaces

#20) Prove that if W is a subspaces of Vector Space, and $w_1, \ldots, w_n \in W$ then

$a_1w_1 + \ldots + a_nw_n \in W \quad \forall a_i \in F$

If W is a Subspace then $\forall w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$ and for any $a \in F$ $a_iw_i \in W$

then $\Rightarrow a_jw_j \in W \Rightarrow a_iw_i$ and $a_jw_j \in W \Rightarrow a_iw_i + a_jw_j \in W$. $i \neq j$ WLOG

then by induction let $a_1w_1 + \ldots + a_{n-1}w_{n-1} \in W$

let $a_n \in F$ and $w_n \in W \Rightarrow a_nw_n \in W$ -

then $a_1w_1 + \ldots + a_{n-1}w_{n-1} + a_nw_n \in W$ by Induction

By theorem 1.5 W is a subspace of $V \Rightarrow \sum a_iw_i \in \text{SPAN}(W) \Rightarrow \text{SPAN}(W) \in W \trianglelefteq V$

#21) Show that the set of convergent sequences $\{a_n\}$; $\lim_{n\to\infty} a_n \to a$ is a subspace of a Vector space V in exercise 20.

Let W be a Vector space s.t. $\forall w \in W$ $\forall a \in F$ then $\sum a_iw_i \in W$

Let $X \subseteq W$ s.t. $\forall \{x_n\} \in X \quad \lim_{n\to\infty} x_n \to x$

Let $\{x_n\} \in X$ s.t. $\lim_{n\to\infty} x_n \to x$ let $\{y_n\} \in X$ s.t. $\lim_{n\to\infty} y_n \to y \in X$

WNTS $\{x_n + y_n\} \to x + y \in X$

$\lim_{n\to\infty}\{x_n + y_n\} = \lim_{n\to\infty}\{x_n\} + \lim_{n\to\infty}\{y_n\} = x + y \in X$

$\lim_{n\to\infty} a x_n = a x \in X$

$\lim_{n\to\infty} 0 = 0 \in X$

So X is a subset of W and is a subspace. by $\boxed{\text{theorem 1.5}}$ $\text{SPAN}(X) \in X$ as a subspace

s.t. $\underbrace{\sum_{i \in J} \lim_{n\to\infty} a_i x_{n_i}}_{\text{SPAN}(X)} = \lim_{n\to\infty} \sum_{i \in J} a_i x_{n_i} = \sum_{i \in J} a_i x_i \Rightarrow$ converges $\Rightarrow \in X$.

$\therefore$ Subspace.

(#23) Let W_1 and W_2 be subspaces of a Vector space V

a) Prove that W_1+W_2 is a subspace of V containing both W_1 and W_2

b) Prove that any subspace of V that contains both W_1 and W_2 must also contain W_1+W_2

Proof a: Let W_1+W_2 be a subset of $V(W_1 \cup W_2, \mathbb{R})$ s.t. $W_1+W_2 = \{x+y \mid x \in W_1, y \in W_2\}$

since $0 \in W_1 \cap W_2 \Rightarrow 0 \in W_1+W_2$ for some $x \in W_1$ $y \in W_2$

$\forall a,b \in W_1+W_2$ then $a+b = (x+y)+(x'+y')$ for $x,x' \in W_1$ $y,y' \in W_2$

$= (x+x')+(y+y')$ for $x+x' \in W_1 \trianglelefteq V$ $y+y' \in W_2 \trianglelefteq V$

$\Rightarrow a+b \in W_1+W_2$

$\forall c \in F$, $\forall a \in W_1+W_2 \Rightarrow ca = c(x+y) = cx+cy$ for $cx \in W_1$, $cy \in W_2$

$\Rightarrow ca \in W_1+W_2$

$W_1+W_2 \trianglelefteq V$

for $0 \in W_2$ $\forall x \in W_1$ $x+0 \in W_1+W_2$ and $W_1 \subseteq W_1+W_2$

for $0 \in W_1$, $\forall y \in W_2$ $0+y \in W_1+W_2$ and $W_2 \subseteq W_1+W_2$

can there be linear Combinations which generate W_1? W_2?

