Analysis

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Text

Principles of Mathematical Analysis (3^{rd} edition)

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Chapter 1

The Real and Complex Numbers

1.1 Ordered Sets

Definition. Let S be any set. An **order** on S is a relation $<$ such that:

(1) For $x, y \in S$, one and only one of the following hold:

$$x < y \qquad x = y \qquad y < x$$

We call this property the **trichotomy law**

(2) $<$ is transitive over S.

We denote the relations $>$ and $\leq$ to mean $x > y$ if and only if $y < x$, and $x \leq y$ if and only if $x < y$, or $x = y$. We call S together with $<$ an **ordered set**.

Example 1.1. Define $<$ on $\mathbb{Q}$ such that for $r, s \in \mathbb{Q}$, $r < s$ implies $< 0s - r$.

Definition. Let S be an ordered set, and let $E \subseteq S$. We say that E is **bounded above** there is some $\beta \in S$ for which $x \leq \beta$, for all $x \in E$. We say that E is **bounded below** if $\beta \leq x$, for call $x \in E$. We say an $\alpha \in S$ is a **least upperbound** of E, if α is an upperbound of E, and for all other upperbounds, γ, of E, $\alpha \leq \gamma$. Likewise, α is a **greatest lowerbound** of E if α is a lowerbound of E, and for all other lowerbounds γ of E, $\gamma \leq \alpha$. We denote the least upperbound, and greatest lowerbound by $\sup E$ and $\inf E$, respectively.

Lemma 1.1.1. *Let S be an ordered set, and let $E \subseteq S$. Then E has (if they exist) a unique least upperbound, and a unique greatest lowerbound.*

Proof. Let $\alpha, \beta \in S$ be least upperbounds of E. Then by definition, we have that $\alpha \leq \beta$, and $\beta \leq \alpha$; thus by the trichotomy law, $\alpha = \beta$. The proof is the same for greatest lowerbounds. ■

Example 1.2. (1) Let $A = \{p \in \mathbb{Q} : p^2 < 2\}$, and $B = \{p \in \mathbb{Q} : p^2 > 2\}$. Clearly, we have that every element of B is an upperbound of A, and every element of A is a lowerbound of B. Now take $p \in \mathbb{Q}$ a positive rational, and take $q \in \mathbb{Q}$ such that $q = p - \frac{p^2-2}{p+2}$. Then

$q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2}$. Now if $p \in A$, then $p^2 - 2 < 0$, which implies that $p < q$, and $q^2 < 2$; thus A has no largest element; similarly, if $p \in B$, then $p^2 - 2 > 0$, which implies that $q < p$ and $q^2 > 2$, which shows that B has no least element. Thus $\sup A$ and $\inf B$ do not exist in $\mathbb{Q}$.

(2) If $\alpha = \sup E \in S$, it may or may not be that $\alpha \in E$. Take $E_1 = \{r \in \mathbb{Q} : r < 0\}$, and $E_2 = \{r \in \mathbb{Q} : r \leq 0\}$. Then $\sup E_1 = \sup E_2 = 0$, but $0 \notin E_1$, where as $0 \in E_2$

(3) Consider the set $\frac{1}{\mathbb{Z}^+} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$. By the well ordering principle, 1 is the least element, and is also an upperbound of all $\frac{1}{n}$ for $n > 1$. Now also notice that as n gets arbitrarily large, then $\frac{1}{n}$ gets arbitratirly small; that is to say $\frac{1}{n}$ "tends" to 0, so $\sup \frac{1}{\mathbb{Z}^+} = 1 \in \frac{1}{\mathbb{Z}^+}$, and $\inf \frac{1}{\mathbb{Z}^+} = 0 \notin \frac{1}{\mathbb{Z}^+}$.

Definition. We say an ordered set S has the **least upperbound property**, if whenever $E \subseteq S$, nonempty, and bounded above, then $\sup E \in S$ exists; likewise, S has the **greatest lowerbound property** if whenever E is nonempty, bounded below then $\inf E \in S$ exists.

Example 1.3. (1) The set of all rationals $\mathbb{Q}$ does not have the least upperbound property, nor the greatest lowerbound property, take A, B as in the previous example. Letting $E = \{1, \frac{1}{2}, \frac{1}{4}\} \subseteq \frac{1}{\mathbb{Z}^+}$, we see that $\frac{1}{\mathbb{Z}^+}$ satisfies both properties, with $\sup E = 1$, and $\inf E = \frac{1}{4}$.

(2) Let $A \subseteq \mathbb{R}$ be nonempty, and be bounded below. Then by the greatest lowerbound property, $\alpha = \inf A \in \mathbb{R}$ exists; Then for all $x \in A$, $\alpha \leq x$, and for all other lowerbounds γ, $\gamma \leq \alpha$. Then $-x \leq -\alpha$, and $-\alpha \leq -\gamma$, then we see that $-\gamma$ and $-\alpha$ are upperounds of $-A$, and that $-\alpha$ is the least upperbound of $-A$

Theorem 1.1.2. *If S is an ordered set with the least upperbound property, then S also inherits the greatest lowerbound property.*

Proof. Let $B \subseteq S$, and let $L \subseteq S$ be the set of all lowerbounds of B. Then we have for any $y \in L$, $x \in B$, $y \leq x$. So every element of B is an upperbound of L, and L is nonempty, hence $\alpha = \sup L \in S$ exists. Now if $\gamma \leq \alpha$, then γ is not an upperbound of L, hence $\gamma \notin B$; thus $\alpha \leq x$ for all $x \in B$, so $\alpha \in L$, and by definition of the greatest lowerbound, we get $\alpha = \inf B$. ∎

1.2 Fields

Definition. A **field** is a set F, together with binary operations $+$ and $\cdot$ (called **addition** and **multiplication**, respectively) such that:

(1) F forms an abelian group under $+$.

(2) $F\backslash\{0\}$ forms an abelian group under $\cdot$ (where 0 is the additive identity of F).

(3) $\cdot$ distributes over $+$.

We now state the following propositions without proof.

Proposition 1.2.1. *For all $x, y, x \in F$:*

(1) $x + y = x + y$ *implies* $y = z$

(2) $x + y = x$ *implies* $y = 0$

(3) $x + y = 0$ *implies* $y = -x$

(4) $-(-x) = x$.

Proposition 1.2.2. *For all $x, y, x \in F\backslash\{0\}$:*

(1) $xy = xy$ *implies* $y = z$

(2) $xy = x$ *implies* $y = 1$

(3) $xy = 1$ *implies* $y = x^{-1}$

(4) $(x^{-1})^{-1} = x$.

Proposition 1.2.3. *For all $x, y, x \in F$:*

(1) $0x = 0$

(2) $x \neq 0$ *and* $y \neq 0$ *implies* $xy \neq 0$

(3) $(-x)y = -(xy) = x(-y)$

(4) $(-x)(-y) = xy$.

Definition. An **ordered field** is a field F that is also an ordered set, such that:

(1) $x + y < x + z$ whenever $y < z$, for $x, yz, z \in F$

(2) $xy > 0$ whenever $x > 0$ and $y > 0$, for $x, y \in F$.

Proposition 1.2.4. *Let F be an ordered field, then for any $x, y, z \in F$, the following hold:*

(1) $x > 0$ *implies* $-x < 0$.

(2) *If* $x > 0$ *and* $y < z$, *then* $xy < xz$.

(3) *If* $x < 0$ *and* $y < z$, *then* $xz < xy$.

(4) *If* $x \neq 0$, *then* $x^2 > 0$, *in particular,* $1 > 0$.

(5) $0 < x < y$ *implies that* $0 < y^{-1} < x^{-1}$.

Proof. (1) If $x > 0$, then $0 = x + (-x) > 0 + (-x)$, so $-x < 0$.

(2) We have $0 < z - y$, so $0 < x(z - y) = xz - xy$, so $xy < xz$.

(3) Do the same as (2),, multiplying $z - y$ by $-x$.

(4) If $x > 0$, we are done. Now suppose that $x < 0$, then $-x > 0$, so $(-x)(-x) = xx = x^2 > 0$; in particular, we also have that $1 \neq 0$, and $1 = 1^2$, so $1 > 0$.

(5) We have $0 < xy^{-1} < yy^{-1} = 1$, then $0 < x^{-1}xy^{-1} = y^{-1} < x^{-1}1 = x^{-1}$

■

1.3 The Field of Real Numbers

Theorem 1.3.1. *There exists an ordered field* $\mathbb{R}$ *with the least upperbound property, such that* $\mathbb{Q} \subseteq \mathbb{R}$.

Definition. We call the field $\mathbb{R}$ the **field of real numbers**,and we call the elements of $\mathbb{R}$ **real numbers**.

Definition. Let S be an ordered field, and let $E \subseteq S$. We say that E is **dense** in S, if for all $r, s \in S$, with $r < s$, there is an $\alpha \in E$ such that $r < \alpha < s$.

Theorem 1.3.2 (The Archimedean Principle)**.** *If* $x, y \in \mathbb{R}$, *and* $x > 0$, *then there is an* $n \in \mathbb{Z}^+$ *such that* $nx > y$.

Proof. Let $A = \{nx : n \in \mathbb{Z}^+\}$, and suppose that $nx \leq y$. Then y is an upperbound of A, abd since A is nonempty, $\alpha = \sup A \in \mathbb{R}$, since $x > 0$, we have $\alpha - x < \alpha$, so $\alpha - x$ is not an upperbound of A. Hence $\alpha - x < mx$ for some $m \in \mathbb{Z}^+$. Then $\alpha < (1 - m)x \in A$, contradicting that α is an upperbound of A. ■

Theorem 1.3.3 (The density of $\mathbb{Q}$ in $\mathbb{R}$)**.** $\mathbb{Q}$ *is dense in* $\mathbb{R}$.

Proof. Let $x < y$ be realnumbers, then $y - x > 0$, so by the Archimedean principle, there is an $n \in \mathbb{Z}^+$ fir which $n(y-x) > 1$. By the Archimedean principle again, we have $m_1, m_2 \in \mathbb{Z}^+$ for which $m_1 > nx$ and $m_2 > -nx$, thus $-m_2 < nx < m_1$, and we also have that there is an $m \in \mathbb{Z}^+$ for which $-m_2 < m < m_1$, and $m - 1 \leq nx < m$. Thus combining inequalities, we get $nx < m < ny$, thus $x < \frac{m}{n} < y$. ■

Theorem 1.3.4 (The existence of n^th roots of positive reals)**.** *For every real number* $X > 0$, *and for every* $n \in \mathbb{Z}^+$, *there is one, and only one positive real number* y *for which* $y^n = x$.

Proof. Let $y > 0$ be a real number; then $y^n > 0$, so there is atmost one such y for which $y^n = x$. Now let $E = \{t : \mathbb{R} : t^n < x\}$, choosing $t = \frac{x}{1+x}$, we see that $0 \leq t < 1$, hence $t^n < t < x$, so E is nonempty. Now if $1 + x < t$, then $t^n \geq x$, so $t \notin E$, and E has $1 + x$ as an upperbound. Therefore, $\alpha = \sup E \in \mathbb{R}$ exists.

Now suppose that $y^n < x$, choose $0 \leq h < 1$ such that $h < \frac{x-y^n}{n(y+1)^{n-1}}$, then $(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)n - 1 < x - y^n$, thus $(y+h)^n < x$, so $y + h \in E$, contraditing that y is an upperbound. On the other hand, if $y^n > x$, choosing $k = \frac{y^n - x}{ny^{n-1}}$, then $0 \leq k < y$, and letting $t \geq y - k$, we get that $y^n - t^n \leq y^n + (y-k)^n < kny_{n-1} = y^n - x^n$, so $t^n \geq x$, making $y - k$ an uppeerbound of E, which contradicts $y = \sup E$. ■

Remark. We denote y as $\sqrt[n]{x}$, or as $x^{\frac{1}{n}}$.

Corollary. *If* $a, b \in \mathbb{R}$, *with* $a, b > 0$, *and* $n \in \mathbb{Z}^+$, *then* $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$.

Proof. Let $\alpha = \sqrt[n]{a}$, and $\beta = \sqrt[n]{b}$. Then $\alpha^n = a$, and $\beta^n = b$, so $ab = \alpha^n\beta^n = (l\alpha\beta)^n$, we are done. ■

Definition. We define the **extended real number system** to be the field $\mathbb{R}$, together with symbols ∞, and $-\infty$, called **positive infinity** and **negative infinity**, such that $-\infty < x < \infty$ for all $x \in \mathbb{R}$. We call elements of the extended real numbers **infinite**, and every other element not in the extended real numbers **finite**.

Lemma 1.3.5. *∞ is an upperbound for every subset E, of $\mathbb{R}$, and $-\infty$ is a lowerbound for every subset E of $\mathbb{R}$. Moreover, if E is not bounded above, then $\sup E = \infty$, and if E is not bounded below, then $\inf E = -\infty$.*

Remark. We make the following assumptions for extended real numbers:

(1) If $x \in \mathbb{R}$, then $x + \infty = \infty$, $x - \infty = -\infty$, and $\frac{x}{\infty} = \frac{x}{-\infty} = 0$.

(2) If $x > 0$, then $x(\infty) = \infty$ and $x(-\infty) = -\infty$.

(3) If $x < 0$, then $x(\infty) = -\infty$ and $x(-\infty) = \infty$.

1.4 The Complex Field

Definition. We define a **complex number** to be a pair of real numbers (a, b). We denote the set of all comlex numbers by $\mathbb{C}$. We define the **addition** and **multiplication** of complex numbers to be the binary operations $+ : \mathbb{C} \to \mathbb{C}$ and $\cdot : \mathbb{C} \to \mathbb{C}$ such that

$$\begin{aligned}(a,b) + (c,d) &= (a+c, b+d) \\ (a,b)(c,d) &= (ac - bd, ad + bc)\end{aligned}$$

Lastly, we define i to be the complex number such that $i = (0, 1)$.

