# Topology

Alec Zabel-Mena

**<u>Text</u>**

Topology ($2^{rd}$ edition)

James Munkres.

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# Chapter 1

# Topological Spaces and Continuous Functions.

## 1.1 Topological Spaces.

**Definition.** A **topology** on a set $X$ is a collection $\mathcal{T}$ of subsets of $X$ such that:

(1) $\emptyset, X \in \mathcal{T}$.

(2) For any subcollection $\{U_\alpha\}$ of subsets of $X$, $\bigcup_\alpha U_\alpha \in \mathcal{T}$.

(3) For any finite subcollection $\{U_i\}_{i=1}^n$ of subsets of $X$, $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

We call the pair $(X, \mathcal{T})$ a **topological space**, and we call the elements of $\mathcal{T}$ **open sets**.

**Example 1.1.** (1) Let $X$ be any set, the collection of all subsets of $X$, $2^X$ is a topology on $X$, which we call the **discrete topology**. We call the topology $\mathcal{T} = \{\emptyset, X\}$ the **indiscrete topology**.

(2) The set of three points $\{a, b, c\}$ has the 9 following topologies in figure 1.1.

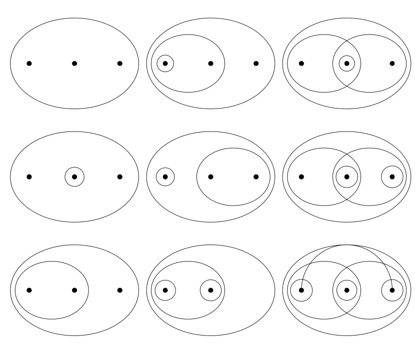

Figure 1.1: The Topologies on $\{a, b, c\}$.

(3) Let $X$ be any set, and let $\mathcal{T}_f = \{U \subseteq X : X \backslash U$ is finite, or $X \backslash U = X\}$. Then $\mathcal{T}_f$ is a topology and called the **finite complement topology**.

(4) Let $X$ be any set, and let $\mathcal{T}_c = \{U \subseteq X : X \backslash U$ is countable, or $X \backslash U = X\}$. Then $\mathcal{T}_c$ is a topology on $X$.

**Definition.** Let $X$ be a set, and let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on $X$. We say that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$, and $\mathcal{T}'$ **finer** than $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{T}'$. If two topologies are either coarser, or finer than each other, we call them **comparable**.

**Example 1.2.** The topologies $\mathcal{T}_f$ and $\mathcal{T}_c$ are comparable, and we see that $\mathcal{T}_f \subseteq \mathcal{T}_f$, so $\mathcal{T}_f$ is coarser than $\mathcal{T}_c$, and $\mathcal{T}_c$ is finer than $\mathcal{T}_f$.

## 1.2 The Basis and Subbasis for a Topology.

**Definition.** If $X$ is a set, the **basis** for a topology on $X$ is a collection $\mathcal{B}$ of subsets of $X$, called **basis elements**, such that:

(1) For every $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$.

(2) For $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

We define the topology $\mathcal{T}$ **generated** by $\mathcal{B}$ to be collection of open sets: $\mathcal{T} = \{U \subseteq X : x \in B$ for some $B \in \mathcal{B}\}$.

**Theorem 1.2.1.** *Let $X$ be a set, and $\mathcal{B}$ a basis of $X$, then the collection of subsets of $X$, $\mathcal{T} = \{U \subseteq X : x \in B$ for some $B \in \mathcal{B}\}$ is a topology on $X$.*

*Proof.* Let $\mathcal{B}$ be a basis for a topology in $X$, and consider $\mathcal{T}$ as defined above. Cleary, $\emptyset \in X$ and so is $X$.

Now let $\{U_\alpha\}$ be a subcollection of subsets of $X$, and let $U = \bigcup U_\alpha$. Then if $x \in U$ for some $\alpha$, there is a $B_\alpha$ such that $x \in B_\alpha \subseteq U_\alpha$, thus $x \in B_\alpha \subseteq U$.

