§1.3 Chapter 1 Notes And definitions

definitions:

subspaces: a subset W of a vector space V over a Field is a subspace of V if W is a vector space of F w/ operations of addition and scalar multiplication defined on V.

i) $x+y \in W \ \forall x,y \in W$ closed under addition

ii) $cx \in W$ for $c \in F$, $x \in W$ closed under multiplication

iii) $0 \in W$

iv) $\forall x \in W \ \exists y \in W$ s.t. $x+y=0$

theorem 1.3 Let V be a vector space and W a subset of V. Then W is a subspace of V iff TFAT for operations on V

i) $0 \in W$

ii) $x+y \in W \ \forall x,y \in W$

iii) $cx \in W \ \forall c \in F, x \in W$

Proof: $W \subseteq V$; and $V(V,F)$ is a vector space

$\Rightarrow$ Let W be a subspace of V. Then W is closed under addition and multiplication

Let $0 \in W$ and $0' \in V$ so $x+0 \in W$ and $x+0' \in W \Rightarrow x+0 = x+0' = 0 = 0'$ by cancelly

$\Leftarrow$ assume i), ii), iii) hold; WNTST W is a Vector Space defined on V

WNTST additive inverse in W

$cx \in W \Rightarrow cx + x \in W$ by ii. Let $c = -1 \in \mathbb{R} \Rightarrow -x+x = 0 \in W$

Definitions

<u>transpose:</u> $A_{ij}^t = A_{ji}$

<u>Symmetric Matrix:</u> $A^t = A$; for square Matrices only.

i) symmetric Matrices are closed under Addition and Multiplication

$\boxed{\text{Example 1:}}$ $P_n(F)$ be all polynomials in $P(F)$ having degree less than or equal to n

i) $P_0(F)$ has degree $-1 \Rightarrow P_0(F) \in P_n(F)$

ii) $P_x(F) + P_y(F)$ s.t. $\deg(x) \leq n$ $\deg(y) \leq n \Rightarrow P_x(F) + P_y(F) \in P_n(F)$

iii) $cP_n(F) \in P_n(F)$

$\Rightarrow P_n(F) \leq P(F)$

$\boxed{\text{Example 2}}$

Let $C(\mathbb{R})$ be set of all cont's functions; $\mathcal{F}(\mathbb{R},\mathbb{R})$ set of all functions $\mathbb{R} \to \mathbb{R}$

$C(\mathbb{R}) \trianglelefteq \mathcal{F}(\mathbb{R},\mathbb{R})$

$f(t) = 0$ is the zero function

$f(t) = 0$ is Cnt's $\in C(\mathbb{R})$

Let $f,g \in C(\mathbb{R}) \Rightarrow (f+g)(t)$ is conti's $\in C(\mathbb{R})$

$f \in C(\mathbb{R})$, $c \in \mathbb{R} \Rightarrow cf(t) \in C(\mathbb{R})$ $\therefore C(\mathbb{R}) \trianglelefteq \mathcal{F}(\mathbb{R},\mathbb{R})$

$\boxed{\text{Example 3}}$ Let $M_{n\times n}(F)$ be the vector space of all $n\times n$ Matrices over F

Let $D_{n\times n}(F) \leq M_{n\times n}(F)$

$D_{n\times n}(0)$ s.t. $\forall i,j$ $D_{ij} = 0$ for $i \neq j \Rightarrow D(0) \in D_{n\times n}(F)$

$A,B \in D_{n\times n}(F) \Rightarrow A+B \in D_{n\times n}(F)$ $A_{ij} + B_{ij} = 0 + 0 = 0$ $i \neq j$

$A \in D_{n\times n}(F), c \in F \Rightarrow cA \in M_{n\times n}(F)$ where $cA_{ij} = 0$ for $i \neq j \Rightarrow cA \in D_{n\times n}(F)$ subspace.

theorem 1.4

Any intersection of subspaces of V; is a subspace of V.