Theorem 1.4.1. *$\mathbb{C}$ forms a field together with $+$ and $\cdot$.*

Theorem 1.4.2. *For $(a, 0), (b, 0) \in C$, $(a, 0) + (b, 0) = (a + b, 0)$, and $(a, 0)(b, 0) = (ab, o)$.*

Proof. This is a straightforward application of the addition and multiplication of complex numbers. ■

Theorem 1.4.3. *$i^2 = -1$.*

Proof. $i^2 = (0, 1)(0, 1) = (0 - 1, 1 - 1) = (-1, 0) = -1$. ■

Theorem 1.4.4. *Let $(a, b) \in \mathbb{C}$, then $(a + b) = a + ib$.*

Proof. $(a, b) = (a, 0) + (0, b) = (a, 0) + (0, 1)(b, 0) = a + ib$. ■

Definition. Let $a, b \in \mathbb{R}$, and let $z \in \mathbb{C}$ such that $z = a + ib$. We define the **complex conjugate** of z to be the complex number $\overline{z} = a - ib$. Moreover, we define the **real part** of z to be a, and the **imaginary part** of z to be b, and we denote them $a = \operatorname{Re} z$, $b = \operatorname{Im} z$

Theorem 1.4.5. *Let $z, w \in \mathbb{C}$. Then*

(1) $\overline{z + w} = \overline{z} + \overline{w}$.

(2) $\overline{zw} = \overline{zw}$.

(3) $z + \overline{z} = 2 \operatorname{Re} z$ *and* $z - \overline{z} = 2i \operatorname{Im} z$.

(4) $z\overline{z}$ *is a nonegative real number.*

Proof. Let $z = a + ib$, and let $w = c + id$. Then $z + w = (a + c) + i(b + d$, so $\overline{z + w} = (a+b) - i(b+d) = (a - ib) + (c - id) = \overline{z} + \overline{w}$; similarly, we get $\overline{zw} = \overline{zw}$. Moreover, we have $(a+ib)+(a-ib) = 2a$, and $(a+ib)-(a-ib) = 2ib$, we also have that $z\overline{z} = (a+ib)(a-ib) = a^2 + b^2 \geq 0$, and $z\overline{z} = 0$ if and only if $a = b = 0$. ■

Definition. Let $z \in \mathbb{C}$. We define the **modulus** of z to be $|z| = \sqrt{z\overline{z}}$.

Remark. $|z|$ exists and is unique.

Theorem 1.4.6. *Let $z, w \in \mathbb{C}$, then:*

(1) $|z| \geq 0$ *and* $|z| = 0$ *if and only if* $z = 0$.

(2) $|\overline{z}| = |z|$.

(3) $|zw| = |z||w|$.

(4) $\operatorname{Re} z \leq |z|$.

(5) $|z + w + \leq |z| + |w|$.

Proof. Let $z = a + ib$, and $w = c + id$. Then $|z| = \sqrt{a^2 + b^2} \geq 0$, and $|z| = 0$ if and only if $a, b = 0$. Moreover, $|\overline{z}| = |a + i(-b)| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$. We also habe $|zw|^2 = (a^2 + b^2)(c^2 + d^) = |z|^2|w|^2$, likewise, $\|rez| = |a + i0| = \sqrt{a^2} \leq \sqrt{a^2 + b^2}$. Finally we prove (5).

We have $|z + w|^2 = (x + w)(\overline{z} + \overline{w}) = z\overline{z} + \overline{z}w + \overline{w}z + w\overline{w} = |z|^2 + w \operatorname{Re} z\overline{w} + |w|^2 \leq |z|^2 + 2|s\overline{w}| + |w|^2 = (|z| + |w|)^2$. ■

Theorem 1.4.7 (The Cauchy Schwarz Inequality)**.** *Let $a_i, b_i \in \mathbb{C}$, for $1 \leq i \leq n$. Then:*

$$|\sum_{i=1}^{n} a_i\overline{b_i}| \leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_j|^2 \tag{1.1}$$

Proof. Let $A = \sum a_j|^2$, $B = \sum |b_i|^2$, and $C = \sum a_i\overline{b_i}$. If $B = 0$, then $b_i = 0$ for $1 \leq i \leq n$, and we are done; so suppose that $B > 0$. Then

$$\begin{aligned}
\sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(B\overline{a_j} - \overline{Cb_j}) \\
&= B \sum |a_j|^2 - B\overline{C} \sum a_j\overline{b_j} - BC \sum \overline{a_j}b_j + |C^2| \sum |b_j|^2 \\
&= (B^2A - B|C|^2) = B(AB - |C|^2) \geq 0
\end{aligned}$$

Since $B > 0$, we get $|C|^2 \leq AB$ as required. ■

1.5 Euclidean Spaces

Definition. Let $k \in \mathbb{Z}^+$, and let $\mathbb{R}^k$ be the set of all ordered k-tuples $(x_1, x_2, \dots, x_k)$, with $x_i \in \mathbb{R}$ for $1 \leq i \leq k$. We call $\mathbb{R}^k$ the **Euclidean space** of **dimension** k; more simply the **Euclidean k-space**. We call elements of $\mathbb{R}^k$ **vectors** or **points**; and we define **vector addition** and **scalar multiplication** to be:

$$(x_1, \dots, x_k) + (y_1, \dots, y_k) = (x_1 + y_1, \dots, x_k + y_k)$$
$$\alpha(x_1, \dots, x_k) = (\alpha x_1, \dots, \alpha x_k)$$

for $(x_1, \dots, x_k), (y_1, \dots, y_k) \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$.

Theorem 1.5.1. $\mathbb{R}^k$ *forms a vector space together with vector addition and scalar multiplication.*

Definition. Let $x, y \in \mathbb{R}^k$. We define the **inner product** of x and y to be the binary operation $\langle , \rangle : \mathbb{R}^k\mathbb{R}^k \to \mathbb{R}$ such that

$$\langle x, y\rangle = \sum_{i=1}^{k} x_i y_i$$

We define the **norm** of x to be $||x|| = \sqrt{\langle x, x\rangle} = \sqrt{\sum x_i^2}$.

Theorem 1.5.2. *Let $x, y \in \mathbb{R}^k$, and $\alpha \in \mathbb{R}$. Then:*

(1) $||x|| \geq 0$ and $||x|| = 0$ if and only if $x_i = 0$ for all $1 \leq i \leq k$.

(2) $||\alpha x|| = |\alpha|||x||$.

(3) $||\langle x, y\rangle|| \leq ||x||||y||$.

(4) $||x + y|| \leq ||x|| + ||y||$, and $||x - z|| \leq ||x - y|| + ||y - z||$

Proof. (1) follows by definition of the norm. We also have that $||\alpha x|| = \sqrt{\sum \alpha^2 x_i^2} = \sqrt{\alpha^2}\sqrt{\sum x_i^2} = |\alpha|||x||$.

Now by the Cauchy Schwarz inequality, we have that $||\langle x, y\rangle||^2 = \sum x_i^2 y_i^2 \leq \sum x_i^2 \sum y_i^2 = ||x||||y||$. Finally we have that $||x + y|| = \langle x + y, x + y\rangle = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \leq ||x||^2 + 2||x||||y|| + ||y^2|| = (||x|| + ||y||)^2$, the last result follows immediately. ■

Chapter 2

Topological Foundations

2.1 Finite, Countable, and Uncountable Sets

Definition. Let A be a set, and let $E \subseteq \mathbb{N}$. We say that A is **finite** if there exists a 1-1 mapping of A ont E, we say A is **countable** if $E = \mathbb{N}$, and we say A is **atmost countable** if A is either finite or countable.

Example 2.1. The set of all integers $\mathbb{Z}$ is countable. Take $f : \mathbb{N} \to \mathbb{Z}$ such that $f(n) = 2$ if n is even, and $f(n) = -n$ if n is odd.

Definition. Let A be a set, and let $E \subseteq \mathbb{N}$. A **sequence** in A is a mapping $f : E \to A$ such that $f(n) = x_n$, for $x_n \in A$. We call the values of f **terms** of the sequence. We denote sequences by $\{x_n\}_{i=1}^{n}$, and when $E = \mathbb{N}$, we denote them simply by $\{x_n\}$.

Theorem 2.1.1. *Every infinite subset of a countable set is countable.*

Proof. Let A be countable, and let $E \subseteq A$ be infinite. Arrange the elements of A into a sequence $\{x_n\}$, and construct a sequence $\{n_k\}$ such that n_1 is the least term for which $\{x_{n_k}\} \in E$, and n_k is the least term greater than n_{k-1} for which $x_{n_k} \in E$. Let $f(k) = x_{n_k}$, and we get a 1-1 mapping of $\mathbb{N}$ onto E. ■

Theorem 2.1.2. *Let $\{E_n\}$ be a sequence of countable sets. Then $S = \bigcup E_n$ is also countable.*

Proof. Arrange every set E_n in a sequence $\{x_{nk}\}$, and consider the infinite array (x_{ij}), in which the elements of E_n form the n-th row. Then (x_{ij}) contains all the elements of S, and we can arrange them is a sequence

$$x_{11}, (x_{21}, x_{12}), (x_{31}, x_{22}, x_{13}), \ldots$$

Moreover, if $E_j \cap E_j \neq \emptyset$, for $i \neq j$, then the elements of $E_i \cap E_j$ appear more than once in the sequence of S; so taking $T \subseteq \mathbb{N}$, we get a 1-1 mapping of T onto S, hence S is atmost countable, and since $E_i \subseteq S$ for $i \in \mathbb{N}$, is infinite, by theorem 2.1.1, S is infinite, thus S is countable. ■

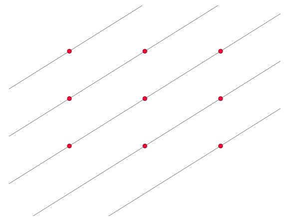

Figure 2.1: The infinite array (x_{ij})

Corollary. *Let A be atmost countable, and suppose for all $\alpha \in A$ that the sets B_α are atmost countable. Then*

$$T = \bigcup_{\alpha \in A} B_\alpha$$

is atmost countable.

Theorem 2.1.3. *Let A be countable, and let B_n be the set of all n-tuples $(a_1, \dots, a_n)$ such that $a_i \in A$ for $1 \leq i \leq n$. Then B_n is countable.*

Proof. By induction on n, we have that $B_1 = A$, which is countable. Now suppose that B_n is countable, and consider B_{n+1} whose elements are of the form (b, a) where $b \in B_n$ and $a \in A$. Fixing b, we get a 1-1 correspondence between the elements of B_{n+1} and A; therefore B is countable. ■

Corollary. $\mathbb{Q}$ *is countable.*

Proof. For every rational $\frac{p}{q} \in \mathbb{Q}$, represent $\frac{p}{q}$ as (p, q). Then the countability of $\mathbb{Q}$ follows from theorem 2.1.3. ■

Theorem 2.1.4. *Let A be the set of all sequences of 0 and 1; then A is uncountable.*

Proof. Let EA be countable, and let E consist of all the sequences of 0 and 1, $s_1, s_2, s_3, \dots$. Construct the sequence s such that if the n-th term of the sequence s_i is 0, then the n-th term of s is 1, and vice versa, for $i \in \mathbb{Z}^+$. Then the sequence s differs from the sequence s_i at atleast one place; thus $s \notin E$, but $s \in A$. Therefore $E \subset A$, which establishes the uncountablitiy of A. ■

2.2 Metric Spaces

Definition. A set X, whose elements we will call **points**, is said to be a **metric space** if there exists a mapping $d : X \times X \to \mathbb{R}$, called a **metric** (or **distance function**) such that for $x, y \in X$

(1) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$.

(3) $d(x, y) \leq d(x, z) + d(z, y)$ (The Triangle Inequality).

Example 2.2. The absolute value, $|\cdot|$ for real numbers, the modulus $|\cdot|$ for complex numbers, and the norm $||\cdot||$ for vectors are all metrics. They turn $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{R}^k$ into metric spaces respectively.

Definition. An **open interval** in $\mathbb{R}$ (or **segment**) is a set of the form $(a, b) = \{a, b \in \mathbb{R} : a < x < b\}$, a **closed interval** in $\mathbb{R}$ is a set of the form $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$; and **half open intervals** in $\mathbb{R}$are sets of the form $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ and $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$.

If $a_i < b_i$, for $1 \leq i \leq k$, the set of all points $(x_1, \ldots, x_k) \in \mathbb{R}^k$ which satisfy the Inequalities $a_i \leq x_i \leq b_i$ is called a **k-cell** in $\mathbb{R}^k$. If $x \in \mathbb{R}^k$, and $r > 0$, we call the set $B_r(x) = \{y \in \mathbb{R}^k : ||x - y|| < r\}$ an **open ball** in $\mathbb{R}^k$, and we call the set $B_r[x] = \in \mathbb{R}^k : ||x - y|| \leq r\}$ a **closed ball** in $\mathbb{R}^k$.

Definition. We call a set $E \subseteq \mathbb{R}^k$ **convex**, if whenever $x, y \in E$, $\lambda x + (1 - \lambda)y \in E$ for $0 < \lambda < 1$.

Lemma 2.2.1. *Open and closed balls, along with k-cells are convex.*

Proof. Let $B_r(x)$ be an open ball; let $y, x \in B_r(x)$, and $0 < \lambda < 1$. Then $||x - (\lambda y + (1 - \lambda)z|| = ||\lambda(x - y) - (1 - \lambda)(x - z)|| \leq \lambda||x - y|| + (1 - \lambda)||x - z|| < \lambda r + (1 - \lambda)r$. The proof is analogous for closed ball.

Now let K be a k-cell for $a_i < b_i$, for $1 \leq i \leq k$, let $x, y \in K$, then $a_i \leq x_i, y_i \leq b_i$, so $\lambda a_i \leq \lambda x_i \leq \lambda b_i$, and $(1 - \lambda)a_i \leq (1 - \lambda)y_i \leq (1 - \lambda)b_i$, since $0 < \lambda < 1$, $a_i \leq a_i + (1 - \lambda)a_i \leq \lambda x_i + (1 - \lambda)y_i \leq \lambda b_i + (1 -)b_i \leq b$. ■

Corollary. *Open and closed intervals, along with half open intervals are convex.*

Proof. We just notice that open and closed intervals are open and closed balls in $\mathbb{R}^1 = \mathbb{R}$, we also notice that half open intervals $[a, b)$ and $(a, b]$ are subsets of the closed interval $[a, b]$, and hence inherit convexity. ■

For the following definitions, let X be a metric space with metric d.

Definition. A **neighborhood** of a point $x \in X$ is the set $N_r(x) = \{y \in X : d(x, y) < r\}$ for some $r > 0$ called the **radius** of the neighborhood. We call x a **limit point** of a set $E \subseteq X$ if every neighborhood of x contains a point $y \neq x$ such that $y \in E$. If $y \in E$, and y is not a limit point, we call y an **isolated point**.

Definition. We call a set $E \subseteq X$ **closed** if every limit point of E is in E. A point $x \in X$ is an **interior point** of E if there is a neighborhood N of x such that $N \subseteq E$. We call E **open** if every point of E is an interior point of E.

