Topology

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Text

Topology (2^{rd} edition)

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Chapter 1

Topological Spaces and Continuous Functions.

1.1 Topological Spaces.

Definition. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X such that:

(1) $\emptyset, X \in \mathcal{T}$.

(2) For any subcollection $\{U_\alpha\}$ of subsets of X, $\bigcup_\alpha U_\alpha \in \mathcal{T}$.

(3) For any finite subcollection $\{U_i\}_{i=1}^n$ of subsets of X, $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

We call the pair $(X, \mathcal{T})$ a **topological space**, and we call the elements of $\mathcal{T}$ **open sets**.

Example 1.1. (1) Let X be any set, the collection of all subsets of X, 2^X is a topology on X, which we call the **discrete topology**. We call the topology $\mathcal{T} = \{\emptyset, X\}$ the **indiscrete topology**.

(2) The set of three points $\{a, b, c\}$ has the 9 following topologies in figure 1.1.

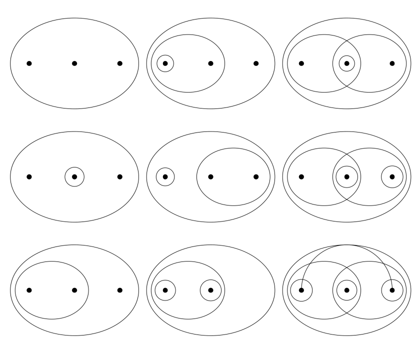

Figure 1.1: The Topologies on $\{a, b, c\}$.

(3) Let X be any set, and let $\mathcal{T}_f = \{U \subseteq X : X\backslash U \text{ is finite, or } X\backslash U = X\}$. Then $\mathcal{T}_f$ is a topology and called the **finite complement topology**.

(4) Let X be any set, and let $\mathcal{T}_c = \{U \subseteq X : X\backslash U \text{ is countable, or } X\backslash U = X\}$. Then $\mathcal{T}_c$ is a topology on X.

Definition. Let X be a set, and let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on X. We say that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$, and $\mathcal{T}'$ **finer** than $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{T}'$. If two topologies are either coarser, or finer than each other, we call them **comparable**.

Example 1.2. The topologies $\mathcal{T}_f$ and $\mathcal{T}_c$ are comparable, and we see that $\mathcal{T}_f \subseteq \mathcal{T}_f$, so $\mathcal{T}_f$ is coarser than $\mathcal{T}_c$, and $\mathcal{T}_c$ is finer than $\mathcal{T}_f$.

1.2 The Basis and Subbasis for a Topology.

Definition. If X is a set, the **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X, called **basis elements**, such that:

(1) For every $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$.

(2) For $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

We define the topology $\mathcal{T}$ **generated** by $\mathcal{B}$ to be collection of open sets: $\mathcal{T} = \{U \subseteq X : x \in B \text{ for some } B \in \mathcal{B}\}$.

Theorem 1.2.1. *Let X be a set, and $\mathcal{B}$ a basis of X, then the collection of subsets of X, $\mathcal{T} = \{U \subseteq X : x \in B \text{ for some } B \in \mathcal{B}\}$ is a topology on X.*

Proof. Let $\mathcal{B}$ be a basis for a topology in X, and consider $\mathcal{T}$ as defined above. Cleary, $\emptyset \in X$ and so is X.

Now let $\{U_\alpha\}$ be a subcollection of subsets of X, and let $U = \bigcup U_\alpha$. Then if $x \in U$ for some α, there is a B_α such that $x \in B_\alpha \subseteq U_\alpha$, thus $x \in B_\alpha \subseteq U$.

Now let $x \in U_1 \cap U_2$, and choose $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Then by definition, there is a B_3 for which $x \in B_3 \subseteq B_1 \cap B_2$. Now suppose for arbitrary n, that $U = \bigcap_{i=1}^{n} U_i \in \mathcal{T}$, for some finite subcollection $\{U_i\}$ of subsets of X. Then by let $B_n, B_{n+1} \in \mathcal{B}$ such that $x \in B_n \subseteq U$ and $x \in B_{n+1} \subseteq U_{n+1}$. Then by our hypothesis, there is a B for which $x \in B \subseteq B_n \cap B_{n+1}$, thus $U \cap U_{n+1} = \bigcap_{i=1}^{n+1} U_i \in \mathcal{T}$. This make $\mathcal{T}$ a topology on X. ∎

Example 1.3. (1) Let $\mathcal{B}$ be the set of all circular regions in the plane $\mathbb{R} \times \mathbb{R}$, then $\mathcal{B}$ satisfies the conditions needed for a basis.

(2) The collection $\mathcal{B}'$ in $\mathbb{R} \times \mathbb{R}$ of all rectangular region also forms a basis for a topology on $\mathbb{R} \times \mathbb{R}$.

(3) For any set X, the set of all 1-point elements of X forms a basis for a topology on X.

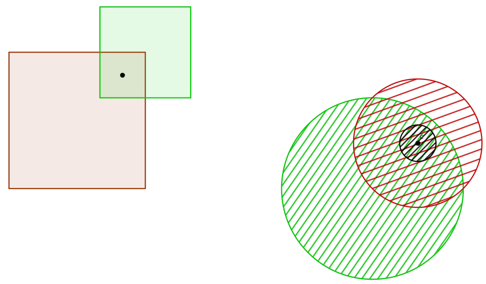

Figure 1.2: The basis for $\mathcal{B}$ and $\mathcal{B}'$ in $\mathbb{R} \times \mathbb{R}$ (see example (2)).

Lemma 1.2.2. *Let X be a set, and $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T} = \{\bigcup B : B \in \mathcal{B}\}$.*

Proof. Given a collection $\{B\}$ of basis elements in $\mathcal{B}$, since they are all in $\mathcal{T}$, their unions are also in $\mathcal{T}$. Conversely, given $U \in \mathcal{T}$, then for every point $x \in U$, choose a $B_x \in \mathbb{B}_x$ such that $x \in B_x \subseteq U$, then $U = \bigcup_{x \in U} B_x$. ∎

Lemma 1.2.3. *Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{C} \subseteq \mathcal{T}$ be a collection of open sets of X such that for every $x \in U$, there is a $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is the basis for a $\mathcal{T}$ on X.*

Proof. Take any $x \in X$, then there is a $C \in \mathcal{C}$ such that $x \in C \subseteq U$, thus the first condition for a basis is satisfied. Now let $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C}$, since $C_1 \cap C_2$ is open in X, there is a $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. Therefore $\mathcal{C}$ is a basis for a topology on X.

Now let $\mathcal{T}_{\mathcal{C}}$ be the the topology generated by $\mathcal{C}$, now for $U \in \mathcal{T}$, we have by the hypothesis, that $U \in \mathcal{T}_{\mathcal{C}}$; and by lemma 1.2.2, $W \in \mathcal{T}_{\mathcal{C}}$ is the union of elements of $\mathcal{C}$, which is a subcollection of $\mathcal{T}$, thus $W \in \mathcal{T}$. Therefore $\mathcal{T}_{\mathcal{C}} = \mathcal{T}$. ∎

Lemma 1.2.4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ on X. Then the $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for all $x \in X$, and all $B \in \mathcal{B}$, there is a $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Proof. Suppose first that $\mathcal{T} \subseteq \mathcal{T}'$, and let $x \in X$, and choose $B \in \mathcal{B}$ such that $x \in B$, then B is open in $\mathcal{T}$, thus it is open in $\mathcal{T}'$, thus there is a $B' \in \mathcal{B}'$ such that $x \in b' \subseteq B$. Conversely, suppose there is a $B' \in \mathcal{B}'$ for which $x \in B' \subseteq B$ for all $x \in X$, $B \in \mathcal{B}$. Take $x \in U \in \mathcal{T}$, since $\mathcal{B}$ generates $\mathcal{T}$, $x \in B \subseteq U$, since $B' \subseteq B$, this implies that $U \in \mathcal{T}'$ and $\mathcal{T} \subseteq \mathcal{T}'$. ∎

Definition. If $\mathcal{B}$ is the collection of open intervals (a, b) in $\mathbb{R}$, we call the topology generated by $\mathcal{B}$ the **standard topology** on $\mathbb{R}$, and we denote it simply by $\mathbb{R}$.

Definition. If $\mathcal{B}$ is the collection of half open intervals $[a, b)$ in $\mathbb{R}$, we call the topology generated by $\mathcal{B}$ the **lower limit topology** on $\mathbb{R}$, and we denote it simply by $\mathbb{R}_l$. If $\mathcal{B}'$ is the collection of all half open intervals $(a, b]$ in $\mathbb{R}$, then we call the topology generated by $\mathcal{B}'$ the **upper limt topology** on $\mathbb{R}$, and denote it $\mathbb{R}_L$.

Definition. If $\mathcal{B}$ is the collection of all open intervals of the form $(a,b)\setminus \frac{1}{\mathbb{Z}^+}$, where $\frac{1}{\mathbb{Z}^+} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$, we call the topology generated by $\mathcal{B}$ the $\frac{1}{\mathbb{Z}^+}$**-topology** on $\mathbb{R}$, and we denote it $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$.

Lemma 1.2.5. *The topologies $\mathbb{R}_l$, $\mathbb{R}_L$, and $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ are all strictly finer than $\mathbb{R}$, but are not comparable with each other.*

Proof. Let (a,b) be a basis element for $\mathbb{R}$, and let $x \in (a,b)$, the basis element $[x,b) \in \mathbb{R}_l$ lies in (a,b) and contains x, however, there can be no interval (a,b) in $[x,b)$ as $x \leq a$, thus $\mathbb{R}\mathbb{R}_l$; a similar argument holds for $\mathbb{R}_L$.

Similarly, for $(a,b) \in \mathbb{R}$, the basis element $(a,b)\setminus\frac{1}{\mathbb{Z}^+}$ of $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ lies in (a,b), however, choose the basis $B = (-1,1)\setminus\frac{1}{\mathbb{Z}^+}$, and choose $0 \in B$, since $\mathbb{Z}^+$ is dense in $\mathbb{R}$, there is no interval (a,b) containing 0 and lying in B, thus $\mathbb{R} \subseteq \mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$.

Now choose $[0,1)$ in $\mathbb{R}_l$, and choose $\frac{1}{k} \in [0,1)$ such that $k \in \mathbb{Z}^+$. Now $(0,1) \subseteq [0,1)]$, so we cannot say that $[0,1)$ is a basis for $\mathbb{R}$, and moreover, $[0,1)\setminus\frac{1}{\mathbb{Z}^+}$ cannot be said to be a basis in $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$, thus $\mathbb{R}_l$ and $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ are incompararble, a similar argument holds for $\mathbb{R}_L$.

Lastly, let (a,b) be in $\mathbb{R}$ and choose $x \in (a,b)$. Then $(a,x]$ and $[x,b)$ are both in (a,b), however it is clear that $(a,x]$ and $[x,b)$ connot be contained in each other, thus $\mathbb{R}_l$ and $\mathbb{R}_L$ are incomparable. ■

Definition. A **subbasis**, $\mathcal{S}$, for a topology on X is a collection of subsets of X whose union equals X. We call the **topology generated by** $\mathcal{S}$ to be the collection of all unions of finite intersections of elements of $\mathcal{S}$, that is:

$$\mathcal{T} = \{\bigcup\bigcap_{i=1}^{n} S_i : S_i \in \mathcal{S} \text{ for } 1 \leq i \leq n\}$$

Theorem 1.2.6. *Let $\mathcal{S}$ be a subbasis for a topology on X. Then the collection $\mathcal{T} = \{\bigcup\bigcap_{i=1}^{n} S_i : S_i \in \mathcal{S}$ for $1 \leq i \leq n\}$ is a topology on X.*

Proof. It is sufficient to show that the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ is a basis for a topology on X. By lemma 1.2.1, for $x \in X$, it belongs to an element S of $\mathcal{S}$, and therefore, to an element of $\mathcal{B}$. Now let $B_1 = \bigcap_{i=1}^{m} S_i$ and $B_2 = \bigcap_{j=1}^{n} S'_j$ be basis elements of $\mathcal{B}$. The intersection $\mathbb{B}_1 \cap B_2$ is a finite intersection of elements of $\mathcal{S}$, and hence also belongs in $\mathcal{B}$, and hence we can take another basis element B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$. ■

1.3 The Order Topology.

Definition. Let X be a set with a simple order relation, and suppose that $|X| > 1$. Let $\mathcal{B}$ be the collection of sets of the following forms:

(1) All open intervalcs $(a,b) \in X$.

