# Advanced Calculus

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**Text**

An Introduction to Analysis ($3^{rd}$ edition)

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# Chapter 1

# Chapter 1: The Real Number System.

## 1.1 The Ordered Field Axioms

We want to study the algebraic structure of the real numbers; the so called set $\mathbb{R}$.We assume standard knowlege of set theory. We display the field axioms below:

**Postulate 1.** *There are binary operations* $+$ *and* $\cdot$ *defined on the set* $\mathbb{R} \times \mathbb{R}$*, satisfying the following properties* $\forall a, b, c \in \mathbb{R}$ :

*(1)* $a + b \in \mathbb{R}$.

*(2)* $a + (b + c) = (a + b) + c$ *and* $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

*(3)* $a + b = b + a$ *and* $a \cdot b = b \cdot a$.

*(4)* $a \cdot (b + c) = (ab) + (a \cdot c)$.

*(5) There exists a unique element* $0 \in \mathbb{R}$ *such that* $\forall a \in \mathbb{R}$, $a + 0 = a$.

*(6) There exists a unique element* $1 \neq 0 \in \mathbb{R}$ *such that* $\forall a \in \mathbb{R}$, $a \cdot 1 = a$.

*(7) For each* $a \in \mathbb{R}$*, there exists a unique element* $-a \in \mathbb{R}$ *such that* $a + (-a) = 0$.

*(8) For each* $a \in \mathbb{R} \backslash \{0\}$*, there exists a unique element* $a^{-1} \in \mathbb{R}$ *such that* $a \cdot a^{-1} = 1$.

We call postulate 1 the **field axioms** for real numbers; in essence what it says is that the pairs $(\mathbb{R}, +)$ and $(\mathbb{R}, \cdot)$ form abelian groups, and that $\cdot$ distributes over $+$. W now get the following properties from the field axioms:

(1) $(-1)(-1) = 1$.

(2) $0a = 0$.

(3) $-(a - b) = b - a$.

(4) For $a, b \in \mathbb{R}$ and $ab = 0$, then either $a = 0$ or $b = 0$.

The proofs for these are relatively easy, and can be done at anytime as an exercise. We now introduce the second postulalte called the **order axioms**:

**Postulate 2.** *There is a relation $<$ on $\mathbb{R} \times \mathbb{R}$ satisfying the following properties:*

*(1) Given $a, b \in \mathbb{R}$, one and only one of the following hold: either $a < b$, $b < a$, or $a = b$.*

*(2) If $a < b$ and $b < c$, then $a < c$.*

*(3) If $a < b$ and $c \in \mathbb{R}$, then $a + c < b + c$.*

*(4) If $a < b$ and $c > 0$, then $ac < bc$; if $c < 0$, then $bc < ac$.*

We may may the following remarks: by $b > a$ we mean that $a < b$, by $a \leq b$, we mean that $a < b$ or $a = b$, and by $a < b < c$, we mean that $a < b$ and $b < c$.

**Definition.** For $a \in \mathbb{R}$, $a$ is called **nonnegative** if $a \geq 0$, $a$ is called **nonpositive** if $a \leq 0$. We call $a$ **positive** if $a > 0$ and we call $a$ **negative** if $a < 0$.

**Example 1.1.** If $a \in \mathbb{R}$, show that $a \neq 0$ implies $a^2 > 0$; in particular that $-1 < 0 < 1$.

*Proof.* If $a \neq 0$, then either $a > 0$ or $a < 0$. If $a > 0$, then clearly $a^2 > 0$ and we are done. If $a < 0$, then $-a > 0$, hence $(-a)(-a) > (-a)0$, and so $a^2 > 0$.

Now we have that $1 \neq 0$, and that $1^2 = 1 > 0$, subtracting we get $1 - 1 > 0 - 1$ so $0 > -1$.s ■

**Example 1.2.** If $a \in \mathbb{R}$, show that $0 < a < 1$ implies that $0 < a^2 < a$, and $a > 1$ implies that $a^2 > a$.

*Proof.* We have already that if $a > 0$, then $a^2 > 0$. Now suppose that $a < 1$ then multiplying by $a$ we get $(a)(a) < 1(a)$, hence $a^2 < 1$. Likewise, by the same reasoning, if $a > 1$ we get $a^2 > a$. ■

**Example 1.3.** Prove that:

(1) $0 \leq a < b$ and $0 \leq c < d$ imply that $ac < bd$.

(2) $0 \leq a < b$ imply that $0 \leq a^2 < b^2$ and $0 \leq \sqrt{a} < \sqrt{b}$.

(3) $0 < a < b$ implies $0 < \frac{1}{b} < \frac{1}{a}$.

*Proof.* (1) Let $0 \leq a < b$ and $0 \leq c < d$. Then $0 \leq ac < bc$, and since $c < d$, then $bc < bd$, hence we have that $0 \leq ac < bd$.

(2) Let $0 \leq a < b$. Then $0 \leq aa < ab$, and notice that $ab < bb$, so $0 \leq aa < bb$, i.e. $0 \leq a^2 < b^2$. Now notice that $a = (\sqrt{a})(\sqrt{a})$ and $b = (\sqrt{b})(\sqrt{b})$, by the previous result, we have $0 \leq (\sqrt{a})(\sqrt{a}) < (\sqrt{b})(\sqrt{b})$, henve we have that $0 \leq \sqrt{a} < \sqrt{b}$.

(3) Let $0 < a < b$, multiplying by $\frac{1}{b}$, we get that $0 < a\frac{1}{b} < b\frac{1}{b}$ and get $0 < a\frac{1}{b} < 1$, now multiplying again by $\frac{1}{a}$, we get $0 < \frac{1}{a}a\frac{1}{b} < \frac{1}{a}$, thus we have that $0 < \frac{1}{b} < \frac{1}{a}$.

■

**Definition.** The **absolute value** of an element $a \in \mathbb{R}$ is a real number $|a|$ defined such that:

$$|a| = \begin{cases} a, \text{ if } a \geq 0 \\ -a, \text{ if } a < 0 \end{cases} \tag{1.1}$$

*Remark.* The absolute value is multiplicative, i.e. $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$.

*Proof.* We do a casewise evaluation.

(1) If $a = 0$, or $b = 0$, then $|ab| = 0 = |a||b|$ (since $|a| = a = 0$ or $|b| = b = 0$).

(2) Let $a, b > 0$. then $|a| = a$ and $|b| = b$. Then $|ab| = ab$ as $ab > 0$. Hence $|ab| = ab = |a||b|$.

(3) Let $a > 0$ and $b < 0$, then $|a| = a$ and $|b| = -b$ si $|a||b| = a(-b) = -ab = |ab|$ as $ab < 0$.

(4) Let $a, b < 0$, then $|a| = -a$ and $|b| = -b$ and $|a||b| = (-a)(-b) = ab = |ab|$, since $ab > 0$.

