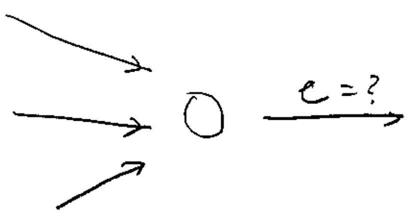
Bufferless Statistical Multiplexing

Example:

1000 sources
each one
generates
data at
rate:



Goal: don't lose any packets.

$\Rightarrow c = 1000$ bps.

1 bps w.p 0.5 } independent
0 bps w.p 0.5 } of each other.

Goal: Allow small probability of overflow.

$X_i =$ rate at which source i generates data.

Find the maximum N such that

$$\mathbb{P}\left(\sum_{i=1}^{N} X_i > c\right) \leq \varepsilon \qquad \underline{X_i \text{ iid}}.$$

Chernoff bound $\quad c = Nc \quad$ (ie. define $c \triangleq c/N$)

$$\mathbb{P}\left(\sum_{i=1}^{N} X_i \geq Nc\right) = \mathbb{P}\left(e^{\theta \sum_{i=1}^{N} X_i} \geq e^{\theta N c}\right). \qquad \theta > 0$$

$$\leq \frac{\mathbb{E}\left(e^{\theta \sum_{i=1}^{N} X_i}\right)}{e^{\theta N c}}$$

Define:

$$M_x(\theta) = \mathbb{E}(e^{\theta X}) \quad \text{(Moment generating function)}$$

$$\Lambda_X(\theta) = \log M_X(\theta) \quad \text{(log moment generating fn)}.$$

then we have,

$$\mathbb{P}\left(\sum_{i=1}^{N} X_i \geq Nc\right) \leq \frac{M_X^N(\theta)}{e^{\theta Nc}} \quad \text{(use independence)}$$

$$= \frac{e^{N \log M_X(\theta)}}{e^{\theta Nc}}$$

$$= e^{-N(\theta c - \Lambda_X(\theta))}.$$

Thus,

$$\mathbb{P}\left(\sum_{i=1}^{N} X_i \geq Nc\right) \leq e^{-N I(c)},$$

where $I(c) = \sup_{\theta > 0} \{ \theta c - \Lambda_X(\theta) \}$.

Lower Bound:

$$\max_{k \geq 1} \mathbb{P}\left(\frac{1}{N} \sum_{i=1}^{N} A_i(k) \geq b + kc \right)$$

$$\approx \max_{k \geq 1} e^{-N I_A(b+kc)} = e^{-N \min_{k \geq 1} I_A(b+kc)},$$

where,

$I_A(\cdot)$ is the rate function of $A_i(k)$

suppose $k^* = \arg\min_{k \geq 1} I_A(b+kc)$. most likely time of overflow

Upper Bound:

$$\approx \sum_{k \geq 1} e^{-N I_A(b+kc)}$$

$$\approx e^{-N \min_{k \geq 1} I_A(b+kc)}$$ for large N (one term dominates).

ie, under appropriate conditions,

$$\lim_{N \to \infty} \frac{1}{N} \ln\left(Q(0) \geq Nb\right) = \min_{k \geq 1} I_k(b+kc).$$

Cramer's Theorem:

Let X_i's iid, then

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(\frac{1}{N} \sum_{i=1}^{N} X_i \geq c \right) = -\sup_{\theta}\left(\theta c - \Lambda(\theta)\right)$$

$$= -I(c)$$

Assumptions:

(i) $c > \mathbb{E}(X_i) \triangleq \mu$

(ii) $\Lambda(\theta)$ is well defined in a neighborhood of $\theta = 0$.

(iii) $\exists\ \theta^*$ in this nbd such that

$$I(c) = \theta^* c - \Lambda(\theta^*)$$

Under these assumptions:

$$e^{-N(I(c)+\varepsilon)} \leq \mathbb{P}\left(\sum_{i=1}^{N} X_i \geq Nc\right) \leq e^{-N I(c)}$$

Given $\varepsilon > 0$, $\exists\ n : \forall\ N > n$ the above is true.

Upper bound :

At $\theta = 0$.

$$\theta c - \Lambda(\theta) = 0$$

If we show that

$$\theta c - \Lambda(\theta) \leq 0 \quad \forall\ \theta < 0,$$

then $\theta \geq 0$ can be removed in the Chernoff bound.