Definition. $E \subseteq X$ is called **prefect** if E is closed, and every point of E is a limit point of E. We call E **dense** if every point of X is either a limit point of E, or a point of E, or both.

Lemma 2.2.2. *If $E \subseteq X$, then E is perfect in X if and only if $\overline{E} = E$.*

Lemma 2.2.3. *If EX is dense in X, then either E is perfect in X, or $X = E$, or both.*

Definition. We call $E \subseteq X$ **bounded** if there is a real number $M > 0$, and a point $y \in X$ such that $d(x, y) < M$ for all $x \in E$.

Theorem 2.2.4. *Let X be a metric space and $x \in X$. Every neighborhood of x is open.*

Proof. Consider the neighborhood $N_r(x)$, and $y \in E$, there is a positive real number h such that $d(x, y) = r - h$, then for $z \in X$ such that $d(y, s) < h$, we have $d(x, s) \leq d(x, y) + d(y, s) < r - h + h = r$, thus $s \in E$, so y is an interior point of E. ∎

Theorem 2.2.5. *If x is a limit point of a set E, then every neighborhood of x contains infinitely many points of E.*

Proof. Let N be a neighborhood of x contai ning only a finite number points of E. Let $y_1, \ldots, y_n$ be points of $N \cap E$ distinct from x and let $r = \min\{d(x, y_i)\}$ for $1 \leq i \leq n$, then $r > 0$, and the neighborhood $N_r(x)$ contains no point y of E for which $y \neq x$, so x is not a limit point; which is a contradiction. ∎

Corollary. *A finite point set has no limit points.*

Proof. By theorem 2.2.5, if x is a limit point in the finite point set E, then evry neoghborhood of contains infinitely many points of E; contradicting its finiteness. ∎

Example 2.3. (1) The set of all $z \in \mathbb{C}$ such that $|z| < 1$ is open, and bounded.

(2) The set of all $z \in \mathbb{C}$ for which $|z| \leq 1$ is closed, perfect, and bounded.

(3) Any nonempty finite set is closed, and bounded.

(4) $\mathbb{Z}$ is closed, but it is not open, perfect, or bounded.

(5) The set $\frac{1}{\mathbb{Z}^+}$ is neither closed, nor open, it is not perfect; but it is bounded..

(6) $\mathbb{C}$ is closed, open, and perfect, but it is not bounded.

(7) The open interval in (a, b) is open (only in $\mathbb{R}$), and bounded.

Theorem 2.2.6. *Let X be a metric space, a set $E \subseteq X$ is open if and only if $X \backslash E$ is closed.*

Proof. Suppose that $X \backslash E$ is closed, let $x \in E$, then $x \notin X \backslash E$, and x is not a limit point of $X \backslash E$. Thus there is a neighborhood N of x such that $N \cap E = \emptyset$, thus $N \subseteq E$, and so x is an interior point of E.

Conversely, suppose that E is open, and let x be a limit point of $X \backslash E$, then every neighborhood of of x contains a point of $X \backslash E$, so x is not an interior point of E, since E is open, it follows that $x \in X \backslash E$, thus $X \backslash E$ is closed. ∎

Corollary. *E is closed if and only if $X \backslash E$ is open.*

Proof. This is the converse of theorem 2.2.5. ∎

Theorem 2.2.7. *Let X be a metric space. The following are true:*

(1) If $\{G_\alpha\}$ is a collection of open sets, then $\bigcup G_\alpha$ is open.

(2) If $\{G_i\}_{i=1}^n$ is a finite collection of open sets, then $\bigcap_{i=1}^n G_i$ is open.

(3) if $\{G_\alpha\}$ is a collection of closed sets, then $\bigcap G_\alpha$ is closed.

(4) If $\{G_i\}_{i=1}^n$ is a finite collection of closed sets, then $\bigcup_{i=1}^n G_i$ is closed.

Proof. Let $G = \bigcup G_\alpha$, then if $x \in G$, $x \in G_\alpha$ for some α, then x is an interior point of G_α, hence an interior point of G, so G is open. Now let $G = \bigcap_{i=1}^n G_i$ For $x \in G$, there are neighborhoods N_i of x, with radii r_i such that $N_i \subseteq G_i$ for $1 \leq i \leq n$. Then let $r = \min\{r_1, \ldots, r_n\}$, and let N be the neighborhood of x with radius r, then $N \subseteq G_i$, hence $N \subseteq G$, so G is open.

The proofs of (3) and (4) are just the converse of the proofs of (1) and (2). ■

Definition. Let X be a metric space, and let $E \subseteq X$, and let E' be the set of all limit points of E. We define the **closure** of E to be the set $\overline{E} = E \cup E'$.

Theorem 2.2.8. *If X is a metric space, and $E \subseteq X$, then the following hold*

(1) $\overline{x}$ is closed.

(2) E is closed if and only if $E = \overline{E}$.

(3) If $F \subseteq X$ such that $E \subseteq F$, and F is closed, then $\overline{E} \subseteq F$.

Proof. If $x \in X$, and $x \notin \overline{E}$, then $x \notin E$, nor is it a limit point of E, thus there is a neighborhood of x that is disjoint from E, hence $X \backslash \overline{E}$ is open.

Now if E is closed, then $E' \subseteq E$, so $\overline{E} = E$, conversely, if $E = \overline{E}$, then clearly E is closed. Now if F is closed and $E \subseteq F$, then $F' \subseteq F$, and $E' \subseteq F$, therfore $\overline{E} \subseteq F$. ■

Theorem 2.2.9. *Let $E\mathbb{R}$ be nonempty and bnounded above, let y $supE$, then $y \in \overline{E}$, hence $y \in E$ if E is closed.*

Proof. Suppose that $y \notin E$, then for every $h > 0$, there exists a point $x \in E$ such that $y - h < x < y$, then y is a limit point of E, thus $y \in \overline{E}$. ■

Theorem 2.2.10. *Let $Y \subseteq X$; a subset E of Y is open in Y if and only if $E = Y \cap G$ for some open subset G of X.*

Proof. Suppose E is open in Y, then for each $x \in E$, there is a $r_p > 0$ such that $d(x, y) < r_p$, if $y \in Y$, that implies that $y \in E$; hence let V_x be the set of all $y \in X$ such that $d(x, y) < r_p$, and define

$$G = \bigcup_{x \in E} V_p$$

Then by theorems 2.2.2 and 2.2.6, G is open in X, and $EG \cap Y$. Now we also have that $V_p \cap YE$, thus $G \cap YE$, thus $E = G \cap Y$. Conversely, if G is open in X, and $E = G \cap Y$, then every $x \in E$ has a neighborhood $v_p \in G$, thus $V_p \cap Y \subseteq E$, hence E is open in Y. ■

2.3 Compact Sets

Definition. Let X be a metric space, and let $E \subseteq X$. An **open cover** of E is a collection $\{G_\alpha\}$ of subsets of X such that $E \subseteq \bigcup G_\alpha$. We call a collection $\{E_\beta\}$ of subsets of X an **open subcover** of E if $\{E_\beta\}$ is a cover of E, and $\bigcup E_\beta \subseteq \bigcup G_\alpha$. We call E **compact** if every open cover of E contains a finite open subcover.

Lemma 2.3.1. *Every finite set is compact.*

Proof. Let K be finite, and let $\{G_\alpha\}$ be an open subcover of K. Since K is finite, there is a 1-1 mapping of K onto the set $\{1, \ldots, n\}$. Let $\{E_i\}_{i=1}^n$ be the finite collection of all subsets of K, clearly, $\{E_i\}$ is an open cover of K. Moreover, if $\bigcup E_i \subseteq \bigcup G_\alpha$, we are done, and if $\bigcup G_\alpha \subseteq \bigcup E_i$, then $\{G_i\}$ is a finite subcollection that covers K, so in either case, K is compact. ■

Theorem 2.3.2. *Let X be a metric space, and let $K \subseteq Y \subseteq X$. Then Y is compact in X if and only if K is compact in Y.*

Proof. Suppose K is compact in Y, and let $\{G_\alpha\}$ be a collection of subsets of Y X that cover K, and let $V_\alpha = Y \cap G_\alpha$, then $\{V_\alpha\}$ is a collection of subsets of X covering K, in which $V_\alpha \subseteq G_\alpha$ for all α, therefore K is compact in Y

conversely, suppose that K is compact in X, and let $\{V_\alpha\}$ be a collection of open sets in Y sucht that $K \subseteq \bigcup V_\alpha$, by theorem 2.2.10, there is a collection $\{G_\alpha\}$ of open sets in Y such that $V_\alpha = Y \cap G_\alpha$, for all α. Then $K \subseteq \bigcup_{i=1}^n G_{\alpha_i}$; therefore, K is compact in Y. ■

Theorem 2.3.3. *Compact subsets of metric spaces are closed.*

Proof. Let X be a metric space, and let K be compact in X and let $x \in X \backslash K$, if $y \in K$, let U and V be neighborhoods of x and y respectively, each of radius $r < \frac{1}{2}d(x, y)$. Since K is compact, there are finitely many points $y_1, \ldots y_n$ such that $K \bigcup_{i=1}^n V_i = V$, where V_i is a neighborhood of y_i for $1 \leq i \leq n$. Let $U = \bigcap_{i=1}^n U_i$, then $V \cap W$ is empty, hence $UX \backslash V$, therefore, $x \in X \backslash K$, therefore K is closed. ■

Theorem 2.3.4. *Closed subsets of compact sets are compact.*

Proof. Let X be a metric space with $F \subseteq KX$, with F closed in X, and K compact. Let $\{V_\alpha\}$ be an open cover of F. If we append $X \backslash F$ to $\{V_\alpha\}$, we get an open cover Θ of K, and since K is compact, there is a finite subcollection Φ which covers K, so Φ is an open cover of F, $X \backslash F \Phi$, then $\Phi \backslash (X \backslash F)$ still covers F, therfore F is compact. ■

Theorem 2.3.5. *Let $\{K_\alpha\}$ be a collection of compact sets of a metric space X, such that every finite subcollection of $\{K_\alpha\}$ is nonempty. Then $\bigcap K_\alpha$ is nonempty.*

Proof. Fix $K_1 \subseteq \{K_\alpha\}$, and let $G_\alpha = X \backslash K_\alpha$. Suppose no point of K_1 is in $\bigcap K_\alpha$, then $\{G_\alpha\}$ covers K_1, and since K is compact, we have $K_1 \bigcup_{i=1}^n G_{\alpha_i}$, for $1 \leq i \leq n$, which implies that $\bigcap K_\alpha$ is empty, a contradiction. ■

Corollary. *If $\{K_\alpha\}$ is a sequence of nonempty compact sets, such that $K_{n+1} \subseteq K_n$, then $\bigcap_{i=1}^\infty K_n$ is nonempty.*

Theorem 2.3.6. *If E is a infinite subset of a compact set K, then E has a limit point in K.*

Proof. Suppose no point of K is a limit point of E, then for all $x \in K$, the neighborhood U_x contains atmost one point in E. Then no finite subcollection of $\{U_x\}$ covers E, which contradicts the compactness on K. ■

Theorem 2.3.7 (The Nested Interval Theorem)**.** *if $\{I_n\}$ is a sequence of intervals in $\mathbb{R}$ such that $I_{n+1} \subseteq I_n$, then $\bigcap_{j=1}^{\infty} I_n$ is nonempty.*

Proof. We let $I_n = [a_n, b_n]$. Letting E be the set of all a_n, E is nonempty and bounded above by b_1. Letting $x = \sup E$, and $m \geq n$, we have $[a_m, b_m] \subseteq [a_n, b_n]$, thus $a_m \leq x \leq b_m$ for all m, thus $x \in I_m = \bigcap_{j=i}^{n} I_j$ ■

Theorem 2.3.8. *Let $k \in \mathbb{Z}^+$, and $\{I_n\}$ be a nonempty sequence of k-cells of $\mathbb{R}^k$ such that $I_{n+1} I_n$. Then $\bigcap_{j=1}^{\infty} I_n$ is nonempty.*

Proof. Let I_n be the set of all points $x \in \mathbb{R}^k$ such that $a_{n,j} \leq x_j \leq b_{n,j}$, and let $I_{n,j} = [a_{n,j}, b_{n,j}]$. Then for each $1 \leq j \leq k$, by the nested interval theorem, $\bigcap_{l=1}^{\infty} I_{l,j}$ is nonempty, hence there are real numbers x'_j such that $a_{n,j} \leq x'_j \leq b_{n,j}$. Letting $x' = (x'_1, \ldots, x'_k)$, we get that $x' \in I \bigcap_{l=1}^{\infty} I_l$ ■

Theorem 2.3.9. *Every k-cell is compact.*

Proof. Let I be a k-cell, and let $\delta = \sqrt{\sum_{j=1}^{k} a(b_j - a_j)^2}$ we get for $x, y \in I$, $||x - y|| \leq \delta$. Now suppose there is an open cover $\{G_\alpha\}$ of I for which no finite subcover is contained. Let $c_j = \frac{a_j + b_j}{2}$, then the closed intervals $[a_j, c_j]$, $[c_j, b_j]$ determine the 2^k k-cells Q_i such that $\bigcup Q_i = I$. Then atleast one Q_i cannot be covered by any finite subcollectio of $\{G_\alpha\}$. Subdividing Q_1, we get a sequence $\{Q_n\}$ such that $Q_{n+1} \subseteq Q_n$, Q_n is not covered by any finite subcollection of $\{G_\alpha\}$, and $||x - y|| \leq \frac{\delta}{2^n}$ for $x, y \in Q_n$. Then by theorem 2.3.8, there is a point $x' \in Q_n$, and for some α, $x' \in G_\alpha$; since G_α is open, there is an $r > 0$ for which $||x - || < r$ implies $y \in G_\alpha$. THen for n sufficiently large, we have that $\frac{\delta}{2^n} < r$, then we get that $Q_n \in G_\alpha$, which is a contradiction. ■

Theorem 2.3.10 (The Heine-Borel Theorem)**.** *If E is a subset of $\mathbb{R}^k$, then the following are equivalent:*

(1) E is closed and bounded.

(2) E is compact.

(3) Every infinite subset of E has a limit point in E.

Proof. Suppose that E is closed and bounded, then $E \subseteq I$ for some k-cell I in $\mathbb{R}^k$, and hence it is compact. By theorem 2.3.4, E is compact. Now suppose that E is compact, then by theorem 2.3.6, every infinite subset of E has a limit point in E.