(2) All half open intervals $[a_0, b)$ where a_0 is the least element (if any) of X.

(3) All half open intervals of the form $(a, b_0]$ where b_0 is the greatest element (if any) of X.

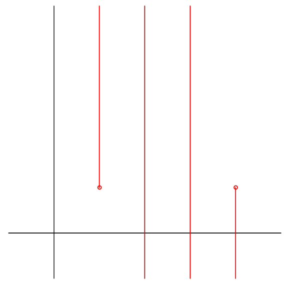

Figure 1.3: The order topology on $\mathbb{R} \times \mathbb{R}$.

Then $\mathcal{B}$ forms the basis for a topology on X called the **order topology**

Theorem 1.3.1. *The collection $\mathcal{B}$ forms a basis.*

Proof. Consider $x \in X$, if x is the least element of X, then it liess in all intervals of type (2), if it is the largest, then it lies in all intervals of type (3). If x is neither the least nor largest element, then $x \in (a_0, b_0)$ with a_0 and b_0 the least and largest elements (if any) of X. If no such elements exist, then $x \in (a, b)$, for some lowerbound a and upperbound b. Thus, in all three cases, there is a basis element containing x.

Now suppose $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$. If B_1 and B_2 are both of type (1), then let $B_1 = (a, b)$, $B_2 = (c, d)$, then $B_1 \cap B_2$ is an open interval of type (1), now fix B_1 to be of type one. If B_2 is of type (2), then letting $B_2 = [a_0, c)$, then $x \in [a_0, d)$ for some $d \in X$. Likewise, if $B_2 = (c, b_0]$, is of type (3), we get a similar result. Moreover, the results are analogous if we fix B_2 and let B_1 range between intervals of the three types. Thusm in all cases, there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. ■

Example 1.4. (1) The standard topology on $\mathbb{R}$ is the order topology on $\mathbb{R}$ induced by the usual order relation. We have that $\mathbb{R}$ under this topology has no intervals of type (2), nor (3), so all bases elements in the standard topology are open intervals in $\mathbb{R}$.

(2) Consider the dictionary order on $\mathbb{R} \times \mathbb{R}$. Since $\mathbb{R} \times \mathbb{R}$ has no intervals of type (2), nor (3), the bases of $\mathbb{R} \times \mathbb{R}$ under the dictionary order are the open intervals of the form $(a \times b, c \times d)$ Where $a \leq c$, and $b < d$.

(3) The positive integers $\mathbb{Z}^+$ with the least element 1 form an ordered set under the usual order. Taking $n > 1$, we see the bases of $\mathbb{Z}^+$ under the order topology are of the form $(n-1, n+1) = \{n\}$ and $[1, n) = \{1, \ldots, n-1\}$. Thus the order topology on $\mathbb{Z}^+$ is the discrete topology.

(4) The set $X = \{1, 2\} \times \mathbb{Z}^+$ over the dictionary order is also an ordered set, with the least elelment 1×1. Denote $1 \times n$ as a_n and $2 \times n$ as b_n. Then X consist of the elements $a_1, a_2, \ldots, b_1, b_2, \ldots$.

Now take $\{b_1\}$, then any open set containing b_1 must have a basis about b_1, and also contains points a_i with $i \in \mathbb{Z}^+$; thus the order topology on X is not the discrete topology.

Definition. Let X be an ordered set, and let $a \in X$. There are two subsets in X, $(a, \infty) = \{x \in X : x > a\}$ and $(-\infty, a) = \{x \in X : x < a\}$ called **open rays** of X. There are also two sets $[a, \infty) = \{x \in X : x \geq a\}$ and $(-\infty, a] = \{x \in X : x \leq a\}$ called **closed rays** of X.

Theorem 1.3.2. *Let X be an ordered set. Then the collection of all open rays in X form a subbasis for the order topology on X.*

Proof. Let $\mathcal{S}$ be the collection of all open rays of X, let (a, ∞) and $(-\infty, b) \in \mathcal{S}$, then $(a, b) = (a, \infty) \cap (-\infty, b)$. Now take:

$$S = \bigcup_{a,b \in X} (a, b)$$

then $S \subseteq X$, likewise, since S runs through all intersections of open rays of X, it contains all open intervals in X, hence $X \subseteq S$, and so $X = S$ as required. ∎

1.4 The Product Topology.

Definition. Let X and Y be topological spaces. We define the **product topology** on $X \times Y$ to be the topology having as basis the collection $\mathcal{B} = \{U \times V \subseteq X \times Y : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$

Theorem 1.4.1. *The collection $\mathcal{B} = \{U \times V \subseteq X \times Y : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$ forms a basis for the product topology on $X \times Y$.*

Proof. Clearly, we have that $X \times Y$ is a basis element of $\mathcal{B}$. Now take $U_1 \times V_1$ and $U_2 \times V_2$ in $\mathcal{B}$. Since $U_1 \times V_1 \cap U_2 \times V_2 = U_1 \cap U_2 \times V_1 \cap V_2$, since $U_1 \cap U_2$ and $V_1 \cap V_2$ are open in X and Y respectively, then we have that $U_1 \times V_1 \cap U_2 \times V_2$ is a basis element as well. ∎

Theorem 1.4.2. *If $\mathcal{B}$ is the basis for a topology on X, and $\mathcal{C}$ is the basis for a topology on Y, then the collection:*

$$\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

Is a basis for the topology on $X \times Y$.

Proof. By lemma 1.2.3, let W be an open set of $X \times Y$, and let $x \times y \in W$. Then there is a basis $U \times V$ such that $x \times y \in U \times V \subseteq W$. Since $\mathcal{B}$ and $\mathcal{C}$ are bases of X and Y respectively, choosing $B \in \mathcal{B}$ and $C \in \mathcal{C}$, we have that $x \in B \subseteq U$, and $y \in C \subseteq Y$, thus $x \times y \in B \times C \subseteq U \times V \subseteq W$. Therefore, $\mathcal{D}$ is the basis for a topology on $X \times Y$. ∎

Example 1.5. The product of the standard topology on $\mathbb{R}$ with itself is called the **standard topology on** $\mathbb{R} \times \mathbb{R}$, and has as basis the collection of all products of open sets in $\mathbb{R}$. By theorem 1.4.2, if we take the collection of all open intervals $(a, b) \times (a, b)$ in $\mathbb{R} \times \mathbb{R}$, we form a basis. Constructing this basis geometrically gives the interior of a rectangle, whose boundaries are the intervals (a, b) and (ac, d).

Definition. Let $\pi_1 : X \times Y \to X$ be defined such that $\pi_1(x, y) = x$, and define $\pi_2 : X \times Y \to Y$ such that $\pi_2(x, y) = y$. We call π_1 and π_2 **projections** of $X \times Y$ onto its first and secnd **factors**; that is onto X and Y, respectively.

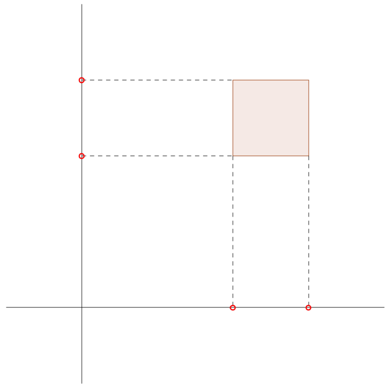

Figure 1.4: A basis element for $\mathbb{R} \times \mathbb{R}$

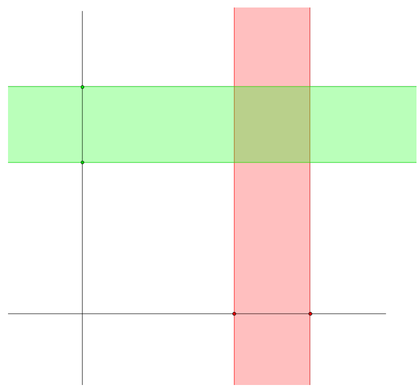

Figure 1.5: The inverse images, $\pi_1^{-1}(U)$ and $\pi_2^{-1}(V)$, of the projections π_1 and π_2 onto the $X \times Y$ plane.

Clearly, π_1 and π_2 are both onto. Now let U be open in X, then $\pi_1^{-1}(U) = U \times Y$ is open in $X \times Y$; similarly, $\pi_2^{-1}(V) = X \times V$ is also open in $X \times Y$, for V open in Y.

Theorem 1.4.3. *The collection* $\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ is open in } Y\}$ *is a subbasis for the product topology on* X.

Proof. Let $\mathcal{T}$ be the product topology on $X \times Y$, and let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$. Since every element of $\mathcal{S}$ is open in $\mathcal{T}$, $\mathcal{T} \subseteq \mathcal{T}'$. Conversely, consider the basis element $U \times V$ of $\mathcal{T}$, then $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times Y \cap X \times V = U \times V$, thus $\mathcal{T} \subseteq \mathcal{T}'$. Therefore, $\mathcal{S}$ is a subbasis for the product topology. ■

1.5 The Subspace Topology.

Theorem 1.5.1. *Let* X *he a topological space with topology* $\mathcal{T}$, *and let* $Y \subseteq X$. *Then the collection:*

$$\mathcal{T}_y = \{Y \cap U : U \in \mathcal{T}\}$$

forms a topology on Y.

Proof. Cleary, $Y \cap \emptyset = \emptyset \in \mathcal{T}_Y$ and $Y \cap X = Y \in \mathcal{T}_Y$. Now consider the collection $\{U_{alpha}\}$. Then $\bigcup Y \cap U_\alpha = Y \cap \bigcup U_\alpha$, similarly, for $\{U_i\}_{i=1}^n$, $\bigcap Y \cap U_i = Y \cap \bigcap U_i$, hence $\mathcal{T}$ is a topology on Y. ■

Definition. Let X be a topological space, and let $Y \subseteq X$. We call the $\mathcal{T}$ defined in theorem 1.5.1 the **subspace topology** on Y. We say that $U \subseteq Y$ is **open in** Y if $U \in \mathcal{T}_Y$.

Lemma 1.5.2. *Let $\mathcal{B}$ be the basis for a topology on X. Then the collection $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$, where $Y \subseteq X$, is a basis for the subspace topology on Y.*

Proof. Let U be open in X, and let $y \in Y \cap U$, and choose $B \in \mathcal{B}$ such that $y \in B \subseteq U$, then $y \in B \cap Y \subseteq U \cap Y$, then by lemma 1.2.2, $\mathcal{B}_y$ is the basis fpr the subspace topology on Y. ■

Lemma 1.5.3. *Let Y be a subspace of X, If $U \subseteq Y$ is open in Y, then U is open in X.*

Proof. The proof is rather trivial, however, it is worth going through the motions. Let $U \in \mathcal{T}_Y$, then for some $V \subseteq X$, $U = Y \cap V$. Now since Y is open in X, and so is V, then it follows that U is also open in X. ■

Remark. What this lemma says is that given a topological space X, and a subspace Y of X, then the subspace topology of Y is courser than the topology on X, i.e. $\mathcal{T}_Y \subseteq \mathcal{T}$.