Now, for $\theta < 0$

$$e^{\theta(\mu - c)} \geq 1 \qquad (\mu < c). \longrightarrow (1)$$

also,

$$E\left(e^{\theta X_i}\right) \geq e^{\theta \mu} \qquad (\mu = E(X_i)) \longrightarrow (2)$$

Jensen's inequality.

$$(1) \ \& \ (2) \Rightarrow e^{\theta(\mu - c)} E\left(e^{\theta X_i}\right) \geq e^{\theta \mu}$$

$$\Rightarrow -\theta c + \Lambda(\theta) \geq 0.$$

$$\theta c - \Lambda(\theta) \leq 0$$

<u>Lower Bound</u> :

$$e^{N I(c)} \mathbb{P}\left(\sum_{i=1}^{N} X_i \geq Nc \right) \geq e^{-N\varepsilon} \qquad (\text{to show})$$

$$e^{N I(c)} \mathbb{P}\left(\sum_{i=1}^{N} X_i \geq Nc \right)$$

$$= \int_{\sum_{i=1}^{N} X_i \geq Nc} e^{N(\theta^* c - \Lambda(\theta^*))} \, d\mathbb{P}_1(x_1)\, d\mathbb{P}_2(x_2) \ldots d\mathbb{P}_N(x_N)$$

$$\geq \int_{N(c+\delta) \geq \sum_{i=1}^{N} X_i \geq Nc} e^{N(\theta^* c - \Lambda(\theta^*))} \, d\mathbb{P}_1(x_1)\, d\mathbb{P}_2(x_2) \ldots d\mathbb{P}_N$$

$$= e^{-N\theta^*\delta} \int_{\substack{N(c+\delta) \geq \sum_{i=1}^{N} x_i \\ \geq Nc}} \frac{e^{N\theta^*(c+\delta)}}{(M(\theta^*))^N} \, d\mathbb{P}_1(x_1) \ldots d\mathbb{P}_N(x_N)$$

Key step $\geq e^{-N\theta^*\delta} \int \prod_{i=1}^{N} \frac{e^{\theta^* x_i}}{M(\theta^*)} \, d\mathbb{P}_i(x_i)$

(if $(c+\delta) \geq \sum x_i$ above is true)

Define $d\mathbb{Q}_i = \frac{e^{\theta^* x_i}}{M(\theta^*)} \, d\mathbb{P}_i$

Y_i is a RV with cdf $\mathbb{Q}$.

Aside: Show that $\mathbb{E}[Y_i] = c$

Proof. $$M_Y(\theta) = \int \frac{e^{\theta y} e^{\theta^* y}}{M(\theta^*)} \, d\mathbb{P}$$

$$= \frac{M(\theta + \theta^*)}{M(\theta^*)}$$

$$\mathbb{E}(Y) = M'_Y(\theta)\Big|_{\theta=0} = \frac{M'(\theta^*)}{M(\theta^*)}$$

Since θ^* maximizes $\theta c - \log M(\theta)$

$$\Rightarrow c = \frac{M'(\theta^*)}{M(\theta^*)}$$

$$\mathbb{P}\left(Nc \le \sum_{i=1}^N Y_i \le N(c+\delta)\right)$$

$$e^{NI(c)} \mathbb{P}\left(\sum_{i=1}^N X_i \ge Nc\right) \geq e^{-N\theta^*\delta} \underset{=}{\mathbb{P}}\left(\sum_{i=1}^N (Y_i - c) \in [0, N\delta]\right)$$

$$= e^{-N\theta^*\delta} \underbrace{\mathbb{P}\left(0 \le \frac{\sum_{i=1}^N (Y_i - c)}{\sqrt{N}} \le \sqrt{N}\delta\right)}$$

By central limit thm, the above converg to $\frac{1}{2}$

ie, given ε, choose N such that. such that.

$$\frac{1}{4} e^{-N\theta^*\delta} \geq e^{-N\varepsilon}$$

Loynes Formula

- $a(k)$ = # of packets arriving in time slot k.
- $A(k) = \sum_{j=-k}^{-1} a(j)$

Wish to estimate $\mathbb{P}(Q(0) \geq b)$.

Assume that $\mathbb{E}(a(1)) < c$

We have,

$$Q(0) = \max(Q(-1) + a(-1) - c, 0)$$

$$= \max(Q(-1) + A(1) - c, 0)$$

$$Q(-1) = \max(Q(-2) + a(-2) - c, 0)$$

$$\Rightarrow Q(0) = \max(\max(Q(-2) + a(-2) - c, 0) + A(1) - c, 0)$$

$$= \max(Q(-2) + A(2) - 2c, A(1) - c, 0)$$

Assume that $Q(-\infty) = 0$.