Now suppose that every infinite subset of E has a limit point in E. If E is not bounded, then $||x_n|| > n$ for some $x_n \in E$ and $n \in \mathbb{Z}^+$. Then the set of all such x_n is infinite, and has no limit point in E, a contradiction; moreover suppose that E is not closed. Then there is a point $x_0 \in \mathbb{R}^k \backslash E$, which is a limit point of E. Then there are points $x_n \in E$ for which $||x_n - x_0|| < \frac{1}{n}$, let S be the set of all such points. Then S is infinite and has x_0 as its only limit point; for if $y \neq x_0 \in \mathbb{R}^k$, then $\frac{1}{2}||x_0 - y|| \leq ||x_0 - y|| - \frac{1}{n} \leq ||x_0 - y|| - ||x_n - x_0|| \leq ||x_n - y||$ for only some n. Thus by theorem 2.2.3, y is not a limit point of S Therefore, if every infinite subset of E has a limit point in E, E must be closed. ∎

Theorem 2.3.11 (The Bolzano-Weierstrass Theorem)**.** *Every bounded infinite subset E of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.*

Proof. We have that $E \subseteq I$, for some k-cell I in $\mathbb{R}^k$. Since k-cells are compact, by the Heine-Borel theorem, E is also compact and has a limit point in I. ∎

2.4 Perfect Sets

Theorem 2.4.1. *If $P \subseteq \mathbb{R}^k$ is a nonempty perfect set, then P is uncountable.*

Proof. Since every point of P is a limit point of P, we gave that P must be infinite. Then suppose that P is countable. For points $x_n \in P$, construct the sequence $\{U_n\}$ of neighborhoods of x_n, for $n \in \mathbb{Z}^+$; now by induction, if U_1 is a neighborhood of x_1, then for $y \in \hat{U_1}$, $||x_1 - y|| \leq r$ for some $r > 0$. Now suppos the neighborhood U_n of x_n has been constructed such that $U_n \cap P$ is nonempty. Then there is a neighborhood U_{n+1} fo x_{n+1} such that $\hat{U_{n+1}} \subseteq U_n$, $x_n \notin \hat{U_{n+1}}$, and $U_{n+1} \cap P$ is nonempty. Therefore there is a nonempty $K_n = U_n \cap P$. Since $\hat{U_n}$ is close and bounded, $\hat{U}$ is compact, and since $x_n \notin K_{n+1}$, $x_n \notin \bigcap_{i=1}^{\infty} K_i$, and since $K_n \subseteq P$, $\bigcap K_i$ is empty, a contradiction. ∎

Corollary. *Let $a < b$ be real numbers. Then the closed interval $[a, b]$ is uncountable. Moreover, $\mathbb{R}$ is uncountable.*

Proof. We have $[a, b]$ is closed, and perfect (since $(a, b)[a, b] isperfect)$, thus $[a, b]$ is uncountable. Moreover, take $f : \mathbb{R} \to [a, b]$, by $f(x) = \frac{a+b}{2}x$; then f is a 1-1 mapping of $\mathbb{R}$ onto $[a, b]$, which makes $\mathbb{R}$ uncountable. ∎

Theorem 2.4.2 (The construction of the Cantor set)**.** *There exists a perfect set in $\mathbb{R}$ which contains no open interval.*

Proof. Let $E_0 = [0, 1]$, and remove $(\frac{1}{3}, \frac{2}{3})$, and let $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Now remove the open intervals $(\frac{1}{9}, \frac{2}{9})$ $(\frac{3}{9}, \frac{6}{9})$, $(\frac{7}{9}, \frac{8}{9})$, and let $E_2 = [0, \frac{2}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{7}{8}, \frac{8}{9}]$. Continuig the remove the middle third of each interval, we obtain the sequence of compact sets $\{E_n\}$, such that $E_{n+1} E_n$, and E_n is the union of 2^n closed intervals of length $\frac{1}{3^n}$. Then let:

$$P = \bigcap_{i=1}^{\infty} E_i \tag{2.1}$$

Then P is nonempty, and compact.

Now let I be the open interval of the form $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})$, with $k, m \in \mathbb{Z}^+$. Then by the construction of P, I has no point in P, we also see that every other open interval contains a subinterval of the form of I; them P contains no open interval.

Now let $x \in P$, and let S be any open interval for which $x \in S$. LEt I_n be the closed interval of E_n such that $x \in I_n$. Choose n sufficiently large such that $I_n S$. If $x_n \neq x$ is an endpoint of I_n, then $x_n \in P$, and so x is a limit point of P. Therefore P is perfect. ■

Definition. The we call the set P constructed in the proof of theorem 2.4.2 the **Cantor set**.

2.5 Connected Sets

Definition. Two subsets A and B of a metric space X are **seperated** if $A \cap \hat{B}$ and $\hat{A} \cap B$ are both empty. We say a subset E of X is **connected**, if E is not the union of two nonepmty speperated sets.

Theorem 2.5.1. *A subset E of $\mathbb{R}$ is connected if and only if $x, y \in E$ and $x < z < y$ imply $z \in E$.*

Proof. Let $x, y \in E$ such that for some $z \in (x, y))$, $z \notin E$. Then $E = A \cup B$, with $A = E \cup (-\infty, z)$ and $B = E \cup (z, \infty)$. Then A and B are seperated, which contradicts the connectedness of E.

Conversely suppose for $x, y \in E$, that $z \in E$ for $z \in (x, y)$. Then there are nonempty seperated sets A and B such that $A \cup B = E$. Choose $x \in A$, $y \in B$ such that $x < y$, and let $z = \sup(A \cap [x, y])$. Then by theorem 2.2.8, $z \in \hat{A}$, so $z\ notinB$. In particular, $x \leq x < y$. Now if $z \notin A$, then $x < z < y$, with $z \notin E$. Now if $z \in A$, then $z \notin \hat{B}$, hence there is a z' such that $z < z' < y$, and $z' \notin B$. Then $x < z' < y$ and $z' \not inE$. ■

Chapter 3

Sequences

3.1 Convergent Sequences

Definition. A sequence $\{x_n\}$ in a metric space X is said to **converge** if there is a point $x \in X$ such that for every $\epsilon > 0$, there is an $N \in \mathbb{Z}^+$ such that $d(x_n, x) < \epsilon$ whenever $n \geq N$. We say $\{x_n\}$ **converges** to x, and we call x the **limit** of $\{x_n\}$ as n approaches ∞. We write $x_n \to x$ as $n \to \infty$, and $\lim_{n\to\infty} x_n = x$ (or $\lim x_n = x$). If $\{x_n\}$ does not converge, we say the $\{x_n\}$ **diverges**, or **is divergent**.

Example 3.1. Consider the following sequences in $\mathbb{C}$.

(1) $\{\frac{1}{n}\}$ is bounded, and $\lim_{n\to\infty} \frac{1}{n} = 0$.

(2) The sequence $\{n^2\}$ us unbounded and diverges.

(3) $1 + \frac{(-1)^n}{n} \to 1$ as $n \to \infty$, and $\{1 + \frac{(-1)^n}{n}\}$ is bounded.

(4) $\{i^n\}$ is bounded and divergent.

(5) $\{1\}$ is bounded and converges to 1.

Theorem 3.1.1. *Let $\{x_n\}$ be a sequence in a metric space, then:*

(1) $\{x_n\}$ converges to $x \in X$ if and only if every every neighborhood of x contains x_n for all but finitely many n.

(2) If $\{x_n\}$ converges to x, and x', then $x = x'$.

(3) If $\{x_n\}$ converges, then x_n is bounded.

(4) If $E \subseteq X$, and x is a limit point of E, then there is a sequence in E that converges to x.

Proof. Suppose $x_n \to x$, and let U be a neighborhood of x. For some $\epsilon > 0$, there is an $N \in \mathbb{Z}^+$ for which $d(x_n, x) < \epsilon$, whenever $n \geq N$, thus $x_n \in U$ for finitely many n. Conversely, suppose that $x_n \in U$ for some $n \geq N$, then letting $\epsilon > 0$, we havae $d(x, x_n) < \epsilon$, hence $x_n \to x$.

Let > 0, then there are $N_1, N_2 \in Z^+$ such that $d(x_n, x) < \frac{\epsilon}{2}$, and $d(x_n, x') < \frac{\epsilon}{2}$. Then choosing $N = \max\{N_1, N_2\}$, and letting ϵ be arbitrarily small, we have $d(x, x') \leq d(x, x_n) + d(x_n, x') < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$; and so we get that $x = x'$.

Let $x_n \to x$, then there is an $N \in \mathbb{Z}^+$ for which $d(x_n, x) < 1$ whenever $n \geq N$. Letting $r = \max\{1, d(x_N, x)\}$, then $d(x_n, x) \leq r$.

Finally, let x be a limit point of E, then for each $n \in Z^+$, there is an $x_n \in E$ such that $d(x, x_n) < \frac{1}{n}$, choose $N > \frac{1}{\epsilon}$, then whenever $n\ geqN$, $d(x, x_n) < \epsilon$; hence $x_n \to x$. ■

Theorem 3.1.2. *Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathbb{C}$, and that $limx_n = x$, $\lim y_n = y$ as $n \to \infty$. Then the following hold as $n \to \infty$:*

(1) $\lim (x_n + y_n) = \lim x_n + \lim y_n = x + y$.

(2) $limx_n y_n = \lim x_n \lim y_n = xy$.

(3) $lim\frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n} = \frac{x}{y}$*; given that* $y_n, y \neq 0$.

Proof. (1) Let > 0, then for $N_1, N_2 \in \mathbb{Z}^+$, $|x_n - x| < \frac{\epsilon}{2}$ and $|y_n - y| < \frac{\epsilon}{2}$. Then choose $N = \max\{N_1, N_2\}$, then whenever $n \geq N$, we have $|(x_n+y_n)-(x+y)| \leq |x_n-x|+|y_n-y| < \epsilon$.

(2) Notice that $x_n y_n - xy = (x_n - x)(y_n - y) + x(y_n - y) + y(x_x - x)$, then for $N_1, N_2 \in \mathbb{Z}^+$, $|x_n - x| < \sqrt{\epsilon}$, and $|y_n - y| < \sqrt{\epsilon}$. Then choosing $N = \max\{N_1, N_2\}$, then $|(x_n - x)(y_n - y)| < \epsilon$, thus we have $|x_n y_n - xy| \leq |(x_n - x)(y_n - y)| + |x(y_n - y)| + |y(x_x - x)| < \epsilon$.

(3) We first show that $\frac{1}{y_n} \to \frac{1}{y}$, given that $y_n, y \neq 0$. Choose m such that $|y_n - y| < \frac{1}{2}|y|$ whenever $n \geq m$, then $|y_n| > \frac{1}{2}|y|$. Then for $\epsilon > 0$, there is an $N > m$ such that whenever $n \geq N$, $|y_n - y| < \frac{1}{2}|y|^2\epsilon$. Then $|\frac{1}{y_n} - \frac{1}{y}| \leq \frac{|y_n - y|}{|y_n y|} < \frac{2}{|y|^2}|y_n - y| < \epsilon$. Then choosing the sequences $\{x_n\}$ and $\{\frac{1}{y_n}\}$, the rest follows.

■

Corollary. *(1) For any $c \in \mathbb{C}$, and a sequene $x_n \to x$, we have $\lim cx_n = c \lim x_n = cx$ and $\lim (c + x_n) = c + \lim x_n = c + x$ as $n \to \infty$.*

(2) Provided that $x, x_n \neq 0$, we have $\lim \frac{1}{x_n} = \frac{1}{\lim x_n} = \frac{1}{x}$, as $n \to \infty$.

Proof. We choose $\{x_n\}$ and $\{y_n\} = \{c\}$ for all n, then the results follow. ■

Theorem 3.1.3. *(1) Let $x_n = (\alpha_{1n}, \ldots \alpha_{kn}) \in \mathbb{R}^k$. Then $\{x_n\}$ converges to x if and only if $\lim \alpha_{jn} = \alpha_j$ for $1 \leq j \leq k$, as $n \to \infty$.*

(2) Let $\{x_n\}, \{y_n\}$ be sequences in $\mathbb{R}^k$, and let $\{\beta_n\}$ be a sequence in $\mathbb{R}$ such that $x_n \to x$, $y_n \to y$, and $\beta_n \to \beta$. Then $\lim (x_n + y_n) = x + y$, $\lim x_n y_n = xy$, and $lim\beta_n x_n = \beta x$.

Proof. If $x_n \to x$, then $|\alpha_{jn} - \alpha_j| \leq ||x_n - x|| < \epsilon$, thus $\lim \alpha_{jn} = \alpha_j$. Conversely, suppose that $\alpha_{jn} \to \alpha_j$. Then for $\epsilon > 0$ there is an $N \in \mathbb{Z}^+$ such that $n \geq N$ implies $|\alpha_{jn} - \alpha_j| < \frac{\epsilon}{\sqrt{k}}$. Then for $n \geq N$,

$$||x_n - x|| = \sqrt{\sum |\alpha_{jn} - \alpha_j|^2} < epsilon$$

To prove (2), we appy part (1) of this theorem together with theorem 3.1.2. ■

Theorem 3.1.4 (The Sandwhich Theorem)**.** *Let $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ be sequences in $\mathbb{R}$, and Suppose that $\lim x_n = \lim y_n = a$ and that there is an $N \in \mathbb{Z}^+$ such hat $x_n \leq w_n \leq y_n$ for all $n \geq N$. Then $\lim_{n\to\infty} w_n = a$.*

Proof. Let $\epsilon > 0$ and let $\{x_n\}$ and $\{y_n\}$ both converge to a. Then by definition there are $N_1, N_2 \in \mathbb{Z}^+$ such that $|x_n - a| < \epsilon$ and $|y_n - a| < \epsilon$ for $n \geq N_1, N_2$. Now choose $N = \max\{N_0, N_1, N_2\}$, if $n \geq N$, we have $-\epsilon < x_n - a < \epsilon$, and we also have $x_n - a < w_n - a < y_n - a$, thus we have that:

$$-\epsilon < x_n - a < w_n - a < y_n - a < \epsilon$$

Thus we have that $|w_n - a| < \epsilon$. ■

Corollary. *If $x_n \to \infty$ as $n \to \infty$, and $\{y_n\}$ is bounded, then $x_n y_n \to 0$ as $n \to \infty$.*

Proof. We have that $\{y_n\}$ is bounded, hence, there is $M > 0$ such that $|y_n| < M$ for all $n \in \mathbb{Z}^+$. And since $\{x_n\}$ converges to 0 we have that for any ϵ there is an $N \in \mathbb{Z}^+$ such that for $n \geq N$, $|x_n - 0| < \frac{\epsilon}{M}$. For $|x_n y_n - 0| = |x_n y_n| < M|x_n| < M\frac{\epsilon}{M} = \epsilon$. Therefore, $x_n y_n \to 0$ as $n \to \infty$. ■

Corollary. *Let $\{x_n\}$, $\{y_n\}$ be sequences such that $0 \leq x_n \leq y_n$ for $n \geq N > 0$. Then if $y_n \to 0$, then $x_n \to 0$ as $n\infty$.*

Proof. This is a special case of the sandwhich theorem. ■

3.2 Subsequences

Definition. Let $\{x_n\}$ be a sequence, and let $\{n_k\}\mathbb{Z}^+$ such that $n_k < n_{k+1}$. We call the sequence $\{x_{n_k}\}$ a **Subsequence** of $\{x_n\}$. If $\{x_{n_k}\}$ converges, we call its limit the **subsequential limit** of $\{x_n\}$.