b) Let $B \trianglelefteq V$ s.t. $W_1, W_2 \subseteq B$ WNTST $W_1+W_2 \subseteq B$

Let $\forall w_1 \in W_1$ and $\forall w_2 \in W_2$

case 1: Let $W_1 \cap W_2 = \{0\}$ where W_1+W_2 is $\{x+y \mid x \in W_1, y \in W_2\}$

$\Rightarrow x \in W_1 \cap B$, $y \in W_2 \cap B$ where $x,y \in B \Rightarrow x+y \in B$ and $x+y \in W_1+W_2$

$\Rightarrow W_1+W_2 \subseteq B$

#30 Let W_1 and W_2 be subspaces of a vector space V. Prove that V is the direct sum W_1 and W_2 iff each vector in V can be **uniquely** written as $x_1 + x_2$ where $x_1 \in W_1$, $x_2 \in W_2$

Proof: $\Rightarrow$ Assume V is a direct sum of W_1, W_2 s.t. $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$

Let $x_1+y_1 = v_1$, ; ... $x_{n-1}+y_{n-1} = v_{n-1}$, $x_n + y_n = v_1$ $\forall x_i, y_i \in W_1, W_2$ respectively

where $W_1, W_2 \trianglelefteq V$. then let $x_1+y_1+x_2+y_2+\ldots+x_n+y_n = (x_1+\ldots+x_n) + (y_1+\ldots+y_n)$

$= 2v_1 + \ldots + v_{n-1}$

where v_1 is a redundant vector $\Rightarrow x_1+y_1+\ldots+x_{n-1}+y_{n-1} = v_1+\ldots+v_{n-1}$

$\Rightarrow W_1 \setminus \{x_n\} + W_2 \setminus \{y_n\} = V$

$\Rightarrow |W_1| = n$ $|W_2| = n$ and $|V| = n-1$ $\Rightarrow W_1, W_2 \not\subseteq V$ #C.

$\Leftarrow$ Let $W_1, W_2 \trianglelefteq V$ and $\forall v \in V$ let $x_i + y_i = v_i$ $\forall x_i \in W_1$, $y_i \in W_2$ and v_i unique.

WNTS $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$

Let $W_1 + W_2 \leq V$, if $W_1 + W_2 \subset V$ $\exists v_j \in V$ s.t. $x_j + y_j \neq v_j$ which contradicts hypothesis. $W_1 + W_2 = V$ and vice versa

Let $W_1 \cap W_2 = \{0, \ldots\}$ s.t. $|W_1 \cap W_2| > 1$

if $\exists a \in W_1 \cap W_2$ s.t. $a \neq 0$ then $a \in W_1$ and $a \in W_2$

then $x + a = v$ where v is unique. $x \in W_1$ and $a \in W_2$

$a = v - x \in W_1$

$a + y = v'$ $\quad a = v' - y$

$x + a = a + y$

$x + v - x = v' - y + y$

$v = v'$ not unique.

Challenge Problem

$\boxed{\#31}$ Let W be a subspace of $V(A,F)$. For $\forall v \in V$ the set $\{v\}+W = \{v+w \mid w \in W\}$ is a coset of W containing v.

trivial proof

If $v=0$ then $\{v\}+W \Rightarrow$
$x+b = (0+0)+(x'+y')$
$\in 0+W$

a) Prove $v+W$ is a subspace of V iff $v \in W$.