■

**Theorem 1.1.1.** *Let $a \in \mathbb{R}$ and let $M$ be nonnegative. Then $|a| \leq M$ if and only if $-M \leq a \leq M$.*

*Proof.* Notice that if $|a| \leq M$, then $-|a| \geq -M$. Suppose then that $a \geq 0$, then $|a| = a \leq M$, and since $-M \leq 0$, we have $-M \leq a \leq M$. Now if $a < 0$, $-|a| = a$, then $a \geq -M$, and since $a < 0$, $a \geq M$.

Conversely suppose that $-M \leq a \leq M$. Then $-M \leq a$ and $a \leq M$. For $-M \leq a$, multiplying by $-1$ we have $M \geq -a$. If $a \leq 0$ then $|a| = a$ and $|a| \leq M$. If $a < 0$ then $|a| = -a \leq M$. Hence we have in both cases that $|a| \leq M$. ■

**Theorem 1.1.2.** *The absolute value satisfies the following three properties For all $a, b \in \mathbb{R}$:*

*(1)* $|a| \geq 0$ *with* $|a| = 0$ *if and only if* $a = 0$.

*(2)* $|a - b| = |b - a|$.

*(3)* $|a + b| \leq |a| + |b|$.

*Proof.* If $a \leq 0$, then clearly $|a| \geq 0$, if $a < 0$, then $|a| = -a \geq 0$. Now if $a = 0$ then $|0| = 0$, and if $|a| = 0$, then $\pm a = 0$; then $a = 1a = (\pm 1)^2 a = (\pm 1)(\pm 1)a = (\pm 1)(\pm a)$, since $\pm a = 0$, $(\pm 1)0 = 0$ hence $a = 0$.

Now we have $(a-b) = -(b-a)$, so $|a-b| = |-1(b-a)| = |-1||b-a| = 1|b-a| = |b-a|$.

Notice for all $x \in \mathbb{R}$, $|x| \leq |x|$. So $-|x| \leq x \leq |x|$, then we have for $a, b \in \mathbb{R}$, $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, adding we get $-(|a|+|b|) \leq a+b \leq |a|+|b|$, thus $|a+b| \leq |a|+|b|$. ■

**Corollary.** *For all $a, b \in \mathbb{R}$, $|a - b| \geq |a| - |b|$ and $||a| - |b|| \leq |a - b|$.*

*Proof.* Now $|a|-|b| = |a+b-b|-|b|$, by the above theorem, $|a|-|b| \leq |a-b|+|b|-|b| = |a-b|$.

Now $||a| - |b|| \leq |a-b|$ implies $-|a-b| \leq |a| - |b| \leq |a-b|$, we need to show that $-|a-b| \leq |a| - |b|$. Then $|a-b| \geq |b| - |a|$ hence $|b-a| \geq |b| - |a|$ which reduces to the second equality. ■

**Example 1.4.** Show that if $-2 < x < 1$ then $|x^2 - x| < 6$.

*Proof.* We have that $-2 < x < 2$, so $|x| < 2$. Then $|x^2 + x| \leq |x^2| + |x| = |x||x| + |x| < (2)(2) + 2 = 6$. ■

**Theorem 1.1.3.** *Let $x, y, a \in \mathbb{R}$.*

*(1) $x < y + \epsilon$ for all $\epsilon > 0$ if and only if $x \leq y$.*

*(2) $x > y - \epsilon$ for all $\epsilon > 0$ if and only if $x \geq y$.*

*(3) $|a| < \epsilon$ for all $\epsilon > 0$ if and only if $a = 0$.*

*Proof.* (1) Suppose that $x < y + \epsilon$ for all $\epsilon > 0$, but that $x > y$. Let $\epsilon = \epsilon_0 = x - y$, then $x = y + \epsilon_0$. Then it is not true that $x < y + \epsilon_0$. A contradiction.

Conversely, suppose that $x \leq y$ and that $\epsilon > 0$. Then either $x < y$ or $x = y$. For $x < y$, $x + 0 < y + 0 < y + \epsilon$, hence $x < y + \epsilon$. Similarly, for $x = y$, $x < y + \epsilon$.

(2) This proof is analgous to the previous. Suppose that $x > y - \epsilon$ for $\epsilon > 0$. Then let $\epsilon = \epsilon_0 = y - x$. Then $x = y - \epsilon_0$, which contradictos our assumption that $x > y - \epsilon$.

Conversely, let $x \geq y$ and let $\epsilon > 0$. Then either $x > y$ or $x = y$. For $x > y$, we have $x - 0 > y - 0 > y - \epsilon$. For $x = y$, we see clearly that $x > y - \epsilon$.

(3) Notice first that if $\epsilon > 0$ then $0 > -\epsilon$. Now suppose that for $a \in \mathbb{R}$, $|a| \leq \epsilon$. Then $-\epsilon < a < \epsilon$, thus by the transitivity of $<$ we have that $0 < a$ and $a < 0$, which cannot happen, so it must be that $a = 0$.

Now let $a = 0$, then clearly, by our assumptions, we see that $a < \epsilon$ and $-\epsilon < a$, thus $|a| \leq \epsilon$.

■

**Definition.** For all $a, b \in \mathbb{R}$, we define the **closed interval** to be the set $[a, b] = \{x : a \leq x \leq b\}$. We define the **open interval** to be the set $(a, b) = \{x : a < x < b\}$. We denote a **half open interval** to be a set of the form $[a, b) = \{x : a \leq x < b\}$ or $(a, b] = \{a < x \leq b\}$.

We denote $(a, \infty) = \{x : a < x\}$, $(-\infty, a) = \{x : x < a\}$ and $(-\infty, \infty) = \{x : x \in \mathbb{R}\}$.

**Definition.** An interval $I$ is **bounded** if and only if it has the form: $[a, b], [a, b), (a, b]$, or $(a, b)$. We call $a$ and $b$ the **endpoints**, or **bounds** of the interval. We call all other intervals **unbounded**.

Now if $a = b$, the two bounds co incide and we call the interval **degenerate**, and if $a < b$, it is called **nondegenerate**. Also notice that an interval $(1, 1) = \emptyset$, but $[1, 1] = \{1\}$. So we notice that a degenerate open interval is the empty set, and a degenerate closed interval is a point. Now for bounded intervals, we call $|a - b|$ is called the **length** of the interval, and sometimes denoted $|I|$.

**Homework.** Do exercises 1, 5, 7, 9, and 10.

## 1.2 The Well Ordering Principle.

So far we have the set of all natrual numbers $\mathbb{N}$, the set of all integers $\mathbb{Z}$, the rationals $\mathbb{Q}$ and the reals $\mathbb{R}$. Now $\mathbb{N}$ is speacial from these, as it has a "minimal" element. We clarify this below.

**Definition.** An element $x \in \mathbb{R}$ is a **least element** in a set $E \subseteq \mathbb{R}$ if and onlt if $x \in E$ and $x \leq \alpha$ for all $\alpha \in E$.