Theorem 3.2.1. *A sequence $\{x_n\}$ converges to a point x if and only if every subsequence $\{x_{n_k}\}$ converges to x.*

Proof. Clearly if $x_n \to x$, then every subsequence $x_{n_k} \to x$, (since subsequences can be thought of as subsets of thier parent sequences). On the other hand, let $x_{n_k} \to x$ for $\{_k\} \subseteq \mathbb{Z}^+$. Then for $\epsilon > 0$, there is a $K \in \mathbb{Z}^+$ for which $d(x_{n_k}, x) < \frac{\epsilon}{2}$ for $k \geq K$. Let $N \in \mathbb{Z}^+$, and choose $n \geq max\{N, K\}$, then $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, d) < \epsilon$. ■

Theorem 3.2.2. *If $\{x_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{x_n\}$ converges to a point x.*

Proof. If $\{x_n\}$ is finite, then thre is an $x \in \{x_n\}$ and a sequence $\{n_k\}$ with $n_k < n_{k+1}$ such that $x_{n_i} = x$ for $1 \leq i \leq k$, then the subsequence converges to x.

Now if $\{x_n\}$ is infinite, there is a limit point $x \in X$ of $\{x_n\}$, then choose n_i such that $d(x, x_i) < \frac{1}{i}$ for $1 \leq i \leq k$. Obtaining $\{n_k\}$ from this, we see that $n_k < n_{k+1}$, and so we get that $\{x_{n_k}\}$ converges to x. ■

Corollary. *Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.*

Theorem 3.2.3. *The subsequential limits of $\{x_n\}$ is a metric space X form a closed subset of X.*

Proof. Let E be the set of all subsequential limits of $\{x_n\}$, and let x be a limit point of E. Choose n_i such that $x_{n_i} \neq x$ and let $\delta = d(x, x_{n_i})$, for $1 \leq i \leq k$. Then consier the sequence $\{n_k\}$, since x is a limit point of E, there is an $x' \in E$ for which $d(x, x') < \frac{\delta}{2^i}$. Thus there is an $N_I > n_i$ such that $d(x', x_{n_i}) < \frac{\delta}{2^i}$, thus $d(x, x_{n_i}) < \frac{\delta}{2^i}$. So $\{x_n\}$ converges to x and $x \in E$. ■

3.3 Cauchy Sequences

Definition. We call a sequence $\{x_n\}$ in a metric space X a **Cauchy sequence** in X, or more simply, **Cauchy** in X if for all $\epsilon > 0$, there is an $N \in \mathbb{Z}^+$ such that $d(x_n, x_m) < \epsilon$ whenever $m, n \geq N$.

Definition. Let E be a nonempty subset of a metrix space X, and lelt $S \subseteq \mathbb{R}$ be the all real numbers $d(x, y)$, with $x, y \in E$. We call $\sup S$ the **diameter** of E, and denote it $\operatorname{diam} E$.

Theorem 3.3.1. *Let $\{x_n\}$ be a sequence, and let E_N be the set of all points p_N such that ${}_N < p_{n+1}$. Then $\{x_n\}$ is Cauchy if and only if $\lim \operatorname{diam} E_N = 0$ as $N \to \infty$.*

Proof. Let $\{x_n\}$ be Cauchy, Let $x_{N_1}, x_{N_2} \in E$ such that $d(x_n, x_{N_1}) < \frac{\epsilon}{2}$, and $d(x_{N_2}, x_m) < \frac{\epsilon}{2}$. Then we see that $d(x_{N_1}, x_{N_2}) \leq d(x_{N_1}, x_n) + d(x_m, x_{N_2}) < \epsilon$, so $\{x_{N_k}\}$ is Cauchy and we see that $\lim \operatorname{diam} E_N = 0$. Now suppose that $\lim \operatorname{diam} E = 0$, then for any $x_n, x_m \in S$, $d(x_n, 0) < \frac{\epsilon}{2}$ and $d(0, x_m) < \frac{\epsilon}{2}$ implies that $d(x_n, x_m) \leq d(x_n, 0) + d(0, x_m) < \epsilon$, whenever $n, m > N$, for $\epsilon > 0$. ■

Theorem 3.3.2. *(1) If $E \subseteq X$, then $\operatorname{diam} \hat{E} = \operatorname{diam} E$.*

(2) If $\{K_n\}$ is a sequence of compact sets in X, such that $K_{n+1} \subseteq K_n$, and if $\lim \operatorname{diam} K_n = 0$ as $n \to \infty$, then $\bigcap_{i=1}^{infty} K_i$ contains exactly one point.

Proof. Clearly $\operatorname{diam} E \leq \operatorname{diam} \hat{E}$. Now let $\epsilon > 0$, and choose $x, y \in \hat{E}$, then there are points $x', y' \in \hat{E}$ such that $d(x, x') < \frac{\epsilon}{2}$ and $d(y, y') < \frac{\epsilon}{2}$. Hence, $d(x, y) \leq d(x, x') + d(x', y') + d(y'y) < \epsilon \operatorname{diam} E$, then choosing ϵ arbitrarily small, $\operatorname{diam} \hat{E} \leq \operatorname{diam} E$.

Now, we also have that by the nested interval theorem that $K = \bigcap K_i$ is nonempty. Now suppose that K contains more that one point. then $\operatorname{diam} K > 0$, and since $K \subseteq K_n$ for all n, $diam K \leq \operatorname{diam} K_n$, a contradiction. Thus K contains exactly one element. ■

Theorem 3.3.3. *(1) In any metric space X, every convergent sequence is a Cauchy sequence.*

(2) If X is compact, and $\{x_n\}$ is Cauchy in X, then $\{x_n\}$ converges to a point in X.

Proof. (1) If $x_n \to x$, and $\epsilon > 0$ such that there is an $N \in \mathbb{Z}^+$ such that $d(x_n, x) < \frac{\epsilon}{2}$ for all $n \geq N$, then for $m \geq N$, we have $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \epsilon$. Thus $\{x_n\}$ is Cauchy.

(2) Let $\{x_n\}$ be Cauchy, and let E_N be the set of all points x_N for which $x_N < x_{N+1}$. Then $\lim \operatorname{diam} \hat{E} = 0$, then being closed in X, each $\hat{E_N}$ is compact in X, and $\hat{E_{N+1}} \subseteq \hat{E_N}$, so by theorem 3.3.2, there is a unique $x \in X$ in all of $\hat{E_N}$. Now for $\epsilon > 0$, there is an $N_0 \in \mathbb{Z}^+$ for which $\operatorname{diam} \hat{E} < \epsilon$. Then for all $x_n \in \hat{E}$, $d(x_n, x) < \epsilon$ whenever $n \geq N_0$. ■

Corollary (The Cauchcy Criterion)**.** *Every Cauchy sequence in $\mathbb{R}^k$ converges to a point in $\mathbb{R}^k$.*

Proof. Let $\{x_n\}$ be Cauchy in $\mathbb{R}^k$, define E_N as in (2), then for some $N \in \mathbb{Z}^+$, $\operatorname{diam} E < 1$, and so $\{x_n\}$ us the union of all E_n, and ther set of points $\{x_1, \dots, x_{N-1}\}$, so $\{x_n\}$ is bounded, and thus has a compact closure, it follows then that $x_n \to x$ for some $x \in \mathbb{R}^k$. ■

Definition. We call a metric space **complete** if every Cauchy sequence in the space converges.

Theorem 3.3.4. *All compact metric spaces, and all Euclidean spaces are complete.*

Example 3.2. Consider $\mathbb{Q}$ together with the metric $|x - y|$. The metric space induced on $\mathbb{Q}$ by $|\cdot|$ is not complete.

Definition. A sequence $\{x_n\}$ in $\mathbb{R}$ is said to be **monotonically increasing** if $x_n \leq x_{n+1}$, $\{x_n\}$ is said to be **monotonically decreasing** if $x_{n+1} < x_n$. We call $\{x_n\}$ **monotonic** if it is either monotonically increasing or monotonically decreasing.

Theorem 3.3.5. *A monotonic sequence converges if and only if it is bounded.*

Proof. Suppose, without loss of generality, that $\{x_n\}$ is monotonically increasing. If $\{x_n\}$ is bounded, then $x_n \leq x$, then for all $\epsilon > 0$, there is an $N \in \mathbb{Z}^+$ for which $x - \epsilon < x_N \leq x$. Then for $n \geq N$, $x_n \to x$. The converse follows from theorem 3.1.2. ■

3.4 Upper and Loweer Limits.

Let $\{_n\}$ be a sequence in $\mathbb{R}$ such that for all $M > 0$, there is an $N \in \mathbb{Z}^+$ for which $n \geq N$ implies that either $x_n \geq M$, or $x_n \leq M$. Then we write $x_n \to \infty$ and $x_n \to -\infty$, respectively.

Definition. Let $\{x_n\}$ be a sequence in $\mathbb{R}$, and let E tbe the set of all extended real numbers x such that $x_{n_k} \to x$ for some subseqence $\{x_{n_k}\}$. Then E contains all subsequential limits of $\{x_n\}$, and possible $\pm\infty$. We then call $\sup E$ the **upper limit** of E, and $\inf E$ the **lower limit** of E.

Theorem 3.4.1. *Let $\{x_n\}$ be a sequence in $\mathbb{R}$, and let E be the set of all extended real numbers x, let $s = \sup E$ and $s' = \inf E$. Then the following hold:*

(1) s,s' $\in E$.

(2) If $x > s$, and $x' > s'$, there is an $N \in \mathbb{Z}^+$ such that $n \geq N$ implies that $x' < x_n < x$.

Proof. We prove the theorem for the case of s, since it is analogous for s'.

(1) If $s = \infty$, then E is not bounded above, so neither is $\{x_n\}$, and there is a subsequence for which $x_n \to \infty$. Now if $s \in \mathbb{R}$, then E is bounded above, and has at least one subsequential limit. Then $s \in E$. Now if $s = -\infty$, then E contains only $-\infty$, and so by definition $x_n \to -\infty$.

(2) Suppose there is an $x > s$, such that $x_n \geq x$ for all n. Then there is a $y \in E$ such that $y \geq x \geq s$, a contradiction of the definition of s.

■

Example 3.3. (1) Let $\{x_n\}$ be a sequence in $\mathbb{Q}$, then every real number is a subsequential limit, and $\limsup x_n = \infty$ and $\liminf x_n = -\infty$.

(2) Let $\{x_n\} = \{\frac{(-1)^n}{1+\frac{1}{n}}\}$; then $\limsup x_n = 1$ and $\liminf X_n = -1$ as $n \to \infty$.

(3) For a sequence $\{x_n\}$ in $\mathbb{R}$, $\lim x_n = x$ if and only if $\limsup x_n = \liminf x_n = x$ as $n \to \infty$.

Theorem 3.4.2. *If $x_n \leq y_n$, for $n \geq N > 0$, then $\liminf x_n \leq \liminf y_n$ and $\limsup x_n \leq \limsup y_n$ as $n \to \infty$.*

3.5 Special Sequences

Theorem 3.5.1. *Let $n, p \in \mathbb{Z}^+$. Then the following hold as $n \to \infty$.*

(1) $\lim \frac{1}{n^p} = 0$.

(2) $\lim \sqrt[p]{n} = 1$.

(3) $\lim \sqrt[n]{n} = 1$.

(4) If $\alpha \in \mathbb{R}$, then $\lim \frac{n^\alpha}{(1+p)^n} = 0$.

(5) If $|x| < 1$, then $\lim x^n = 0$.

Proof. (1) Let $n > [p]\frac{1}{\epsilon}$; then $|\frac{1}{n^p}| < \epsilon$.

(2) If $p = 1$, we are done. If $p > 1$, let $x_n = \sqrt[n]{p} - 1$, then $x_n > 0$. By the binomial theorem, $1 + nx_n \leq (1 + x_n)^p = p$, hence $0 \leq x_n \leq \frac{p-1}{p}$. Now if $1 > p > 0$, then $\frac{1}{p} > 0$, so we notice that $0 \leq \frac{1}{x_n} \leq \frac{1}{\frac{p-1}{n}}$.

(3) Let $x_n = \sqrt[n]{n} - 1$, then $x_n \geq 0$, then by the binomial theorem again, $n = (1 + x_n)^n \geq \frac{n(n-1)}{2}x_n^2$, then $0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$.

(4) Let $k \in \mathbb{Z}^+$ such that $k > \alpha$. Then $n > 2k$,let $(1+p)^n > \binom{n}{k}p^k > \frac{n^k p^k}{2^k k!}$. So $0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k}n^{-k}$, since $\alpha - k < 0$, $n^{\alpha-k} \to 0$ and we are done.

(5) Take $\alpha = 0$, and let $x = \frac{1}{1+p}$, then the result follow.

■

Chapter 4

Continuity

4.1 Limits of Functions.

Definition. Let X, and Y be metric spaces, and let $E \subseteq X$, and let $f : E \to Y$ be a function. We say that f **converges** to a point $q \in Y$, as x **approaches** a limit point $p \in X$ if for every $\epsilon > 0$, there is a $\delta > 0$ for which $d_Y(f(x), q) < \epsilon$, whenever $0 < d_X(x, p) < \delta$. We say that q is the **limit** of f at p and we write $f \to q$ as $x \to p$, and $\lim_{x \to p} f(x) = q$, or more simply, $\lim f = q$.

Example 4.1. (1) Let $X = Y = \mathbb{R}$, under the absolute value $|\cdot|$, and let $I \subseteq \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$. Then f has a limit L as x approaches a limit point $c \in \mathbb{R}$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$. We call functions that map into $\mathbb{R}$ **real valued**.