Theorem 1.5.4. *If A is a subspace of X, and B is a subspace of Y, then the product topology on $A \times B$ is the topology that $A \times B$ inherits as a subspace of $X \times Y$.*

Proof. We have that $U \times V$ is the basis element for $X \times Y$, with U open in X, and V open in Y. Thus $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$ is a basis element for the subspace topology on $X \times Y$. Since $U \cap A$ and $V \cap B$ are open in the subspace topologies of A and B respectively, then $(U \cap A) \times (v \cap B)$ is a basis for the product topology on $A \times B$. ■

Example 1.6. (1) Consider $[0,1] \subseteq \mathbb{R}$. In the subspace topology of $[0,1]$, we have as basis elements of the form $(a,b) \cap [0,1]$, with $(a,b) \subseteq \mathbb{R}$. If we have that $(a,b) \subseteq [0,1]$, then $(a,b) \cap [0,1] = (a,b)$. On the other hand, if $a \in [0,1]$ or $b \in [0,1]$, then we get $(a,b) \cap [0,1] = (a,1]$ or $(a,b) \cap [0,1] = [0,b)$, lastly if neither a nor b are in $[0,1]$, then we have $(a,b) \cap [0,1] = [0,1]$ only if $[0,1] \subseteq (a,b)$, and $(a,b) \cap [0,1] = \emptyset$ otherwise.

Now each of these sets are open in $\mathbb{R}$, under the standard topology, except for $(a,1]$ and $[0,b)$.

(2) For $[0,1) \cup \{2\} \subseteq \mathbb{R}$, the singletomn $\{2\}$ is open in the subspace topology on $[0,1) \cup \{2\}$; for observe, that $(\frac{3}{5}, \frac{5}{2}) \cap ([0,1) \cup \{2\}) = \{2\}$, however, in the order topology, on that same set, $\{2\}$ is not open. Any basis element on $[0,1) \cup \{2\}$ containing 2 is of the form $(a,2]$, where $a \in [0,1) \cup \{2\}$.

(3) The dictionary order on $[0,1] \times [0,1]$ is a restriction of the dictionary order on $\mathbb{R} \times \mathbb{R}$. Now the set $\{\frac{1}{2}\} \times (\frac{1}{2}, 1]$ is open in the subspace topology on $[0,1] \times [0,1]$, but it is not open in the dictionary order on the same set.

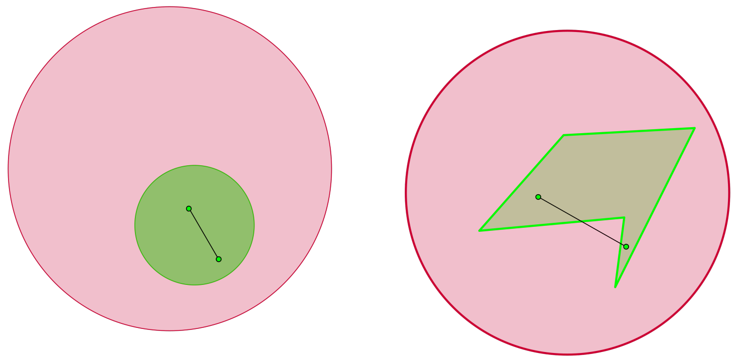

Figure 1.6: A convex set, and a con convex set.

Definition. We call the set $[0,1] \times [0,1]$ on the dictionary odere the **ordered square**, and we denote it by I_0^2.

Definition. Let X be an ordered set. We say that a nonempty subset $Y \subset X$ is **convex** in X if for each pair of points $a, b \in Y$, with $a < b$, then the open interval $(a,b) \subseteq X$ is also contained in Y.

Example 1.7. Let X be any ordered set. Then by definition, all open intervals and rays in X are convex in X.

Theorem 1.5.5. *Let X be an ordered set on the order toplogy, and let $Y \subseteq X$ be convex in X. Then the order topology on Y is the same as the subspace topology on Y.*

Proof. Consider $(a,) \subseteq X$. If $a \in Y$, then $(a,\infty) \cap Y = \{x \in Y : x > a\}$, which is by definition an open ray on Y. Now if $a \notin Y$, then a is either a lowerbound, or an upperbound. Then $(a,\infty) \cap Y = \emptyset$ and $(-\infty, a) \cap Y = Y$ if a is an upperbound, similarly, if a is a lowerbound we get $(a,\infty) \cap Y = Y$ and $(-\infty, a) \cap Y = \emptyset$.

Since $(a,\infty)Y$ and $(-\infty, a) \cap Y$ form a subbasis on the subspace topology on Y, and since they are also open in the order topology, then the order topology contains the subspace topology.

Now if (a,∞) is an open ray in Y, then $(a,\infty) = (b,\infty) \cap Y$, with (b,∞) some open ray in X, hence (a,∞) is open in the subspace topology of Y, and since it also forms the subbasis for the order topology, we have that the ordertopology is contained within the subspace topology. Thus both topologies are equal. ■

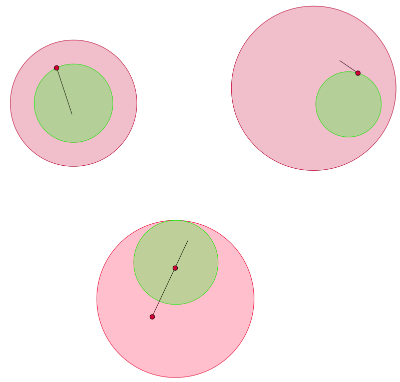

Figure 1.7: An illustration of theorem 1.5.5.

1.6 Closed Sets and Limit Points.

Definition. A subset A of a topological space X is said to be **closed** if $X\backslash A$ is open.

Example 1.8. (1) Consider $[a,b]\subseteq\mathbb{R}$, we have that $\mathbb{R}\backslash[a,b]=(-\infty,a)\cup(b,\infty)$ which is open in $\mathbb{R}$. So $[a,b]$ is closed.

(2) In $\mathbb{R}\times\mathbb{R}$, the set $A=\{x\times y : x,y\geq 0\}$ (i.e the first quadtant of the plane) is closed, for $\mathbb{R}\times\mathbb{R}\backslash A=(-\infty,0)\times\mathbb{R}\cup\mathbb{R}\times(-\infty,0)$, which is open in $\mathbb{R}\times\mathbb{R}$.

(3) Consider the finite complement topology $\mathcal{T}_C$ on a set X. We have that $X\backslash X=\emptyset\in\mathcal{T}$, so X is closed, similarly, $\emptyset$ is also closed. Likewise, if $A\subseteq X$ is a finite set, then $X\backslash A$ is also finite, and hence A is also closed. Thus, we have that all the closed sets of $\mathcal{T}_C$ are those finite subsets of X. As a consequence, this examle also illustrates that sets can be both closed and open.

(4) In the discrete topology 2^X, every open set is closed. This is another example where open sets are also closed sets.

(5) Consider $[0,1]\cup(2,3)$ in the subspace topology on $\mathbb{R}$. We have that $[0,1]$ is open ($[0,1]=[0,1]\cup(2,3)\cap(-\frac{2}{3}),\frac{3}{2})$, similarly, $(2,3)$ is also open. Now taking $[0,1]\cup(2,3)\backslash(2,3)=[0,1]$, which is open, so $[0,1]$ is closed in the subspace topology on $\mathbb{R}$, bu the same reasoning, so is $(2,3)$.

Theorem 1.6.1. *Let X be a topological space. Then:*

(1) $\emptyset$ and X are closed.

(2) Arbitrary intersections of closed sets are closed.

(3) Finite unions of closed sets are closed.

Proof. We have that $X\backslash\emptyset=X$ and $X\backslash X=\emptyset$, both of which are open in X, so they are also closed in X. Now let $\{U_\alpha\}$ be a collection of closed sets of X. We have that:

$$X\backslash\bigcap_\alpha U_\alpha=\bigcup_\alpha X\backslash U_\alpha.$$

Similarly, for $\{U_i\}_{i=1}^n$, we have

$$X \backslash \bigcup_{i=1}^{n} U_i = \bigcap_{i=1}^{n} X \backslash U_i.$$

Both of which are open in X. This completes the proof. ■

Definition. If Y is a subspace of X, we say that A is **closed in** Y if $A \subseteq Y$ and A is closed in the subspace topology of Y.

Theorem 1.6.2. *Let Y be a subspace of X. Then A is closed in Y if and only if A equals the intersection of a closed set of X with Y.*

Proof. Suppose that A is closed in Y, then $Y \backslash A$ is open in Y, hence we have that $Y \backslash A = U \cap Y$ for some open set U of X. Now $X \backslash U$ is closed in X, and with $A \subseteq Y$, we have that $A = Y \cap X \backslash U$.

Conversely, suppose that $A = C \cap Y$, with C closed in X. Then $X \backslash C$ is open in X, hence $X \backslash C \cap Y$ is open in Y, now since $X \backslash C \cap Y = Y \backslash A$, which is open, we have that A is closed in Y. ■

Theorem 1.6.3. *Let Y be a subspace of X. If A is closed in Y, and Y is closed in X, then A is closed in X; that is, closure is transitive.*

Proof. By theorem 1.6.2, if A is closed in Y, then $A = C \cap Y$ with C closed in X, now since Y is closed in X, then $Y = D \cap X$ with D closed in X. Thus $A = (C \cap D) \cap X$, therefore, A is closed in X. ■

We now go over the concepts of the closure, and the interior of a set.

Definition. Let $A \subseteq X$, with X a topological space. The **interior** of A is defined to be the union of all open sets in A. The **closure** of A is defined to be the intersection of all closed sets containing A. We denote the interior and the closure of A as $\operatorname{Int} A$ and $\overline{A}$ respectively

We have by the very definitions that $\operatorname{Int} A \subseteq A \subseteq \overline{A}$

Lemma 1.6.4. $\operatorname{Int} A = A$ *only when A is open, and $\overline{A} = A$ only when A is closed.*

Proof. Now, if A is open, then it is in the union of all open sets of A, hence $A \subseteq \operatorname{Int} A$, likewise, if A is closed, then since $\overline{A}$ is the intersection of all closed sets containing A, we get $\overline{A} \subseteq A$. ■

Corollary. *A is closed and open if and only if* $\operatorname{Int} A = \overline{A}$.

Theorem 1.6.5. *Let Y be a subspace of X, and let $A \subseteq Y$, and let $\overline{A}$ be the closure of A. Then $\overline{A} \cap Y$ is the closure of A in Y.*

Proof. Let $\hat{A}$ be the closure of A in Y. Since $\overline{A}$ is closed in X, by theorem 1.6.2, $\overline{A} \cap Y$ is closed in Y, now we have that $A \subseteq \overline{A} \cap Y$, and since $\hat{A} = \bigcap U$, then $\hat{A} \subseteq \overline{A} \cap Y$.

Conversely, suppose that $\hat{A}$ is closed in Y, again by theorem 1.6.2, we have that $\hat{A} = C \cap Y$, where C is closed in X, since $A \subseteq \hat{A}$, then $A \subseteq C$, and since C is closed, then $\overline{A} \subseteq C$, thus $\overline{A} \cap Y \subseteq \hat{A}$. ■

Definition. Let X be a topological space, and let $x \in X$. We call an open set U of X a **neighborhood** of x if $x \in U$.

Theorem 1.6.6. *If $A \subseteq X$, with X a topological space, then $\overline{A}$ is a neighborhood of $x \in X$ if and only if for every neighborhood U of x, $A \cap U \neq \emptyset$.*

Proof. We prove the contrapositve. If $x \notin \overline{A}$, then $U = X \backslash \overline{A}$ is an open set containing A, disjoint from A. Conversely, suppose there is a neighborhood U of x, with U disjoint from A, then $X \backslash U$ is closed, and therefore contains the closure of A, thus $x \notin \overline{A}$ ■

Corollary. *$\overline{A}$ is a neighborhood of x if and only if for every basis element B of X, containing x, intersects A. endcorollary*

Proof. This is a direct application of theorem 1.6.6, since basis elements are open sets. ■

Example 1.9. (1) We have the closure of $(0,1]$ in $\mathbb{R}$ is the closed interval $[0,1]$, since every neighborhood of 0 intersects $(0,1]$. Now every point outside of $[0,1]$ has a neighborhood disjoint from $[0,1]$ (take the neighborhood $(2,3)$ of 2).