$\Rightarrow$ Let $\{v\}+W \trianglelefteq V$ and $v \neq 0$ (trivial)

$v+0w \in v+W$ and $(-1)v \in v+W$ by definition of subspace; $0 \in W$, scalar multiplication closure

$w+v = -v$ for some $w \in W$ by defn of subspace

$w=-2v \quad \Rightarrow (-\frac{1}{2})w = v \quad \Rightarrow v \in W$

$\Leftarrow$ Let $v \in W$ then $v+v \in \{v\} \trianglelefteq W$

i) then $0 \in v+W$ since $v \in W \trianglelefteq V \Rightarrow v+(-v) = 0 \in v+W$

ii) $\forall x,y \in W \quad x+y \in W$ for some $x=v \in W \Rightarrow v+y \in W$

and $v+y \in \{v\}+W$ then $\forall x,y \in \{v\}+W$ we have $x+y = (v+v)+x'+y' \in W$

$= (2)v + (x'+y') \in v+W$

iii) $\forall c \in F \quad cx \in W$ for $x=v, y \in W \quad cv \in W$ and $\underbrace{cv+0w}_{=cv} \in \{v\}+W$

$\therefore \{v\}+W \trianglelefteq V$

b) Prove that $v_1+W = v_2+W$ iff $v_1-v_2 \in W$

$\Leftarrow$ Let $v_1-v_2 \in W$ then $v_1-v_2+W \trianglelefteq V$ is a subspace

$\Rightarrow v_1-v_2+w \in v+W \trianglelefteq V$

$\Rightarrow (-1)(v_1-v_2)+w \in W \trianglelefteq V$ by hypothesis using defn of subspace

so $v_1+w = v_1+(v_2-v_1+w)$ for some $w \in W$

$v_1+w = v_2+w \Rightarrow v_1+w \in v_2+W$. Similarly $v_2+w \in v_1+W \Rightarrow$ equal.

$\Rightarrow$ If $v_1+W = v_2+W$ Note for showing equality it suffices for double membership

b) Prove $v_1+W = v_2+W$ iff $v_1 - v_2 \in W$

$\Rightarrow$ $v_1+w = v_2+w$ by Hypothesis

then $v_1 = v_1 + 0_w \in v_1+W$

$\Rightarrow v_1 + w_1 + w_2$ for some $w_1, w_2 \in W \leq V$

$v_1 + w = (v_1 + w_1) + w_2 = v_2 + w + w_2$

VECTORS

$v_1 - v_2 = (w + w_2)$ SINCE $W \trianglelefteq V$ Let $w + w_2 = \hat{w} \in W \trianglelefteq V$, and using vector addition

$v_1 - v_2 = \hat{w}$

$\Rightarrow v_1 - v_2 \in W$.

c) Show if $v_1+W = v_1'+W$ and $v_2+W = v_2'+W$ then $(v_1+W) + (v_2+W) = (v_1'+W) + (v_2'+W)$

and $a(v_1+W) = av_1' + W$

Proof: Since $v_1+W = v_1'+W \Rightarrow$ that $v_1 - v_1' \in W \trianglelefteq V$ and $(v_1 - v_1') + W \trianglelefteq V$

Similarly for $(v_2 - v_2') \in W$ and $(v_2 - v_2') + W \trianglelefteq V$ as subspaces

where for $S = \{v + W \mid v \in V\}$ and $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$ as cosets

Let $v_1 + W, v_2 + W \in S$

then for some $w \in S$ $(v_1 + w) + (v_2 + w) = (v_1 + v_1' - v_1 + w) + (v_2 + v_2' - v_2 + w)$

$= (v_1 + w) + (v_2 + w) = (v_1' + w) + (v_2' + w)$ as cosets

$= (v_1 + v_2) + W = (v_1' + v_2') + W$ for $v_1 - v_1' \in W \leq V$

since they are cosets and $(-1)(v_1 - v_1') \in W$

$= v_1' - v_1 + w \in W \trianglelefteq V$

for some $w \in W$

$c(v_1 + w) = c(v_1 + v_1' - v_1 + w) = c(v_1' + w) \in S \Rightarrow cv_1' + W \in V + W$

where $v_1 + v_1' - v_1 = v_1'$ by vector addition $\forall v \in V$

where $v_1 - v_1' \in W$ then W is a subspace $\Rightarrow \lambda(v_1 - v_1') \in W \Rightarrow \lambda v_1 - \lambda v_1' \in W$

d) Prove that the set S is a vector space w/ the operations defined in part c)

this vector space is called the quotient space of V modulo W denoted as V/W.