Proof: Let $Y, W \trianglelefteq V$ consider $Y \cap W$

i $0 \in Y$ and $0 \in W \Rightarrow 0 \in Y \cap W$

i) Let $\forall x, z \in Y$, $\forall x, z \in W \Rightarrow x, z \in Y \cap W$

if $\forall x, z \in Y$ and $x, z \notin W \Rightarrow x, z \notin Y \cap W$

if $Y \cap W = \emptyset$ / so then $\{0\} \in Y \cap W$ is a subspace by definition

Assume $\exists x, z \in Y$, and $W \Rightarrow x, z \in Y \cap W$

Since Y, W are subspaces $\Rightarrow x+z \in Y$ and $x+z \in W \Rightarrow x+z \in Y \cap W$

similarly for $cz \in Y$, $cz \in W \Rightarrow cz \in Y \cap W$

If Y W are disjoint then these above hold for all possible values and vectors

including 0

§1.4 Linear Combinations

Linear Combination: Let V be a vector space and let S be a nonempty subset. If there exists a finite number of vectors $u_1, \ldots, u_n \in S$ and finite number of scalars in F $a_1, \ldots, a_n$ s.t. $v = a_1u_1 + \ldots + a_nu_n$ we say v is a linear combination of vectors in S.

Linear Systems: determine if a system can be written as a linear combination

$v = (2,6,8)$ $u_1 = (1,2,1)$, $u_2 = (-2,-4,-2)$ $u_3 = (0,2,3)$, $u_4 = (2,0,-3)$ $u_5 = (-3,8,16)$

$a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 + a_5u_5 = v$

$a_1(1,2,1) + a_2(-2,-4,-2) + a_3(0,2,3) + a_4(2,0,-3) + a_5(-3,8,16) = (2,6,8)$

$1a_1 \; -2a_2 + 2a_4 + -3a_5 = 2$ $1a_1 - 2a_2 + 2a_4 - 3a_5 = 2$ $1a_1 - 2a_2 + 2a_4 - 3a_5 = 2$

$2a_1 \; -4a_2 + 2a_3 + 0a_4 + 8a_5 = 6 \rightarrow$ $0a_1 + 0a_2 + 2a_3 - 4a_4 + 14a_5 = 2 \Rightarrow$ $1a_3 - 2a_4 + 7a_5 = 1$

$1a_1 \; -2a_2 + 3a_3 + -3a_4 + 16a_5 = 8$ $0a_1 + 0a_2 + 3a_3 - 5a_4 + 19a_5 = 6$ $3a_3 - 5a_4 + 19a_5 = 6$

$1a_1 - 2a_2 + 2a_4 - 3a_5 = 2$ $1a_1 - 2a_2 + 0a_4 + a_5 = -4$ $a_1 = -4 + 2a_2 - a_5$

$1a_3 - 2a_4 + 7a_5 = 1$ $1a_3 + 3a_5 = 7$ $\Rightarrow$ $a_3 = 7 - 3a_5$

$+ 1a_4 - 2a_5 = 3 \rightarrow$ $1a_4 - 2a_5 = 3$ $a_4 = 3 + 2a_5$

$(a_1, a_2, a_3, a_4, a_5) =$

Defn: Row Echelon Form

i) First coefficient is a one

ii) If an unknown coefficient is the first entry in the equation, it is zero in all others

iii) If a coefficient is one for a given index, it is zero for all others preceding it.

SPAN: Let S be a nonempty subset of a vector space V, the span(S) is the set consisting of all linear combinations of the vectors in S.

theorem 1.5: The span of any subset of S of a vector space V is a subspace of V. Any subspace of V that contains S must also contain the span of S.

Proof:

to prove a subset: Just prove that an arbitrary element in A is in $B \Rightarrow A \subseteq B$

Generates (or Spans): A subset of S of a vector space V generates or SPANS V if $\text{SPAN}(S) = V$. then the vectors in S generate V.

§1.5 Linear Algebra

definition: A subset S of a vector space is linear dependent if $\exists$ a finite number of distinct vectors $u_1, \ldots, u_n \in S$ and scalars $a_1, \ldots, a_n$ not all zeroes s.t. $a_1u_1 + \ldots + a_nu_n = 0$, must be non-empty

defn: TRIVIAL Representation: If $a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0$ if $a_1 = \ldots = a_n = 0$ $\forall a_i$ this is the trivial representation of 0, as a linear combination of $\{u_1, \ldots, u_n\}$

Note: any subset of a vector space containing 0 is a linearly dependent since $\exists$ nontrivial representation $0 = 1 \cdot 0$ (this leads to Null space)

defn: Linear Independent: if only the trivial representat $\exists$, then LI

i) the empty set is LI

ii) A set $\{u\}$ of single non-zero vector is LI

theorem 1.6: Let V be a vector space & $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent then S_2 is LD

corollary: If S_2 is linearly Independent then S_1 is LI

defn: Minimum Generating Spanning Sets

finding a basis which generates the SPAN(S)

for if a proper subset of S generates the SPAN(S), then S is LD.