Now let $x \in U_1 \cap U_2$, and choose $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Then by definition, there is a $B_3$ for which $x \in B_3 \subseteq B_1 \cap B_2$. Now suppose for arbitrary $n$, that $U = \bigcap_{i=1}^n U_i \in \mathcal{T}$, for some finite subcollection $\{U_i\}$ of subsets of $X$. Then by let $B_n, B_{n+1} \in \mathcal{B}$ such that $x \in B_n \subseteq U$ and $x \in B_{n+1} \subseteq U_{n+1}$. Then by our hypothesis, there is a $B$ for which $x \in B \subseteq B_n \cap B_{n+1}$, thus $U \cap U_{n+1} = \bigcap_{i=1}^{n+1} U_i \in \mathcal{T}$. This make $\mathcal{T}$ a topology on $X$. ∎

**Example 1.3.** (1) Let $\mathcal{B}$ be the set of all circular regions in the plane $\mathbb{R} \times \mathbb{R}$, then $\mathcal{B}$ satisfies the conditions needed for a basis.

(2) The collection $\mathcal{B}'$ in $\mathbb{R} \times \mathbb{R}$ of all rectangular region also forms a basis for a topology on $\mathbb{R} \times \mathbb{R}$.

(3) For any set $X$, the set of all 1-point elements of $X$ forms a basis for a topology on $X$.

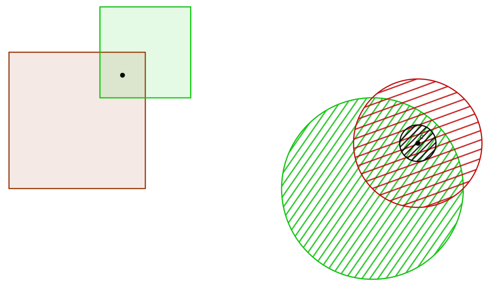

Figure 1.2: The basis for $\mathcal{B}$ and $\mathcal{B}'$ in $\mathbb{R} \times \mathbb{R}$ (see example (2)).

**Lemma 1.2.2.** *Let $X$ be a set, and $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on $X$. Then $\mathcal{T} = \{\bigcup B : B \in \mathcal{B}\}$.*

*Proof.* Given a collection $\{B\}$ of basis elements in $\mathcal{B}$, since they are all in $\mathcal{T}$, their unions are also in $\mathcal{T}$. Conversely, given $U \in \mathcal{T}$, then for every point $x \in U$, choose a $B_x \in \mathbb{B}_x$ such that $x \in B_x \subseteq U$, then $U = \bigcup_{x \in U} B_x$. ■

**Lemma 1.2.3.** *Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{C} \subseteq \mathcal{T}$ be a collection of open sets of $X$ such that for every $x \in U$, there is a $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is the basis for a $\mathcal{T}$ on $X$.*

*Proof.* Take any $x \in X$, then there is a $C \in \mathcal{C}$ such that $x \in C \subseteq U$, thus the first condition for a basis is satisfied. Now let $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C}$, since $C_1 \cap C_2$ is open in $X$, there is a $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. Therefore $\mathcal{C}$ is a basis for a topology on $X$.

Now let $\mathcal{T}_{\mathcal{C}}$ be the the topology generated by $\mathcal{C}$, now for $U \in \mathcal{T}$, we have by the hypothesis, that $U \in \mathcal{T}_{\mathcal{C}}$; and by lemma 1.2.2, $W \in \mathcal{T}_{\mathcal{C}}$ is the union of elements of $\mathcal{C}$, which is a subcollection of $\mathcal{T}$, thus $W \in \mathcal{T}$. Therefore $\mathcal{T}_{\mathcal{C}} = \mathcal{T}$. ■

**Lemma 1.2.4.** *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ on $X$. Then the $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for all $x \in X$, and all $B \in \mathcal{B}$, there is a $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

*Proof.* Suppose first that $\mathcal{T} \subseteq \mathcal{T}'$, and let $x \in X$, and choose $B \in \mathcal{B}$ such that $x \in B$, then $B$ is open in $\mathcal{T}$, thus it is open in $\mathcal{T}'$, thus there is a $B' \in \mathcal{B}'$ such that $x \in b' \subseteq B$. Conversely, suppose there is a $B' \in \mathcal{B}'$ for which $x \in B' \subseteq B$ for all $x \in X$, $B \in \mathcal{B}$. Take $x \in U \in \mathcal{T}$, since $\mathcal{B}$ generates $\mathcal{T}$, $x \in B \subseteq U$, since $B' \subseteq B$, this implies that $U \in \mathcal{T}'$ and $\mathcal{T} \subseteq \mathcal{T}'$. ■

**Definition.** If $\mathcal{B}$ is the collection of open intervals $(a, b)$ in $\mathbb{R}$, we call the topology generated by $\mathcal{B}$ the **standard topology** on $\mathbb{R}$, and we denote it simply by $\mathbb{R}$.

