Matroid Theory

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Text -

Chapter 1

Fundamental of Matroid Theory.

1.1 Definitions

We go over some fundamental definitions and theorems for matroids.

Definition 1.1.1 (The Independence Axioms)**.** We define a **matroid** M to be a set S, called the **ground set**, together with a collection $\mathcal{I} \subseteq 2^S$ of subsets of S which we call **independent sets** to such that;

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$. (Inheretence)

(I2) If $X, Y \in \mathcal{I}$ such that $|Y| < |X|$, then ther is an $x \in X \backslash Y$ such that $Y \cup x \in \mathcal{I}$. (Augmentation)

There is one immediate example for a matroid.

Let V be a finite vector space and let $\mathcal{I}$ be the collection of all linearly independent subsets of vectors of V. Clearly $\emptyset \in V$, and if X is linearly independent, and $Y \subseteq X$, then Y is also linearly independent; so inheretence is satisfied.

Now let X be linearly independent, then $\operatorname{span} S$ must also be linearly independent. Now take $\beta \in V \backslash \operatorname{span} S$ and for $\alpha_1, \ldots, \alpha_m \in S$ take

$$c_1\alpha_1 + \cdots + c_2\alpha_2 + b\beta = 0$$

If $b \neq 0$, then

$$\beta = (-\frac{c_1}{b})\alpha_1 + \cdots + (-\frac{c_m}{b})\alpha_m$$

putting $\beta \in \operatorname{span} S$, a contradiction, hence $b = 0$, and so $S \cup \beta$ is also linearly independent. Thus the augmentation axiom is satisfied and V is a matroid together with $\mathcal{I}$.

We define some additional concepts, all of which can be used in the definition of a matroid.

Definition 1.1.2. A **base** of a matroid M is a maximally independent subset of S. We denote the collection of bases of M by $\mathcal{B}$. We say a subset of S is **spanning** in M if it contains a base.

Definition 1.1.3. We define the **rank function** of a matroid to be the map rank : $2^S \to \mathbb{Z}$ defined by

$$\operatorname{rank} A = \max\{|X| : X \in \mathcal{I} \text{ and } X \subseteq A\} \tag{1.1}$$

We define the **rank** of the matroid to be $\operatorname{rank} M = \operatorname{rank} S$. We say $A \subseteq S$ is **closed**, or a **flat**, or a **subspace** of M if for all $x \in S \backslash A$, $\operatorname{rank} A \cup x = \operatorname{rank} A + 1$, and we say x **depends** on A.

Definition 1.1.4. We define the **closure operator** of a matroid to be the map cl : $@^S \to 2^S$ defined such that cl $A$is the set of all elements which depend on A; that is

$$\operatorname{cl} A = \{x \in S \backslash A : \operatorname{rank} A \cup x = \operatorname{rank} A + 1\}. \tag{1.2}$$

Definition 1.1.5. A **dependent set** of a matroid is a subset $D \subseteq S$ which is not independent; this is $D \notin \mathcal{I}$. A **circuit** of a matroid is a minimally dependent set, and we denote the collection of all circuits as $\mathcal{C}$.

Now one thing that makes matroids so interesting, is that they can be axiomatically defined in various ways. We can not only define them in terms of independence, bu also in terms of bases, circuits, rank, and closure. We give the theorems below that establish the axioms.

Theorem 1.1.1 (The Base Axioms). *A nonempty collection $\mathcal{B}$ of subsesets of a set S forms a set of bases of a matroid M on S if, and only if for:*

(B1) $B_1, B_2 \in \mathcal{B}$*, and* $x \in B_1 \backslash B_2$*, there is a* $y \in B_2 \backslash B_1$ *such that* $(B_1 \cup y) \backslash x \in \mathcal{B}$.

Theorem 1.1.2 (The Circuit Axioms). *A nonempty collection $\mathcal{C}$ of subsesets of a set S forms a set of circuits of a matroid M on S if, and only if:*

(C1) If $Y \in \mathcal{C}$ *and* $X \neq Y$*, then* $X \not\subseteq Y$.

(C2) If $C_1, C_2 \in \mathcal{C}$ *are distinct, and* $z \in C_1 \cap C_2$*, then there is a circuit* $C_3 \in \mathcal{C}$ *such that* $C_3 \subseteq (C_1 \cup C_2) \backslash z$.

Theorem 1.1.3 (The First Rank Axioms). *Let S be a set. A map* rank : $2^S \to \mathbb{Z}$ *is the rank function of a matroid on S if and only if for $X \subseteq S$ and $y, z \in S$*

(R1) $\operatorname{rank} \emptyset = 1$.

