Notes on Combinatorics

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Chapter 1

Essential Problems

There are some essential problems to discuss, but we first give some counting principles.

Axiom 1.0.1 (The Sum Rule)**.** *Suppose $S_1, S_2, \dots, S_m$ are mutually disjoint finite sets and that $|S_i| = n_i$ for $1 \leq i \leq n$. There are $n_1 + n_2 + \cdots + n_m = \sum_{i=1}^{m} n$ ways to select one element from any of the sets S_i*

Axiom 1.0.2 (The Product Rule)**.** *Suppose $S_1, S_2, \dots, S_m$ are finite sets, (not necessarily mutually disjoint) for $1 \leq i \leq n$. Provided that the selections are made independently, there are $n_1 n_2 \dots n_m =_{i=1}^{m} n_i$ ways to select one element from the set S_i followed by an element from S_{i+1}.*

The following problems can now be discussed, and are all solved by the product rule.

Problem 1. How many ways are there to order n different elements in a given n element set?

Solution. Let S be a set with $|S| = n$ and choose one element $s_1 \in S$ and take $S_1 = S \backslash s_1$, since s_1 is arbitrary, by the sum rule, there are n choices for elements in S Now we need to choose elements from S_1 which has $|S_1| = n - 1$, by the same reasoning, choose $s_2 \in S_2$ and take $S_2 = S_1 \backslash s_2$. Since s_2 was arbitrary, by the sum rule again, there are $n - 1$ ways to choose elements from S_2. Continuing along this construction, take $s_i \in S_{i-1}$ and take $S_o = S_{i-1} s_i \backslash$, by the same reasoning there are $n - i$ ways to choose elements from S_i, where $1 \leq i \leq n$. Then by the product rule, there are ${}_{i=1}^{n} n - i = n(n-1) \dots 2 \cdot 1 \cdot 0! = n!$ ways to order n elements of S.

Problem 2. How many ways are there to order k elements from an n element set?

Solution. Let S be a set with $|S| = n$ and choose an arbitrary subset $T \subseteq S$ with $|T| = k$. Now there are $n!$ ways to order the elements of S and $(n-k)!$ ways to order elements from $S \backslash T$, hence there are $\frac{n!}{(n-k)!}$ ways to order k elements of T from S.

Problem 3. How many ways are the to select k elements, regardless of order, from an n element set?

Solution (1)**.** Let S be a set with $|S| = n$ and $T \subseteq S$ with $|T| = k$. We have there are $n!$ ways to order the elements of S, $k!$ ways to order the elements of T and $\frac{n!}{(n-k)!}$ ways to order the elements of $S \backslash T$. Now since order is irrelevant, the ordering of the elements of T does not matter. Hence there are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ ways to select k elements from S in no particular order.

There is an another way of arriving to this solution.

Definition 1.0.1. We define the **falling factorial power** $x^{\underline{k}}$, of x is the product

$$x^{\underline{k}} = x(x-1)(x-2)\dots(x-k+1) = \prod_{i=1}^{k-1} x - i \tag{1.1}$$

We define the **rising factorial power** of x to be

$$x^{\bar{k}} = x(x+1)(x+2)\dots(x+k-1) = \prod_{i=1}^{k-1} x - i \tag{1.2}$$

Ande we define $n^{\underline{0}=n^{\bar{0}}=0!=1}$.

Solution (2)**.** Notice that $\frac{n!}{(n-k)!} = n(n-1)\dots(n-k+1) = n^{\underline{k}} = n^{\bar{k}}$ and $n^{\underline{n}} = 1^{\bar{n}} = n!$. So there are $\frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$ ways to choose k elements from S in no particular order.

Remark. Note that this solution is not unique.

Chapter 2

Binomial Coefficients

Definition 2.0.1. Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. We define the **binomial coefficient** of n **choose** k to be

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & \text{, if } 0 \leq k \leq n \\ 0 & \text{, if } k > n \text{ or } k < 0 \end{cases} \tag{2.1}$$

and where $\binom{n}{0} = \binom{n}{n} = 1$.