WNTST $\forall s_i \in S = \{v + W \mid v \in V\}$ is a vector space

(VS 1) WNTST $\forall x, y \in S$ $x + y = y + x \rightarrow (v_1 + W) + (v_2 + W) = (v_2 + W) + (v_1 + W)$

by part b) $(v_1 + v_2) + W = (v_2 + v_1) + W \iff (v_1 + v_2) - (v_2 + v_1) \in W$

since $0 \in W \trianglelefteq V$ by definition of a subspace

Let $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W = (v_1 + v_2) + W + 0 + 0_W$

$= (v_1 + v_2 + 0) + W$ by defintion of cosets

then for some $w \in W \Rightarrow (v_1 + v_2 + v_2 + v_1 - v_1 - v_2) + W$

$= (v_2 + v_1 + 0) + W = (v_2 + v_1) + W$

and by part b) $0 = v_2 + v_1 - v_1 - v_2 \in W$ if $(v_1 + v_2) + W = (v_2 + v_1) + W$

then for some $w \in W \Rightarrow (v_1 + v_2) + (v_2 + v_1 - v_1 - v_2 + w) = v_2 + v_1 + W = v_1 + v_2 + W$

(VS 2) $\forall x, y, z \in S$ WNTST $(x + y) + z = x + (y + z)$

$= (v_1 + W + v_2 + W) + v_3 + W = (v_1 + v_2) + W + v_3 + W = (v_1 + v_2 + v_3) + W$

$= v_1 + W + (v_2 + W + v_3 + W) = v_1 + W + (v_2 + v_3 + W) = (v_1 + v_2 + v_3) + W$

where this is well defined proven in b) and c)

VS3) $\forall x \in S \ \exists\, 0 \in S$ s.t. $x + 0 = x$

then $v_1 + W \in S$ since $0 \in W \trianglelefteq V$ then $v_1 - v_1 = 0 \in W$

$\Rightarrow v_1 + W + 0 + 0W = (v_1 + 0) + W$ by coset vector addition

$\Rightarrow v_1 + W = v_1 + v_1 - v_1 + W = (v_1 + 0) + W$

VS4) $\forall x \in S \; \exists \; y \in S$ s.t. $x+y=0$

Let $x = v_1 + W$, $y = v_2 + W$ s.t. $v_1 + W + v_2 + W = (v_1 + v_2) + W$

WNTST $(v_1+v_2)+W = 0 + 0w$

$\Rightarrow v_1 + W = -(v_2 + W)$ iff $v_1 + v_2 \in W$ by proof of 6)

then if $v_1 + v_2 \in W \rightarrow -(v_1+v_2) \in W$

s.t. $(v_1 + v_2) + -(v_1 + v_2) = 0$

if $v_1 + v_2 \in W$ then $0 \in W \leq V$ by definition of subspace

s.t. $v_1, v_2 \in V$ where V is a vector space

So $\forall v \in V \; \exists \; v' \in V$ s.t. $v + v' = 0$

so for some $v_1 \in V \; \exists \; v_1' \in V$ s.t. $v_1 + W + v_1' + W = (v_1 + v_1') + 0W = 0 + 0_W$

VS5) $\forall x \in S \; 1x = x$

Let $x = v_1 + W$ for some $v_1 \in V$. $v_1 + W = 1(v_1 + W) = 1v_1 + W$

where $1v_1 - v_1 = 0 \in W \Rightarrow v_1 + W = 1v_1 + W$ since $v_1 \in V(V, F)$

is a vector space.