**Postulate 3** (The Well Ordering Principle.)**.** *Every nonempty subset of* $\mathbb{N}$ *has a least element.*

Now this propertie does not apply to $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$; if one takes the subset $\mathbb{Q}$, one sees immediately that $\mathbb{Q}$ has no least element. Another thing is that the well ordering principle implies the principle of mathematical induction.

**Theorem 1.2.1.** *Suppose that for each $n \in \mathbb{N}$, that $A(n)$ is a propostision such that:*

*(1) $A(1)$ is true.*

*(2) For every $k \in \mathbb{N}$ for which $A(n)$ is true, then $A(k+1)$ is also true.*

*Then $A(n)$ is true for all $n \in \mathbb{N}$/*

*Proof.* Suppose there is some $n$ for whcih $A(n)$ is false. Then the set $E = \{n \in \mathbb{N} : A(n) \text{ is false.}\} \neq \emptyset$. Then by the well ordering principle, $E$ has a least element $x$. Then $A(x)$ is false. However, since $A(1)$ is true, $x \neq 1$, then $x - 1 \in \mathbb{N}$ and $x - 1 < x$, since $x$ is the least element, then $A(x-1)$ is true. By the second condition, we get that $A(x)$ is true, contradicting that $x \in E$. Therefore, $E$ must be empty, and $A(n)$ is true for all $n \in \mathbb{N}$. ■

**Example 1.5.** Show that $\sum_{k=1}^{n}(3k-1)(3k+2) = 3n^3 + 6n^2 + n$.

*Proof.* For $k = 1$ we see that $(3(1)-1)(3(1)+2) = 10 = 2(1)^3 + 6(1)^2 + 1$.

Now suppose that the proposition is true for $n \geq 1$. Then:

$$\begin{aligned}\sum_{k=1}^{n+1}((3k-1)(3k+2)) &= \sum_{k=1}^{n}((3k-1)(3k+2)) + (3(n+1)-1)(3(n+1)+2) \\ &= 3n^3 + 6n^2 + n + (3(n+1)-1)(3(n+1)+2) \\ &= 3(n+1)^3 + 6(n+1)^2 + (n+1)\end{aligned}$$

■

*Remark.* If $m, n \in \mathbb{N}$, then $m + n \in \mathbb{N}$ and $mn \in \mathbb{N}$. This also implies that $m + n, mn \in \mathbb{Z}$.

**Definition.** Let $a, b \in \mathbb{Z}$. We call a **binomial** an expression of the form $(a+b)^n$ for some $n \in \mathbb{N}$.

We wish to study binomials further.

**Definition.** We define, for $n, k \in \mathbb{N}$ the **binomial coefficient** $\binom{n}{k}$ is defined such that:

(1) $\binom{0}{0} = 1$

(2) $\binom{n}{k} = \frac{n!}{(n-k!)k!}$.

The binomial coefficient is itself a natural number, and can be visualized via pascal's triangle.

**Lemma 1.2.2.** $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ *For $n, k \in \mathbb{N}$ and $1 \leq k \leq n$.*

*Proof.*

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{(n-k)!k!} + \frac{n!}{(n-k+1)!(k-1)!} \tag{1.2}$$

$$= n!(\frac{1}{(n-k)!k!} + \frac{1}{(n-k-1)!(k-1)!}) \tag{1.3}$$

$$= n!(\frac{1}{(k-1)!}(\frac{1}{(n-k)!k} + \frac{1}{(n-k-1)!})) \tag{1.4}$$

$$= \frac{n!}{(k-1)!(n-k)!}(\frac{1}{k} + \frac{1}{n-k+1}) \tag{1.5}$$

$$= \frac{(n+1)!}{(n-k+1)!k!} \tag{1.6}$$

$$= \binom{n+1}{k} \tag{1.7}$$

■

**Theorem 1.2.3** (The Binomial Theorem)**.** *If $a, b \in \mathbb{R}$, and $n \in \mathbb{N}$, then:*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{1.8}$$

*Proof.* We use induction in the proof. We see that for $n = 1$, $(a+b)^1 = a+b = \binom{1}{0}a^{1-0} + \binom{1}{1}b^1$. Now suppose that the theorem is true for all $n \geq 1$, that is $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$; and now consider $n+1$.Then:

$$(a+b)^{n+1} = (a+b)(a+b)^n \tag{1.9}$$

$$= (a+b)(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k) \tag{1.10}$$

$$\tag{1.11}$$

by the distributive law:

$$= a\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k) + b\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k) \quad = \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^k) + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}). \tag{1.12}$$

Listing the terms we get $\binom{n}{0}a^{n+1} + \binom{n}{1}a^n b + \binom{n}{2}a^{n-1}b^2 + \cdots + \binom{n}{n+1}ab^n + \binom{n+1}{n+1}b^n$; adding like terms, and by lemma 1.2.2 we get: $\binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^n b + \binom{n+1}{2}a^n b^2 + \cdots + \binom{n+1}{n}ab^n + \binom{n+1}{n+1}b^n = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n-k+1} b^k$. ■

**Homework.** Exercises: 1,2,3,5,6 on page 17. These exercises practice the principle of mathematical induction.

## 1.3 The Axiom of Completeness.

**Definition.** Let $E \subseteq \mathbb{R}$ be nonempty. The set $E$ is **bounded above** if and only if there is an $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in E$; we call $M$ the **upperbound**. A number $s$ is called a **supremum** or **least upperbound** of $E$ of $s$ is an upper bound, and for all other upper bounds $M$, $s \leq M$. We denote the least upper bound as $\sup E$.

**Example 1.6.** If $E = [0, 1]$ prove that $\sup E = 1$.

*Proof.* By definition of $[0, 1]$, 1 is an upperbound. Now let $M \in E$ be an upper bound, then $x \leq M$ for all $x \in E$. Now $1 \in E$, so $1 \leq M$, so by definition, we get that $1 = \sup E$. ■

*Remark.* If a set has one upperbound, then it has infinitely many upper bounds.

*Proof.* If $M_0$ is an upperbound, then for any $M > M_0$, $M$ is an upper bound. ■

*Remark.* If a set has a least upperbound, then that least upperbound is unique.

*Proof.* Asuume that $s_1, s_2 \in \mathbb{E}$ are least upperbounds. Then $s_1$ is an upperbound and so $s_2 \leq s_1$. Likewise, $s_2$ is an upperbound, so $s_1 \leq s_2$, hence it must be that $s_1 = s_2$. ■

**Theorem 1.3.1** (Approximation Property for Least Upperbounds)**.** *If $E$ has a least upperbound, and $\epsilon > 0$ is a positive number, then there is an element $a \in E$ such that $\sup E - \epsilon < a \leq \sup E$.*

*Proof.* Suppose not, that is there is some $\epsilon_{>}0$ for all $a \in E$ for which $a$ does not lie between $\sup E - \epsilon_0$ and $\sup E$. Now $a \leq \sup E$, hence $a \leq E - \epsilon_0$, so $\sup E -_0$ is an upperbound of $E$. So $\sup E \leq \sup E - \epsilon_0$, implying $\epsilon_0 \leq 0$, a contradiction. ■

Now it is not always true that for some set $E$, that $\sup E \in E$.

