Convexity

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<u>**Text**</u>

A Course in Convexity

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Chapter 1

Convex Sets.

1.1 Definitions and Examples

Definition. We call the d-dimensional vector space $\mathbb{R}^d$ the **Euclidean space**, and it is the set of all vectors (also called points) $(x_1, \ldots, x_d)$, where $x_i \in \mathbb{R}$ for $1 \leq i \leq d$. We define the **Euclidean norm** to be the function $|| \cdot || : \mathbb{R}^d \to \mathbb{R}$ such that for $x = (x_1, \ldots, x_m) \in \mathbb{R}^d$, $||x|| = \sqrt{x_1^2 + \cdots + x_m^2}$; and we define the distance of of two points $x, y \in \mathbb{R}^d$ to be the function $\Delta : \mathbb{R}^d \mathbb{R}^d \to \mathbb{R}$ such that $\Delta(x, y) = ||x - y||$.

Definition. Let $x_1, x_2, \ldots, x_m$ be points in $\mathbb{R}^d$. We call a point $x \in \mathbb{R}^d$, of the form $x = \sum_{i=1}^m \alpha_i x_i$, with $\sum \alpha_i = 1$, a **convex combination** of x. We call the set of all convex combinations of a subset $A \subseteq \mathbb{R}^d$ the **convex hull** of A, and denote it $\operatorname{conv} A$.

Definition. Let $x, y \in \mathbb{R}^d$. We call the set $[x, y] = \{\alpha x + (1 - \alpha) y : 0 \leq \alpha \leq 1\}$ of all convex combinations of x and y an **interval** with **endpoints** x, y.

We call a set $A \subseteq \mathbb{R}^d$ **convex** if whenever $x, y \in A$, $[x, y] \in A$.

Example 1.1. The empty set, regular polyhedra, and open balls in $\mathbb{R}^d$ are all convex.

Lemma 1.1.1. *The convex hull of a convex set is convex.*

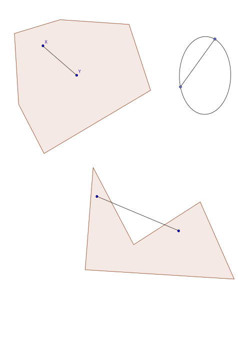

Figure 1.1: Two convex set, and a non-convex set

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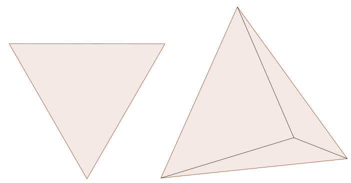

Figure 1.2: Regular polyhedra in $\mathbb{R}^2$ and $\mathbb{R}^3$
fig1.2

Proof. Let $A \subseteq \mathbb{R}^d$, and let $x, y \in \operatorname{conv} A$. Then by definition the set of all convex combinations of x and y is in $\operatorname{conv} A$, thus $[x, y] \in \operatorname{conv} A$ ■

Definition. Let $c_1, \ldots, c_m \in \mathbb{R}^d$ and let $\beta_1, \ldots, \beta_m \in \mathbb{R}$; we call the set $A = \{x \in \mathbb{R}^d : \langle c_i, x\rangle \leq \beta_i, \text{ for } 1 \leq i \leq m\}$ a **regular polyhedron**.

Lemma 1.1.2. *Regular polyhedra in $\mathbb{R}^d$ are convex.*

Proof. Let $A \subseteq \mathbb{R}^d$ be a regular polyhedron and let $x, y \in A$. Then $\langle c_i, x\rangle, \langle c_i, y\rangle \leq \beta_i$ for $c_i \in \mathbb{R}^d$, $\beta_i \in \mathbb{R}$ for $1 \leq i \leq m$.Then by the scalar linearity of the innerproduct, $\langle c_i, \alpha x + (1-\alpha)y\rangle \leq \beta_i$. Thus $[x, y] \in A$. ■

Example 1.2. Let $V = \{v_1, \ldots, v_m\} \subseteq \mathbb{R}^d$, $\rho_1, \ldots, \rho_m \in \mathbb{R}^d$, define $f : \mathbb{R}^d \to \mathbb{R}$ by $f(x) = \sum \rho_i e^{\langle x, v_i\rangle}$, and define $H : \mathbb{R}^d \to \mathbb{R}^d$ by

$$H(x) = \frac{\sum \rho_i v_i e^{\langle x, v_i\rangle}}{f(x)}$$

Letting $x, y \in \mathbb{R}^d$, and choosing $0 \leq \alpha \leq 1$, using the scalar linearity of the inner product, we see that:

$$H(\alpha x_(1-\alpha)y) = \frac{\sum \rho_i v_i e^{\langle \alpha x + (1-\alpha)y, v_i\rangle}}{f(\alpha x + (1-\alpha)y)}$$

so $H(\mathbb{R}^d) \in \operatorname{conv} V$, which also implies that $H(\mathbb{R}^d)$ inherits convexity.

Example 1.3. Consider the function H in the example above. Let $y \in V$, and choose $x = (1-\alpha_m)y' \in \mathbb{R}^d$ such that $||H(x)|| < \frac{\epsilon}{2}$; for $\epsilon > 0$. Then since $y < x$, $||H(y)|| < ||H(x)|| < \frac{\epsilon}{2}$. Thus we have that $||H(x) - y|| \leq ||H(x)|| + ||y|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Moreover, we see that H is 1-1, for suppose there is no nonzero vector $c \in \mathbb{R}^d$ such that $\langle c, v_i\rangle = \alpha$ for some α. If $H(x) = H(y)$, then $\langle x, v_i\rangle = \langle y, v_i\rangle = \alpha$, and so by our supposition, $x = y = 0$.

Example 1.4. Let $q_1, q_2 : \mathbb{R}^n \to \mathbb{R}$ be quadratic forms, and let S^{n-1} be the unit ball in $\mathbb{R}^n$. Define $T : \mathbb{R}^n \to \mathbb{R}^2$ by $T(x) = (q_1(x), q_2(x))$. Then $T(S^{n-1})$ is convex in R^2, provided that $n > 2$.

Theorem 1.1.3 (The Shur-Horn Theorem)**.** *Let* $A = (\alpha_{ij})$ *be an nn matrix, with diagonal* $\operatorname{diag} A = (\alpha_{11}, \ldots \alpha_{nn})$ *and let* $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$*. Let* $X \subseteq \mathbb{R}$ *be the set of all diagonals of* $n \times n$ *matrices with eigenvalues* $\lambda_1, \ldots, \lambda_n$*, Then* X *is a convex set. Morever, if* $l = (\lambda_1, \ldots, \lambda_n)$ *is the vecotr of eigenvalues, and* σ *is a permutation on* $\{1, \ldots, n\}$*, then* $X = \operatorname{conv}\{l^\sigma : 1 \leq i \leq n\}$*.*