(2) Let $X = Y = \mathbb{C}$, under the modulus $|\cdot|$, and let $D \subseteq \mathbb{R}$ be an domain, and $f : D \to \mathbb{R}$. Then f has a limit L as z approaches a limit point $w \in \mathbb{R}$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |z - w| < \delta$. We call functions that map into $\mathbb{C}$ **complex valued**.

(3) Let $X = Y = \mathbb{R}^k$, under the norm $||\cdot||$, and let $D \subseteq \mathbb{R}^k$ be an domain, and $f : D \to \mathbb{R}^k$. Then f has a limit L as x approaches a limit point $c \in \mathbb{R}^k$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that $||f(x) - L|| < \epsilon$ whenever $0 < ||x - c|| < \delta$. We call functions that map into $\mathbb{R}^k$ **vector valued**.

Theorem 4.1.1 (The Sequential Criterion)**.** *Let X and Y be metric spaces, and let $E \subseteq X$, and $f : E \to Y$ be a function, and $p \in E$ be a limit point. Then $\lim f(x) = q$ as $x \to p$ if and only if $\lim f(x_n) = q$ as $n \to \infty$ for any sequence $\{x_n\} \in E$, such that $x_n \neq p$ and $\lim x_n = p$.*

Proof. Suppose that $\lim f(x) = q$ as $x \to p$, and choose $\{x_n\} \subseteq E$ such that $x_n \neq p$ and $\lim x_n = p$ as $n \to \infty$. Then for $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ whenever $0 < d_X(x, p) < \delta$, and since $d_X(x_n, p) < \delta$ whenever $n \geq N$ for some $N > 0$, we have $d_Y(f(x_n), q) < \epsilon$ whenever $d_X(x_n, p) < \delta$.

Conversely, suppose that $\lim f \neq q$, that is for some $\epsilon > 0$, $d_Y(f(x), q) > \geq \epsilon$ whenevr $d_X(x, p) < \delta$ for all $\delta > 0$. Then choose $\delta = \frac{1}{n}$, for $n \in \mathbb{Z}^+$, then we have $\lim x_n = p$, but $\lim f(x_n) \neq q$. ■

The importance of the sequential criterion is that it lets us translate theorems about limits of sequences into theorems about limits of functions.

Corollary. *If f has a limit at p, then the limit of f is unique.*

Definition. Letting $f, g : E \to Y$, we define the **sum**, **product**, **scalar product** and the **quotient** of f and g to be the functions from E into Y:

(1) $f + g(x) = f(x) + g(x)$.

(2) $fg(x) = f(x)g(x)$.

(3) $(\lambda f)(x) = \lambda f(x)$ for $\lambda \in X$.

(4) $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, provided that $g(x) \neq 0$.

It is well known that the set of all functions from E into Y form an algebra under these operations.

Theorem 4.1.2. *Let $E \subseteq X$ a metric space, and let $p \in E$ be a limit point. Let $f, g : E \to Y$ be functions, such that $\lim f = A$ and $\lim g = B$ as $x \to p$. Then the following hold as $x \to p$.*

(1) $\lim\,(f + g) = \lim f + \lim g = A + B$.

(2) $\lim fg = \lim f \lim g = AB$.

(3) $\lim \frac{f}{g} = \frac{\lim f}{\lim g} = \frac{A}{B}$, *provided that* $B \neq 0$.

Corollary. *The following hold:*

(1) $\lim \lambda f = \lambda \lim f = \lambda A$, *and* $\lim\,(\lambda + f) = \lambda + \lim f = \lambda + A$.

(2) $\lim \frac{1}{f(x)} = \frac{1}{\lim f} = \frac{1}{A}, provided that A \neq 0$.

Theorem 4.1.3 (The Sandwich Theorem)**.** *Let f, g, and h be real valued functions defined on $\mathbb{R}$ such that $limf = \lim g = A$ as $x \to p$, and suppose that $f(x) \leq h(x) \leq g(x)$ for all $x \in \mathbb{R}$. Then $\lim h = A$ as $x \to p$.*

Corollary. *Let f, g be real valued functions defined on $\mathbb{R}$ such that $0 \leq f(x) \leq g(x)$ for all $x \in \mathbb{R}$. Then if $g \to 0$ as $x \to p$, then $f \to 0$.*

The proofs of all these are the result of appling the sequential criterion.

4.2 Continuous Functions.

Definition. Let X and Y be metric spaces and let $p \in E \subseteq X$, and $f : E \to Y$ be a function. We say that f is **continuous** at p if for every $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ whenever $0 < d_X(x,p) < \delta$. If f is continuous at every point in X, we say that f is **continuous on** X.

Theorem 4.2.1. *If $E \subseteq X$ a metric space, and if f is a function defined on X, and $p \in E$ is a limit point, then f is continuous if and only if $\lim f(x) = f(p)$ as $x \to p$.*

Theorem 4.2.2. *Suppose X, Y, and Z are metric spaces, and that $f : E \to Y$, $g : Y \to Z$, are functions (with $E \subseteq X$) such that f is continuous at p and g is continuous at $f(p)$. Then $g \circ f$ is continuous at p.*

Proof. For every $\epsilon > 0$, we have $\delta_1, \delta_2 > 0$ such that $d_Y(f(x), f(p)) < \epsilon$, when $0 < d_X(x,p) < \delta_1$, and $d_Z(g(y), g(f(p))) < \epsilon$ whenever $d_Y(y, f(p)) < \delta_2$. Then choose $\delta = \min\{\delta_1, \delta_2\}$,a nd we see that $d_Z(g(f(x)), g(f(p))) < \epsilon$ whenever $0 < d_X(x,p) < \delta$. ■

Theorem 4.2.3. *A mapping f of a metric space X into a metric space Y is continuous if and only if for every open set $V \subseteq Y$, $f^{-1}(V)$ is open in X.*

Proof. Let f be continuous on X, and let V be open in Y. For $p \in X$, $f(p) \in V$, and since V is open, there is an $\epsilon > 0$ such that $y \in V$ when $d_Y(y, f(p)) < \epsilon$. Since f is continuous, there is a $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$, whenever $0 < d_x(x,p) < \delta$. Thus $f^{-1}(V)$ is open in X.

Conversly, suppose that $f^{-1}(V)$ is open in X for V open in Y. Let $p \in X$ and $\epsilon > 0$, and let $V = \{y \in Y : d_Y(y, f(p)) < \epsilon\}$; V is open in Y, so $f^{-1}(V)$ is open in X, thus there is a $\delta > 0$ such that $x \in f^{-1}(V)$ when $0 < d_X(x,p) < \delta)$, then $f(x) \in V$, so $d_Y(f(x), f(p)) < \epsilon$; therefore, f is continuous at p. ■

Corollary. *A mapping f from X into Y is continuous if and only if $f^{-1}(C)$ is closed in X, whenever C is closed in Y.*

Proof. This is the converse of the previous theorem. ■

Theorem 4.2.4. *Let $f, g : X \to \mathbb{C}$ be continuous complex valued functions defined on a metric space X, then $f + g$, fg, and $\frac{f}{g}$ are continuous.*

Proof. This follows from theorem 4.1.2 and the sequential criterion. ■

Theorem 4.2.5. *Let $f_1, \ldots, f_k$ be realvalued functions defined on a metric space X, and define $f : X \to \mathbb{R}^k$ by $f(x) = (f_1(x), \ldots, f_k(x))$ for all $x \in X$. Then f is continuous if and only if f_i is continuous for $11 \leq i \leq k$. Moreover, if $g : X \to \mathbb{R}^k$ and f are continuous, then so is $f + g$ and fg.*

Proof. Notice that $|f_i(x) - f_i(y)| \leq ||f(x) - f(y)|| = \sqrt{\sum |f_i(x) - f_i(y)|^2}$ for $1 \leq i \leq k$. If follows then that f is contiuous if and only f_i is. Moreover, if $g : X \to \mathbb{R}^k$ is also contiuous, then by the previous theorem, so is $f + g$ and fg. ■

Example 4.2. (1) Let $x \in \mathbb{R}^k$, define the functions $\phi_i : \mathbb{R}^k \to \mathbb{R}$ by $\phi_i(x) = x_i$ for all $1 \leq i \leq k$, then ϕ_i is contiuous on $\mathbb{R}^k$

(2) The monomials $x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$, with $n_i \in \mathbb{Z}^+$ for $1 \leq i \leq k$ are contiuous on $\mathbb{R}^k$. So are all constant ultiples, thus the polynomial $\sum c_{n_1,\dots,n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$ is also continuous on $\mathbb{R}^k$.

(3) We have $||||x|| - ||y|||| \leq ||x - y||$ for all $x, y \in \mathbb{R}^k$, thus the mapping $x \to ||x||$ is contiuous on $\mathbb{R}^k$.

4.3 Continuity and Compactness.

Definition. A mappinf $f : E \to \mathbb{R}^k$ is said to be **bounded** if there is a real number $M > 0$ such that $||f|| \leq M$ for all $x \in E$.

Theorem 4.3.1. *Let f be a cn=ontiuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact in Y.*

Proof. Let $\{V_\alpha\}$ be an open cover of $f(X)$, since f is contiuous, then $f^{-1}(V_\alpha)$ is open in X, and since X is compact, $X \subseteq \bigcup_{i=1}^n V_{\alpha_i}$, and $f(f^{-1}(E)) \subseteq E$, we have that $f(X) \subseteq \bigcup_{i=1} 6n V_{\alpha_i}$. ■

Theorem 4.3.2. *If $f : X \to \mathbb{R}^k$ is continuous, where X is a compact metric space, then $f(X)$ is closed and bounded; in particular, f is bounded.*

Proof. From theorem 4.3.1, we have that $f(X)$ is compact in $\mathbb{R}^k$, therefore, it is closed and bounded. ■

Theorem 4.3.3 (The Extreme Value Theorem)**.** *Suppose f is a continuous, realvalued function on a metric space X, and that $M = \sup f$, and $m = \inf f$. Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.*

Proof. By theorem 4.3.2, $f(X)$ is closed and bounded, thus $M, m \in f(X)$. ■

Theorem 4.3.4. *Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping $f^{-1} : Y \to X$ is a Continuous mapping of Y onto X.*

Proof. By theorem 4.2.3, it suffices to show that $f(V)$ is open in Y whenever V is open in X. We have that $X \backslash V$ is closed in X, and compact, thus $f(X \backslash V)$ is closed and compact in Y, thus $f(V) = Y \backslash f(X \backslash V)$ is open in Y. ■

Definition. Let f be a mapping of a metric space X into a metric space Y. We say that f is **uniformly continuous** on X if for every $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(q), f(p)) < \epsilon$, for all $p, q \in X$ for which $d_X(p, q) < \delta$.

Lemma 4.3.5. *If f is uniformly continuous, then f is contiuous.*

Theorem 4.3.6. *Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X*

Proof. Let $\epsilon > 0$, by the continuity of f, we can associate for each $p \in X$ a number $\phi(p) > 0$ such that for $q \in X$, $d_X(p, q) < \phi(p)$ implies $d_Y(f(p), f(q)) < \frac{1}{2}\phi(p)$. Now let $J(p) = \{q \in X : d_X(p, q) < (p)\}$. Clearly, $p \in J(p)$, so $J(p)$ is an open cover of X, and since X is compact, there are $p_1, \dots, p_n$ for which $X \subseteq \bigcup_{i=1}^{n} J(p_i)$, then take $\delta = \min\{\phi(p_1), \dots, \phi(p_n)\}$; we have $\delta > 0$. Now let $p, q \in X$ such that $d_X(p, q) < \delta$. Then there is an $m \in \mathbb{Z}^+$ with $1 \leq m \leq n$ such that $p \in J(p_m)$, thus $d_X(p, q) < \frac{1}{2}\phi(p_m)$, by the triangle inequality, we get $d_(q, p_m) \leq d_X(q, p) + d_X(p, p_m) < \delta + \frac{1}{2}\phi(p_m) = \phi(p_m)$, for $1 \leq m \leq n$. Therefore, $d_Y(f(p), f(q)) \leq d_Y(f(p), f(p_m)) + d_Y(f(p_m), f(q)) < \epsilon$. Thus, f is uniformly contiuous. ■

Remark. What this theorem says, is that in any compact metric space, continuity and uniform continuity are equivalent.

Theorem 4.3.7. *Let $E \subseteq \mathbb{R}$ be noncompact, then:*

(1) There exists a continuous function on E which is not bounded.

(2) There is a bounded, continuous function on E which has no maximum.

(3) If E is bounded, there exists a continuous function on E that is not uniformly continuous.

Proof. Suppose first that E is bounded. Then there is a limit point $x_0 \notin E$ of E. Consider the function

$$f(x) = \frac{1}{x - x_0} \text{ for all } x \in E$$

Then f is continuous on E, but not bounded. Then let $\epsilon > 0$ and $\delta > 0$, and choose $x \in E$ such that $|x - x_0| < \delta$, then taking t arbitrarily close to x_0, we can get $|f(x) - f(t)| \geq \epsilon$, even though $|x - t| < \delta$. Thus f is not uniformly continuous.

Now choose

$$g(x) = \frac{1}{1 + (x - x_0)^2} \text{ for all } x \in E$$

g is continuous, and bounded on E (0¡g¡1), then $\sup g = 1$, and since $g(x) < 1$ for all x, we see that g attains no maximum.

Lastly, suppose that E is unbounded, then the functions $f(x) = x$ and $h(x) = \frac{x^2}{1+x^2}$ for all $x \in E$ establish (1) and (2). ■

Example 4.3. Let f be the mapping of the interval $[0, 2\pi)$ onto the unit circle. That is $f(t) = (\cos t, t)$ for $0 \leq t < 2\pi$. Then f is a continuous 1-1 mapping of $[0, 2\pi)$ onto the unit circle, however, the inverse mapping, f^{-1} fails to be continuous at the point $f(0) = (1, 0)$.