(2) $\overline{\frac{1}{\mathbb{Z}^+}} = \{0\} \cup \frac{1}{\mathbb{Z}^+}$ and $\overline{\{0\} \cup (1,2))} = \{0\} \cup [1,2]$.

(3) $\overline{\mathbb{Q}} = \mathbb{R}$, $\overline{\mathbb{Z}^+} = \mathbb{Z}^+$, $\overline{\mathbb{R}^+} = \mathbb{R}^+ \cup \{0\}$. This first follows from the density of $\mathbb{Q}$ in $\mathbb{R}$. Every neighborhood $n \in \mathbb{Z}^+$ intersects $\mathbb{Z}^+$, so $\overline{Z^+} \subseteq \mathbb{Z}^+$, and we have that the neighborhood $(0,1)$ of 0 intersects $\mathbb{R}^+$, so $\overline{\mathbb{R}^+} \subseteq \mathbb{R}^+ \cup \{0\}$.

Definition. If $A \subseteq X$, with X a topological space, and if $x \in X$, we say that x is a **limit point** of A if every neighborhood of x intersects A at some distinct point. That is: $x \in \overline{X \backslash \{x\}}$.

Example 1.10. (1) Consider $(0,1]$, we have that $0 \in [0,1] = \overline{(0,1]} =$

$\{0\}, so 0 is a limit point of$ (0,1]$, the same can be said for any$ x $\in (0,1]$.

(2) For $\frac{1}{\mathbb{Z}^+}$, 0 is once again a limit point. Let $x \in \mathbb{R}$ be nonzero, and let $[x,b)$ be the neighborhood of x in the lower limit topology. Then $[x,b) \cap \frac{1}{\mathbb{Z}^+} = \emptyset$ or $\{x\}$, hence, 0 is the only limit point of $\frac{1}{\mathbb{Z}^+}$.

(3) $\overline{\{0\} \cup (1,2)} = \{0\} \cup [1,2]$ has all of its limit points in $[1,2]$. Likewise, every point in $\mathbb{R}$ is a limit point of $\mathbb{Q}$. $\mathbb{Z}^+$ has no limit points in $\mathbb{R}$, and the limit points of $\mathbb{R}^+$ are all the points of $\overline{\mathbb{R}^+}$.

Theorem 1.6.7. *Let $A \subseteq X$, X a topological space, and let A' be the set of all limit points in A. Then $\overline{A} = A \cup A'$.*

Proof. Let $x \in A'$, then every neighborhood of x intersects A at some distinct point x', by definition, so by theorem 1.6.6, $x \in \overline{A}$, hence $A' \subseteq \overline{A}$, so $A \cup A' \subseteq \overline{A}$. Now, let $x \in \overline{A}$. If $x \in A$, we are done. Otherwise, since every neighborhood of x intersects A, we have that they intersect at distinct points, thus $x \in A'$, therefore $\overline{A} \subseteq A \cup A'$. ■

Corollary. *$A \subseteq X$ is closed if and only if $A' \subseteq A$.*

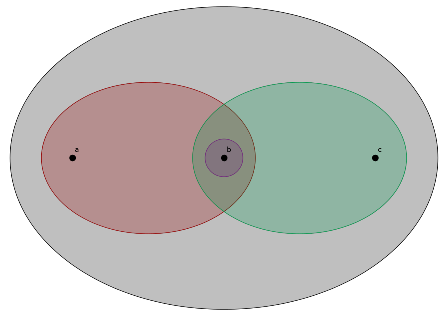

Figure 1.8: A topology on $\{a, b, c\}$, which turns out to be a Hausdorff space.

Proof. If A is closed, then $\overline{A} = A = A \cup A'$, thus $A' \subseteq A$. The converse is obvious. ■

Definition. Let X be a topological space. A sequence $\{x_n\}$ is said to **converge** to a point $x \in X$ if for every neighborhood U of x, there is an $N \in \mathbb{Z}^+$ such that $x_n \in U$ for all $n \geq N$.

Example 1.11. Consider the following topological space on $\{a, b, c\}$ in figure 1.8, and define the sequence $\{x_n\}$ by $x_n = b$ for all $n \in \mathbb{Z}^+$. The neighborhoods of a, b, and c are $U_a = \{a, b\}$, $U_b = \{b\}$, and $U_c = \{b, c\}$. Now let $N > 0$, then we see that for all $n \geq N$, that $b \in U_b, U_a, U_c$, thus b converges to a and to c, and itself,

Definition. A topological space X is called a **Hausdorff space** if for each pair of distinct points x_1 , and x_2, there are neighborhoods U_1 and U_2 of x_1 and x_2 respectively such that U_1 and u_2 are disjoint.

Example 1.12. The topology of the previous example in figure **??** is not a Hausdorff space.

Theorem 1.6.8. *Every finite point set in a Hausdorff space is closed.*

Proof. Let X be a Hausdorff space, and let $x_0 \in X$. We have that $\overline{\{x_0\}} = \bigcap_{\{x_0\} \in U} U$. Now let $x \neq x_0 \in X$. Since $x \in \{x_0\}$, and X is Hausdorff, the inters of the neighborhoods of x and x_0 is empty, thus $x \notin \overline{\{x_0\}}$, therefore $\overline{\{x_0\}} = \{x_0\}$. ■

Remark. We can extend this proof to finite point sets of size n by induction.

Now the condition that finite point sets be closed need not depend on whether or not X is a Hausdorff space. In fact, we can assume the following for some topoltopological spaces.

Axiom 1.6.1 (The T_1 Axiom)**.** *In any topological space, every finite point set of X is closed.*

Theorem 1.6.9. *Let X be a topological space satisfying the T_1 axiom, and let AX. Then a point x is a limit point of A if and only if every neighborhood of x contains infinitely manu points of A.*

Proof. Let U_x be a neighborhood of x. If U_x intersects A at infinitely many points of A, then it intersects A at a point distinct from x, thus x is a limit point of A.

Conversely suppose that x is a limit point of A, and let $U_x \cap A$ be finite, then $U_x \cap A \backslash \{x\}$. Now let $U_x \cap A \backslash \{x\} = \{x_1, \dots x_m\}$. By the T_1 axiom, $\{x_1, \dots, x_m\}$ is closed, so $X \backslash \{x_1, \dots, x_m\}$ is open, thus $U_x \cap X \backslash \{x_1, \dots, x_m\}$ is a neighborhood of x that does not intersect $A \backslash \{x\}$, which contradicts that x is a limit point. ∎

Theorem 1.6.10. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point in X.*

Proof. Let $\{x_n\}$ be a sequence of points converging to x, and let $y \neq x$ and let U_x and U_y be neighborhoods of x and y respectively. Then $U_x \cap U_y = \emptyset$. Now since $\{x_n\}$ converges to x, we have that for $N > 0$, $x_n \in U_x$ whenever $n \geq N$. Then $x_n \notin U_y$, and so $\{x_n\}$ cannot converge to y. ∎

Definition. Let $\{x_n\}$ be a sequence in a Hausdorff space X. If $\{x_n\}$ converges to a point $x \in X$, we call x the **limit** of $\{x_n\}$ and we write $\lim x_n = x$ or $\{x_n\} \to x$.

Theorem 1.6.11. *The following are true:*

(1) Every simply oredered set under the order topology is Hausdorff.

(2) The product of two Hausdorff spaces is Hausdorff.

(3) The subspace of a Hausdorff space is Hausdorff.

Proof. (1) Let X be an ordered set under the order topology. Take $x, y \in X$ distinct, and suppose without loss of generality that $x < y$. Then consider the neighborhoods $(-\infty, x]$ and $[y, \infty)$ of x and y respectively. Then $(-\infty, x] \cap [y, \infty) = \emptyset$.

(2) Let X and Y be Hausdorff, and consider $X \times Y$ in the product topology. Let $x_1 \times y_1$ and $x_2 \times y_2$ be distinct points, and let U_{x_1}, U_{x_2}, V_{y_1} and V_{y_2} be basis elements of x_1, x_2, y_1, and y_2 respectively. Then they are neighborhoods of those elements respectively.

Now we have that $U_{x_1} \times V_{y_1}$ and $U_{x_2} \times V_{y_2}$ are basis elements of $x_1 \times y_1$ and $x_2 \times y_2$, respectively, and hence neighborhoods of those elements respectively.Then we have $(U_{x_1} \times V_{y_1}) \cap (U_{x_2} \times V_{y_2}) = (U_{x_1} \cap U_{x_2}) \times (V_{y_1} \cap V_{y_2}) = \emptyset \times \emptyset = \emptyset)$.

(3) Let X be Hausdorff, and let Y be a subspace of X. Let x_1 and x_2 be distinct points, and let U_{x_1} and U_{x_2} be their neighborhoods. Since Y is open in X, then so are $Y \cap U_{x_1}$ and $Y \cap U_{x_2}$, so they are also neighborhoods of x_1 and x_2 respectively. Then $Y \cap U_{x_1} \cap Y \cap U_{x_2} = Y \cap (U_{x_1} \cap U_{x_2}) = \emptyset$.

∎

1.7 Continuous Functions.

Definition. Let X and Y be topological spaces. We say that a mapping $f : X \to Y$ is **continuous** if for each open set V in Y, $f^{-1}(V)$ is open in X.

Now if $f : X \to Y$ is continuous, the for every open set V of Y, $f^{-1}(V)$ is open in X. Now suppose that $\mathcal{B}$ is a basis of Y, then $V = B_\alpha$, hence $f^{-1}(B_\alpha) = f^{-1}B_\alpha$, which is open in X, thus B_α must also be open in X.

Similarly, if $\mathcal{S}$ is a subbasis of Y, then for any basis element B of Y, $B = \bigcap_{i=1}^n S_i$, which then implies that $f^{-1}(B) = \bigcap_{i=1}^n f^{-1}(S_i)$, thus S_i is also open in X for $1 \leq i \leq n$.

Example 1.13. (1) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous realvalued function. THen for each open interval $I \subseteq \mathbb{R}$, $f^{-1}(I)$ is an open interval in $\mathbb{R}$, so take $x_0 \in \mathbb{R}$ and $\epsilon > 0$, and let $I = (f(x_0) - \epsilon, f(x_0) + \epsilon)$, then since $x_0 \in f^{-1}(I)$, there is a basis $(a, b) \subseteq f^{-1}(I)$ about x_0. Then take $\delta = \min\{x_0 - a, x_0 - b\}$, then $x \in (a, b)$ whenever $0 < |x - x_0| < \delta$, and we get that $f(x) \in I$, that is, $|f(x) - f(x_0)| < \epsilon$. This is the definition of continuity defined in the real analysis. We can prove that the converse holds also.

If $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point x_0, then for every $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $0 < |x - x_0| < \delta$. Then we notice that x and x_0 are distinct, furthermore, $x_0 - \delta < x < x_0 + \delta$, hence $x \in (x_0 - \delta, x_0 + \delta)$ implies that $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$. Letting $V_\delta(x_0) = (x_0 - \delta, x_0 + \delta)$ and $V_\epsilon(f(x_0)) = (f(x_0) - \epsilon, f(x_0) + \epsilon)$, we have that whenever $x \in V_\delta(x_0)$, then $f(x) \in V_\epsilon(f(x_0)) \subseteq f^{-1}(V_\delta(x_0))$. And so the topological definition of continuity is equivialent to the real analytic definition of continuity.