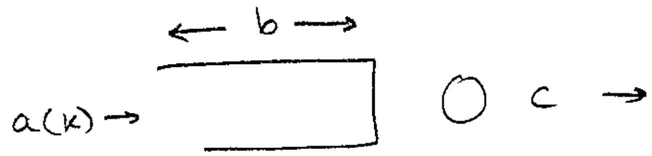
Then,

$$Q(0) = \max\{\max_{k \geq 1}\{A(k) - kc\}, 0\}.$$

ie,

$$\mathbb{P}(Q(0) \geq b) = \mathbb{P}\left(\max_{k \geq 1}(A(k) - kc) \geq b\right).$$

SINGE SOURCE LARGE BUFFER.



$$\lim_{b \to \infty} \frac{1}{b} \log \mathbb{P}(Q(0) \geq b) \leq -\min_{t \geq 0} t \, I_\infty (c + 1/t)$$

<u>Proof</u>:

$$\mathbb{P}(Q(0) \geq b) = \mathbb{P}\left\{ \max_{k \geq 1} A(k) - kc \geq b \right\}$$

(Loyne's formula)

$$= \mathbb{P}\left\{ \max_{t \in \{1/b, \frac{2}{b}, \dots\}} A(bt) \geq b + btc \right\}$$

$$= \mathbb{P}\left\{ \max_{t \in \{..\}} A(bt) \geq b(1 + tc) \right\}$$

$$\leq \sum_{t \in \{1/b, 2/b, ..\}} \mathbb{P}\left(A(bt) \geq b(1+tc) \right)$$

$$\leq \sum_{t \in \{1/b, 2/b, ..\}} \frac{\mathbb{E}\left(e^{\theta A(bt)}\right)}{e^{\theta b (1+tc)}}$$

Define,

$$\Lambda_{bt} = \frac{1}{bt} \log \mathbb{E}\left(e^{\theta \sum_{i=-bt}^{-1} a(i)} \right)$$

LHS

$$= \sum_{t \in \{1/b, 2/b, \ldots\}} e^{-bt\left(\theta(c+1/t) + \Lambda_{bt}(\theta) \right)}$$

$$\leq \sum_{t \in \{1/b, \ldots\}} e^{-bt\, I_{bt}(c+1/t)}$$

$$\lim_{b\to\infty} \frac{1}{b} \log\left(\mathbb{P}(Q(0)>b)\right) \leq -\min_{t \geq 0} t\, I_\infty(c+1/t)$$

$\square$

Claim :

$$\min_{t \geq 0} \; t \, I_\infty (c + 1/t) \geq \varepsilon \quad \left(\left(\mathbb{P}(Q(0) > b) \leq e^{-\varepsilon b} \right) \right)$$

$$\Updownarrow$$

$$\frac{\Lambda(\varepsilon)}{\varepsilon} \leq c.$$ ← effective bandwidth of the source.

Proof :

$$\inf_{t \geq 0} t \, I_\infty (c + 1/t) = \inf_{x \geq 0} \frac{1}{x} I(c+x)$$

$$\frac{1}{x} I_\infty (c+x) \geq \varepsilon \quad \forall x$$ (by assumption

$$\Rightarrow \quad I_\infty (c+x) \geq \varepsilon x \quad \forall x$$

$$\Rightarrow \quad \varepsilon x - I_\infty (c+x) \leq 0 \quad \forall x$$

$$\Rightarrow \quad \varepsilon (c+x) - I_\infty (c+x) \leq c\varepsilon \quad \forall x$$

$$\Rightarrow \quad \sup_{c+x} \varepsilon (c+x) - I_\infty (c+x) \leq c\varepsilon$$

$$\Rightarrow \quad \frac{\Lambda(\varepsilon)}{\varepsilon} \leq c$$

Similarly in other direction.

Properties of effective bandwidth.

$$\frac{\Lambda(\varepsilon)}{\varepsilon} = \lim_{K\to\infty} \frac{1}{K\varepsilon} \ln \mathbb{E}\left(e^{\varepsilon \sum_{i=-K}^{-1} a(i)}\right)$$

(1) $$\lim_{\varepsilon\to 0} \frac{\Lambda(\varepsilon)}{\varepsilon} = \mathbb{E}(a(i)).$$

"Proof": $$\frac{1}{K\varepsilon} \log \mathbb{E}\left(e^{\varepsilon \sum_{i=1}^{K} a(i)}\right) = \frac{1}{K\varepsilon} \log\Big(\mathbb{E}\Big(1 + \varepsilon \textstyle\sum a(i) + \frac{\varepsilon^2 (\sum a_i)^2}{2!} \dots\Big)$$

$$\approx \frac{1}{K\varepsilon} \log \mathbb{E}\left(1 + \varepsilon \textstyle\sum a_i\right)$$

$$\approx \frac{\cancel{\varepsilon}\mathbb{E}(\sum a_i)}{K \cancel{\varepsilon}} = \mathbb{E}[a(i)].$$