*Remark.* If $E \subseteq \mathbb{N}$ has a least upperbound, then $\sup E \in E$.

*Proof.* Let $s = \sup E$. By the approximation property, for $\epsilon = 1$, there is an $x_0$ in $E$ such that $s - 1 < x_0 \leq s$. Now if $x_0 = s \in E$, then we are done.

Otherwise, we have $s - 1 < x < s$. Then appying the approximation property again for $\epsilon = s - x_0$, $s - \epsilon = s - s - x_0 = x_0$, then there is an $x_1 \in E$ for which $x_0 < x_1 \leq s$. Again if $x_1 = s$ we are done. Now if $x_1 < s$ we get $0 < x_1 - x_0 < s - x_0 < s - (s - 1) = 1$. We also know that $x_1 - x_0 \geq 1$ for different integers $x_1 > x_0$, a contradiction. ■

This brings us to the axiom of completeness.

**Postulate 4** (The Axiom of Completeness)**.** *If $E$ is a nonempty subset of $\mathbb{R}$ that is bounded above, then $E$ has a finite least upperbound.*

This does not apply to $\mathbb{Q}$, for example, take $\pi$. We can take the set $E$ to be $E = \{3.14, 3.141, 43.1415, \}$. We see that $\pi$ is an upperbound, so is 4; however, $\pi$ is the least upperbound. If we only consider $E \in \mathbb{Q}$, then $E$ has upperbounds, but no least upperbound (as $\pi \notin \mathbb{Q}$). In essence, $\mathbb{Q}$ has a "hole"; infact it has many "holes" that is $\mathbb{Q}$ is not complete. But if $E \in \mathbb{R}$, we see that $\sup E = \pi$. In essence, $\mathbb{R}$ is complete as $\mathbb{R}$ has no "holes".

**Theorem 1.3.2** (The Archimedean Principle)**.** *Given positive real numbers $a$, and $b$; there is an integer $n$ such that $b < na$.*

*Proof.* We want to build a nonempty set of integers such that it is bounded above with a largest integer. The set $E = \{k \in \mathbb{Z} : ka \leq b\}$ has an upper bound; hence it has a least upperbound $k_0$. Then $k_0 + 1 \notin E$, hence $(k_0 + 1)a > b$. ∎

**Example 1.7.** Let $A = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$ and $B = \{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots\}$ Show that $\sup A = \sup B = 1$.

*Proof.* First, 1 is an upperbound for both $A$ and $B$. Now for $A$ it is clear that 1 is the least upperbound. Now let $M$ be an upperbound for $B$ and $M < 1$, so $1 - M > 0$ and $\frac{1}{1-M} > 0$. Then by the Archimedean principle, for $\frac{1}{1-M}$ and 1, there is an $n \in \mathbb{Z}$ such that $\frac{1}{1-M} < n$. Now we have that $n < 2^n$; hence it follows that $1 - \frac{1}{2^n} > M$. So $M$ is not an upperbound; a contradiction. Hence $\sup B = 1$. ∎

**Theorem 1.3.3.** *If $a, b \in \mathbb{R}$ such that $a < b$, then there exists a rational $q \in \mathbb{Q}$ such that $a < q < b$.*

*Proof.* If $b - a > 0$, we have a pari of positive numbers 1 and $b - a$; so by the Archimedean principle, there is an $n \in \mathbb{N}$ such that $n(b - a) > 1$.

Now suppose that $b > 0$. Consider the set $E = \{k \in N : b \leq \frac{k}{n}\}$. Clearly $E \neq \emptyset$ as $1 \in E$. Then by the well ordering principle, $E$ has a least element $k_0$. Now let $m = k_0 - 1$ and let $q = \frac{m}{n}$. Now $m < k_0$, so $m \notin E$; now there are twon possible cases: either $m \leq 0$ or $b > \frac{m}{n} = q$, either way we have that $b > q$. And $k_0 \in \mathbb{E}$, so $b \leq \frac{k_0}{n}$ so $a = b - (b - a) < \frac{k_0 - 1}{n} = q$

Now suppose that $b \leq 0$, then $-b > 0$; applying the Archimedean principle for 1 and $b$, there is a $k \in \mathbb{N}$ such that $-b < (k)(1)$, thus $k + b > 0$. Now if $b = 0$, clearly $1 + 0 > 0$. By the first case, there is a rational $q \in \mathbb{Q}$ such that $a + k < q < b + k$. Subtracting $k$, we get $a < q - k < b$, which finishes the proof as $q - k \in \mathbb{Q}$. ∎

*Remark.* If $x > 1$ and $x \notin \mathbb{N}$, then there is a natrual number $n \in \mathbb{N}$ such that $n < x < n + 1$.

*Proof.* For the pair $(x, 1)$ by the Archimedean principle, there is a $k \in \mathbb{N}$ such that $E = \{m \in \mathbb{N} : x < m\}$ is nonempty. Then by the well ordering principle, there is a least element $m_0$, hence $x < m_0$, Let $n = m_0 - 1$, then $n \notin E$. Now either $n \leq 0$, or $n < x$. As $x \notin \mathbb{N}$, $x \neq n$. Now if $n \leq 0$, then $n < x < m_0 = n + 1$; and if $n < x$, then $n < x < m_0 = n - 1$. ∎

*Remark.* If $n \in \mathbb{N}$ is not a perfect square, then $\sqrt{n}$ is irrational.

*Proof.* Assume that $\sqrt{n}$ is rational for a nonperfect square $n \in N$. Then $\sqrt{n} = \frac{p}{q}$ for $p, q \in \mathbb{Z}$. Now since 1 is a perfect square, let $n \geq 2$. Then $\sqrt{n} > 1$. Then there is an $m_0 \in \mathbb{N}$ such that $m_0 < \sqrt{n} < m_0 + 1$. Consider the set $E = \{k \in \mathbb{N} : k\sqrt{n} \in \mathbb{N}\}$. Now $q\sqrt{n} = p \in \mathbb{N}$, so $E \neq \emptyset$. By the well ordering principle, $E$ has a least element $n_0$. Then $n_0\sqrt{n} \in \mathbb{N}$ and $n_0 m_0 \in \mathbb{N}$. Now $n_0(\sqrt{n} - m_0) = x \in \mathbb{N}$. Now $0 < \sqrt{n} - m_0 < 1$, then $0 < x < n_0$. Now $x \notin E$, on the other hand, $x\sqrt{n} = n_0(\sqrt{n} - m_0)\sqrt{n}$, so $x \in E$; a contradiction. ∎

**Definition.** Let $E \subseteq \mathbb{R}$ be nonempty. The set $E$ is **bounded below** if and only if there is an $m \in \mathbb{R}$ such that $a \geq m$ for all $a \in E$. We call $m$ a **lowerbound** of $E$. A nuumber $t$ is called an **infimum** or a **greatest lowerbound** if and only if $t$ is a lowerbound of $E$ and $t \geq m$ for all lowerbounds $m$ of $E$. We denote the greatest lowerbound as $t = \inf E$.