(2) Let $f : \mathbb{R} \to \mathbb{R}_l$ be defined such that $f(x) = x$ for all $x \in \mathbb{R}$. Take $[a, b) \subseteq \mathbb{R}_l$, we have that $f^{-1}([a, b)) = [a, b)$, which is not open in $\mathbb{R}$ (under the standard topology), hence f is not continuous. However, the map $g : \mathbb{R}_l \to \mathbb{R}$ defined the same way is continuous since $g^{-1}((a, b))$ is open in $\mathbb{R}_l$.

Theorem 1.7.1. *Let X and Y be topological spaces, and let $f : X \to Y$ be a mapping of X into Y. Then the following are equivalent:*

(1) f is continuous.

(2) For every $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

(3) For every closed set $B \subseteq Y$, $f^{-1}(B)$ is closed in X.

(4) For each $x \in X$ and each neighborhood V of $f(x)$, tgere us a neighborhood U of x such that $f(U) \subseteq V$.

Proof. Let f be continuous and let $A \subseteq X$. Consider the neighborhood V of $f(x)$, then $f^{-1}(V)$ is open in X, and intersects A at a point y. Then $V \cap f(A) = f(y)$, thus $f(x) \in \overline{f(A)}$.

Now let B be closed in Y, and let $A = f^{-1}(B)$. THen we have that $F(A) = f(f^{-1(B)}) \subseteq B$, thus $x \in \overline{A}$.

Now let V be open in Y, so that $B = Y \backslash V$ is closed in Y, and $f^{-1}(B) = f^{-1}(Y) \backslash f^{-1}(V) = X \backslash f^-1(V)$ which is closed in X, hence $f^{-1}(V)$ is open in X.

Now let $x \in X$, and let V be a neighborhood of $f(x)$. Then $U = f^{-1}(V)$ is a neighborhood of x for which $f(U) \subseteq V$. Finally let V be open in Y, and let $x \in f^{-1}(V)$, then $f(x) \in V$, so there is a neighborhood U_x of x for which $f(U_x) \subseteq V$, then $U_x \subseteq f^{-1}(V)$, then $f^{-1}(V)$ is a union of open sets, and hence open in X. ∎

Definition. Let X and Y be topological spaces, and $f : X \to Y$ be a $1-1$ mapping of X onto Y. We call f a **homeomorphism** if both f and f^{-1} are continuous.

Lemma 1.7.2. *Let X and Y be topological spaces and let $f : X \to Y$ be a homeomorphism. Then $F(U)$ is open if and only if U is open.*

Proof. We have that both $f : X \to Y$ and $f^{-1} : Y \to X$ are contiuous $1-1$ of X and Y onto each other (respectively). Now let U be open in X, then $U = f^{-1}(V)$, for some set V open in Y. Notice then, that $f(U) = f(f^{-1}(V)) = V$, thus $f(U)$ is open in Y. Conversely, let $V = f(U)$ be open in Y for some open set U in X, then $U = f^{-1}(V)$, so by definition of continuity, U is open in X. ■

Definition. Let X and Y be topological spaces and let $f : X \to Y$ be a contiuous $1-1$ maping of X into Y, and consider $f(X)$ as a subspace of Y. We call $f : X \to f(X)$ a **topological imbedding** if f is a homeomorphism of X onto $f(X)$.

Example 1.14. (1) The map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 1$ is a homeomorphism whose inverse is $f^{-1}(y) - \frac{1}{3}(y - 3)$, both f and f^{-1} are contiuous.

(2) The map $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) = \frac{x^2}{1-x^2}$ has as its inverse the map $f : \mathbb{R} \to (-1, 1)$ defined by $f^{-1}(y) = \frac{2y}{1+\sqrt{1+4y^2}}$. Both f and f^{-1} are contiuous, so f is a homeomorphism.

(3) Teh map $g : \mathbb{R}_l \to \mathbb{R}$ defined by $g(x) = x$ is not a homeomorphism, despite being contiuous, as $g^{-1}(1)$ is undefined.

(4) Let S^1 be the unit circle in $\mathbb{R}$, which is a subspace of $\mathbb{R}$, and define $f : [0, 1) \to S^1$ by $f(t) = (\cos(t), \sin(t))$. Clearly f is $1-1$ onto S^1, and contiuous, however f^{-1} is not contiuous as $f([0, \frac{1}{4}))$ is not open in S^1 as $f(0)$ is in no open set of $\mathbb{R}^2$ such that $U \cap S^1 = f([0, 1))$.

(5) Consider the mappings $g : [0, 1) \to \mathbb{R}^2$ by $f(t) = (\cos(2t\pi), \sin(2t\pi)))$. Now g is $1-1$ and continuous, and we have that $g([0, 1)) \subseteq S^1$, however since g is not a homeomorphism, g fails to be a topological embedding.

Theorem 1.7.3 (Constructions for continuous functions.)**.** *Let X and Y be topological spaces, then:*

(1) (Constant construction) If $f : X \to Y$ maps $x \to y_0$ for all $x \in X$, then f is continuous.

(2) (Inclusion) If $A \subseteq X$ is a subspace, then the inclusion mapping $\iota : A \to Y$ is continuous.

(3) (Construction by composition) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $f \circ g : X \to Z$ is also continuous.

(4) (Domain restriction) If $f : X \to Y$ is continuous and $A \subseteq X$, then $f : A \to Y$ is continuous.

(5) *(Range restriction) if $f : X \to Y$, and $Z \subseteq Y$ such that $f(X) \subseteq Y$, then $f : X \to Z$ is continuous.*

(6) *(Range exapnsion) If $f : X \to Y$ is continuous, and $Y \subseteq Z$ is a subspace of Z, then $f : X \to Z$ is continuous.*

(7) *(Local Formulation) The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f : U_\alpha \to Y$ is continuous for all α.*

Proof. (1) Let $f(x) = y_0$ for all $x \in X$, and let V be open in Y, then $f^{-1}(V) = X$ or $\emptyset$ depending on if $y_0 \in V$ or noy. In either case, $f^{-1}(V)$ is open.

(2) If U us open in X, then $f^{-1}(U) = U \cap A$ which is open in the subspace topology of X.

(3) If U is open in Z, $g^{-1}(U)$ is open in Y, hence $f^{-1(g^{-1(U)})}$ is open in X.

(4) Notice that $f_A = \iota \circ f = f : A \to Y$ which is continuous by (2) and (3).

(5) Let $f : A \to Y$ be continuous and let $f(X) \subseteq Z \subseteq Y$. Let B be open in Z, so $B = Z \cap U$ for some U open in Y. Now by hypothesis, we have that $f^{-1}(U) \subseteq f^{-1}(B)$, hence $f^{-1}(B)$ is open in X, thus $f : XZ$ is coontinuous.

(6) Let f be as in (5), and let $Y \subseteq Z$ be a subspace of Z. Then the mapping $h : X \to Z$ defined by $h = \iota \circ f$ is continuous.

(7) Let $X = U_\alpha$ where U_α is open in X, and $f : U_\alpha \to Y$ is continuous for all α. LEt V be open in Y, then $f^{-1}(V) \cap U_\alpha = f_U^{-1}\alpha(V)$, and since f is continuous on $U_{\alpha]}$, then $f^{-1}(V) = f_U^{-1}{}_{[}\alpha](V)$ is open in X.

■

Theorem 1.7.4 (The pasting lemma)**.** *Let $X = A \cap B$ with A and B closed in X, and let $f : A \to Y$ and $g : BX$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, then we can construct a mapping $h : X \to Y$ defined by $h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$. Then h is continuous.*

Proof. Let C be closed in Y, then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Since f and g are continuous, then $f^{-1}(C)$ and $g^{-1}(C)$ are closed in A and B, respectively. Thus $h^{-1}(C)$ is closed in X. ■

Example 1.15. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = \begin{cases} x, & x \leq 0 \\ \frac{x}{2}, & x \geq 0 \end{cases}$. We have that x and $\frac{x}{2}$ are continuous on their respective domains, intersecting at 0, i.e. $x : (-\infty, 0] \to \mathbb{R}$, $\frac{x}{2} : [0, \infty \to \mathbb{R}$, and $\{0\} = (-\infty, 0] \cap [0, \infty)$. Thus h is continuous on $\mathbb{R}$.

However, $k, l : \mathbb{R} \to \mathbb{R}$ defined by $k(x) = \begin{cases} x - 2, & x \leq 0 \\ x + 2, & x \geq 0 \end{cases}$ and $l(x) = \begin{cases} x - 2, & x < 0 \\ x + 2, & x \geq 0 \end{cases}$ are not continuous. We have that their domains intersect at 0, but that $k(0) = \pm_2$, (so k isn't even a function). Likewise, $(-\infty, 0) \cap [0, \infty) = \emptyset$, which is open in $\mathbb{R}$ see 1.9.

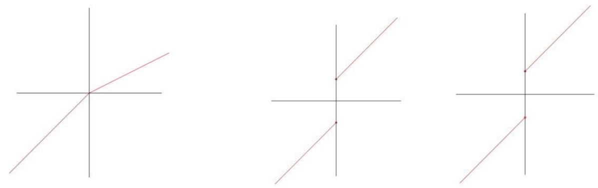

Figure 1.9: The mappings h, k, and l.

Theorem 1.7.5. *Let $f : A \to X \times Y$ be defined by $f(a) = (f_1(a), f_2(a))$, where $f_1 : A \to X$ and $f_2 : A \to Y$. Then f is continuous if and only if f_1 and f_2 are continuous.*

Proof. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be projections onto X and Y respectively. Since $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$ are both open in $X \times Y$, π_1 and π_2 are continuous. Then notice that $f_1(a) = \pi_1 \circ f(a)$ and $f_2(a) = \pi_2 \circ f(a)$, both of which are continuous.

Now suppose that f_1 and f_2 are continuous..We have that $a \in f^{-1}(U \times V)$ if and only if $f_1(a) \in U$ and $f_2(a) \in V$, then $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in A, hence so is $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$. ■

Definition. We define the **parametrized curve** of the plane $\mathbb{R}^2$ to be the continuous function $f : [a, b] \to \mathbb{R}^2$ defined by $f(t) = (x(t), y(t))$. If f is in a vector field, then wwe define $f(t) = x(t)i + y(t)j$ where $i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Example 1.16. The function $f(t) = ((\cos(t)), \sin(t))$ is a parametrization of the curve $x^2 + y^2 = 1$, i.e. the unit circle S^1.

Chapter 2

More on Topological Spaces

2.1 The Product Topology.

We now explore more about the product topology.

Definition. Let J be an indexed set, and let X be a set. We define a J**-tuple** of elements of X to be a map $x : J \to X$, where if $\alpha \in J$, then $x(\alpha) = x_\alpha$, and we call it the α**-th coordinate** of x. We write $(x_\alpha)_\alpha \in J$, or just simply (x_α)

Definition. Let $\{A_\alpha\}$ be an indexed family, and let $X = \bigcup_{\alpha \in J} A_\alpha$. We define the **cartesian product** of $\{A_\alpha\}$, $\prod_{\alpha \in J} A_\alpha$ to be the set of all J-tuples (x_α) of elements of X, where $x_\alpha \in A_\alpha$

Theorem 2.1.1. *Let $\{X_\alpha\}$ be a family of topological spaces, and consider the cartesian product $\prod X_\alpha$. Then the collection of all cartesian products $\prod U_\alpha$, where U_α is open in X_α, for all α, forms a basis for the topology on $\prod X_\alpha$.*

Proof. Clearly $\prod X_\alpha$ itself is a basis element by the first condition. Now consider $\prod U_\alpha$ and $\prod V_\alpha$, then $\prod U_\alpha \cap \prod V_\alpha = \prod U_\alpha \cap V$, which is also a basis element. ■

Definition. Let $\{X_\alpha$ be a family of topological spaces, and take as basis the collection of all products $\prod U_\alpha$ where U_α, where U_α is open in X_α. We call the topology generated by this basis the **box topology** on $\prod X_\alpha$.