**Definition.** If $\mathcal{B}$ is the collection of half open intervals $[a, b)$ in $\mathbb{R}$, we call the topology generated by $\mathcal{B}$ the **lower limit topology** on $\mathbb{R}$, and we denote it simply by $\mathbb{R}_l$. If $\mathcal{B}'$ is the collection of all half open intervals $(a, b]$ in $\mathbb{R}$, then we call the topology generated by $\mathcal{B}'$ the **upper limt topology** on $\mathbb{R}$, and denote it $\mathbb{R}_L$.

**Definition.** If $\mathcal{B}$ is the collection of all open intervals of the form $(a,b)\setminus \frac{1}{\mathbb{Z}^+}$, where $\frac{1}{\mathbb{Z}^+} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$, we call the topology generated by $\mathcal{B}$ the $\frac{1}{\mathbb{Z}^+}$**-topology** on $\mathbb{R}$, and we denote it $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$.

**Lemma 1.2.5.** *The topologies $\mathbb{R}_l$, $\mathbb{R}_L$, and $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ are all strictly finer than $\mathbb{R}$, but are not comparable with each other.*

*Proof.* Let $(a,b)$ be a basis element for $\mathbb{R}$, and let $x \in (a,b)$, the basis element $[x,b) \in \mathbb{R}_l$ lies in $(a,b)$ and contains $x$, however, there can be no interval $(a,b)$ in $[x,b)$ as $x \leq a$, thus $\mathbb{R}\mathbb{R}_l$; a similar argument holds for $\mathbb{R}_L$.

Similarly, for $(a,b) \in \mathbb{R}$, the basis element $(a,b)\setminus\frac{1}{\mathbb{Z}^+}$ of $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ lies in $(a,b)$, however, choose the basis $B = (-1,1)\setminus\frac{1}{\mathbb{Z}^+}$, and choose $0 \in B$, since $\mathbb{Z}^+$ is dense in $\mathbb{R}$, there is no interval $(a,b)$ containing 0 and lying in $B$, thus $\mathbb{R} \subsetneq \mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$.

Now choose $[0,1)$ in $\mathbb{R}_l$, and choose $\frac{1}{k} \in [0,1)$ such that $k \in \mathbb{Z}^+$. Now $(0,1) \subsetneq [0,1)]$, so we cannot say that $[0,1)$ is a basis for $\mathbb{R}$, and moreover, $[0,1)\setminus\frac{1}{\mathbb{Z}^+}$ cannot be said to be a basis in $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$, thus $\mathbb{R}_l$ and $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ are incompararble, a similar argument holds for $\mathbb{R}_L$.

Lastly, let $(a,b)$ be in $\mathbb{R}$ and choose $x \in (a,b)$. Then $(a,x]$ and $[x,b)$ are both in $(a,b)$, however it is clear that $(a,x]$ and $[x,b)$ connot be contained in each other, thus $\mathbb{R}_l$ and $\mathbb{R}_L$ are incomparable. ■

**Definition.** A **subbasis**, $\mathcal{S}$, for a topology on $X$ is a collection of subsets of $X$ whose union equals $X$. We call the **topology generated by** $\mathcal{S}$ to be the collection of all unions of finite intersections of elements of $\mathcal{S}$, that is:

$$\mathcal{T} = \{\bigcup\bigcap_{i=1}^{n} S_i : S_i \in \mathcal{S} \text{ for } 1 \leq i \leq n\}$$

**Theorem 1.2.6.** *Let $\mathcal{S}$ be a subbasis for a topology on $X$. Then the collection $\mathcal{T} = \{\bigcup\bigcap_{i=1}^{n} S_i : S_i \in \mathcal{S}$ for $1 \leq i \leq n\}$ is a topology on $X$.*

*Proof.* It is sufficient to show that the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ is a basis for a topology on $X$. By lemma 1.2.1, for $x \in X$, it belongs to an element $S$ of $\mathcal{S}$, and therefore, to an element of $\mathcal{B}$. Now let $B_1 = \bigcap_{i=1}^{m} S_i$ and $B_2 = \bigcap_{j=1}^{n} S'_j$ be basis elements of $\mathcal{B}$. The intersection $\mathbb{B}_1 \cap B_2$ is a finite intersection of elements of $\mathcal{S}$, and hence also belongs in $\mathcal{B}$, and hence we can take another basis element $B_3$ such that $x \in B_3 \subseteq B_1 \cap B_2$. ■

## 1.3 The Order Topology.

**Definition.** Let $X$ be a set with a simple order relation, and suppose that $|X| > 1$. Let $\mathcal{B}$ be the collection of sets of the following forms:

(1) All open intervalcs $(a,b) \in X$.