(R2) $\operatorname{rank} X \leq \operatorname{rank} X \cup y \leq \operatorname{rank} X + 1$.

(R3) If $\operatorname{rank} X \cup y = \operatorname{rank} X \cup z = \operatorname{rank} X$*, then* $x \cup y \cup z = \operatorname{rank} X$.

Theorem 1.1.4 (The Second Rank Axioms). *Let S be a set. A map* rank : $2^S \to \mathbb{Z}$ *is the rank function of a matroid on S if and only if for $X, Y \subseteq S$*

(R′1) $0 \leq \operatorname{rank} X \leq |X|$.

(R′2) If XY*, then* $\operatorname{rank} X \leq \operatorname{rank} Y$

(R3̀) $\operatorname{rank} X \cup Y + \operatorname{rank} X \cap Y \leq \operatorname{rank} X + \operatorname{rank} Y$.

Theorem 1.1.5 (The Closure Axioms)**.** *Let S be a set. A map $\mathrm{cl} : 2^S \to 2^S$ is the closure operator of a matroid on S if and only if for $X, Y \subseteq S$, and $x, y \in S$*

(S1) $X \subseteq \mathrm{cl}\, X$.

(S2) If $Y \subseteq X$, *then* $\mathrm{cl}\, Y \subseteq \mathrm{cl}\, X$.

(S3) $\mathrm{cl}\, X = \mathrm{cl}\,(\mathrm{cl}\, X)$.

1. *If* $y \notin \mathrm{cl}\, X$ *and* $y \in \mathrm{cl}\,(X \cup x)$, *then* $x \in \mathrm{cl}\,(X cupy)$.

We defer their proofs to the relevant sections.
We can already provea fact about matroids.

Proposition 1.1.6. *If an element of a matroid belongs to every base, then it can belong to no circuit of the matroid.*

Proof. Let M be a matroid and let $\mathcal{B}$ be the collection of all bases of M, $\mathcal{C}$ the collection of all circuits of M, and $\mathcal{I}$ the collection of all independent sets of M. Take $x \in X = \bigcap_{B \in \mathcal{B}} B$ and suppose that $x \in C$ for $C \in \mathcal{C}$. By theorem 1.1.1, we have that $X \neq \emptyset$, moreover since $x \in C$, $C \subseteq X$, i.e $C \in \mathcal{B}$. mow notice that since C is a circuit, it is dependent, so $C \notin \mathcal{I}$, but we have that $C \in \mathcal{B}$ which makes it a base, and hence independent; so $C \in \mathcal{I}$, a contradiction. □

Definition 1.1.6. We say that two matroids M_1 and M_2 on S_1 and S_2 respectively are **isomorphic** if there is a $1-1$ map $\phi : S_1 \to S_2$ of S_1 onto S_2 such that if $X \subseteq S_1$ is independent in M_1, then $\phi(X) \subseteq S_2$ is independent in M_2. We write $M_1 \simeq M_2$ to denote isomorphism.

We list all nonisomorphic matroids on a set of n elements for $n = 1, 2, 3$.

$\underline{n=1}$ For $S = \{1\}$, we have $M_1 = \emptyset$ and $M_2 = 2^S$. There are $2^1 = 2$ matroids on S.

$\underline{n=2}$ On $S = \{1, 2\}$we have

$$
\begin{aligned}
M_1 &= \emptyset \\
M_2 &= \{\emptyset, \{1\}\} \\
M_3 &= \{\emptyset\{1\}, \{2\}\} \\
M_4 &= \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} = 2^S
\end{aligned}
$$

there are $2^2 = 4$ matroids on S

$\underline{n=3}$ For $S = \{1, 2, 3\}$ we have

$$
\begin{aligned}
M_1 &= \emptyset \\
M_2 &= \{\emptyset, \{1\}\} \\
M_3 &= \{\emptyset\{1\}, \{2\}\} \\
M_4 &= \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \\
M_5 &= \{\emptyset, \{1\}, \{2\}, \{3\}\} \\
M_6 &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}\} \\
M_7 &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\} \\
M_8 &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} = 2^S
\end{aligned}
$$

There are $2^3 = 8$ matroids on S.

Now one might be tempted to generalize that there are a total of 2^n matroids on a given n element set, however a quick check for $n = 4$ concludes that that is not the case.