Definition. Let $\pi_\beta : \prod X_\alpha \to X_\beta$ be defined by $\pi_\beta((x_\alpha)) = x_\beta$. We call this map the **projection mapping** of $\prod X_\alpha$ onto X_β

Theorem 2.1.2. *Let $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) : U_\beta$ is open in $X_\beta\}$, and let $\mathcal{S} = \bigcup \mathcal{S}_\beta$. Then $\mathcal{S}$ forms the basis for a topology on $\prod X_[\alpha]$.*

Proof. Since U_β is open in X_β, $\pi_\beta^{-1}(U_\beta)) \subseteq \prod X_\alpha$. Taking $\bigcup \mathcal{S}$, we get that $\bigcup \pi_\beta^{-1}(U_\beta)) = \prod X_\beta$ for all β. Thus $\mathcal{S}$ is a subbasis. ■

Definition. Let π_β be a projection mapping of $\prod X_\alpha$ onto $X_[\beta]$, and take as subbasis the collection of all $\pi_\beta^{-1}(U_\beta)$ where U_β is open in X_β. We call the topology generated by this subbasis the **product space topology**, or more generally the **product topology** on $\prod X_\alpha$.

Theorem 2.1.3. *The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α, and the product space topology has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α, and $U_\alpha = X_\alpha$ except only for finitely many α.*

Proof. That the box topology has as a basis all sets of the form $\prod U_\alpha$ is clear. Now consider the basis $\mathcal{B}$ that $\mathcal{S}$ generates, and let $\beta_1, \ldots \beta_n$ be a finite set of distinct indices and let U_{β_i} be open in X_{β_i}, and $U_\alpha = X_\alpha$ for all other α. Since $B \in \mathcal{B}$ is a finite intersection of elements of $\mathcal{S}$, we have that $B = \bigcap_{i=1}^{n} \text{-}\beta_i{}^{-1}(U_{\beta_i})$.

Now a point $x = (x_\alpha) \in B$ if and only if the β_i-th coordinate is in $U_{[}\beta_i]$, for $1 \leq i \leq n$, hence membership depends only on a finite number of α, thus $B = \prod U_\alpha$ where $U_\alpha = X_\alpha$ for $\alpha \neq \beta_i$ for $1 \leq i \leq n$. ∎

Corollary. *The box topology on $\prod X_\alpha$ is finer than the product topology on $\prod X_\alpha$; moreover, if $\{X_i\}_{i=1}^n$ is a finite family of topologies, then the box toplogy, and the product topology on $\prod_{i=1}^n X_i$ are equal.*

For convention, from now on when we consider the product $\prod X_\alpha$, we assume that it is under the product space topology.

Theorem 2.1.4. *Suppose the topology on X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets $\prod B_\alpha$ where $B_\alpha \in \mathcal{B}_\alpha$ for each α is a basis for the box topology on $\prod X_\alpha$.*

The same collection for a finite number of α, and where $B_\alpha = X_\alpha$ for all other α forms a basis for the product space topology on $\prod X_\alpha$.

Proof. Let $\mathcal{B}$ be the collection of all $\prod B_\alpha$, where $B_\alpha \in \mathcal{B}_\alpha$. Now each X_α is already its own basis, hence so is $\prod X_\alpha$. Now let $\prod U_\alpha$ and $\prod V_\alpha$ be basis elements. Since $\prod U_\alpha \cap \prod V_\alpha = \prod U_\alpha \cap V_\alpha$, for finite alpha, and since $\prod U_\alpha \cap \prod V_\alpha = \prod X_\alpha$ for all other α (in the case of the product space topology), we get another basis element. Hence $\mathcal{B}$ is a basis for the box topology, and, provided the necessary condidition, is also a basis for the product topology. ∎

Theorem 2.1.5. *Let A_α be a subspace of X_α. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ under both the box and product space topologies.*

Proof. Since $\prod A_\alpha \cap \prod U_\alpha = \prod A_\alpha \cap U_\alpha$, and $A_\alpha \cap U_\alpha$ is a basis element for X_α under the subspace topology, then it follows that $\prod A_\alpha \cap U_\alpha$ is a basis element for the same topology on $\prod X_\alpha$, thus $\prod A_\alpha$ is a subspace. ∎

Theorem 2.1.6. *If X_α is a Hausdorff space, then so is $\prod X_\alpha$ under both the box and product space topologies.*

Proof. Since X_α is a Hausdorff space, a sequence of points of X_α, $\{x_{\alpha_n}\}$ converges to atmost one point. Now construct a sequence $\{x_n\}$ where $x_i = x_{\alpha_i}$ and x_{α_i} is the i-th term of (x_α), we see that $\{x_{\alpha_n}\}$ is a subsequence of $\{x_n\}$, by deifinition, and hence $\{x_n\}$ must also converge at atmost one point. ∎

Example 2.1. For Euclidean space $\mathbb{R}^n$, a basis consists of all products of the form $(a_1, b_1) \times \cdots \times (a_n, b_n)$ where (a_i, b_i) is an open interval for all $1 \leq i \leq n$. Since $\mathbb{R}^n$ is a finite product space, both the box and product topologies on $\mathbb{R}^n$ are the same.

Theorem 2.1.7. *If $A_\alpha \subseteq X_\alpha$, then $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$*

Proof. Let $x = (x_\alpha) \in \prod \overline{A_\alpha}$ and let $U = \prod U_\alpha$ be a basis element (for either topology) Choosing $y_\alpha \in U_\alpha \cap A_\alpha$, for each α, let $y = (y_\alpha)$. Then $y \in U$, and $y \in \prod A_\alpha$, hence $x \in \overline{\prod A_\alpha}$.

Now suppose that $x \in \overline{\prod A_\alpha}$ (in either topology). Let V_β be an open set of X_β containing x_β. Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ (in either topology), it containts a point $y = (y_\alpha)$ of $\prod A_\alpha$. Then $y_\beta \in V_\beta \cap A_\beta, hence x \in \overline{A_\beta}$. ■

Theorem 2.1.8. *Let $f : A \to \prod X_\alpha$ be defined by $f(a) = (f_\alpha(a))$, where $f_\alpha : A \to X_\alpha$. Letting $\prod X_\alpha$ have the product space topoplogy, f is continous if and only if f_α is continous for each α.*

Proof. We know that the projection mapping π_β is continous. Now suppose that f is continuous, and notice that $f_\beta = \pi_\beta \circ f$, which makes f_β continuous for each β.

On the other hand, suppose that f_β is continuous for each β. Notice that $f_\beta^{-1} = f^{-1} \circ \pi_\beta^{-1}$, since $\pi_\beta^{-1}(U_\beta)$ is open in $\prod X_\alpha$, then so is $f^{-1} \circ \pi_\beta^{-1}(U_\beta) = f_\beta^{-1}(U_\beta)$. This makes f continuous. ■

Example 2.2. Theorem 2.1.8 holds only for the product space topology and fails in general for the box topology. Consider $\mathbb{R}^\omega$ and define the map $f : \mathbb{R} \to \mathbb{R}^\omega$ by $f(t) = (t, t, t, \dots)$. We have that $f_n(t) = t$ is continuous, which makes f continuous under the product topology. Now consider the box topology: let $B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \dots$, and supppose that $f^{-1}(B)$ were open. Then it contains some interval $(-\delta, \delta)$, about 0, thus $\pi_\beta \circ f((-\delta, \delta)) = f_\beta((-\delta, \delta)) = (-\delta, \delta) \subseteq (-\frac{1}{n}, \frac{1}{n})$, which is absurd. Thus the only implication of the theorem that holds for the box topology is that f_α is continuous only when f is continuou is continuous.

2.2 The Metric Topology

Definition. A **metric** (or **distance function**) on a set X is a map $d : X \times X \to \mathbb{R}$ satisfying the following for all $x, y, z \in X$:

(1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$.

(3) (The Triangle Inequality) $d(x, y) \leq d(x, z) + d(z, y)$.

We call $d(x, y)$ the **distance** between x and y, and given $\epsilon > 0$, we define the **ϵ-ball centered about** x to be the set $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$.

Lemma 2.2.1. *Let d be a metric on X. For $x, y \in X$, and $B_d(x, \epsilon)$ an ϵ-ball centered about x, there is a δ-ball centered about y, $B_d(y, \delta)$ such that $B_d(y, \delta) \subseteq B_d(x, \epsilon)$.*

Proof. Let $y \in B_d(x, \epsilon)$ and let $\delta = \epsilon - d(x, y)$, and take $z \in B_d(y, \delta)$, then we have that $d(y, z) < \epsilon - d(x, y)$, thus $d(x, x) \leq d(x, y) + d(y, z) < \epsilon$ which complete the proof. ■

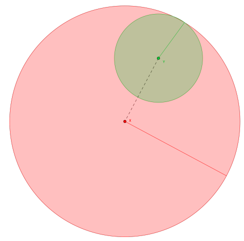

Figure 2.1: All ϵ-balls centered about x are open in the metric topology by lemma 2.2.1.

Theorem 2.2.2. *Let d be a metric on X. Then the colletion of all ϵ-balls about x, for some $x \in X$ forms the basis for a topology on X.*

Proof. Clearly $x \in B_d(x, \epsilon)$, by definition, so it remains to show that the intersection of two ϵ-balls contains an ϵ-ball. Let B_1 and B_2 be ϵ-balls about x, and let $y \in B_1 \cap B_2$. By lemma 2.2.1, there are $\delta_1, \delta_2 > 0$ such that $B_d(y, \delta_1) \subseteq B_1$ and $B_d(y, \delta_2) \subseteq B_2$. Now take $\delta = \min\{\delta_1, \delta_2\}$, then we see that $B_d(y, \delta) \subseteq B_1 \cap B_2$. ■

Definition. If d is a metric on X, we call the topology having as basis the collection of all ϵ-balls centered about x, for some $x \in X$ and $\epsilon > 0$, the **metric topology** induced by d.

Corollary. *A set U is open in the metric topology induced by d if and only if for each $y \in U$, and $\delta > 0$, there is a δ-ball centered about y contained in U.*

Example 2.3. (1) Define $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Clearly d is a metric on X, and induces the discrete topology on X. The basis $B_d(x, 1) = \{x\}$

(2) The standard metric on $\mathbb{R}$ is defined to be $d(x, y) = |x - y|$ and is a metric on $\mathbb{R}$ (that is, the absolute $|cdot|$ is a metric on $\mathbb{R}$). This metric induces the standard topology on $\mathbb{R}$ as we see that it has basis $B_d(x, \epsilon) = \{y \in \mathbb{R} : |x - y| < \epsilon\} = \{y \in \mathbb{R} : y - \epsilon < x < y + \epsilon\} = (y - \epsilon, y + \epsilon)$.

Definition. If X is a topological space, we call X **metrizable** if there is a metric d which induces the topology on X. A **metric space** is a metrizable space X together with the metric inducing the topology of X.

Definition. Let X be a metric space with metric d. A subset $A \subseteq X$ is said to be **bounded** if there is an $M > 0$ such that $d(a_1, a_2) \leq M$ for all $a_1, a_2 \in X$. We define the **diameter** of a bounded set A to be $\operatorname{diam} A = \sup\{d(a_1, a_2) : a_1, a_2 \in A\}$.

It is easy to see that boundedness of a set does not depend on the topology of X, but on the metric.

Theorem 2.2.3. *Let X be a metric space with metric d and define $\overline{d} : X \times X \to \mathbb{R}$ by $\overline{d} = \min\{d(x, y), 1\}$ for all $x, y \in X$. Then $\overline{d}$ is a metric on X that induces the same topology as d.*

Proof. Clearly we have that $0 \leq \overline{d}(x,y) \leq 1$, and that $\overline{d}(x,y) = \min\{d(x,y),1\} = \min\{d(y,x),1\} = \overline{d}(y,x)$. It remains to show the triangly inequality.