If no proper subset of S generates the SPAN(S), then S is LI

theorem 1.7: Let S be a LI subset of a vector space of V, and let $v \in V$ s.t. $v \notin S$ then $S \cup \{v\}$ is LD iff $v \in$ SPAN(S).

$\Rightarrow$ if $S \cup \{v\}$ is linear dependent then $\exists\, u_1, \dots, u_n$ and

$a_1, \dots, a_n$ not all zero s.t. $a_1u_1 + \dots + a_nu_n + a_{n+1}v = 0$

$\Rightarrow v = (a_1u_1 + \dots + a_nu_n)a_{n+1}^{-1}$

$v \in \mathrm{SPAN}(S)$

$\Leftarrow$ Let $v \in \mathrm{SPAN}(S)$ where $\exists\, u_1, \dots, u_n \in S$ s.t. $v = a_1u_1 + \dots + a_nu_n$ $\quad u_i \in S$

where $a_1u_1 + \dots + a_nu_n = 0$ iff $\forall a_i = 0$

$\Rightarrow a_1u_1 + \dots + a_nu_n + (-1)v = 0 \quad \forall_{i=1:n}\, a_i = 0$

$= 0 + (-1)v = 0$

where $v \neq 0$ $\quad \Rightarrow \{v\}$ is LD $\quad \Rightarrow S \cup \{v\}$ is LD

§1.6 Bases and Dimension

<u>defn</u>: A basis β for V vector space, is a linearly independent subset of that generates V.

basis does not need to BE finite.

<u>theorem 1.8</u>: Let V be a vector space and $\beta = \{u_1, u_2, \ldots, u_n\}$ be a subset of V. Then β is a basis for V iff $\forall v \in V$ can be uniquely expressed as a linear combination of vectors of β; that is $\forall v \in V \ \exists\ a_1, \ldots, a_n \in F$ s.t. $v = a_1u_1 + \ldots + a_nu_n$; where $a_1, \ldots, a_n$ are unique.

<u>Proof</u>: Let $\beta = \{u_1, \ldots, u_n\}$ be a basis for V vector space then $\text{SPAN}(\beta) = V$

then $v \in \text{SPAN}(\beta) \Rightarrow v = a_1u_1 + \ldots + a_nu_n$ and $v = b_1u_1 + \ldots + b_nu_n$

WNTST $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$

$\Rightarrow 0 = (a_1 - b_1)u_1 + \ldots + (a_n - b_n)u_n$. where $a_i - b_i = 0 \ \forall i$ since β is a basis

$\Rightarrow a_i = b_i \ \forall i$

$\Leftarrow$ Let $v \in V$ can be expressed uniquely as a linear combination of vectors in $\beta = \{u_1, \ldots, u_n\}$

WNTST β is a basis (LI, generates)

then $\forall v \in V \ \exists$ unique scalars $(a_1, \ldots, a_n)$ s.t. $v = a_1u_1 + \ldots + a_nu_n$

then by hypothesis $V \subseteq \text{SPAN}(\beta)$.

Conversely each n-tuple $(a_1, \ldots, a_n)$ determines a unique linear combination expressed as $v \in V$ $\quad \text{SPAN}(\beta) \subseteq V \quad \Rightarrow \text{SPAN}(\beta) = V$.

Theorem 1.9 If a vector space V is generated by a finite set of S then some subset of S is a basis for V. V then has a finite basis.

"Any spanning subset for a vector space can be reduced to a basis for V."

Theorem 1.10: Let V be a vector space that is generated by a set G w/ exactly n vectors, and let L be a LI subset of V w/ m vectors. then $m \leq n$ and $\exists$ a subset $H \subseteq G$ containing exactly $n-m$ vectors s.t. $G = L \cup H$ generates V.

