# Notes on Combinatorics

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# Chapter 1

# Essential Problems

There are some essential problems to discuss, but we first give some counting principles.

**Axiom 1.0.1** (The Sum Rule). *Suppose $S_1, S_2, \ldots, S_m$ are mutually disjoint finite sets and that $|S_i| = n_i$ for $1 \leq i \leq n$. There are $n_1 + n_2 + \cdots + n_m = \sum_{i=1}^{m} n$ ways to select one element from any of the sets $S_i$*

**Axiom 1.0.2** (The Product Rule). *Suppose $S_1, S_2, \ldots, S_m$ are finite sets, (not necessarily mutually disjoint) for $1 \leq i \leq n$. Provided that the selections are made independently, there are $n_1 n_2 \ldots n_m =_{i=1}^{m} n_i$ ways to select one element from the set $S_i$ followed by an element from $S_{i+1}$.*

The following problems can now be discussed, and are all solved by the product rule.

**Problem 1.** How many ways are there to order $n$ different elements in a given $n$ element set?

**Solution.** Let $S$ be a set with $|S| = n$ and choose one element $s_1 \in S$ and take $S_1 = S \backslash s_1$, since $s_1$ is arbitrary, by the sum rule, there are $n$ choices for elements in $S$ Now we need to choose elements from $S_1$ which has $|S_1| = n - 1$, by the same reasoning, choose $s_2 \in S_2$ and take $S_2 = S_1 \backslash s_2$. Since $s_2$ was arbitrary, by the sum rule again, there are $n - 1$ ways to choose elements from $S_2$. Continuing along this construction, take $s_i \in S_{i-1}$ and take $S_o = S_{i-1} s_i \backslash$, by the same reasoning there are $n - i$ ways to choose elements from $S_i$, where $1 \leq i \leq n$. Then by the product rule, there are $_{i=1}^{n} n - i = n(n-1) \ldots 2 \cdot 1 \cdot 0! = n!$ ways to order $n$ elements of $S$.

**Problem 2.** How many ways are there to order $k$ elements from an $n$ element set?

**Solution.** Let $S$ be a set with $|S| = n$ and choose an arbitrary subset $T \subseteq S$ with $|T| = k$. Now there are $n!$ ways to order the elements of $S$ and $(n-k)!$ ways to order elements from $S \backslash T$, hence there are $\frac{n!}{(n-k)!}$ ways to order $k$ elements of $T$ from $S$.

**Problem 3.** How many ways are the to select $k$ elements, regardless of order, from an $n$ element set?

**Solution** (1)**.** Let $S$ be a set with $|S| = n$ and $T \subseteq S$ with $|T| = k$. We have there are $n!$ ways to order the elements of $S$, $k!$ ways to order the elements of $T$ and $\frac{n!}{(n-k)!}$ ways to order the elements of $S \backslash T$. Now since order is irrelevant, the ordering of the elements of $T$ does not matter. Hence there are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ ways to select $k$ elements from $S$ in no particular order.

There is an another way of arriving to this solution.

**Definition 1.0.1.** We define the **falling factorial power** $x^{\underline{k}}$, of $x$ is the product

$$x^{\underline{k}} = x(x-1)(x-2)\dots(x-k+1) = \prod_{i=1}^{k-1} x - i \tag{1.1}$$

We define the **rising factorial power** of $x$ to be

$$x^{\bar{k}} = x(x+1)(x+2)\dots(x+k-1) = \prod_{i=1}^{k-1} x - i \tag{1.2}$$

Ande we define $n^{\underline{0}=n^{\bar{0}}=0!=1}$.

**Solution** (2)**.** Notice that $\frac{n!}{(n-k)!} = n(n-1)\dots(n-k+1) = n^{\underline{k}} = n^{\bar{k}}$ and $n^{\underline{n}} = 1^{\bar{n}} = n!$. So there are $\frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$ ways to choose $k$ elements from $S$ in no particular order.

*Remark.* Note that this solution is not unique.