Now if $d(x,z) \leq 1$ and $d(z,y) \leq 1$, then by the triangle inequality $d(x,y) \leq 1$ and $\overline{d}(x,y) \leq \overline{d}(x,z) + \overline{d}(z,y)$. Now if $d(x,z) < 1$ and $d(z,y) < 1$, we get the same conclusion. Thus we see that $\overline{d}$ is a metric on X.

Now take as basis the collection of all ϵ-balls with $0 < \epsilon < 1$, and any basis element of x contains such an ϵ-ball, thus $\overline{d}$ induces the same topology as d. ■

Definition. We call $\overline{d}$ the **standard bounded metric** corresponding to d.

Definition. Let $x \in \mathbb{R}^n$. We define the **norm** of x, $||x||$, to be $||x|| = \sqrt{x_1^2 + \cdots + x_n^2}$. We define the **square metric** ρ on $\mathbb{R}^n$ to be $\rho(x,y) = \{|x_1 - y_1|, \ldots |x_n - y_n|\}$.

Before we show that $||\cdot||$ and ρ are metrics, we introduce the following:

Definition. Let $x, y \in \mathbb{R}^n$. We define the **inner product** of x and y to be:

$$\langle x, y\rangle = x_1y_1 + \ldots x_ny_n \tag{2.1}$$

Lemma 2.2.4. *For $x, y \in \mathbb{R}^n$, $\langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$ and $|\langle x,y\rangle| \leq ||x||||y||$.*

Proof. W have $\langle x, y+z\rangle = x_1(y_1+z_1) + \cdots + x_n(y_n+z_n) = (x_1y_1 + \cdots + x_{nyn}) + (x_1z_1 + \cdots + x_nz_n) = \langle x,y\rangle + \langle x,z\rangle$.

Now if $x = 0$ and $y = 0$, then $|\langle x,y\rangle| = ||x||||y|| = 0$, so suppose that both $x, y \neq 0$, and let $a = \frac{1}{||x||}$ and $b = \frac{1}{||y||}$. Notice that $||ax+by|| \geq 0$ and $||ax-by|| \geq 0$, then $||ax+by||^2||ax-by||^2 = ||x||^2||y^2|| - |\langle x,y\rangle|^2 \geq 0$, hence $|\langle x,y\rangle| \leq ||x||||y||$. ■

Remark. We call the last relation in the lemma the **Cauchy-Schwarz inequality**.

Theorem 2.2.5. *Both the norm and square metrics make $\mathbb{R}^n$ into a metric space.*

Proof. We start with the norm. Now clearly, since $\sqrt{x} \geq 0$ (for real numbers), $||x-y|| \geq 0$, and if $||x-y|| = 0$ then $(x_1-y_1)^2 + \ldots (x_n-y_n)^2 = 0$ hence $x_i = y_i$ for all $1 \leq i \leq n$, and if $x_i = y_i$, then clearly $||x-y|| = 0$. We also see that $||x-y|| = ||y-x||$.

Now consider $z \in \mathbb{R}^n$, notice that $||x+y||^2 = \langle x+y, x+y\rangle = \langle x+y, x\rangle + \langle x+y, y\rangle = \langle x,x\rangle + \langle y,y\rangle \leq ||x||^2 + ||y||^2$. Using this we have $||x-z|| + ||z-y|| \geq ||x-y||$ (square the left hand side and evaluate), so $||\cdot||$ is a metric on $\mathbb{R}^n$.

Now consider the square metric. Clearly we have that $\rho(x,y) \geq 0$ and that $\rho(x,y) = 0$ if and only if $x = y$ (snce $|\cdot|$ is also a metric), we also see that $\rho(x,y) = \rho(y,x)$.

Now let $x \in \mathbb{R}^n$, and we have for all $1 \leq i \leq n$ that $|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$, hence by definition $|x_i - y_i| \leq \rho(x,z) + \rho(z,y)$, thus $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$ and ρ is a metric. ■

Lemma 2.2.6. *Let d and d' be metrics on X and let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies induced by d and d' respectively. $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for each $x \in X$, and $\epsilon > 0$, there is a $\delta > 0$ such that $B_{d'}(x,\epsilon) \subseteq B_d(x,\epsilon)$.*

Proof. Suppose that $\mathcal{T} \subseteq \mathcal{T}'$, and take $B_d(x,\epsilon)$ in $\mathcal{T}$, then by lemma 2.2.1, there is a B' in $\mathcal{T}'$ or which $B' \subseteq B_d(x,\epsilon)$, hence there is a δ-ball about x for which $B_{d'}(x,\delta) \subseteq B'$.

Conversly, suppose for $x \in X$ and $\epsilon > 0$, that ther is a $\delta > 0$ for which $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$. Given a basis B of $\mathcal{T}$, ther is an ϵ-ball about x contained in B, hence $B_d(x,\epsilon)$ is also in B, thus we have that $\mathcal{T} \subseteq \mathcal{T}'$. ■

Theorem 2.2.7. *The norm and the square metric both induce the product topology on $\mathbb{R}^n$.*

Proof. Notics that $\rho(x,y) \leq ||x-y|| \leq \sqrt{n}\rho(x,y)$. This first inequality shows that $B_{||\cdot||}(x,\epsilon) \subseteq B_\rho(x,\epsilon)$, and the second shows that $B_\rho(x,\frac{\epsilon}{\sqrt{n}}) \subseteq B_{||\cdot||}(x,\epsilon)$, thus both $||\cdot||$ and ρ induce the same topology.

Now let $B = (a_1,b_1) \times \cdots \times (a_n,b_n)$ be a basis for the product topology on $\mathbb{R}^n$. Since, for each i, there is an $\epsilon_i > 0$ such that $(x_i - \epsilon_i, x_i + epsilon_i) \subseteq (a_i,b_i)$, choosing $\epsilon = \min\{\epsilon_1,\ldots,\epsilon_n\}$, we see that $B_\rho(x,\epsilon) \subseteq B$. Conversely, given $y \in B_\rho(x,\epsilon)$, notice that $B_\rho(x,\epsilon) = (x_1-\epsilon,x_1+\epsilon) \times \cdots \times (x_n-\epsilon,x_n+\epsilon)$, thus $B \subseteq B_\rho(x,\epsilon)$. Thus the topologies are the same. ∎

Definition. Given an index set J and given points $x = (x_\alpha)$ and $y = (y_\alpha)$ fo $\mathbb{R}^J$, we define the **uniform metric** $\overline{\rho}$ on $\mathbb{R}^J$ by $\overline{\rho}(x,y) = \sup\{\overline{d}(x_\alpha,y_\alpha) : \alpha \in J\}$, where $\overline{d}$ is the standard bounded metric on $\mathbb{R}^J$. We call the topology induced by $\overline{\rho}$ the **uniform topology**.

Theorem 2.2.8. *The uniform topology on $\mathbb{R}^J$ is finer than the product topology on $\mathbb{R}^J$ and coarser than the box topology on $\mathbb{R}^J$, and all three topologies are different if J is infinite.*

Proof. Let $x = (x_[\alpha])$ and $\prod U_\alpha$ be a basis element for the product topology, and let $\alpha_1,\ldots,\alpha_n$ be the indices for which $U_\alpha \neq \mathbb{R}$, and for each i, choose $\epsilon_i > 0$ such that the ϵ_i-ball about x_{α_i}, $B_{\overline{d}}(x_{\underline{\alpha_i}},\epsilon_i) \subseteq U_\alpha$. Let $\epsilon = \min\{\epsilon_1,\ldots,\epsilon_n\}$, and let $z = (z_\alpha) \in \mathbb{R}^J$ be such that $\overline{\rho}(x,z) < \epsilon$. Then $\overline{d}(x_\alpha,z_\alpha) < \epsilon$ for all α. Hence $B_{\overline{\rho}} \subseteq \prod U_\alpha$ for all α; so the uniform topology is finer.

Likewise, consider B the ϵ-ball about x in the $\overline{\rho}$ metric. Then the box $U = \prod (x_\alpha - \frac{\epsilon}{2}, x_\alpha + \frac{\epsilon}{2})$ and $y \in U$ if $\overline{d}(x_\alpha, y_\alpha < \frac{\epsilon}{2})$, then $\overline{\rho}(x,y) \leq \frac{\epsilon}{2}$, so $U \subseteq B$, and the uniform topology is coarser.

Now in the case where J is infinite, if J is uncountable, we are done, since there is no way to map the indices of J onto $\mathbb{Z}^+$. So consider the case where J is countable, and map $J \to \mathbb{Z}^+$ by $\alpha_i \to i$. $\overline{\text{Let } U =} \prod (x_i - \epsilon, x_i + \epsilon)$ and consider a base $B_{\overline{\rho}}(x,\epsilon)$. We have that for $y \in B_{\overline{\rho}}(x,\epsilon)$ that $\overline{d(x_\alpha,y_\alpha)} = \min\{\rho(x_i,y_i),1\}$, we have that $\overline{d}(x_i,y_i) = \overline{\rho}(x_i,y_i)$ or 1, and if we choose $0 < \epsilon < 1$, then $\overline{\rho}$ fails to put $B_{\overline{\rho}}(x,\epsilon)$ inside of U. Likewise, the basis $\prod U_\alpha$ (in the prodct topology) failes to be contained in $B_{\overline{\rho}}(x,\epsilon)$ by the same argument. Thus the uniform topology is not necessarily finer than the box topology, nor coarser than the product topology in $\mathbb{R}^J$, when J is infinite. ∎

Remark. Clearly the box, product and uniform topologies on $\mathbb{R}^J$ are the same when J is finite, as the box and product topologies are the same for finite product spaces.

Theorem 2.2.9. *Let $\overline{d}(a,b) = \min|a-b|,1\}$ be the standard bounded metric on $\mathbb{R}$, and for $x,y \in \mathbb{R}^\omega$, define:*

$$D(x,y) = \sup\{\frac{\overline{d}(x_i,y_i)}{i}\} \tag{2.2}$$

. Then D is a metric that induces the product topology on $\mathbb{R}^\omega$. \$$B_{\overline{\rho}}(x,\epsilon)$

Proof. Since $\overline{d}$ is a metric, D satisfies the conditions for a metric space, it is worth looking into the case for the triangle inequality however. Notice that for all i, $\frac{\overline{d}(x_i,y_i)}{i} \leq \frac{\overline{d}(x_i,z_i)}{i} + \frac{\overline{d}(z_i,y_i)}{i} \leq D(x,z) + D(z,y)$, thus $D(x,y) \leq D(x,z) + D(z,y)$.

Now let U be open in the metric topology induced by D and let $x \in U$. Choose an ϵ-ball $B_D(x,\epsilon) \subseteq U$, and choose $N > 0$ large enough that $\frac{1}{N} < \epsilon$, and let $V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \cdots \times \mathbb{R} \times \ldots$. Given $y \in \mathbb{R}^\omega$, we have $\frac{\overline{d}(x_i,y_i)}{i} \leq \frac{1}{N}$, thus $D(x,y) \leq \max\{\overline(d)(x_1,y_1), \ldots, \overline(d)(x_i,y_i)i, \frac{1}{N}\} forall i \geq N$. Now if $y \in V$, then the expression is less than ϵ, so $V \subseteq B_D(x,\epsilon)$. Convsersely, let $U = \prod U_i$ with U_i open in $\mathbb{R}$ for $i = \alpha_1, \ldots, \alpha_n$, and $U_i = \mathbb{R}$ for all other indices. Given $x \in U$, for $i = \alpha_1, \ldots, \alpha_n$ and let $\epsilon = \min\{\frac{\epsilon_i}{i} : i = \alpha_1, \ldots, \alpha_n\}$. Letting $B_d(x,\epsilon)$, we see that $\frac{\overline{d}(x_i,y_i)}{i} \leq D(x,y) < \epsilon$. Now if $i = \alpha_1, \ldots, _n$, then $\overline{d}(x_i,y_i) < \epsilon_i \leq 1$, hence $|x_i - y_i| < \epsilon_i$, thus $y \in U$. ■

2.3 More on Metric Spaces

We go more in depth on metric spaces here.