(2) All half open intervals $[a_0, b)$ where $a_0$ is the least element (if any) of $X$.

(3) All half open intervals of the form $(a, b_0]$ where $b_0$ is the greatest element (if any) of $X$.

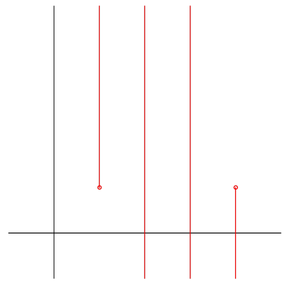

Figure 1.3: The order topology on $\mathbb{R} \times \mathbb{R}$.

Then $\mathcal{B}$ forms the basis for a topology on $X$ called the **order topology**

**Theorem 1.3.1.** *The collection $\mathcal{B}$ forms a basis.*

*Proof.* Consider $x \in X$, if $x$ is the least element of $X$, then it liess in all intervals of type (2), if it is the largest, then it lies in all intervals of type (3). If $x$ is neither the least nor largest element, then $x \in (a_0, b_0)$ with $a_0$ and $b_0$ the least and largest elements (if any) of $X$. If no such elements exist, then $x \in (a, b)$, for some lowerbound $a$ and upperbound $b$. Thus, in all three cases, there is a basis element containing $x$.

Now suppose $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$. If $B_1$ and $B_2$ are both of type (1), then let $B_1 = (a, b)$, $B_2 = (c, d)$, then $B_1 \cap B_2$ is an open interval of type (1), now fix $B_1$ to be of type one. If $B_2$ is of type (2), then letting $B_2 = [a_0, c)$, then $x \in [a_0, d)$ for some $d \in X$. Likewise, if $B_2 = (c, b_0]$, is of type (3), we get a similar result. Moreover, the results are analogous if we fix $B_2$ and let $B_1$ range between intervals of the three types. Thusm in all cases, there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. ∎

**Example 1.4.**

(1) The standard topology on $\mathbb{R}$ is the order topology on $\mathbb{R}$ induced by the usual order relation. We have that $\mathbb{R}$ under this topology has no intervals of type (2), nor (3), so all bases elements in the standard topology are open intervals in $\mathbb{R}$.

(2) Consider the dictionary order on $\mathbb{R} \times \mathbb{R}$. Since $\mathbb{R} \times \mathbb{R}$ has no intervals of type (2), nor (3), the bases of $\mathbb{R} \times \mathbb{R}$ under the dictionary order are the open intervals of the form $(a \times b, c \times d)$ Where $a \leq c$, and $b < d$.

(3) The positive integers $\mathbb{Z}^+$ with the least element 1 form an ordered set under the usual order. Taking $n > 1$, we see the bases of $\mathbb{Z}^+$ under the order topology are of the form $(n-1, n+1) = \{n\}$ and $[1, n) = \{1, \dots, n-1\}$. Thus the order topology on $\mathbb{Z}^+$ is the discrete topology.

(4) The set $X = \{1, 2\} \times \mathbb{Z}^+$ over the dictionary order is also an ordered set, with the least elelment $1 \times 1$. Denote $1 \times n$ as $a_n$ and $2 \times n$ as $b_n$. Then $X$ consist of the elements $a_1, a_2, \dots, b_1, b_2, \dots$.

Now take $\{b_1\}$, then any open set containing $b_1$ must have a basis about $b_1$, and also contains points $a_i$ with $i \in \mathbb{Z}^+$; thus the order topology on $X$ is not the discrete topology.

**Definition.** Let $X$ be an ordered set, and let $a \in X$. There are two subsets in $X$, $(a, \infty) = \{x \in X : x > a\}$ and $(-\infty, a) = \{x \in X : x < a\}$ called **open rays** of $X$. There are also two sets $[a, \infty) = \{x \in X : x \geq a\}$ and $(-\infty, a] = \{x \in X : x \leq a\}$ called **closed rays** of $X$.