It should be worthwhile to say that the least element of a set, and it's greatest lowerbound are not the same.

**Definition.** A nonempty set $E\mathbb{R}$ is said to be **bounded** if it is bounded above and bounded below. That is there exist $m, M \in \mathbb{R}$ such that $m \leq a \leq M$ for all $a \in E$. Is $\sup E \in E$ then we write $\sup E = \max E$ and call it the **maximum** of $E$. If $\inf E \in E$, then we write $\inf E = \min E$ and call it the **minimum** of $E$.

**Definition.** The **reflection** of a set $E \in \mathbb{R}$ is defined to be the set $-E = \{xx = -a : \text{ for some } a \in E\}$.

**Theorem 1.3.4.** *Let $E \subseteq \mathbb{R}$ be a nonempty set. Then $E$ has a greatest upperbound if and only if $-E$ has a greates lowerbound; in which case we have that* $\inf(-E) = -\sup E$. *Likewise, $E$ has a greatest lowevound if and only if $-E$ has a least upperbound, in which case* $\sup(-E) = -\inf E$.

*Proof.* Suppose that $E$ has a least upperbound $s$, and let $t = -s$. Now we have that $-a \geq -s = t$, for all $a \in E$, so $t$ is a lowerbound of $E$. Now suppose $m$ is any lowerbound of $E$. Then $m \leq -a$ for all $a \in E$. Now, $-m \geq a$ for all $a \in E$, so $-m$ is an upperbound,; now since $s$ is the least upperbound, $s \leq -m$, hence $t = -s \geq m$

Conversely, suppose that $-E$ has a greatest lowerbound $t$; then $tleq - a$ for all $a \in E$, hence $-t \geq a$ for all $a \in E$, so $-t$ is an upperbound. Let $M$ be any upperbound of $E$, then $M \geq a$ for all $a \in E$. Then $-M \leq -a$ and $-M$ is a lowerbound. Then $-M \leq t$, hence $M \geq -t = s$, so $s$ is the least upperbound of $E$. ■

**Homework.** Exercises 2,3,4,5($a$), and 6 on page 23.

**Theorem 1.3.5** (Monotone property)**.** *Let $A \subseteq B$ be nonempty subsets of $\mathbb{R}$. Then:*

*(1) If $B$ has a least upper bound, then* $\sup A \leq \sup B$.

*(2) If $B$ has a greatest lower bound then* $\inf A \geq \inf B$

*Proof.*

1. We have that $A \subseteq B$, so any upper bound of $B$ is an uppper bpund pf $A$. Therefore $\sup A$ is an upper bound of $A$; By completeness, $\sup A$ exisists, and more over, $\sup A \leq \sup B$

2. Consider $-A \subseteq -B$, by part (1), we have $\sup -A \leq \sup -B$, and we also have that $\sup -A = -\inf A$ and $\sup -B = -\inf B$; thus $-\inf A \leq -\inf B$ therefore, $\inf A \geq \inf B$.

■

To consider $\sup A$ and $\inf A$, we need nonempty and bounded sets of $\mathbb{R}$. Now we would like to talk about the least upper and greates lower bounds for any subset of $\mathbb{R}$; not just the bounded ones.

**Definition.** An **extended real number** $x$ is real number such that either $x = \infty$ or $x = -\infty$.

Now if $E$ is not bounded above, then we define the least upper bound of $E$ to be $\sup E = \infty$. If $E$ is not bounded below we define $\inf E = -\infty$. Now for the empty set $\emptyset$, we define $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. It is worth considering why this phenomena occurs with the empty set.

## 1.4 Functions, Countability, and The Algebra of Sets.

**Definition.** Given a mapping $f : E \to \mathbb{R}$, we call $f$ a **real valued function** of a **real variable**. If we are given $f : \mathbb{R}^n \to \mathbb{R}$ we call the mapping a real valued **multivariable fucntion**. If we are given $f : \mathbb{R}^m \to \mathbb{R}^m$, we call it a **vector valued** multivariable function.

We are interested in a whole slew of functions; the trigonometric functions: sin, cos, tan, cot, sec, and csc. Other functions of interest are the natural logarithmic function, and exponential function: log and $e^x$, as well as arbitrary power functions $x^\alpha$. We can define the power function to be $x^\alpha = e^{\alpha \log x}$; where $x > 0$ and $\alpha \in \mathbb{R}$.

The derivitives of these functions carry over, and will be examined with more scrutiny later on.

**Definition.** A mapping $f : X \to Y$ is called **1-1** if and only if for $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. $f$ is called **pnto** if for every $y \in Y$, there exists an $x \in X$ such that $y = f(x)$.

**Example 1.8.** $f(x) = x^2$ is 1-1 on the interval $[0, \infty)$, but it is not 1-1 on $(0, \infty)$.

**Theorem 1.4.1.** *Let $X$, and $Y$ and let $f : X \to Y$ be a mapping. The $f$ is 1-1 from $X$ onto $X$ if there is a unique mapping $g : Y \to X$ that satisfies: $f(g(y)) = y$ and $g(f(x)) = x$.*

*Proof.* Suppose that $f$ is 1-1 and onto. Then for each $y \in Y$, choose the unique $x \in X$ such that $f(x) = y$. We define $g(y) = x$, Then it is clear to see that $g$ takes $Y$ onto $X$, and $g$ is also 1-1. Then $g(y) = g(f(x)) = x$ and $f(x) = f(g(y)) = y$.

Conversely, suppose that there exists a mapping $g : Y \to X$ satisfying the $g(y) = g(f(x)) = x$ and $f(x) = f(g(y)) = y$. Now let $x_1, x_2 \in X$ and let $f(x_1) = f(x_2)$. Then $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$; so $f$ is 1-1. For any $y \in Y$, let $x = g(y)$, then $f(x) = f(g(y)) = y$; thus $f$ is onto.

Now suppose ther is another mapping $h : Y \to X$satisfying the conditions that $g$ satisfies. Let $y \in Y$ and choose $x \in X$ such that $f(x) = y$. Then $h(yh(f(x)) = x = g(f(x)) = g(y)$; then it must be that $g = h$. ■

**Definition.** if $f : X \to Y$ is 1-1 and onto, we say that $f$ has a **inverse mapping** and denote it $f^{-1}$.

If $f : E \to \mathbb{R}$ is a real valued function, then the pair $(x, f(x))$ is on the graph of $f$; $(f^{-1}(y), y)$ is also on the graph of $f$. However the pair $(y, f^{-1}(y))$ is on the graph of $f^{-1}$. The graphs of $f$ and $f^{-1}$ are symmetric with respect to the line $y = x$.