Proof: By induction let $m=0$, s.t. $L = \emptyset$. $\Rightarrow H = G$.

Now by induction hypothesis assume it is true for some $m \in \mathbb{N}$.

WNTST $m+1$ is true.

Let $L = \{v_1, \ldots, v_{m+1}\}$ be LI by corollary T1.6 $\overset{L'}{\{v_1, \ldots, v_m\}}$ is LI

by hypothesis $m \leq n$ and $\exists\, U \subseteq G$ s.t. $\{u_1, \ldots, u_{n-m}\}$ not necessarily LIN IND. s.t. $\text{SPAN}(L' \cup H) = V$

By hypothesis $\{v_1, \ldots, v_m\} \cup \{u_1, \ldots, u_{n-m}\}$ generates V.

If $\{v_{m+1}\} \in \text{SPAN}(L \setminus \{v_{m+1}\})$ then $\underbrace{L' \cup \{v_{m+1}\}}_{=L}$ is LD by T1.7 contradicting the hypothesis. where $n-m > 0$. Since $v_{m+1} \notin L \setminus \{v_{m+1}\}$ and $v_{m+1} \in \text{SPAN}(L)$

then $n > m \Rightarrow n \geq m+1$

$\Rightarrow \exists\, a_1, \ldots, a_m, b_1, \ldots b_{n-m}$ s.t. $v_{m+1} = a_1v_1 + \ldots + a_mv_m + b_1u_1 + \ldots + b_{n-m}u_{n-m}$ by hypothesis assumption

$u_1 = (-b_1^{-1}a_1)v_1 + (-b_1^{-1}a_2)v_2 + \ldots + (b_1^{-1})v_{m+1} + (-b_1^{-1}b_2)u_2 + \ldots + (-b_1^{-1}b_{n-m})u_{n-m}$

$H = \{u_2, \ldots, u_{n-m}\}$. then $u_1 \in \text{SPAN}(L \cup H) \Rightarrow \{v_1, \ldots, v_m, u_1, \ldots, u_{n-m}\} \subseteq \text{SPAN}(L \cup H)$ By arbitrary $u_1 \in L' \cup H'$

Since $L' \cup H'$ generates V; by T1.5 Since $L \cup H \subseteq V \Rightarrow \text{SPAN}(L \cup H) \leq V$

$\Rightarrow \text{SPAN}(L \cup H) = V_*$ $\quad$ * where $\text{SPAN}(L' \cup H') \subseteq \text{SPAN}(L \cup H) \Rightarrow V \subseteq \text{SPAN}(L \cup H)$.

by hypothesis $\{v_1,\dots,v_m,u_1,\dots,u_{n-m}\}$ generates V

$\Rightarrow \text{span}(L\cup H) = \text{span}(\{v_1,\dots,v_{m+1},u_2,\dots,u_{n-m}\}) = V$

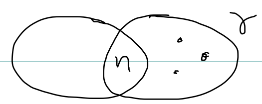
then $H \subseteq G$

where $|H| = n-m-1 = n-(m+1)$. ← by construction

Corollary 1: Let V be a vector space w/ a finite basis. Then every basis for V contains the same number of vectors.

Note: there can be many basis which generate the space.

Let β be a finite basis w/ n vectors. Let γ has more than n vectors, s.t. γ is a basis.
↑ generating set



$\{n_1,\dots,n_n\}=\beta$ $\text{span}(\beta)=V$

$\{m_1,\dots,m_{m+n}\}=\gamma$ is LI Let $|\gamma|>n$

so $\exists\, S\subseteq\gamma$ s.t. $S=\{m'_1,\dots,m'_{n+1}\}\subseteq\gamma$ $m+n$ $n-m$

S is LI, and β generates V $|\beta|=n$

but $|S|=n+1>n$ contradiction. ← RT then γ is finite and $|\gamma|\le n=|\beta|$

then reversing the argument yields $|\beta|\le|\gamma| \Rightarrow |\beta|=|\gamma|$

Finite Dimensional: A vector space is finite dimensional if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis is called dimension of V denoted $\dim(V)$.

A vector space not finite dimensional is infinite-dimensional.

Generates: A subset $S \subseteq V$ generates (or spans V) if $\text{span}(S) = V$
in this case the vectors of S generate or spans V.
$\text{span}(S)$ is not necessarily LI; you can have LD vectors in a set which has $\text{span}(S) = V$.