**Theorem 1.3.2.** *Let $X$ be an ordered set. Then the collection of all open rays in $X$ form a subbasis for the order topology on $X$.*

*Proof.* Let $\mathcal{S}$ be the collection of all open rays of $X$, let $(a, \infty)$ and $(-\infty, b) \in \mathcal{S}$, then $(a, b) = (a, \infty) \cap (-\infty, b)$. Now take:

$$S = \bigcup_{a,b \in X} (a, b)$$

then $S \subseteq X$, likewise, since $S$ runs through all intersections of open rays of $X$, it contains all open intervals in $X$, hence $X \subseteq S$, and so $X = S$ as required. ∎

## 1.4 The Product Topology.

**Definition.** Let $X$ and $Y$ be topological spaces. We define the **product topology** on $X \times Y$ to be the topology having as basis the collection $\mathcal{B} = \{U \times V \subseteq X \times Y : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$

**Theorem 1.4.1.** *The collection $\mathcal{B} = \{U \times V \subseteq X \times Y : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$ forms a basis for the product topology on $X \times Y$.*

*Proof.* Clearly, we have that $X \times Y$ is a basis element of $\mathcal{B}$. Now take $U_1 \times V_1$ and $U_2 \times V_2$ in $\mathcal{B}$. Since $U_1 \times V_1 \cap U_2 \times V_2 = U_1 \cap U_2 \times V_1 \cap V_2$, since $U_1 \cap U_2$ and $V_1 \cap V_2$ are open in $X$ and $Y$ respectively, then we have that $U_1 \times V_1 \cap U_2 \times V_2$ is a basis element as well. ∎

**Theorem 1.4.2.** *If $\mathcal{B}$ is the basis for a topology on $X$, and $\mathcal{C}$ is the basis for a topology on $Y$, then the collection:*

$$\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

*Is a basis for the topology on $X \times Y$.*

*Proof.* By lemma 1.2.3, let $W$ be an open set of $X \times Y$, and let $x \times y \in W$. Then there is a basis $U \times V$ such that $x \times y \in U \times V \subseteq W$. Since $\mathcal{B}$ and $\mathcal{C}$ are bases of $X$ and $Y$ respectively, choosing $B \in \mathcal{B}$ and $C \in \mathcal{C}$, we have that $x \in B \subseteq U$, and $y \in C \subseteq Y$, thus $x \times y \in B \times C \subseteq U \times V \subseteq W$. Therefore, $\mathcal{D}$ is the basis for a topology on $X \times Y$. ∎

**Example 1.5.** The product of the standard topology on $\mathbb{R}$ with itself is called the **standard topology on** $\mathbb{R} \times \mathbb{R}$, and has as basis the collection of all products of open sets in $\mathbb{R}$. By theorem 1.4.2, if we take the collection of all open intervals $(a, b) \times (a, b)$ in $\mathbb{R} \times \mathbb{R}$, we form a basis. Constructing this basis geometrically gives the interior of a rectangle, whose boundaries are the intervals $(a, b)$ and $(ac, d)$.

**Definition.** Let $\pi_1 : X \times Y \to X$ be defined such that $\pi_1(x, y) = x$, and define $\pi_2 : X \times Y \to Y$ such that $\pi_2(x, y) = y$. We call $\pi_1$ and $\pi_2$ **projections** of $X \times Y$ onto its first and secnd **factors**; that is onto $X$ and $Y$, respectively.

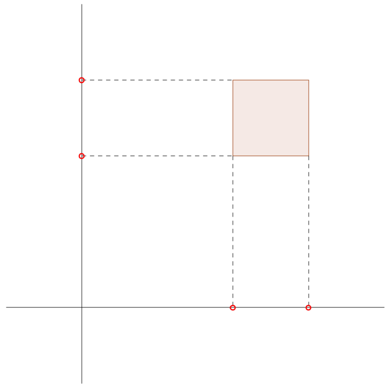

Figure 1.4: A basis element for $\mathbb{R} \times \mathbb{R}$

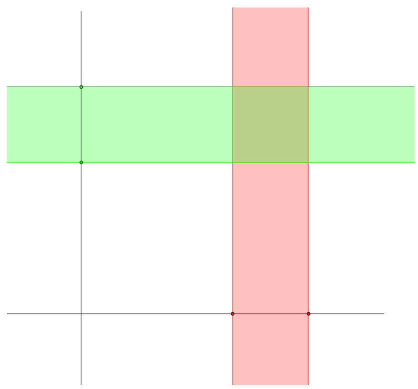

Figure 1.5: The inverse images, $\pi_1^{-1}(U)$ and $\pi_2^{-1}(V)$, of the projections $\pi_1$ and $\pi_2$ onto the $X \times Y$ plane.

Clearly, $\pi_1$ and $\pi_2$ are both onto. Now let $U$ be open in $X$, then $\pi_1^{-1}(U) = U \times Y$ is open in $X \times Y$ ; similarly, $\pi_2^{-1}(V) = X \times V$ is also open in $X \times Y$, for $V$ open in $Y$.