*Remark.* If $f$ is defferentiable on an open interval $I$ and $f'(x) \neq 0$ for all $x \in I$.

This remark is a sufficient, but not a necessary condition. The concept of "differentiability" must also be discussed. We can prove this remark with the "mean value theorem" which will be discussed later.

**Example 1.9.** Prove that $f(x) = e^x - e^{-x}$ is 1-1 and find $f^{-1}$.

*Proof.* By the chain rule, we have $f'(x) = e^x + e^{-x} > 0$ fr all $x \in \mathbb{R}$. Now let $y = e^x - e^{-x}$, then $e^x y = e^{2x} - 1$, by the quadratic formula we get:

$$e^x = \frac{y}{2} + \frac{\sqrt{y^2+4}}{2}$$

So taking the natural logarithm:

$$y = \frac{x}{2} + \frac{\sqrt{x^2}+4}{2} = f^{-1}(x)$$

■

**Definition.** A set $E$ is **finite** if and only if $e =$ or there is a 1-1 mapping $f$ between $\{, \dots, n\}$ onto $E$. $E$ is said to be **countable** if and only if there is a 1-1 mapping from $\mathbb{N}$ onto $E$. $E$ is said to be **at most countable** if and only if $E$ is either finite or countable. $E$ is said to be **uncountable** if and only if $E$ is neither finite, nor countable.

To show a set to be countable, we need a 1-1 mapping from $\mathbb{N}$ onto $E$. For example, let $2\mathbb{N} = \{2, 4, 6, 8, \dots\}$, and defining $f(n) = 2n$, then $f : \mathbb{N} \to 2\mathbb{N}$ is 1-1 and onto. Therefore the set $2\mathbb{N}$ is countable. We know that if a set $B$ is finite, then any proper subset $A$ of $B$ has strictly less numbeer of elements.

**Definition.** A set $B$ is an **infinite** set if and only if there is a proper subset $A$ so that $A$ and $B$ have the same number of elements.

Now all countable sets are infinite, but not all infinite sets are countable.

**Theorem 1.4.2** (Cantor's Diagonalization Argument)**.** *The open interval* $(0, 1)$ *is uncountable.*

*Proof.* Suppose that the interval $(0, 1)$ is countable; then by definintion, there exists a mapping $f : \mathbb{N} \to (0, 1)$ that is 1-1 and onto. Then this mapping exhausts all the elements of $(0, 1)$. We wish to construct an element $xin(0, 1)$ such that $x \neq f(n)$ for all $n \in \mathbb{N}$. For each $f(n)$ we have:

$$f(n) = 0.\alpha_{n_1}\alpha_{n_2}\dots$$

This expansion may not be unique, consider $0.1 = 0.0999\dots$. We require that the decimal expansion does not terminate in 9's. Then $f(n)$ and $0.\alpha_{n_1}\alpha_{n_2}\dots$ are 1-1:

$$f(1) = 0.\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}\dots \tag{1.13}$$
$$f(2) = 0.\alpha_{21}\alpha_{22}\alpha_{23}\alpha_{24}\dots \tag{1.14}$$
$$f(3) = 0.\alpha_{31}\alpha_{32}\alpha_{33}\alpha_{34}\dots \tag{1.15}$$
$$f(4) = 0.\alpha_{41}\alpha_{42}\alpha_{43}\alpha_{44}\dots \tag{1.16}$$
$$\vdots \tag{1.17}$$

So we get an infinite matrix; an all $\alpha_{ij}$ are digits in $\{0,1,\dots,9\}$, and none of them terminate in 9's. Now let $x = 0.\beta_1\beta_1\beta_1\dots$ with:

$$\beta_k = \begin{cases} \alpha_{kk} + 1 \text{ if } \alpha_{kk} \leq 5 \\ \alpha_{kk} - 1 \text{ if } \alpha_{kk} > 5 \end{cases}$$

By this construction, we have that $\beta_1 \neq \alpha_{11}$, $\beta_2 \neq \alpha_{22}$, $dots$, $\beta_n \neq \alpha_{nn}$; so $x \neq f(n)$ for all $n \in \mathbb{N}$ and where $x \in (0,1)$. This is a contradiction of the fact that we assumed $f$ to be onto. Therefore $(0,1)$ is uncountable. ■

Now we want to study the countability of the sets $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. We sstudy some general properties first:

**Lemma 1.4.3.** *A nonempty set $E$ is at most countable if and only if there is a mapping $g : \mathbb{N} \to E$ that is onto.*

*Proof.* Assume that $E$ is at most countable, then by definition, we are done. If $E$ is finite, by definition there is an $n \in \mathbb{N}$ and a 1-1 mapping $f$ taking $\{1,\dots,n\}$ onto $E$. Now define:

$$g(j) = \begin{cases} f(j) \text{ if } j \leq n \\ f(1) \text{ if } j > n \end{cases}$$

Then $g$ takes $\mathbb{N}$ onto $E$.

Conversely, suppose that $g$ takes $\mathbb{N}$ onto $E$, we need to construct a 1-1 $f$ from $\mathbb{N}$ or a subset of $\mathbb{N}$ onto $E$. Let $k_1 = 1$ and $E_! = \{k \in \mathbb{N} : g(k) \neq g(k_1)\}$. Now if $E = \emptyset$, we are done. Else, by the well ordering principle, there is a least element $k_2$ and define $E_2 = \{k \in \mathbb{N} : g(k) \neq g(k_1), \text{ and } g(k) \neq g(k_2)\}$. Now we have that $k_2 > k_1$ and $k_2 \geq 2$. If $E_2 = \emptyset$ then $E = \{g(k_1), g(k_2)\}$ is finite and we are done. Now if $E_2 \neq \emptyset$, then we continue along eoth our methpd until we exhaust all possible set until we reach $E_n = \emptyset$ for some $n$.

If this process ever terminates, then $E = \{g(k_1),\dots,g(k_n)\}$ and is finite; now if it never terminates, then we have a sequence $k_1 < k_2 < k_2, \dots$, and so $k_{j+1}$ is the least element of $E_j$ and $k_j \geq j$. We define $f(j) = g(k_j)$ and we show that $f$ is 1-1. For $j < l$ we have $k_j < k_l$, so $k_j \leq k_{l-1}$; by construction, $g(k_l) \in E_{l-1}$, so $g(k_l) \neq g(k_j)$, f is 1-1. Now for any $x \in E$ there is an $n \in \mathbb{N}$ such that $x = g(n)$, and there is an $l$ such that $x = g(n) = g(k_l)$; and so $E = \bigcup_{l=1}^{\infty}$, and so $f$ is onto. Therefore, $E$ is countable. ■

**Theorem 1.4.4.** *Suppose that $A$ and $B$ are sets and that $A \subseteq B$ and $B$ is at most countable. Then $A$ is at most countable. Likewise if $A \subseteq B$ and $A$ is uncountable, then $B$ is uncountable; in particular, $\mathbb{R}$ is uncountable.*

*Proof.* If $B$ is at most countable, then there is a function $g : \mathbb{N} rightarrow B$. If $A$ is empty, we are done. If $A$ is not empty, then choose an element $a_0 \in A$ and define $f : \mathbb{N} \to A$ such that $f(n) = g(n)$ if $h(n) \in A$, and $f(n) = a_0$ if $g(n) \notin A$. Therefore, $f$ is onto, and $A$ is atmost countable.