VS6) WNTST $\forall (a,b) \in F$, $\forall x \in S$ $ab(x) = ab(v_1 + W) = a(bv_1 + W)$

by coset multiplication $ab(v_1 + W) = abv_1 + W$

and $a(bv_1 + W) = abv_1 + W$.

then $abv_1 + W = a(bv_1 + W)$ s.t. $abv_1 - a(bv_1) \in W$ since

$v_1 \in V(V, F)$ a vector space. so $ab(v_1) = a(bv_1)$

VS7) $\forall a \in \underline{F}$ and $\forall (x,y) \in S$ WNTST $a(x+y) = ax + ay$

$a(v_1 + W + v_2 + W) = a((v_1+v_2) + W)$ by Coset addition

$= a(v_1+v_2) + W$ by Coset Multiplication

$= av_1 + av_2 + W$ by vector Multiplication since we have closure of $v_1, v_2 \in V(V, F)$

then $a(v_1+v_2) + W = (av_1 + av_2) + W$

since $a(v_1+v_2) - av_1 + av_2 = 0 \in W$ by closure of scalar multiplication in V.

VS8) Skipping Proof.

notes cosets are sets in the same space yet co-linear in some way to the original space.

consider a line on the z-axis $L = t(0,0,1)$ a coset to L will be a displacement "parallel" to L $(x,y) \to (x,y,z) + t(0,0,1)$

where the z coordinate is not important $(x,y) + L = (x,y,z) + t(0,0,1)$

A co-set in the plane is a plane that is merely shifted, not necessarily rotated

s.t. consider $P = t(1,0,0) + s(0,1,0) = (s,t,0)$

a coset is shifting up the z-axis $z + P \Rightarrow (x,y,z) + P = (x,y,z) + (s,t,0)$

where all that is important is the z-coordinate.

§1.4 Linear Combination Friedburg

$\boxed{\#4}$ For Each list of Polynomials in $P_3(\mathbb{R})$ determine whether the first Polynomial can be expressed as a LC of the other two.

i) $x^3-3x+5,\ x^3+2x^2-x+1,\ x^3+3x^2-1$

$(1,0,-3,5) = a(1,2,-1,1) + b(1,3,0,-1)$

$1 = a+b$ $a=3$

$0 = 2a+3b$ $b=-2$ yes

$-3 = -a$

$5 = a-b$ $3(1,2,-1,1) - 2(1,3,0,-1) = (1,0,-3,5)$ ✓

ii) $4x^3+2x^2-6,\ x^3-2x^2+4x+1,\ 3x^3-6x^2+x+4$

$(4,2,0,-6) = x(1,-2,4,1) + y(3,-6,1,4)$

$4 = x+3y$

$2 = -2x-6y$ $x = 4-3y$ $29-30=1$

$0 = 4x+y$ $4-3y+4y=-6$ $-58+60=2$ ✓

$-6 = x+4y$ $4+y=-6$ $116-10\neq 0$

$y=-10$ $29-40\neq -6$

$2-60=-2x$

$\frac{-58}{-2} = \boxed{x=29}$ NO.

#8) Show that $P_n(F)$ is generated by $\{1, \dots, x^n\}$

$\Rightarrow$ Let $v = a_n x^n + \dots + a_1 \in SPAN(S)$ be arbitrary LC of S

then degree of $v = n$; $v \in P_n(F)$

$\Leftarrow$ $\forall x \in P_n(F)$ $degree(x) = n$ by definition

Let $y = a_n x^n + \dots + a_1 \in P_n(F)$ w\ degree n

So $\exists (a_n, \dots, a_1) \in F^n$ s.t.

$(a_n, \dots, a_1) \cdot (x^n, \dots, x, 1) = a_n x^n + \dots + a_1 = y$

$\Leftarrow (x^n, \dots, 1) \in S \Rightarrow y \in SPAN(\quad)$

$\Rightarrow y \in SPAN(S)$

$S =$

#9 Show that $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ generates $M_{2\times2}(F)$

WNTST for $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_{2\times2}(F)$ $SPAN(S) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

For $\forall a \in F$ let $a = (a_1, a_2, a_3, a_4) \cdot S = a_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} + \dots + \begin{pmatrix} 0 & 0 \\ 0 & a_4 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ SINCE each $s \in M_{2\times2}(F)$ and closure under addition.

