Notes on Combinatorics

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Chapter 1

Essential Problems

There are some essential problems to discuss, but we first give some counting principles.

Axiom 1.0.1 (The Sum Rule). *Suppose $S_1, S_2, \ldots, S_m$ are mutually disjoint finite sets and that $|S_i| = n_i$ for $1 \leq i \leq n$. There are $n_1 + n_2 + \cdots + n_m = \sum_{i=1}^{m} n$ ways to select one element from any of the sets S_i*

Axiom 1.0.2 (The Product Rule). *Suppose $S_1, S_2, \ldots, S_m$ are finite sets, (not necessarily mutually disjoint) for $1 \leq i \leq n$. Provided that the selections are made independently, there are $n_1 n_2 \ldots n_m =_{i=1}^{m} n_i$ ways to select one element from the set S_i followed by an element from S_{i+1}.*

The following problems can now be discussed, and are all solved by the product rule.

Problem 1. How many ways are there to order n different elements in a given n element set?

Solution. Let S be a set with $|S| = n$ and choose one element $s_1 \in S$ and take $S_1 = S \backslash s_1$, since s_1 is arbitrary, by the sum rule, there are n choices for elements in S Now we need to choose elements from S_1 which has $|S_1| = n - 1$, by the same reasoning, choose $s_2 \in S_2$ and take $S_2 = S_1 \backslash s_2$. Since s_2 was arbitrary, by the sum rule again, there are $n - 1$ ways to choose elements from S_2. Continuing along this construction, take $s_i \in S_{i-1}$ and take $S_o = S_{i-1} s_i \backslash$, by the same reasoning there are $n - i$ ways to choose elements from S_i, where $1 \leq i \leq n$. Then by the product rule, there are $_{i=1}^{n} n - i = n(n-1) \ldots 2 \cdot 1 \cdot 0! = n!$ ways to order n elements of S.

Problem 2. How many ways are there to order k elements from an n element set?

Solution. Let S be a set with $|S| = n$ and choose an arbitrary subset $T \subseteq S$ with $|T| = k$. Now there are $n!$ ways to order the elements of S and $(n-k)!$ ways to order elements from $S \backslash T$, hence there are $\frac{n!}{(n-k)!}$ ways to order k elements of T from S.

Problem 3. How many ways are the to select k elements, regardless of order, from an n element set?

Solution. Let S be a set with $|S| = n$ and $T \subseteq S$ with $|T| = k$. We have there are $n!$ ways to order the elements of S, $k!$ ways to order the elements of T and $\frac{n!}{(n-k)!}$ ways to order the elements of $S \backslash T$. Now since order is irrelevant, the ordering of the elements of T does not matter. Hence there are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ ways to select k elements from S in no particular order.