Examen 1

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Mate 6540

Tarea 1

Problema 1.	Show that if d is a metric for X, then $$d'(x, y) = d(x, y)/(1 + d(x, y))$$ is a bounded metric that gives the topology of X. [*Hint:* If $f(x) = x/(1+x)$ for $x > 0$, use the mean-value theorem to show that $f(a+b) - f(b) \leq f(a)$.]
Problema 2.	If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?
Problema 3.	Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.
Problema 4.	Show that the collection $$\mathcal{C} = \{[a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$ is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.
Problema 5.	Sea $X = \{f \mid f : [0,1] \to [0,1] \text{ es una función}\}$. Para cada subconjunto A de $[0,1]$, defina $$B_A = \{f \in X \mid f(x) = 0, \quad \forall x \in A\}.$$ Demuestre que $\mathscr{B} = \{B_A \mid A \subseteq [0,1]\}$ es una base para una topología sobre X.

(1) Let d be a metric on X, and define $d' : X \times X \to \mathbb{R}$ by $d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$. We would like to show that d' is a metric on X.

Lemma 0.0.1. *d' is a metric on X.*

Proof. We have that $d(x,y) \geq 0$, and $d(x,y) = 0$ if and only if $x = y$. Then notice that $1 + d(x,y) \geq 1 > 0$, and that $\frac{d(x,y)}{1+d(x,y)} \geq 0$. Moreover, $\frac{d(x,y)}{1+d(x,y)} = 0$ if and only if $x = y$. Notice as well that

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = d'(y,x)$$

Finally, letting $x, y, z \in X$, we have:

$$\begin{aligned} d'(x,z) &= \frac{d(x,z)}{1+d(x,z)} \\ &\leq \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \\ &\leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \\ &= d'(x,y) + d'(y,z) \end{aligned}$$

Hence, the triangle inequality is satisfied, and d' is a metric on X. ∎

We have a further property of this metric.

Corollary. *d' is a bounded metric.*

Proof. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x}{1+x}$, which is continous on an interval $[a,b]$ and differentiable on (a,b). Then by the mean value theorem, we have that there is a point $x \in (a,b)$for which

$$\frac{b}{1+b} - \frac{a}{1+a} = \frac{b-a}{x^2}$$

Then we have that

$$\frac{a+b}{1+a+b+ab} = \frac{b-a}{x^2} \leq \frac{a+b+2ab}{1+a+b+ab}$$

Hence we have that $f(a+b) \leq f(a)+f(b)$. Then composing f with d, we get $d' = f \circ d$, and so this shows that d' is a bounded metric. ∎

Corollary. *d' induces the same topology as d.*

Proof. Consider the ϵ-ball $B_d(x, \epsilon)$. By definition, we have that $d'(x, y) \leq d(x, y)$, Then $d'(x, y) = \frac{d(x,y)}{1+d(x,y)} \leq d(x, y) < \epsilon$, it follows that there is a δ-ball $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.

On the other hand, we have $d'(x, y) + d(x, y)d'(x, y) = d(x, y)$, suppose then, that $y \in B_d(x, \epsilon)$, then $d'(x, y) + d(x, y)d'(x, y) < \epsilon$, then we also have that $d'(x, y) < \epsilon$; that is to say $B_d(x, \epsilon) \subseteq B_{d'}(x, \epsilon)$. ■

(2)

Lemma 0.0.2. *Let $\{\mathcal{T}_\alpha\}$ be a collection of topological spaces on X. Then $\mathcal{T} = \bigcap \mathcal{T}_\alpha$ is also a topological space on X.*

Proof. Let $\{\mathcal{T}_\alpha\}$ be a collection of topological spaces on a set X, and consider $\mathcal{T} = \bigcap \mathcal{T}_\alpha$. Clearly, $\emptyset, X \in \mathcal{T}$, as $, X \in \mathcal{T}_\alpha$ for all α. Now consider a collection $\{U_\alpha\}$, where U_α is open in X under $\mathcal{T}_\alpha$. Then $u_\alpha \in \mathcal{T}_\alpha$, hence so is $\bigcup U_\alpha$. This implies that $\bigcup U_\alpha \in \mathcal{T}$.

Similarly let $\{U_i\}_{i=1}^n$ be a collection with U_i open in X under $\mathcal{T}_i$, for $1 \leq i \leq n$, then by similar reasoning, $U_i \in \mathcal{T}_i$, and hence so is $\bigcap_{i=1}^n U_i$, hence $\bigcap_{i=1}^n U_i \in \mathcal{T}$. This makes $\mathcal{T}$ a topology on X. ■

Remark. Consider $X = \{a, b, c\}$ and the topologies $\mathcal{T}_1 = \{\emptyset, \{a\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, \{b\}, X\}$. Then $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, \{a\}, \{b\}, X\}$, which is not a topology for it does not contain $\{a, b\}$. In general, $\mathcal{T} = \bigcup \mathcal{T}_\alpha$ is not a topology on arbitrary X.