**Theorem 1.4.3.** *The collection* $\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ is open in } Y\}$ *is a subbasis for the product topology on* $X$*.*

*Proof.* Let $\mathcal{T}$ be the product topology on $X \times Y$, and let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$. Since every element of $\mathcal{S}$ is open in $\mathcal{T}$, $\mathcal{T} \subseteq \mathcal{T}'$. Conversely, consider the basis element $U \times V$ of $\mathcal{T}$, then $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times Y \cap X \times V = U \times V$, thus $\mathcal{T} \subseteq \mathcal{T}'$. Therefore, $\mathcal{S}$ is a subbasis for the product topology. ∎

## 1.5 The Subspace Topology.

**Theorem 1.5.1.** *Let* $X$ *he a topological space with topology* $\mathcal{T}$*, and let* $Y \subseteq X$*. Then the collection:*

$$\mathcal{T}_y = \{Y \cap U : U \in \mathcal{T}\}$$

*forms a topology on* $Y$*.*

*Proof.* Cleary, $Y \cap \emptyset = \emptyset \in \mathcal{T}_Y$ and $Y \cap X = Y \in \mathcal{T}_Y$. Now consider the collection $\{U_{alpha}\}$. Then $\bigcup Y \cap U_\alpha = Y \cap \bigcup U_\alpha$, similarly, for $\{U_i\}_{i=1}^n$, $\bigcap Y \cap U_i = Y \cap \bigcap U_i$, hence $\mathcal{T}$ is a topology on $Y$. ■

**Definition.** Let $X$ be a topological space, and let $Y \subseteq X$. We call the $\mathcal{T}$ defined in theorem 1.5.1 the **subspace topology** on $Y$. We say that $U \subseteq Y$ is **open in** $Y$ if $U \in \mathcal{T}_Y$.

**Lemma 1.5.2.** *Let $\mathcal{B}$ be the basis for a topology on $X$. Then the collection $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$, where $Y \subseteq X$, is a basis for the subspace topology on $Y$.*

*Proof.* Let $U$ be open in $X$, and let $y \in Y \cap U$, and choose $B \in \mathcal{B}$ such that $y \in B \subseteq U$, then $y \in B \cap Y \subseteq U \cap Y$, then by lemma 1.2.2, $\mathcal{B}_y$ is the basis fpr the subspace topology on $Y$. ■

**Lemma 1.5.3.** *Let $Y$ be a subspace of $X$, If $U \subseteq Y$ is open in $Y$, then $U$ is open in $X$.*

*Proof.* The proof is rather trivial, however, it is worth going through the motions. Let $U \in \mathcal{T}_Y$, then for some $V \subseteq X$, $U = Y \cap V$. Now since $Y$ is open in $X$, and so is $V$, then it follows that $U$ is also open in $X$. ■

*Remark.* What this lemma says is that given a topological space $X$, and a subspace $Y$ of $X$, then the subspace topology of $Y$ is courser than the topology on $X$, i.e. $\mathcal{T}_Y \subseteq \mathcal{T}$.

**Theorem 1.5.4.** *If $A$ is a subspace of $X$, and $B$ is a subspace of $Y$, then the product topology on $A \times B$ is the topology that $A \times B$ inherits as a subspace of $X \times Y$.*

*Proof.* We have that $U \times V$ is the basis element for $X \times Y$, with $U$ open in $X$, and $V$ open in $Y$. Thus $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$ is a basis element for the subspace topology on $X \times Y$. Since $U \cap A$ and $V \cap B$ are open in the subspace topologies of $A$ and $B$ respectively, then $(U \cap A) \times (v \cap B)$ is a basis for the product topology on $A \times B$. ■

**Example 1.6.** (1) Consider $[0, 1] \subseteq \mathbb{R}$. In the subspace topology of $[0, 1]$, we have as basis elements of the form $(a, b) \cap [0, 1]$, with $(a, b) \subseteq \mathbb{R}$. If we have that $(a, b) \subseteq [0, 1]$, then $(a, b) \cap [0, 1] = (a, b)$. On the other hand, if $a \in [0, 1]$ or $b \in [0, 1]$, then we get $(a, b) \cap [0, 1] = (a, 1]$ or $(a, b) \cap [0, 1] = [0, b)$, lastly if neither $a$ nor $b$ are in $[0, 1]$, then we have $(a, b) \cap [0, 1] = [0, 1]$ only if $[0, 1] \subseteq (a, b)$, and $(a, b) \cap [0, 1] = \emptyset$ otherwise.