$\underline{n=4}$ For $S = \{1, 2, 3, 4\}$ we have

$$
\begin{aligned}
M_1 &= \emptyset \\
M_2 &= \{\emptyset, \{1\}\} \\
M_3 &= \{\emptyset\{1\}, \{2\}\} \\
M_4 &= \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \\
M_5 &= \{\emptyset, \{1\}, \{2\}, \{3\}\} \\
M_6 &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}\} \\
M_7 &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\} \\
M_8 &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \\
M_9 &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}\} \\
M_{10} &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}\} \\
M_{11} &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}\} \\
M_{12} &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\} \\
M_{13} &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}\} \\
M_{14} &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\} \\
M_{15} &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\{1, 3, 4\}\} \\
M_{16} &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\{1, 3, 4\}, \\
&\quad \{2, 3, 4\}\} \\
M_{17} &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\{1, 3, 4\}, \\
&\quad \{2, 3, 4\}, \{1, 2, 3, 4\}\} = 2^S
\end{aligned}
$$

Notice that $S = \{1, 2, 3, 4\}$ has $17 = 2^4 + 1$ matroids.

1.2 Examples

We define some matroids, and observe the properties of a peculiar one.

Proposition 1.2.1. *Let S be a set with $|S| = n$ and define $\mathcal{I} = \{X \subseteq S : |X| \leq k\}$ for $k \leq n$. Then S is a matroid on S.*

Proof. Clearly $\emptyset \in \mathcal{I}$, and if X is independent, and $Y \subseteq X$, then $|Y| \leq |X| \leq k$, hence $Y \in \mathcal{I}$.

Now take $Y, X \in \mathcal{I}$ with $|Y| < |X|$. Now if $|Y| + 1 = |X| = k$, the result is clear. Otherwise, choose an $x\imath X \backslash Y$, then since $|Y| < k$, $|Y \cup x| \leq k$, and hence is independent. $\square$

Definition 1.2.1. We call the matroids on a set S, generated by the collection $\mathcal{I} = \{X \subseteq S : |X| \leq\}$ the **uniform matroid** of rank k on S; and we denote it $U_{n,k}$.

We discuss some properties of the uniform matroid.

Corollary. $\mathcal{B}(U_{n,k}) = \{X \subseteq S : |X| = k\}$ *and* $\mathcal{C}(U_{n,k}) = \{X \subseteq S : |X| = k + 1\}$

Proof. If $B \in \mathcal{I}(U_{n,k})$ is a base, then $|B| \leq k$ and by the maximality of B for any $x \in S \backslash B$, $B \cup x \notin \mathcal{I}$, i.e. $|B \cup x| = |B| + 1 > k$. It follows that $|B| = k$. Likewise by the same reasoning we see by the minimality of any circuit $C \in \mathcal{C}$ that $|C| = k + 1$ $\square$

Remark. Since any base, and any dependent set in $U_{n,k}$ has size $\geq k$, it is easy to see that any set A with $|A| \geq k$ is spanning in $U_{n,k}$.

Corollary. *For any* $A \subseteq U_{n,k}$ $\operatorname{rank} A = \begin{cases} |A|, & \text{if } |A| \leq k \\ k, & \text{if } |A| > k \end{cases}$ *and* $\operatorname{cl} A = \begin{cases} A, & \text{if } |A| < k \\ S, & \text{if } |A| \geq k \end{cases}$

Proof. By definition, we have that $\operatorname{rank} A = \max\{|X| : X \subseteq A, |X| \leq k\}$. Now if A is independent, then $\operatorname{rank} A = A$. If A is dependent, well since every dependent set of $U_{n,k}$ is spanning, choose a base $B \subseteq A$. Then $\operatorname{rank} A = \operatorname{rank} B = k$.

Now by the closure axioms, $A \subseteq \operatorname{cl} A \subseteq S$. Suppose that $|A| < k$, and take $x \in \operatorname{cl} A$, then $\operatorname{rank} A \cup x = \operatorname{rank} A + 1 = |A| + 1$, hence $\operatorname{rank} A = |A|$, thus $x \in A$, so $\operatorname{cl} A = A$. Now if $|A| \geq k$, then for any $x \in S \backslash A$, $\operatorname{rank} A \cup x = \operatorname{rank} A + 1 \geq k + 1 > |A| + 1$, this puts $x \in \operatorname{cl} A$, and bu consequence $S \subseteq \operatorname{cl} A$. Therefore $\operatorname{cl} A = S$. $\square$

Corollary. $\operatorname{rank} U_{n,k} = k$.