(3)

Lemma 0.0.3. *Let $\{\mathcal{T}_\alpha\}$ be a collection of topologies on X. Then there exists a unique coarsest topology on X containing all $\mathcal{T}_\alpha$, and there exists a unique finest toplogy on X contained in all of $\mathcal{T}_\alpha$.*

Proof. From lemma 0.0.2, we have that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Now define $\mathcal{T}_s$ to be the topology on X generated by the subbasis $\mathcal{S} = \bigcup \mathcal{T}_\alpha$ (which contains all $\mathcal{T}_\alpha$), and suppose there is another topology $\mathcal{T}_s'$ on X generated by $\mathcal{S}$, then $\mathcal{S} \in \mathcal{T}_s'$. Now take $U \in \mathcal{T}_s$ with $U \not\subseteq \mathcal{S}$, then either $U = \bigcup S_\alpha$ or $U = \bigcap_{i=1}^n S_i$ (with S_α, S_i subbasis elements open in X). Since $U \not\subseteq \mathcal{S}$, and if $U = \bigcup U_\alpha$, then $U \notin \mathcal{T}_s'$, a contradiction. On the otherhand, if $U = \bigcap_{i=1}^n U_i$, then $U \subseteq \mathcal{S}$, another contradiction. Hence $\mathcal{T}_s$ is the coarsest such topology.

Now let $\mathcal{T}_l = \bigcap \mathcal{T}_\alpha$, and suppose there is another topology $\mathcal{T}_l'$ for which $\mathcal{T}_l \subseteq \mathcal{T}_l'$ and $\mathcal{T}_l' \subseteq \mathcal{T}_\alpha$ for all α. Then take $U_\alpha \in \mathcal{T}'$, then $U_\alpha \in \mathcal{T}_\alpha$, for all α hence $U_\alpha \in \bigcap \mathcal{T}_\alpha = \mathcal{T}_l$, thush $\mathcal{T}_l' \subseteq \mathcal{T}_l$. Then $\mathcal{T}_l$ is the finest such topology. ■

(4)

Lemma 0.0.4. *The collection $\mathcal{C} = \{[a, b) : a, b \in \mathbb{Q} \text{ and } a < b\}\}$ forms a basis generating a topology different from the lower limit topology $\mathbb{R}_l$.*

Proof. First, consider an element $[c, d) \in \mathcal{C}$, by the nested interval theorem, we have that there is a $[c', d') \subseteq [c, d)$, then taking $\mathcal{C}' = \{[c', d') \in \mathcal{C} : [c', d') \subseteq [c, d)\}$ We get that for $U \neq \emptyset$ open in $\mathbb{R}_l$ that $U = \bigcap_{C \in \mathcal{C}'} C$. Thus $\mathcal{C}$ forms a basis for a topology on $\mathbb{R}_l$.

Now consider $\mathbb{R}_l$ in the lower limit topology, and let $[a, b) \in \mathbb{R}_l$ be a basis element. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a $[c, d) \subseteq [a, b)$; however, $\mathbb{R}$ is not dense in $\mathbb{Q}$, so $[a, b) \subseteq [c, d)$ does not in general hold, for $c, d \in \mathbb{Q}$. So $\mathcal{C}$ forms a basis that generates a topology different from that of $\mathbb{R}_l$. ■

(5)

Lemma 0.0.5. *Let* $X = \{f : f : [0, 1] \to [0, 1]\}$, *and for each subset* $A \subseteq [0, 1]$, *let* $B_A = \{f \in X : f(x) = 0 \textit{for all } x \in A\}$, *and define* $\mathcal{B} = \{B_A : A \subseteq [0, 1]\}$. *Then* $\mathcal{B}$ *is a basis for a topology on* X.

Proof. Since $A \subseteq [0, 1]$, by the nested interval theorem, there is an interval C such that $C \subseteq A \subseteq [0, 1]$. Then for some $x \in C$, we have $f(x) = 0$, that is $B_C \subseteq B_A$. Now take $\mathcal{B}' = \{B_C : C \subseteq A\}$, then $\mathcal{B}' \subseteq \mathcal{B}$, and given $U \neq \emptyset$ open in X, since x was arbitrary, $U = \bigcup_{B \in \mathcal{B}'} B$. Therefore $\mathcal{B}$ is a basis for a topology on X. ■