Now each of these sets are open in $\mathbb{R}$, under the standard topology, except for $(a, 1]$ and $[0, b)$.

(2) For $[0, 1) \cup \{2\} \subseteq \mathbb{R}$, the singletomn $\{2\}$ is open in the subspace topology on $[0, 1) \cup \{2\}$; for observe, that $(\frac{3}{5}, \frac{5}{2}) \cap ([0, 1) \cup \{2\}) = \{2\}$, however, in the order topology, on that same set, $\{2\}$ is not open. Any basis element on $[0, 1) \cup \{2\}$ containing 2 is of the form $(a, 2]$, where $a \in [0, 1) \cup \{2\}$.

(3) The dictionary order on $[0, 1] \times [0, 1]$ is a restriction of the dictionary order on $\mathbb{R} \times \mathbb{R}$. Now the set $\{\frac{1}{2}\} \times (\frac{1}{2}, 1]$ is open in the subspace topology on $[0, 1] \times [0, 1]$, but it is not open in the dictionary order on the same set.

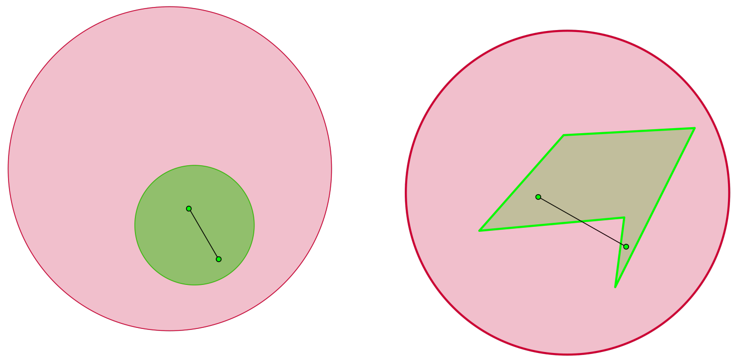

Figure 1.6: A convex set, and a con convex set.

**Definition.** We call the set $[0,1]\times[0,1]$ on the dictionary odere the **ordered square**, and we denote it by $I_0^2$.

**Definition.** Let $X$ be an ordered set. We say that a nonempty subset $Y \subset X$ is **convex** in $X$ if for each pair of points $a, b \in Y$, with $a < b$, then the open interval $(a,b) \subseteq X$ is also contained in $Y$.

**Example 1.7.** Let $X$ be any ordered set. Then by definition, all open intervals and rays in $X$ are convex in $X$.

**Theorem 1.5.5.** *Let $X$ be an ordered set on the order toplogy, and let $Y \subseteq X$ be convex in $X$. Then the order topology on $Y$ is the same as the subspace topology on $Y$.*

*Proof.* Consider $(a,) \subseteq X$. If $a \in Y$, then $(a,\infty)\cap Y = \{x \in Y : x > a\}$, which is by definition an open ray on $Y$. Now if $a \notin Y$, then $a$ is either a lowerbound, or an upperbound. Then $(a,\infty)\cap Y = \emptyset$ and $(-\infty,a)\cap Y = Y$ if $a$ is an upperbound, similarly, if $a$ is a lowerbound we get $(a,\infty)\cap Y = Y$ and $(-\infty,a)\cap Y = \emptyset$.

Since $(a,\infty)Y$ and $(-\infty,a)\cap Y$ form a subbasis on the subspace topology on $Y$, and since they are also open in the order topology, then the order topology contains the subspace topology.

Now if $(a,\infty)$ is an open ray in $Y$, then $(a,\infty) = (b,\infty)\cap Y$, with $(b,\infty)$ some open ray in $X$, hence $(a,\infty)$ is open in the subspace topology of $Y$, and since it also forms the subbasis for the order topology, we have that the ordertopology is contained within the subspace topology. Thus both topologies are equal. ■

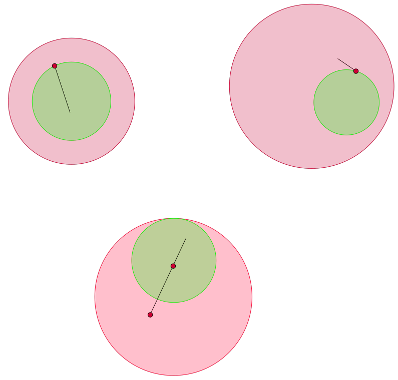

Figure 1.7: An illustration of theorem 1.5.5.