The solution to problem 3 is a sufficient proof for the relation; and the fact that $\binom{n}{k} = 0$ for $n < k$ and $k < 0$ is evident since there can be no k element subsets of an n element set under those conditions. What follows are some fundamental lemmas about the binomial coefficient.

Lemma 2.0.1 (Symmetry)**.** $\binom{n}{k} = \binom{n}{n-k}$

Proof. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}$. □

Lemma 2.0.2 (Pascal's Lemma)**.** *If* $n \in \mathbb{N}$ *and* $k \in Z$ *then*

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \tag{2.2}$$

Proof. For $k < 0$ or $k > n$, the result is obvious by definition. Now suppose that $0 \leq k \leq n$, then $\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} = (n-1)!\frac{(n-k)+k}{k!(n-k)!} = \binom{n}{k}$. □

Remark. We can use Pascal's lemma to construct Pascal's triangle in figure 2.1.

Theorem 2.0.3 (The Binomial Theorem)**.** *If* $n \in \mathbb{N}$*, then*

$$(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k} \tag{2.3}$$

Proof. We use induction on n. Notice that $(x+y)^0 = 1 = \binom{0}{0}x^0y^0$, and $(x+y)^1 = x + y = \binom{1}{0}x^0y^1 + \binom{1}{1}x^1y^0$ □

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Figure 2.1: Pascal's triangle up to $n = 6$.

Proof. We use induction on n. Notice that $(x+y)^0 = \binom{0}{0}x^0y^0$ and $(x+y)^1 = x + y = \binom{1}{0}x^0y^1 + \binom{1}{1}x^1y^0$. Now suppose that the theorem holds for $n \geq 1$. Then $(x+y)^{n+1} = (x+y)\sum\binom{n}{k}x^ny^{n-k} = \sum\binom{n}{k}x^{k+1}y^{n-k} + \sum\binom{n}{k}x^ky^{n+1-k} = \sum(\binom{n}{k-1}\binom{n}{k})x^ky^{n+1-k} = \sum\binom{n+1}{k}x^ky^{n+1-k}$, by Pascal's lemma. $\square$

Corollary. *$\sum_k \binom{n}{k} = 2^n$ and there are exactly 2^n subsets of an n element set.*

Proof. Exapand $(1+1)^n$, also notice that since there are $\binom{n}{k}$ possible k element subsets of a given n element set, then all we need to do is take the sum of $\binom{n}{k}$ over k. $\square$

Corollary. $\sum_k (-1)^k\binom{n}{k} = \begin{cases} 0 & , \textit{ if } n \geq 11 \\ , \textit{ if } n = 0 & \end{cases}$

Proof. Expand $(-1+1)^n$. $\square$

Lemma 2.0.4. *If $m, n \in \mathbb{N}$, then $\sum_k \binom{k}{m} = \binom{n+1}{m+1}$.*

Proof. For $n = 0$, the result follows, $\binom{0}{0} = 1 = \binom{1}{1}$, and if $m > 0$, then each side is 0. Now suppose for $n \geq 0$ that the lemma holds. Then $\sum_{k=0}^{n+1}\binom{k}{m} = \binom{n+1}{m} + \binom{n+1}{m+1} = \binom{n+1}{m+1}$ by Pascal's lemma. $\square$

Theorem 2.0.5 (Vandermonde's Convolution)**.** *If $m, n \in \mathbb{N}$ and $l, p \in \mathbb{Z}$, then*

$$\sum_k \binom{m}{p+k}\binom{n}{l-k} = \binom{m+n}{l+p} \tag{2.4}$$

Proof. By induction on n, for $n = 0$, $\binom{m}{l+p} = \sum\binom{m}{p+k}$. Now suppose that the theorem holds for $n \geq 0$, then by Pascal's lemma we have that $\binom{m+n+1}{p+l} = \binom{m+n}{p+l} + \binom{m+n}{p+l-1} = \sum\binom{m}{p+k}\binom{n}{l-k} = \binom{m}{p+k}\binom{n+1}{0} + \sum\binom{m}{p+k}\binom{n+1}{l-k} = \sum\binom{m}{p+k}\binom{n+1}{l-k}$. $\square$

Chapter 3

Multinomial Coefficients