Assume that $B$ is atmost countable, then by the above, since $A \subseteq B$, $A$ is also atmost countable. This is a contradiction by assumption, thus, $B$ must also be uncoutable.

Now, since the set $(0,1)$ is uncountable, and $(0,1) \subseteq \mathbb{R}$, then by the above $\mathbb{R}$ is uncountable. ■

**Theorem 1.4.5.** *Let $A_1, A_2, \dots$ be atmost countable sets, then:*

*(1)*

$$A_1 A_2$$

*is atmost countable.*

*(2) If $E = \bigcup_{j\in\mathbb{N}} A_j$, then $E$ is atmost countable.*

*Proof.* $A_1$ and $A_2$ are atmost countable, then there exist onto functions $\phi_1 : \mathbb{N} \to A_1$, and $\phi : \mathbb{N} \to A_2$, (more over we notice that since $A_n$ is also atmost countable, then there exists an onto function $\phi_n : \mathbb{N} \to A_n$).

Now define $f : \mathbb{N} \times \mathbb{N} A_1 \times A_2$ by $f(n, m) = (\phi_1(m), \phi_2(n))$, clearly $f$ is onto, now if we can define another onto function $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, then the compostion $f \circ g : \mathbb{N} \to A_1 \times A_2$ is also onto; now $g(1) = (1, 1)$, $g(2) = (2, 1)$, $g(3) = (1, 2)$, $g(4 = (3, 1)$, ..., by observing the behaviour f $g$ as $g$ moves through $\mathbb{N}$, we can deduce a formula. Assume that $g(j)$ lies on the $n$-th line, so $g(1) = (1, n + 1)$, $g(2) = (2, n - 1)$, $g(3) = (3, n - 2)$, and so on. Thus we deduce that for $l \in \mathbb{N}$ that $g(j) = (l, n + 1 - 1)$. Now what is the relation between $j$ and $n$? We have $j > 1 + 2 + 3 + \cdots + (n - 1) = \frac{n(n-1)}{2}$. Now $j \geq 1$, so $j + 1 \geq 2$, hence $\frac{j+1}{2} \geq 1$, to $\frac{f(f+1)}{2} \geq j$. Now let $E = \{k\imath\mathbb{N} : j \leq \frac{k(k+1)}{2}\}$ which is nonempty, hence $E$ has a least element $n$. so $j \leq \frac{n(n+1}{2}$ Thus $l = j - \frac{n(n-1)}{2}$, therefore $j = l + \frac{n(n+1)}{2}$.

Now we have for arbitrary $j \in \mathbb{N}$ that $A_j$ is countable, thus there is an onto function $\phi_j : \mathbb{N} \to A_j$. Define $f : \mathbb{N} \times n \to \bigcup_{j\in\mathbb{N}} A_j$ by $f(m, n) = \phi_n(m)$, then $f$ is also onto. Therefor $\bigcup_{j\in\mathbb{N}} A_j$ is atmost countable. ■

*Remark.* We have $\mathbb{Z} = \mathbb{N} \cup -\mathbb{N}$ is countable, and $\mathbb{Q} = \bigcup_{n\in\mathbb{N}}\{\frac{p}{n} : n \in \mathbb{Z}\}$ is also countable. Now $\mathbb{R}$ is uncountable, if $\mathbb{Q}^*$ is also countable, then we get $\mathbb{Q}^* \cup \mathbb{Q} = \mathbb{R}$ is countable, contradicting the uncountability of $\mathbb{R}$, hence $\mathbb{Q}^*$ must also be uncountable.

**Definition.** A collection of sets $\mathcal{E}$ is said to be indexed by a set $A$ if and only if there is a function $F$ from $A$ onto $\mathcal{E}$. We call $A$ the **index set** of $\mathcal{E}$. We may write $\mathcal{E} = \{E_\alpha\}_{\alpha\in A}$.

**Definition.** Let $\{E_\alpha\}_{\alpha\in A} = \mathcal{E}$ be a collection of sets. Then:

(1) $\bigcup E_{\alpha\in A} = \{x : x \in E_\alpha \text{ for some } \alpha \in A\}$

(2) $\bigcap E_{\alpha\in A} = \{x : x \in E_\alpha \text{ for all } \alpha \in A\}$

**Theorem 1.4.6** (DeMorgan's Law)**.** *Let $\mathcal{X}$ be a set and let $\{E_\alpha\}_{\alpha\in A}$ be a collection of subsets of $\mathcal{X}$. For each $E \subseteq \mathcal{X}$*

*then:*

$(\bigcap E_\alpha)^C = \bigcup E_\alpha^C$

*where $E^C = \mathcal{X} \backslash E$.*

**Definition.** Let $X$ and $Y$ be sets and let $f : X \to Y$ be function. Then the **image** of $X$ under $f$ is the set $f(X) = \{f(x) : x \in X\}$. The **inverse image** is the set $f^{-1}(E) = \{x \in X : y = f(x)$ for some $y \in E\}$.

For the inverse image, we do not require that the mapping $f$ be $1-1$ as we require it for the existence of $f^{-1}$. Now let $X, Y$ be sets and let $fX \to Y$. If $\{E_\alpha\}_{\alpha \in A}$ for some index set $A$ then:

$$f(\bigcup_{\alpha \in A} E_\alpha) = \bigcup_\alpha f(E_\alpha) \tag{1.18}$$

and

$$f(\bigcap_{\alpha \in A} E_\alpha) = \bigcap_\alpha f(E_\alpha) \tag{1.19}$$

Now if $B$ and $C$ are subsets of $X$, then:

$$f(B) \backslash f(C) \subseteq f(B \backslash C) \tag{1.20}$$

and if $B, C \subseteq Y$ then:

$$f^{-1}(B \backslash C) = f{-}1(B) \backslash f^{-1}(C) \tag{1.21}$$

and if $\{E_\alpha\}_{\alpha \in A}$ then:

$$f^{-1}(\bigcup_{\alpha \in A} E_\alpha) = \bigcup_\alpha f^{-1}(E_\alpha) \tag{1.22}$$

and

$$f^{-1}(\bigcap_{\alpha \in A} E_\alpha) = \bigcap_\alpha f^{-1}(E_\alpha) \tag{1.23}$$

Finally, if $E \subseteq f(X)$, then $f(f^{-1}(E)) = E$, but if $E \subseteq X$, then $E \subseteq f^{-1}(f(E))$. We accept these properties witout proof as they can be demonstrated trough elementary se theory.