$\Leftarrow$ Let $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_{2\times2}(F)$ then by Vector Space Properties

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{12}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$= (a_{11}, a_{12}, \dots, a_{22}) \cdot \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \in SPAN(S)$

$\therefore$

#11 WE NEED TO Show that $SPAN(\{x\}) = \{ax : a \in F\}$, interpret Geometrically

Let $x \in V(V, F)$ some vector space.

the $SPAN(\{x\})$ is linear combination of $\{x\}$.

Let $v = a_1 x + a_2 x + \dots + a_n x$ for $(a_1, \dots, a_n) \in F$

$v = (a_1 + \dots + a_n) \cdot x$ where $a_1 + \dots + a_n = a' \in F$ SINCE F is a field

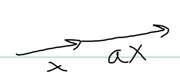
$= v = a'x \Rightarrow SPAN(\{x\}) \subseteq \{ax : a \in F\}$

$\Rightarrow$ Let $a_1 x \in \{ax : a \in F\}$ then we can Express $a_1 = a_1' + a_2' + \dots + a_n'$

$= a_1' + \dots + a_n'(x) = a_1'x + \dots + a_n'x \in SPAN\{x\}$

$\Rightarrow SPAN(\{x\}) = \{ax : a \in F\}$

Geometrically this is taking vectors of x and "stretching" By $a \in F$



this doesn't incorporate a displacement, or shift.

$\boxed{\#12}$ Show that a subset W of a vector space V is a subspace of V iff $SPAN(W)=W$

Proof: $\Leftarrow$ Assume that $SPAN(W)=W$ for $W \subseteq V$ for $V(V,F)$

By theorem 1.5 the $SPAN(W)$ is a subspace of V for $W \subseteq V$

$\Rightarrow SPAN(W) \trianglelefteq V \rightarrow W \trianglelefteq V$

$\Rightarrow$ Assume that $W \subseteq V$ and $W \trianglelefteq V$

then by theorem 1.5 $SPAN(W) \subseteq W \trianglelefteq V$

then since $W \trianglelefteq V$ $\forall x,y \in W \Rightarrow x+y \in W$ and $a_1x_1 + a_2x_2 \in W$ by subspace defnt

then let $v = a_1w_1 + a_2w_2 + \ldots + a_nw_n \rightarrow v \in W$

and $v \in SPAN(W)$ since v is general

$\Rightarrow W \subseteq SPAN(W) \rightarrow W = SPAN(W)$

By theorem 1.5 $SPAN(W) \trianglelefteq V$ and Clearly $W \subseteq SPAN(W)$

By T1.5 $SPAN(W) \subseteq W \subseteq SPAN(W)$

$\Rightarrow SPAN(W) = W$

$\boxed{\#13}$ Show that if S_1, S_2 are subsets of vector space V, s.t. $S_1 \subseteq S_2$ then $SPAN(S_1) \subseteq SPAN(S_2)$. In Particular, if $S_1 \subseteq S_2$ s.t. $SPAN(S_1) = V \Rightarrow SPAN(S_2) = V$

Proof: Let $S_1 \subseteq S_2 \subseteq V$ for $V(V, F)$

then $SPAN(S_2) \trianglelefteq V$ by T1.5

where $S_2 \subseteq SPAN(S_2)$ by Identity

$\Rightarrow S_1 \subseteq SPAN(S_2)$

$\Rightarrow$ By T1.5 $SPAN(S_1) \subseteq SPAN(S_2)$

where if $SPAN(S_1) = V$

$\Rightarrow SPAN(S_2) \trianglelefteq V$

$\Rightarrow SPAN(S_2) \subseteq V$

$\Rightarrow SPAN(S_1) \subseteq SPAN(S_2) \subseteq V$

$\Rightarrow V \subseteq SPAN(S_2) \subseteq V$

$SPAN(S_2) = V$

$\boxed{\#14}$ Show that if S_1, S_2 are arbitrary subsets of $V(V, F)$ then

$SPAN(S_1 \cup S_2) = SPAN(S_1) + SPAN(S_2) = \{ x+y \mid x \in SPAN(S_1), y \in SPAN(S_2) \}$