## 1.6 Closed Sets and Limit Points.

**Definition.** A subset $A$ of a topological space $X$ is said to be **closed** if $X \backslash A$ is open.

**Example 1.8.** (1) Consider $[a, b] \subseteq \mathbb{R}$, we have that $\mathbb{R} \backslash [a, b] = (-\infty, a) \cup (b, \infty)$ which is open in $\mathbb{R}$. So $[a, b]$ is closed.

(2) In $\mathbb{R} \times \mathbb{R}$, the set $A = \{x \times y : x, y \geq 0\}$ (i.e the first quadtant of the plane) is closed, for $\mathbb{R} \times \mathbb{R} \backslash A = (-\infty, 0) \times \mathbb{R} \cup \mathbb{R} \times (-\infty, 0)$, which is open in $\mathbb{R} \times \mathbb{R}$.

(3) Consider the finite complement topology $\mathcal{T}_C$ on a set $X$. We have that $X \backslash X = \emptyset \in \mathcal{T}$, so $X$ is closed, similarly, $\emptyset$ is also closed. Likewise, if $A \subseteq X$ is a finite set, then $X \backslash A$ is also finite, and hence $A$ is also closed. Thus, we have that all the closed sets of $\mathcal{T}_C$ are those finite subsets of $X$. As a consequence, this examle also illustrates that sets can be both closed and open.

(4) In the discrete topology $2^X$, every open set is closed. This is another example where open sets are also closed sets.

(5) Consider $[0, 1] \cup (2, 3)$ in the subspace topology on $\mathbb{R}$. We have that $[0, 1]$ is open $([0, 1] = [0, 1] \cup (2, 3) \cap (-\frac{2}{3}), \frac{3}{2})$, similarly, $(2, 3)$ is also open. Now taking $[0, 1] \cup (2, 3) \backslash (2, 3) = [0, 1]$, which is open, so $[0, 1]$ is closed in the subspace topology on $\mathbb{R}$, bu the same reasoning, so is $(2, 3)$.

**Theorem 1.6.1.** *Let $X$ be a topological space. Then:*

*(1) $\emptyset$ and $X$ are closed.*

*(2) Arbitrary intersections of closed sets are closed.*

*(3) Finite unions of closed sets are closed.*

*Proof.* We have that $X \backslash \emptyset = X$ and $X \backslash X = \emptyset$, both of which are open in $X$, so they are also closed in $X$. Now let $\{U_\alpha\}$ be a collection of closed sets of $X$. We have that:

$$X \backslash \bigcap_\alpha U_\alpha = \bigcup_\alpha X \backslash U_\alpha.$$

Similarly, for $\{U_i\}_{i=1}^n$, we have

$$X \backslash \bigcup_{i=1}^{n} U_i = \bigcap_{i=1}^{n} X \backslash U_i.$$

Both of which are open in $X$. This completes the proof. ■

**Definition.** If $Y$ is a subspace of $X$, we say that $A$ is **closed in** $Y$ if $A \subseteq Y$ and $A$ is closed in the subspace topology of $Y$.

**Theorem 1.6.2.** *Let $Y$ be a subspace of $X$. Then $A$ is closed in $Y$ if and only if $A$ equals the intersection of a closed set of $X$ with $Y$.*

*Proof.* Suppose that $A$ is closed in $Y$, then $Y \backslash A$ is open in $Y$, hence we have that $Y \backslash A = U \cap Y$ for some open set $U$ of $X$. Now $X \backslash U$ is closed in $X$, and with $A \subseteq Y$, we have that $A = Y \cap X \backslash U$.

Conversely, suppose that $A = C \cap Y$, with $C$ closed in $X$. Then $X \backslash C$ is open in $X$, hence $X \backslash C \cap Y$ is open in $Y$, now since $X \backslash C \cap Y = Y \backslash A$, which is open, we have that $A$ is closed in $Y$. ■

**Theorem 1.6.3.** *Let $Y$ be a subspace of $X$. If $A$ is closed in $Y$, and $Y$ is closed in $X$, then $A$ is closed in $X$; that is, closure is transitive.*

*Proof.* By theorem 1.6.2, if $A$ is closed in $Y$, then $A = C \cap Y$ with $C$ closed in $X$, now since $Y$ is closed in $X$, then $Y = D \cap X$ with $D$ closed in $X$. Thus $A = (C \cap D) \cap X$, therefore, $A$ is closed in $X$. ■