Now to illustrate an example of these properties, t $f : \mathbb{R} \to \mathbb{R}$ be defined by $x \to x^2$, let $E_1 = \{1\}$ and $E = \{-1\}$. Then $E_1 \cap E_2 = \emptyset$ , then $f(E_1 \cap E_2) =, emptyset$; but $f(E_1) = \{1\}$ and $f(E_2) = \{1\}$, hence $f(E_1) \cap f(E_2) = \{1\}$. It is intersting to note that if $f$ is $1-1$, then all the relations estabku=ished above are equalities.

**Homework.** Exercises 6, 9, 10, and 11 on page 33 of the book.

# Chapter 2

# Sequences in $\mathbb{R}$.

## 2.1 Limits of Sequences.

**Definition.** A **sequence** is a mapping $f : \mathbb{N} \to \mathbb{R}$, whose terms are $x_n = f(n)$. We denote a sequence as $\{x_n\}_n \in \mathbb{N}$ or just simply $\{x_n\}$.

**Definition.** A sequence of real numbers $\{x_n\}$ is said to **converge** to a real number $a \in \mathbb{R}$ if and only if for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ (in general depending on $\epsilon$) such that for all $n \geq N$, $|x_n - a| < \epsilon$. We say that $\{x_n\}$ (or $x_n$) **converges** to $a$, and we write:

$$\lim_{n\to\infty} x_n = a \tag{2.1}$$

or simply, $x_n \to a$ as $n \to \infty$. We call $a$ the **limit** of the sequence. A sequence which does not cpnverge is said to **diverge**.

We can consider $x_n$ as a sequence of "approximations" to $a$ and $\epsilon$ as an upperbound for the "error" of those approximations. $N \to \infty$ as $\epsilon \to 0$, thay is $N$ gets larger as $\epsilon$ gets smaller.

Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$

*Proof.* Let $\epsilon > 0$, by the Archimedean principle, there is an $N \in \mathbb{N}$ such that $\frac{1}{\epsilon} < N$. Then $\frac{1}{N} < \epsilon$. For all $n \geq N$, $\frac{1}{n} < \frac{1}{N} < \epsilon$, hence, $|\frac{1}{n} - 0| < \epsilon$. ■

The sequence $\{(-1)^n\}_{n\in\mathbb{N}}$ does not converge

*Proof.* Assume that $(-1)^n \to a$ as $n \to \infty$. By definition, we have that for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|(-1)^n - a| < \epsilon$ for $n \geq N$. Choose $\epsilon = \frac{}{2}$. then for $n$ od we have $(-1)^n = -1$, and for $n$ even, $(-1)^n = 1$. Then $|(-1)^n - (-1)^{n+1}| = |((-1^n) - a) - ((-1)^{n+1}) - a|$; by the triangle inequality, we have $|(-1)^n - (-1)^{n+1}| \leq |(-1)^n - a| + |(-1)^{n+1} - a| < \frac{1}{2} < 1$, when $n \geq N$. But $|(-1)^n - (-1)^{n+1}| = 2$, a contradiction. Thus $\{(-1)^n\}$ does not converge. ■

*Remark.* A sequence can have at most one limit.

*Proof.* Suppose it has atleast two limits, that is $x_n \to a$ and $x_n \to b$ (with $a \neq b$) as $n \to \infty$. For $\epsilon > 0$ there is an $N_1$ such that for all $n \geq N_1$ $|x_n - a| \leq \epsilon$, and there is an $N_2$ such that $|x_n - b| \leq \epsilon$. Choose $N = \max(N_1, N_2)$, then for $n \geq N$ we have:

$$|x_n - a| < \epsilon \qquad \text{and} \qquad |x_n - b| < \epsilon \tag{2.2}$$
$$\tag{2.3}$$

Then $|a - b| = |(a - x_n) - (b - x_n)| \leq |a - x_n| + |b - x_n| < 2\epsilon$ for every $\epsilon$. Now choose $\epsilon = |a - b|/4 > 0$, then $2\epsilon = |a - b|/2$ hence $|a - b| < |a - b|/2$, which is a contradiction. Then $x_n$ converges to at most one limit. ■

**Definition.** A **subsequence** of a sequence $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of the form $\{x_{n_k}\}_{k\in\mathbb{N}}$ where $n_k \in \mathbb{N}$ and $n_1 < n_2 < \dots$.

Recall that the sequence $\{-1)^n\}$ does not converge, however, suppose we only select even terms and form $\{(-1)^{2n}\}$, then this latter sequence converges. Like wise the sequence $\{\frac{1}{n}\} \to 0$ rather slowly, but forming the subsequence $\{\frac{1}{2^n}\}$, the latter converges rather quickly. So the immediate use of subsequences is in the correction of sequences that behave badly (i.e they don't converge), or to make them converge quikly. Now if $n_k = k$, then the subsequence is the original sequence, if $n_k > k$, then the subsequence is a **proper** subsequence.

*Remark.* If $\{x_n\}$ converges to $a$ and $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$, then $\{x_{n_k}\}$ also converges to $a$.

*Proof.* For any $\epsilon > 0$, there is an $N > 0$ such that for all $n \geq N$, $|x_n - a| < \epsilon$. Now if $n_k = k$, we are done, so suppose that $n_k > k$, so for $k = N$, $n_k > N$ then $|x_{n_k} - a| < \epsilon$. ■

**Definition.** Let $\{x_n\}$ be a sequence of real numbers. Then we say that $\{x_n\}$ is **bounded above** if there is an $M \in \mathbb{R}$ such that $x_n \leq M$ for all $n \in \mathbb{N}$. Similarly, $\{x_n\}$ is **bounded below** if there is an $m \in \mathbb{R}$ such that $m \leq x_n$ for all $n \in \mathbb{N}$. We say $\{x_n\}$ is **bounded** if it is both bounded above and bounded below

Now $\{x_n\}$ is bounded if and only if there is some $C \in \mathbb{R}$ with $C > 0$ such that $|x_n| \leq C$ for all $n \in \mathbb{N}$.

**Theorem 2.1.1.** *Every convergent sequence is bounded.*

*Proof.* Let $\{x_n\}$ converge to $a$. Then for every $\epsilon > 0$, there is an $N > 0$ such that for every $n \geq N$, $|x_n - a| < \epsilon$. Choosing $\epsilon = 1$, we have $|x_n - a| < 1$, so $|x_n| = |x_n - a + a| \leq |x_n - a| + |a| < 1 + |a|$. Now let $C = |x_1| + |x_2| + \dots + |x_{N-1}| + |a| + 1$. Then we see that $|x_n| \leq C$ for all $n \in \mathbb{N}$. Thus $\{x_n\}$ is bounded. ■

The converse of this theorem is not true; take $\{(-1)^n\}$ which is bounded by 1, but $\{(-1)^n\}$ diverges.

**Homework.** Exercises 1, 4, 5, and 6 on page 38.

**Lemma 2.1.2.**