Key concepts: reducing a generating set to a basis.

Corollary 2: i) Let V have dimension n. Any generating set must have at least n vectors. A generating set w/ exactly n vectors is a basis.
ii) Any LI set w/ exactly n vectors is a basis.
iii) Every LI subset of V can be extended to a basis for V.

Theorem 1.11: Let W be a subspace of a finite dimensional vector space V. then W is finite dimensional s.t. $\dim(W) \le \dim(V)$.
if $\dim(W) = \dim(V) \Rightarrow W = V$.

Corollary: If W is a subspace of V of a finite dimensional vector space V, then any basis of W can be extended to a basis for V by Corollary 2 of Replacement theorem.

The Lagrange Interpolation Formula: the polynomials $f_0(x), f_1(x), \ldots, f_n(x)$ can be defined as:
$$f_i(x) = \frac{(x-c_0)(x-c_1)\ldots(x-c_{i-1})(x-c_{i+1})\ldots(x-c_n)}{(c_i-c_0)\ldots(c_i-c_{i-1})(c_i-c_{i+1})\ldots(c_i-c_n)}$$
$$= \prod_{\substack{k=0 \\ k\neq i}}^{n} \frac{x-c_k}{c_i-c_k}$$
n = degree of polynomial desired construction

these are defined as Lagrange Polynomials, for each $f_i(x)$ of degree n

is in $P_n(F)$ by regarding $f_i(x)$ as a polynomial function

$f_i(x): \quad F \to F$

$f_i(c_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

the property of LP is that $\{f_0, f_1, \dots, f_n\}$ is a linearly Ind. subset of $P_n(F)$

suppose that the zero function is defined as this suppose zero property:

$\sum_{i=0}^{n} a_i f_i = 0 \quad \forall a_i \text{ s.t. } (a_0, \dots, a_n)$

then $\sum_{i=0}^{n} a_i f_i(c_j) = 0$ for $j = 0, \dots, n$

by definition of LP $\sum_{i=0}^{n} a_i f_i(c_j) = a_j$ for $i = j$

where $a_j = 0 \Rightarrow \{f_0, \dots, f_n\}$ is Linearly Ind. w\ $n+1$ vectors s.t $\dim(P_n(F))$

$= n+1. \Rightarrow \beta$ is basis.

since β is a basis any polynomial function $g \in P_n(F)$ can be expressed

in terms of β as LC.

Suppose $g = \sum_{i=0}^{n} b_i f_i$

then $g(c_j) = \sum_{i=0}^{n} b_i f_i(c_j) = b_j$ for $i=j$ by defn of LP

$\Rightarrow g = \sum_{i=0}^{n} g(c_i) f_i$

where for $b_0, \dots, b_n$ are $n+1$ scalars not necessarily distinct;

then $g(c_j) = b_j$ is the unique representation by LI of β

this shows the unique polynomial w\ degree$(f) \leq n$ that has unique

values at b_j at points c_j in domain $(j = 0, 1, \dots, n)$

Note: if $f \in P_n(F)$ & $f(c_i) = 0$ for $n+1$ distinct scalars $(c_0, \dots, c_n)$ in F

then f is the zero function: AKA (the kernel)

§1.7 Maximal Linearly Independent Subsets.

$n=1 \quad \{f_0(x), f_1(x)\} \quad \{c_0, c_1\}$

$$f_0(x) = \prod_{\substack{k=0 \\ k \neq 0}}^{1} \frac{x-c_k}{c_0-c_k} = \frac{x-c_1}{c_0-c_1}$$

$$f_1(x) = \prod_{\substack{k=0 \\ k \neq 1}}^{1} \frac{x-c_k}{c_1-c_k} = \frac{x-c_0}{c_1-c_0}$$

$f_0(c_0) = 1 \qquad f_0(c_1) = \frac{c_1-c_1}{c_0-c_1} = 0$

$f_1(c_0) = 0 \quad f_1(c_1) = 1$

$$a_0 f_0(c_0) + a_1 f_1(c_1) + a_1 f_0(c_1) + a_0 f_1(c_0) = a_0 f_0(c_1) + a_1 f_1(c_1) = a_0 + a_1$$