4.4 Continuity and Connectedness.

Theorem 4.4.1. *If f is a continuous mapping of a metric space X into a metric space Y, and if $E \subseteq X$ is Connected, then so is $f(E)$.*

Proof. Suppose that $f(E) = A \cup B$ with $A, B \subseteq Y$ nonempty and seperated. Let $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$, then $E = G \cup H$, and G and H are both nonempty. THen since $A \subseteq \overline{A}$, $G \subseteq f^{-1}(\overline{A})$, and since f is continuous, $f^{-1}(\overline{A})$ is closed, so $\overline{G} \subseteq f^{-1}(\overline{A})$, thus $f(\overline{G}) \subseteq \overline{A}$. Since $f(H) = B$, and $\overline{A} \cap B$ is empty, $\overline{G} \cap H$, and $H \cap \overline{H}$ are alsp empty, which contradicts the connectedness of E. ∎

Theorem 4.4.2 (The Intermediate Value Theorem). *Let $f[a, b] \to \mathbb{R}$ be a realvalued function. If $f(a) < f(b)$, and $c \in \mathbb{R}$ such that $f(a) < c < f(b)$, then there is an $x \in (a, b)$ such that $f(x) = x$.*

Proof. We have that $[a, b]$ is connected in $\mathbb{R}$, thus by theorem 4.4.1, $f([a, b])$ is connected in $\mathbb{R}$, hence there is an $x \in (a, b)$ for which $f(x) = c$. ∎

Corollary. *If $f : [a, b] \to \mathbb{R}$ is a realvaulued function such that $f(a) < 0 < f(b)$, then there is an $x \in (a, b)$ such that $f(x) = 0$.*

4.5 Discontinuities.

Definition. Let X and Y be metric spaces, and let $f : E \to Y$ for $E \subseteq X$. If there is a point $x\ inE$ for which f is not continuous, we say that f is textbfdiscontinuous at x, and we say that f has a **discontinuity** at x.

Definition. Let f be defined on (a, b), and let x be such that $a \leq x < b$. We write $f(x+) = q$ if $f(t_n) \to q$ for all sequences $\{t_n\}$ in (x, b) such that $t_n \to x$. Similarly, if x is such that $a < x \leq b$, we write $f(x-) = q$ if $f(t_n) \to q$ for all sequences $\{t_n\}$ in (a, x) such that $t_n \to x$. We call $f(x+)$ and $f(x-)$ the **right handed limit** and **left handed limit** of f at x respectively, and write $\lim_{t \to x^+} f = f(x+)$ and $lim_{t \to x^-} f = f(x-)$.

Theorem 4.5.1. *If $x \in (a, b)$, then $\lim f$ exists as $t \to x$ if and only if, $f(x+) = f(x-) = lim f$.*

Proof. Suppose that $\lim f$ exists, by the uniqueness of the limit, and the sequential criterion, we get that $f(x+) = f(x-) = \lim f$. Conversely, suppose that $f(x+) = f(x-) = q$. Then $f(t_n) \to q$ for all sequences $\{t_n\}$ in (x, b) and (a, x), then $f(t_n) \to q$ for all sequences $\{t_n\}$ in (a, b), thus by the sequential criterion again, $\lim f$ exists, and $\lim f = q$. ∎

Definition. Let f be defined on (a, b). If f is discontinuous at a point x, and $f(x+)$ and $f(x-)$ exists, we say that f has a **removable discontinuity** at x, otherwise, we say the f has an **infinite discontinuity**.

Example 4.4. (1) The function $f(x) = 1$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \in \mathbb{R} \backslash \mathbb{Q}$ has an infinite discontinuity at every point x.

(2) The function $f(x) = x$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \in \mathbb{R} \backslash \mathbb{Q}$ is continuous at $x = 0$, and has an infinite discontinuity at every other point x.

(3) The function $f(x) = \sin \frac{1}{x}$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$, has an infinite discontinuity at $x = 0$.

(4) The function $f(x) = x + 2$ for $-3 < x < -2$ and $0 \leq x < 1$ and $f(x) = -x - 2$ for $-2 \leq x < 0$ has a removable discontinuity at $x = 0$, and is continuous everywhere else.

Remark. The discontinuities in examples (1) and (2) are the result of $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ being dense in $\mathbb{R}$.

4.6 Monotonic Functions.

Definition. Let f be a realvalued function on an interval (a, b). We say that f is **monotonically increasing** on (a, b) if $a < x < y < b$ implies $f(x) \leq f(y)$. We say that f is **monotonically decreasing** on (a, b) if $a < x < y < b$ implies $f(y) \leq f(x)$. We say f is **monotonic** if it is either monotonically increasing or monotonically decreasing.

Theorem 4.6.1. *Let f be monotonic on (a, b) then $f(x+)$ and $f(x-)$ exist at every point of (a, b) and $\sup f = f(x-)$ and $\inf f = f(x+)$, and the following hold:*

(1) If f is monotonically increasing $f(x-) \leq f(x) \leq f(x+)$

(2) If f is monotonically decreasing $f(x+) \leq f(x) \leq f(x-)$

Proof. We prove only (1), since (2) is analogous. Suppose that f is monotonically increasing, clearly, f has an upperbound A for which $A \leq f$. Now let $\epsilon > 0$, then there is a $\delta > 0$ for which $a < x - \delta < x$, and $A - \epsilon < f(x - \delta) \leq A$. Then we have $f(x - \delta) < f(t) \leq A$ for all $x - \delta < t < x$, then we get $|f(t) - A| < \epsilon$, hence $f(x-) = A$, Similarly, we get $f(+) = -\inf f$. Now since $\sup f \leq f \leq \inf f$, we get the desired result. ∎

Corollary. *Monotonic functions have no infinite discontinuities.*

Theorem 4.6.2. *Let f be monotonic on (a, b), then the set of all points of (a, b) for which f is discontinuous is atmost countable.*

Proof. Suppose, without loss of generality that g is monotonically increasing, and let E be the set of all points of (a, b) for which f is discontinuous. By the density of $\mathbb{Q}$ in $\mathbb{R}$, for each $x \in E$ associate $r(x) \in \mathbb{Q}$ such that $f(x+) < f(x) < f(x-)$. Since $x_1 < x_2$ implies $f(x_1+) \leq f(x_2-)$, then $r(x_1) \neq r(x_2)$, thus $x_1 \neq x_2$, and so r is a 1-1 mapping of E into $\mathbb{Q}$. ∎

Now, given a countable E in an interval (a, b), we can construct a monotonic function f that is discontinuous at every point in E and continuous everywhere else. Arrange the points of E into a sequence $\{x_n\}$ and let $\{c_n\}$ be a sequence such that $c_n > 0$ for all $n \in \mathbb{Z}^+$, such that $\sum c_n$ converges. Define $f(x) = \sum_{x_n < x} c_n$, for $x \in (a, b)$. Then we have that

(1) f is monotonically increasing on (a, b).

(2) f is discontinuous at every point in E with $f(x_n+) - f(x_n-) = c_n$.

(3) f is continuous at every point in $(a, b)\backslash E$.

Definition. Let f be a realvalued function defined on an interval (a, b). We say that f is **continuous form the right** if $f(x+) = f(x)$, and we say f is **continuous from the left** if $f(x-) = f(x)$.

4.7 Infinite Limits and Limits at Infinity.

Definition. For any $c \in \mathbb{R}$, the set of all real numbers x such that $x > c$ is called the **neighborhood of** ∞, and denoted (c, ∞). The set of all real numbers x such that $x > c$ is called the **neighborhood of** $-\infty$, and denoted $(-\infty, c)$.

Definition. Let $f : E \to \mathbb{R}$ be a realvalued function. We say that $f(t) \to A$ as $t \to x$, with A, and x extended real numbers if for every neighborhood of U A, there is a neighborhood V of x such that $V \cap E$ is nonempty, and $f(t) \in U$ for all $t \neq x \in V \cup E$.

Theorem 4.7.1. *Let $f, g : E \to \mathbb{R}$ be realvalued functions such that $f \to A$, and $g \to B$ as $t \to x$, for extended real nuumbers A, B, and x. Then the following hold as $t \to x$.*

(1) $f \to A'$ implies $A = A'$.

(2) $f + g \to A + B$.

(3) $fg \to AB$.

(4) $\frac{f}{g} \to \frac{A}{B}$. Provided that (1), (2), *and* (3) *are not of the forms $\infty - \infty$, 0 $\cdot\infty$, $\frac{\infty}{\infty}$, and $A_{\bar{0}}$, respectively.*

Proof. This is a direct application of the sequential criterion using the appropriate definition. ∎

Chapter 5

Differentiation

5.1 The Derivative of Real valued Functions.

Definition. Let $f : [a, b] \to \mathbb{R}$ be a realvalued function defined on $[a, b]$. The **derivative** of f at a point $x \in (a, b)$ is the function $f' : (a, b) \to \mathbb{R}$ defined by

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \tag{5.1}$$

If f' is defined at $x \in [a, b]$, then we say that f is **differentiable** at x, and if f' is defined for all $x \in (a, b)$, we say that f is **differentiable** on (a, b).

Theorem 5.1.1. *Let $f : [a, b] \to \mathbb{R}$ be a realvalued function. If f is differentiable at a point $x \in (a, b)$, then f is continuous.*

Proof. As $t \to x$, we get $|f(t) - f(x)| = |\frac{f(t)-f(x)}{t-x}||t - x| \to f'(x)0 = 0$, thus $f(t) \to f(x)$. ■

Theorem 5.1.2. *Suppose $f, g : [a, b] \to \mathbb{R}$ are realvalued functiond differentiable at a point $x \in (a, b)$. Then $f + g$, fg, and $\frac{f}{g}$ are differentiable at x, and as $t \to x$:*

(1) $(f + g)' = f' + g'$.

(2) $(fg)' = f'g + fg'$.

(3) $(\frac{f}{g})' = \frac{f'g-fg'}{g^2}$, provided that $g(x) \neq 0$.

Proof. (1) follows directly from the definiton. Now notice that $fg(t) - fg(x) = f(t)(g(t) - g(x)) + g(t)(f(t) + f(x))$, then dividing by $t - x$, the result follows by definition.

Now also notice that $\frac{\frac{f}{g}(t)-\frac{f}{g}(x)}{t-x} = \frac{1}{g(t)g(x)}(g(x)\frac{f(t)-f(x)}{t-x} - f(x)\frac{g(t)-g(x)}{t-x})$, and the result again follows by definition. ■

Example 5.1. (1) The derivative of constant functions are alway 0, and the derivative of the identity function is always 1.

(2) Let $f(x) = x^n$, for $n \in \mathbb{Z}$, and $x \neq 0$ for $n < 0$, then f is differentiable and $f'(x) = nx^{n-1}$.

(3) Polynomial functions are differentiable, and so are rational functions $\frac{p}{q}$, provided that $q \neq 0$.

Theorem 5.1.3 (Caratheodory's Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous realvalued function. Then f is differentiable at a point $x \in (a, b)$ if and only if there is a continuous function $\phi : (a, b) \to \mathbb{R}$ such that $f(t) - f(x) = \phi(t)(t - x)$; moreover, $\phi = f'$.*

Proof. Suppose f' exists at x, and define $\phi : (a, b) \to \mathbb{R}$ by $\phi(t) = \frac{f(t)-f(x)}{t-x}$ when $t \neq x$, and $\phi(t) = f'(x)$ at $t = x$. Then by the contiunity of f, ϕ is continuous at x, moreover, at $t \neq x$ we see that $f(t) - f(x) = \phi(t)(x - t)$.

Conveersesly, sup[ose there is a ϕ, continuous at x such that $f(t) - f(x) = \phi(t)(x - t)$, then clearly, $\lim \phi = f'(x)$ as $t \to x$, and since ϕ is continuous, $\phi(x) = f'(x)$. ∎

Theorem 5.1.4 (The Chain Rule)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are continuous, where $f([a, b]) \subseteq I \subseteq [a, b]$, and suppose that f is differentiable at x, and that g is differentiable at $f(x)$. Then $g \circ g$ is differentiable at x, and $(g \circ f)' = (g' \circ f)f'$.*

Proof. We have by Caratheodory's theorem that $f(t) - f(x) = (t - x)(f'(x) - u(t))$, and $g(s) - g(y) = (s - y)(g'(y) - v(s))$. Then letting $y = f(x)$, and $s \to y$ as $t \to x$, we see that $u, v \to 0$, and we get that $g(f(t)) - g(f(x)) = g'(f(t)f(t)) - g'(f(x))f(x)$, dividing by $t - x$ give the desired result. ∎

Example 5.2. (1) Let $f(x) = \sin\frac{1}{x}$ at $x \neq 0$, and $f(x) = 0$ at $x = 0$. We have at $x \neq 0$, that $f'(x) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}$, but at $x = 0$, we must appeal to the definition, and we get $f(t) = \sin\frac{1}{t}$, which diverges at $t \to 0$, thus $f'(0)$ does not exist.

(2) Let $f(x) = x^2 \sin\frac{1}{x}$ at $x \neq 0$, and $f(x) = 0$ at $x = 0$. For $x \neq 0$, we get $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$, and at $x = 0$, we notice that $|t\sin\frac{1}{t}| \leq |t|$, so by the sandwhich theorem, $f'(0) = 0$ as $t \to 0$.

5.2 Mean Value Theorems.

Definition. Let $f : X \to \mathbb{R}$ be defined on a metric space X. We say that f has a **local maximum** at a point $p \in X$, if there is a $\delta > 0$ for which $f(q) \leq f(p)$ whenever $d(q, p) < \delta$. Likewise f has a **local minimum** at a point $p \in X$, if there is a $\delta > 0$ for which $f(q) \leq f(p)$ whenever $d(q, p) < \delta$. We call local maxima and local minumums **local extrema**.

Theorem 5.2.1. *Let $f : [a, b] \to \mathbb{R}$ be a realvalued function, and suppose that f has a local extremum at $x \in (a, b)$. If f' exists, then $f'(x) = 0$.*

Proof. Suppose, without loss of generality that f has a local maximum at x. Choosse $\delta > 0$ such that $a < x - \delta < x < x + \delta < b$. Then if $x - \delta < t < x$, we have $|t - x + \delta| < \delta$, so $f(t) \leq f(x)$, thus $\frac{f(t)-f(x)}{t-x} \leq 0$. Similarly, for $x < t < x + \delta$, we get $\frac{f(t)-f(x)}{t-x} \geq 0$, hence, as $t \to x$, we get $0 \leq f'(0) \leq 0$, thus $f'(x) = 0$. ∎

Theorem 5.2.2 (The Generalized Mean Value Theorem)**.** *If $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$, and differentiable on (a, b), then there is a point $x \in (a, b)$ such that $(f(b) - f(a))g('x) = (g(b) - g(a))f'(x)$.*

Proof. Let $h(t) = (f(b) - f(a))g('x) - (g(b) - g(a))f'(x)$, for $t \in [a, b]$, then h is continuous on $[a, b]$, and differentiable on (a, b), moreover, we have $h(b) = f(b)g(a) - f(a)g(b) = h(b)$. Now if h is constant, then $h' = 0$ for all t and we are done., Now suppose that $h(a) \leq h(t)$, and let $x \in (a, b)$, be a local minimum of h, then $h'(x) = 0$, and we are done; the same result follows for local minima of h. ■

Corollary (The Mean Value Theorem). *LEt $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, and differentiable on (a, b). Then there is an $x \in (a, b)$ such that $f(b) - f(a) = (b - a)f'(x)$.*

Proof. Take $g(t) = t$. ■

Theorem 5.2.3. *Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b). Then the following hold for all $x \in (a, b)$:*

(1) If $f' \geq 0$, then f is monotonically increasing.