Theorem 2.3.1. *If A is a subspace of a metric space X, with metric d, then d restricted to $A \times A$ makes A into a metric space.*

Proof. Clearly $d : A \times A \to \mathbb{R}$ is a metric. So consider the ϵ-ball about x, $B_d(x,\epsilon)$ in X; restricting d to $A \times A$, cpnsider $A \cup B_d(x,\epsilon)$. For $y \in A$, there is a δ-ball about y such that $B_d(y,\delta) \subset B_d(x\epsilon)$;; then $B_d(y,\delta) \subseteq B_d(x,\epsilon)$. This makes A as a subspace, into a metric space. ■

Theorem 2.3.2. *The Hausdorff axiom is satisfied in every metric space.*

Proof. If $x, y \in X$ are distinct, let $\epsilon = \frac{1}{2}d(x,y)$, by the triangle inequality, we have that $B_d(x,\epsilon)$ and $B_d(y,\epsilon)$ are disjoint. ■

Theorem 2.3.3. *Countable products of metrizable spaces are metrizable.*

Proof. Let X be a metric space with metrix d. Define $\overline{d}(x,y) = \min\{d(x,y), 1\}$ on X and define $D(x,y) = \sup\{\frac{\overline{d}(x_i,y_i)}{i}\}$ on X^ω. It is clear that both $\hat{d}$ and D are metrics on X and X^ω respectively. We would like to show that D induces the product topology on X^ω.

Let U be open and let $x \in U$. Choose $B_D(x,\epsilon) \subseteq U$ and choose N large enough such that $\frac{1}{N} < \epsilon$. Now let $V = (x_1 - \epsilon, x_1 + \epsilon) \times \ldots (x_N - \epsilon, x_N + \epsilon) \times X \times \ldots$ be a basis in the product topology on X^ω. Given $y \in X^\omega$, such that $\frac{\overline{d}(x_i,y_i)}{i} \leq \frac{1}{N}$, we have by definition that $D(x,y) \leq \max\{\overline{d}(x_1,y_1), \frac{\overline{d}(x_2,y_2)}{2}, \ldots, \frac{\overline{d}(x_N,y_N)}{N}, \frac{1}{N}\}$. IF $y \in V$, we get that $V \subseteq B_D(x,\epsilon)$ and we are done.

Conversely let $U = U_i$ be a basis of the product topology where U_i is open in X for $i = 1, \ldots, n$ and $U_i = X$ for all other indices. Now let $x \in U$ and choose an interval about x_i, $(x_i - \epsilon_i, x_i + \epsilon_i)$ lying in U_i with $0 < \epsilon_i \leq 1$ for all i. Choose $\epsilon = \min\{\epsilon_1, \frac{\epsilon_2}{2}, \ldots, \frac{\epsilon_n}{n}\}$. Then $x \in B_D(x,\epsilon) \subseteq U$, for if $y \in B_D(x,\epsilon)$, we have that $\frac{\overline{d}(x_i,y_i)}{i} \leq D(x,y) < \epsilon$, hence $\epsilon \leq \frac{\epsilon_i}{i}$ and $d(x_i,y_i) < \epsilon_i$, and so $y \in U_i$. Therefore D induces the product space topology ■

Remark. This theorem generalizes theorem 2.2.9 for any countable product space of a metric space X. Hence we can take theorem 2.2.9 as a corollary to this theorem.

We would now like to study continuous functions in metric spaces, which brings us into the realm of analu=ysis. We show that the "ϵ-δ" definition, and the sequence definiton of contiuity carry over.

Theorem 2.3.4. *Let $f : X \to Y$ with X and Y metric spaces with metric d_X and d_Y respectively. Then f is continuous if and only if for $x \in X$,a nd $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), f(y)) < \epsilon$ whenever $d_X(x, y) < \delta$.*

Proof. Suppose that f is continous and consider $f^{-1}(B(f(x), \epsilon))$ open in X. Then it contains a δ-ball $B(x, \delta)$ about x. If $y \in B(x, \delta)$, then $f(y) \in B(f(x), \epsilon)$, as is required.

Now suppose that for $x \in X$ and $\epsilon > 0$, that there is a $\delta > 0$ such that $d_Y(f(x), f(y)) < \epsilon$ whenever $d_X(x, y) < \delta$, for $x \in X$. Let Vbe open in Y and let $x \in f^{-1}(V)$. Then $f(x) \in V$, hence there is an ϵ-ball $B(f(x), \epsilon) \subseteq V$. By hypothesis, there is a $\delta > 0$ such that $f(B(x, \delta)) \subseteq B(f(x), \epsilon)$, hence $B(x, \delta) \subseteq f^{-1}(V)$ which makes $f^{-1}(V)$ open. ■

Lemma 2.3.5 (The Sequence Lemma)**.** *Let X be a topological space and let $A \subseteq X$. If there is a sequence of points og A converging to $x \in X$, then $x \in \overline{A}$. The converse holds if X is metrizable.*

Proof. Suppose for some sequence $\{x_n\} \subseteq A$ that $x_n \to x$. By theorem 1.6.6, we have every neighborhood of x contains points of A, hence $x \in \overline{A}$. Conversely, suppose that X is metrizable with metric d, and let $x \in \overline{A}$. For $n \in \mathbb{Z}^+$, take $B_d(x, \frac{1}{n})$ and take $\{x_n\} = B_d(x, \frac{1}{n}) \cap A$. Then $x_n \to n$, for: any open set U of x contains an ϵ-ball about x, $B_d(x, \epsilon)$, so choose N large enough so that $\frac{1}{N} < \epsilon$, hence U contains x_i for all $i \geq N$. ■

Theorem 2.3.6 (The Sequential Criterion)**.** *Let $f : X \to Y$ be continuous, then for every convergent sequence $\{x_n\}$ converging to $x \in X$, the sequence $\{f(x_n)\}$ converges to $f(x)$. the converse holds if X is metrizable.*

Proof. Let f be continuous and suppose that $x_n \to x$. Let V be a neighborhood of $f(x)$, then $f^{-1}(V)$ is a neighborhood of x; hence there is an $N > 0$ such that $x_n \in f^{-1}(V)$ whenever $n \geq N$, thus $f(x_n) \in V$ whenever $n \geq N$.

Conversely suppose that for every $\{x_n\}$ converging to x, that $\{f(x_n)\}$ converges to $f(x)$, and let $A \subseteq X$. if $x \in \overline{A}$, by the sequence lemma, there is a sequence $\{x_n\} \subseteq A$ converging to X. Now since $f(x_n) \to f(x)$. and $f(x_n) \in f(A)$, by the sequence lemma again, $f(x) \in \overline{f(A)}$. Thus $f(\overline{A}) \subseteq \overline{f(A)}$ and we are done. ■

We now consder methods for constructing continous functions on metric spaces.

Lemma 2.3.7. *The additions, subtraction, and multiplication operations are continous from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. The quotient operation is continous from $\mathbb{R} \times \mathbb{R} \backslash \{0\}$ to $\mathbb{R}$.*

Theorem 2.3.8. *If X is a topological space and if $f, g : X \to \mathbb{R}$ are continous, then $f + g$, $f - g$, a nd fg are continuous; moreover if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is also continuous.*

Proof. The map $h : X \to \mathbb{R} \times \mathbb{R}$ defined by $h(x) = f(x) \times g(x)$ is continuous. Then notice that $f + g(x) = +(f(x), g(x)) = + \circ h(x)$, so by the above lemma, we get that $f + g$ is contincontinuous. We also have that $f - g$ iss continuous for $f - g(x) = +(f(x), -g(x))$. The same argument holds for fg and $\frac{f}{g}$. ■

Definition. Let $f_n : X \to Y$ be a sequence of functions from X to the metric space Y, with metric d. We say that the sequence $\{f_n\}$ **converges uniformly** to the function $f : X \to Y$ if for $\epsilon > 0$, there is an integer $N > 0$ such that $d(f_n(x), f(x)) < \epsilon$ whenever $n \geq N$, for all $x \in X$.

Theorem 2.3.9. *Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If $\{f_n\}$ converges uniformly to f, then f is continuous.*

Proof. Let V be open in Y and let $x_0 \in f^{-1}(V)$. Let $y_0 = f(x_0)$ and choose $\epsilon > 0$ such that $B(y_0, \epsilon) \subseteq V$. By uniform convergence, choosing $N > 0$ so that whenever $n \geq N$, $d(f_n(x), f(x)) < \frac{\epsilon}{3}$ for all $x \in X$. By the continuity of f_N, choose a neighborhood U of x_0 such that $f_N(U) \subseteq B(f_N(x_0), \frac{\epsilon}{3})$. Then if $x \in U$, we have $d(f(x), f_N(x)) < \frac{\epsilon}{3}$, $d(f_N(x), f_N(x_0)) < \frac{\epsilon}{3}$, and $d(f_N(x_0), f(x_0)) < \frac{\epsilon}{3}$ by the triangle inequality we get $d(f(x), f(x_0)) < \epsilon$ which completes the proof. ■

Example 2.4. (1) $\mathbb{R}^\omega$ is not metrizable in the box topology. Let $A = \{(x_1, x_2, \dots) \in \mathbb{R}^\omega : x_i > 0\}$ and consider $0 = (0, 0, \dots) \in \mathbb{R}^\omega$. $-0 \in \overline{A}$ if for any basis elemebt $B = (a_1, b_1) \times (a_2, b_2) \times \dots$, $0 \in B$; then $B \cap A \neq \emptyset$ (take the point $\frac{1}{2}b \in \mathbb{R}^\omega$). Now let $\{a_n\}$ be a sequence of points of A with $a_n = (x_{1n}, x_{2n}, \dots, x_{in}, \dots)$, since $x_{in} > 0$, construct a basis element $B' = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \dots$. Then $0 \in B'$, but $\{a_n\} \nsubseteq B'$ for the point $x_{nn} \notin (-x_{nn}, x_{nn})$. Thus $a_n \not\to 0$.

(2) An uncountable product of $\mathbb{R}$ with itself is not metrizable. Let J be uncountable, and let $A = \{(x_\alpha) \in \mathbb{R}^J : x_\alpha = 1 \text{ for all but finitely many } \alpha\}$. Consider $0 \in \mathbb{R}^J$ and let U be a basis for containing 0. Now $U_\alpha \neq \mathbb{R}$ for $\alpha_1, \dots, \alpha_n$, so let $(x_\alpha) \in A$ be definedby letting $x_\alpha = 0$ for $\alpha_1, \dots, \alpha_n$ and $x_{\alpha=1}$ for all other indices. Then $(x_\alpha) \in A \cap U_\alpha$.

Nowe let $\{a_n\} \subseteq A$ and for $n \in \mathbb{Z}^+$ elt $J_n = \{\alpha \in J : \alpha_{\alpha n} \neq 1\}$. Then we see that $\bigcup J_n$ is a countable union of finite sets, and hence countable itself. Now since J is uncountable, there is a $\beta \in J$ for which $\beta \notin \bigcup J_n$, so $a_{\beta n} \neq 1$. Lettng $U_\beta = (-1, 1)$ in $\mathbb{R}$ let $U = \pi_\beta^{-1}(U_\beta)$ in $\mathbb{R}^J$. Then U is a neighborhood of 0 not containing any points of $\{a_n\}$, so $a_n \not\to 0$.

2.4 The Quotient Topology