WNTST $SPAN(S_1 \cup S_2) \subseteq SPAN(S_1) + SPAN(S_2)$

By definition $Z = SPAN(S_1) + SPAN(S_2) \trianglelefteq V$

$\Rightarrow S_1, S_2 \subseteq Z$

$\Rightarrow S_1 \cup S_2 \subseteq Z$

$SPAN(S_1 \cup S_2) \subseteq Z$ by T1.5

$\boxed{\#14}$ cont'd

by definition $SPAN(S_1), SPAN(S_2) \trianglelefteq V$

let $Z = SPAN(S_1) + SPAN(S_2) = \{x+y \mid x \in SPAN(S_1), y \in SPAN(S_2)\}$

WNTST $Z \trianglelefteq V$ WLOG $Z \subseteq V$

i) since $0 \in SPAN(S_1)$ & $0 \in SPAN(S_2) \Rightarrow 0 \in Z$

ii) $\forall x, y \in Z = SPAN(SPAN(S_1), SPAN(S_2))$

Let $x = a_1 s_1 + a_2 s_2 + \ldots + a_n s_n \in SPAN(S_1)$ & $y = t_1 s_1' + \ldots t_n s_n' \in SPAN(S_2)$

$x+y = a_1 s_1 + \ldots + a_n s_n + t_1 s_1' + \ldots + t_n s_n' \in SPAN(x, y)$

$\exists (a_1, \ldots, a_n)$ s.t. $(a_1, \ldots, a_n) \circ (s_1, \ldots, s_n) \in SPAN(S_1)$ as arbitrary

since $SPAN(S_2) \trianglelefteq V \Rightarrow 0 \in SPAN(S_2)$

$\Rightarrow a_1 s_1 + \ldots + a_n s_n + 0\, SPAN(S_2) = a_1 s_1 + \ldots + a_n s_n \in SPAN(SPAN(S_1), SPAN(S_2))$

$\Rightarrow SPAN(S_1) \subseteq Z$ same for $SPAN(S_2) \subseteq Z$

$\Rightarrow S_1 \subseteq Z, S_2 \subseteq Z$

then $(S_1 \cup S_2) \subseteq Z$

$SPAN(S_1 \cup S_2) \subseteq Z$ by T1.5

$\Leftarrow$ WNTST $SPAN(S_1) + SPAN(S_2) \subseteq SPAN(S_1 \cup S_2)$

$SPAN(S_1) \subseteq SPAN(S_1 \cup S_2)$ & $SPAN(S_2) \subseteq SPAN(S_1 \cup S_2)$ by T1.5

since $SPAN(S_1 \cup S_2) \trianglelefteq V \Rightarrow S_1 \cup S_2 \subseteq SPAN(S_1 \cup S_2) \rightarrow S_2, S_1 \subseteq SPAN(S_1 \cup S_2)$

let $x \in SPAN(S_1), y \in SPAN(S_2)$ s.t. $x, y \in SPAN(S_1 \cup S_2)$

$\Rightarrow x+y \in SPAN(S_1 \cup S_2)$ by subspace

then $\forall x, y$ $x+y \in SPAN(S_1 \cup S_2) \Rightarrow SPAN(S_1) + SPAN(S_2) \subseteq SPAN(S_1 \cup S_2)$

$\Rightarrow SPAN(S_1 \cup S_2) = SPAN(S_1) + SPAN(S_2)$