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(2) Suppose that

$$P = \inf \{a : F(a) = 1\}$$ ie. $P =$ peak rate

$$\lim_{\theta\to\infty} \frac{\Lambda(\theta)}{\theta} = P.$$

Proof: Assume $a = k_i$ w.p π_i

$$k_1 < k_2 < \dots < k_N = P$$

$$\frac{\mathbb{E}(e^{a\theta})}{e^{\theta P}} = \pi_N + \pi_{N-1} e^{\theta(K_{N-1} - P)} + \cdots + \pi_1 e^{\theta(K_1 - P)}$$

For large θ,

$$\mathbb{E}[e^{a\theta}] \approx \pi_N e^{\theta P}$$

$$\Rightarrow \quad \frac{\log \mathbb{E}[e^{a\theta}]}{\theta} = \frac{P\theta + \log \pi_N}{\theta}$$

$$\longrightarrow P$$

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(3) $\dfrac{\Lambda(\theta)}{\theta}$ is non-decreasing in θ.

$$f(\theta) = \frac{\Lambda(\theta)}{\theta}, \quad a = \sum_{i=1}^{K} a_i$$

$$\frac{f(\theta_2)}{f(\theta_1)} = \frac{\ln \mathbb{E}[e^{\theta_2 a}]\ \theta_1}{\ln \mathbb{E}[e^{\theta_1 a}]\ \theta_2}$$

$$= \frac{\ln\left(\mathbb{E}[e^{\theta_2 a}]\right)^{\theta_1/\theta_2}}{\ln \mathbb{E}[e^{\theta_1 a}]}.$$

If $\theta_2 > \theta_1$, we want to show that

$$\frac{f(\theta_2)}{f(\theta_1)} \geq 1$$

Now, if $\theta_2 > \theta_1$, then

x^{θ_1/θ_2} is concave for $x \in \mathbb{R}$.

Let $y = e^{\theta_2 a}$, then

$$\frac{\ln\left[\left(\mathbb{E}[y]\right)^{\theta_1/\theta_2}\right]}{\ln \mathbb{E}[e^{\theta_1 a}]} \geq \frac{\ln \mathbb{E}[y^{\theta_1/\theta_2}]}{\ln \mathbb{E}[e^{\theta_1 a}]}$$

$$= \frac{\ln \mathbb{E}\left[e^{\cancel{\theta_2} a \, \theta_1/\cancel{\theta_2}}\right]}{\ln \mathbb{E}[e^{\theta_1 a}]}$$

$$= 1$$

$$I(x) = \sup_\theta \; \theta x - \Lambda(\theta).$$

1. $I(x)$ is convex

Proof.

$$I(\alpha x_1 + (1-\alpha)x_2)$$

$$= \sup_\theta \; \theta(\alpha x_1 + (1-\alpha)x_2) - (\alpha\Lambda(\theta) + (1-\alpha)\Lambda(\theta))$$

$$\le \alpha \sup_\theta \; \theta x_1 - \Lambda(\theta) + (1-\alpha)\sup_\theta \; \theta x_2 - \Lambda(\theta).$$

$$= \alpha I(x_1) + (1-\alpha) I(x_2)$$

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2. $I(x) = \sup_\theta \; \theta x - \Lambda(\theta)$

$$\Updownarrow$$

$$\Lambda(\theta) = \sup_x \; \theta x - I(x)$$

$\Lambda(\theta)$ is a convex transform

Proof:

$$I(x) = \sup_\theta \; \theta x - \Lambda(\theta)$$

$$\frac{dI(x)}{d\theta} = x - \Lambda'(\theta)$$

i.e. $I(x) = \theta_x x - \Lambda(\theta_x) \longrightarrow (1)$

here $\boxed{\Lambda'(\theta_x) = x} \longrightarrow (2)$

$$\frac{dI}{dx}(x) = x\frac{d\theta_x}{dx} + \theta_x - \Lambda'(\theta_x)\frac{d\theta_x}{dx}$$

From ②, $\Lambda'(\theta_x) = x$

$$\Rightarrow \frac{dI(x)}{dx} = \theta_x \quad \text{or} \quad \boxed{I'(x) = \theta_x} \longrightarrow ③$$

Now, we have to show,

$$\sup_x \theta x - I(x) = \Lambda(\theta)$$

<u>LHS</u>.

To find sup. diff wrt x.

$$\theta - I'(x_0) = 0 \Rightarrow I'(x_0) = \theta \longrightarrow ④$$

and,

$$LHS = \theta x_0 - I(x_0) \longrightarrow ⑤$$

$$= \theta x_0 - (\theta_{x_0} x_0 - \Lambda(\theta_{x_0})) \quad (\text{from } ①$$

From ③, $\theta_{x_0} = I'(x_0)$

From ④ $I'(x_0) = \theta$

$\Rightarrow \theta_{x_0} = \theta$.

Thus, LHS $\theta x_0 - \theta x_0 - \Lambda(\theta)$

$= \Lambda(\theta)$