(2) If $f' = 0$, then f is constant.

(3) If $f' \leq 0$, then f is monotonically decreasing.

Proof. Let $x_1, x_2 \in (a, b)$, then by the mean value theorem, there is an $x \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = (x_2 - x_1)f'(x)$. Then if $f'(x) = 0$, we get $f(x_2) = f(x_1)$, and that f is constant. If $f'(x) \geq 0$, we get $f(x_2) \geq f(x_1)$, making f monotonically increasing, similarly, if $f'(x) \leq 0$, we get f monotonically decreasing. ■

5.3 The Continuity of Derivatives.

Theorem 5.3.1. *Let $f : [a, b] \to \mathbb{R}$ be differentiable on all of $[a, b]$, and suppose that $f'(a) < \lambda < f'(b)$. Then there is an $x \in (a, b)$ such that $f'(x) = \lambda$.*

Proof. Let $g(t) = f(t) - \lambda t$, then $g'(a) < 0$ and $g'(b) > 0$. Then for $t_1, t_2 \in (a, b)$, $g(t_1) < g(a)$, and $g(b) < g(t_2)$. Then by the extreme value theorem, g attains a maximum at a point $x \in (t_1, t_2)$, hence $g'(x) = 0$, hence $f'(x) = \lambda$. ■

Corollary. *If $f : [a, b] \to \mathbb{R}$ is differentiable, then f cannot have any removable discontinuities, nor jump discontinuities.*

Remark. f' may have infinite discontinuities.

5.4 L'Hosptal's Rule.

Theorem 5.4.1 (L'Hospital's Rule). *Suppose f and g are realvalued functions differentiable on (a, b), and that g' neq0 for all $x \in (a, b)$, where $-\infty \leq a < b \leq \infty$, and suppose that $\frac{f'}{g'} \to A$ as $x \to a$. If $f, g \to 0$, or if $g \to \pm\infty$, as $x \to a$, then $\frac{f}{g} \to A$ as $x \to a$.*

Proof. Suppose first that $-\infty \leq A < \infty$, and choose $q, r \in \mathbb{R}$ such that $A < r < q$. By hypothesis, there is a $c \in (a, b)$ for which $a, x < c$ implies $\frac{f}{g} < r$. If $a < x < y < c$, then by the generalized mean value theorem, $\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)} < r$, thus letting $x \to a$, we see hat $\frac{f(y)}{g(y)} \leq r < q$. Now suppose, without loss of generality, that $g \to \infty$. Fixing y, and choosing $c_1 \in (a, y)$ such that $g(x) > g(y)$, and $g(x) > 0$, if $a < x < c_1$, then $\frac{f(x)}{g(x)} < r - r\frac{g(y)+f(y)}{g(x)}$, then as $x \to a$, there is a $c_2 \in (x, c_1)$ such that $\frac{f}{g} < q$.

Likewise, if we suppose that $-\infty < A \leq \infty$, by the same reasoning, we can choose a $p < A$ and $c_3 \in (a, b)$ such that $p < \frac{f}{g}$ as $x \to a$. Since $p < A < q$, by the sandwhich theorem, we get $\frac{f}{g} = A$ as $x \to a$. ■

5.5 Taylor's Theorem.

Definition. If f has a derivative f' on an interval, and f' is differentiable, we denote f'' to be $(f')'$ and call it the **second derivative** of f; likewise, if f'' is differentiable, we denote the **third derivative** by $f^{(3)} = (f'')'$. More generally, for $n \in \mathbb{Z}^+$, we define recursively the n**th derivative** to be:

(1) $f^{(0)} = f$ and $f^{(1)} = f'$.

(2) $f^{(n+1)} = (f^{(n)})'$, given that $f^{(n)}$ is differentiable.

We call f n**th differentiable** if $f^{(n)}$ exists.

Theorem 5.5.1 (Taylor's Theorem). *Suppose $f : [a, b] \to \mathbb{R}$ is a realvalued function, that is nth differentiable, and let $n \in \mathbb{Z}^+$ be such that $f^{(n-1)}$ is continuous on $[a, b]$, and that $f^{(n)}$ exists on (a, b). LEt $\alpha, \beta \in [a, b]$ be distinct, and define:*

$$p(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k \tag{5.2}$$

Then there exists a point $x \in (\alpha, \beta)$ such that $f(\beta) = p(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$.

Proof. For $n = 1$, this reduces to the mean value theorem, so suppose that $n > 1$. Let $M \in \mathbb{R}$ be such that $f(\beta) = p(\beta) + M(\beta - \alpha)^n$, and let $g(t) = f(t) - p(t) + M(\beta - \alpha)^n$, for $t \in [a, b]$. Then g is nth differentiable, and we get $g^{(n)} = f^{(n)} - n!M$ for $t \in (a, b)$. We wish to show that $f^{(n)} = n!M$.

We have that $p^{(k)} = f^{(k)}(\alpha)$ for $0 \leq k \leq n - 1$, then $g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0$, and our choice of M shows that $g(\beta) = 0$. So $g'(x_1) = 0$ for $x_1 \in (\alpha, \beta)$, so by the mean value theorem, since $g'(\alpha) = 0$, then $g''(_2) = 0$ for $x_2 \in (\alpha, x_2)$. Proceeding inductively, we then get that $g_{(n)(x_n)=0}$ for $x_n \in (\alpha, x_{n-1})$, hence we get that $n!M = f^{(n)}(x)$. ■

Definition. We call the series in equation (5.2) the **Taylor series** (or **Taylor polynomial**) of f about α. We call the realnumber M such tat $n!M = f^{(n)}(x)$ the **tail**, (or **error**) of the Taylor series.

5.6 Derivatives of vector valued functions.

Definition. Let $f : [a, b] \to \mathbb{C}$ be a complex valued function, such that $f(t) = f_1(t) + if_2(t)$. We say that f is **differentiable** at a point x if and only if f_1 and f_2 are differentiable, and we denote the **derivative** of f to be the function $f : (a, b) \to \mathbb{C}$ such that $f' = f_1' + if_2'$

Definition. Let $f : [a, b] \to \mathbb{R}^k$ be a vectorvalued function for $k \in \mathbb{Z}^+$. f is said to be **differentiable** at $x \in (a, b)$ if there is some point $f'(x) \in \mathbb{R}^k$ such that:

$$\lim_{t\to x} ||\frac{f(t) - f(x)}{t - x} - f'(x)|| = 0 \tag{5.3}$$

We define the **derivative** of f at x to be the function $f' : (a, b) \to \mathbb{R}$ such that the values of f' statisfy equat ion (5.3)

Remark. If $f : [a, b] \to \mathbb{R}^k$ is defined by $f = (f_1, \ldots, f_k)$, then f is differentiable at a point $x \in (a, b)$ if and only if f_i is differentiable at x for $1 \leq i \leq k$, and we have that $f' = (f_1', \ldots, f_k')$.

Theorem 6.1.1 follows natrually, and so does theorem 5.1.2(a) and (2), where we define fg as $\langle f, g\rangle$, howeever, the mean value theorem in general does not hold.

Example 5.3. (1) Define $f : \mathbb{R} \to \mathbb{C}$ by $f(x) = e^{ix} = \cos x+i \sin x$. Then $f(2\pi)-f(0) = 0$, however, $f'(x) = ie^{ix} \neq 0$ for all x (moreover, $|f'| = 1$), so the generalized mean value theorem fails here.

(2) Define $f, g : (0, 1) \to \mathbb{C}$ by $f(x) = x$ and $g(x) = x + x^2e^{\frac{i}{x^2}}$ for all x. Since $|e^{it}| = 1$, we have that $\lim \frac{f}{g} = 1$ as $x \to 0$. Now $g'(x) = 1 + (2x - i\frac{2}{x})e^{\frac{1}{x^2}}$ on $(0, 1)$, hence $|g'| = |2x - i\frac{2}{x}| - 1 \geq \frac{2}{x} - 1$, so $|\frac{f'}{g'}| \leq \frac{x}{2-x} \to 0$ as $x \to 0$, so L'Hospital's rule fails in $\mathbb{C}$ as well, and hence in $\mathbb{R}^2$ (as $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$).

Theorem 5.6.1. *Suppose $f : [a, b] \to \mathbb{R}^k$, for $k \in \mathbb{Z}^+$ is continuous, and that f is differentiable on (a, b). Then there is an $x \in (a, b)$ for which $||f(b) - f(a)|| \leq (b - a)||f'(x)||$.*

Proof. Let $z = f(b) - f(a)$, and define $\phi = \langle f, g\rangle$ for all $t \in [a, b]$, then ϕ is a realvalued function continuous on $[a, b]$, moreover it is differentiable on (a, b); therefore, by the mean value theorem, $\phi(b) - \phi(a) = (b - a)\phi'(a) = (b - a)\langle z, f'(x)\rangle$ for $x \in (a, b)$. On the other hand, we have that $\phi(b) - \phi(a) = \langle z, z\rangle = ||z||^2$, hence, by the Cauchy Schwarz inequality, we have that $||z||^2 = (b - a)\langle z, f'\rangle \leq ||z||||f'||$, which gives the desired result. ∎

Chapter 6

Integration

6.1 The Riemann-Stieltjes Integral.

Definition. Let $[a, b]$ be an interval. A **partition** of $[a, b]$ is a set of points $P = \{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < \cdots < x_n = b$, and we write $\Delta x_i = x_i - x_{i-1}$. Now let $f : [a, b] \to \mathbb{R}$ be a bounded realvalued function, and for each partition P of $[a, b]$ let $M_i = \sup f$ and $m_i = \inf_f$ for all $x_{i-1} \leq x \leq x_i$. We define the **upper Riemann sum** and the **lower Riemmann sum** to of f with respect to be:

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i \tag{6.1}$$

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i \tag{6.2}$$

respectively. We also define the **upper Riemann integral** and the **lower Riemann integral** of f over $[a, b]$ to be:

$$\overline{\int_a^b} f(x)dx = \inf U(f, P) \tag{6.3}$$

$$\underline{\int_a^b} f(x)dx = \sup L(f, P) \tag{6.4}$$

Respectively.

If $\overline{\int_a^b} f = \underline{\int_a^b} f$, then we say that f is **Riemann integrable** on $[a, b]$, and we its value the **Riemann integral**, and denote it to be:

$$\int_a^b f(x)dx = bar \int_a^b f(x)dx = \underline{\int_a^b} f(x)dx \tag{6.5}$$

Lemma 6.1.1. *$\overline{\int_a^b} f$, and $\underline{\int_a^b} f$ are defined for every bounded realvalued function f over $[a, b]$.*

Proof. Let f be bounded on $[a, b]$, then there are m and M such that $m \leq f \leq M$ for all $a \leq x \leq b$. Now let P be a partition of $[a, b]$. Since $\inf f \leq \sup f$, we have that

$m \leq m_i = \inf f \leq M_i = \sup f \leq M$, thus $m(b-a) \leq L(f,P) \leq U(f,P) \leq M(b-a)$, hence L and U form a bounded set, and we are done. ■

Corollary. *$L(f,P) \leq U(f,P)$ for every bounded function f.*

Now the question of the integrability of f is a very delicate matter, and requires a closer scrutiny on the concepts of upper and lower sums. Infact, it turns out that the Riemann integral is a consequence of a more general class of integrals. Developng this more general situation will allow us to discern facts about the Riemann integral.

Definition. Let α be a bounded monontonically increasing function on $[a,b]$, and let P be a partition of $[a,b]$ and let $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. For any realvalued, bounded function on $[a,b]$, defined the **upper sum** and the **lower sum** of f with respect to P and α to be:

$$U(f,P,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \tag{6.6}$$

$$L(f,P,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i \tag{6.7}$$

Where $M_i = \sup f$ and $m_i = \inf f$ for all $x_{i-1} \leq x \leq x_i$, and again, define the **upper integral** and **lower integral** of f with respect to α on $[a,b]$ to be:

$$\overline{\int_a^b} f(x)d\alpha = \inf U(f,P,\alpha) \tag{6.8}$$

$$\underline{\int_a^b} f(x)d\alpha = \sup L(f,P,\alpha) \tag{6.9}$$

If $\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha$, we call the value:

$$\int_a^b f(x)d\alpha = bar\int_a^b f(x)d\alpha = \underline{\int_a^b} f(x)d\alpha \tag{6.10}$$

the **Riemann-Stieltjes integral** of f with respect to f on $[a,b]$. If such an integra exists, we say that f is **integrable** with respect to on $[a,b]$.

Example 6.1. Let $\alpha(x) = \alpha$, be defined over $[a,b]$. Then α is monontonically increasing, and our defintitions reduces to those for the Riemann integral. Here $U(f,P,x) = U(f,P)$ and $L(f,P,x) = L(F,P)$.

We are now in a position to investigate the properties of integrability, in the Riemann-Stieltjes sense.

Definition. Let $a,b]$ be an interval, and let P and Q be partitions of $[a,b]$. We say that Q is a **refinment** of P if PQ, and we also say that Q is **finer** than P. Now if neither P nor Q is a refinment of the other, we say that the two partitions are **noncomparable**.

Lemma 6.1.2. *Let P and Q be partitions of and interval $[a, b]$, then $P \cup Q$ is a partition of $[a, b]$, and is a refinment of both P and Q.*

Proof. If P is a refinment of Q, or viceversa, then we are done; so suppose that P and Q are noncomparable. Let $P = \{x_0, x_1, \dots, x_n\}$ and $Q = \{y_0, y_1, \dots, y_m\}$ with $a = x_0 < x_1 < \dots x_n = b$ and $a = y_0 < y_1 < \dots y_m = b$. Then $P \cup Q = \{x_0, y_0, x_1, y_1, \dots, x_n, y_m\}$ and $a = x_0 = y_0 < x_1, y_1 < \cdots < x_n = y_m = b$, thus $P \cup Q$ is a partition of $[a, b]$, that it is a refinment of P and Q follows trivially. ∎