Real Analysis

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Text

An Introduction to Real Analysis (4^{th} edition)
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Chapter 1

Preliminaries.

1.1 A Note on Finite and Infinite Sets

We go over the notions of "finite" and "infinite" sets.

Definition. (1) The emptyset $\emptyset$, is said to have 0 **elements**.

(2) If $n \in \mathbb{N}$, a set S is said to have n **elements** if there is a bijection from the set $\{1, 2, , \ldots, n\}$ onto S.

(3) A set S is **finite** if it is either the emptyset, or has n elements.

(4) A set S is **infinite** if it is not finite

We assume two theorems which will help us study finite and infinite sets.

Theorem 1.1.1. *If S is a finite set, then the number of elements of S is a natural number.*

Theorem 1.1.2. *The set $\mathbb{N}$ is infinite.*

Theorem 1.1.3. *(1) If A has m elements, and B has n elements, and $A \cap B = \emptyset$,then $A \cup B$ has $m + n$ elements.*

(2) If A has m elements and CA is a set with 1 element, then $C \backslash A$ has $m - 1$ elements.

(3) If C is an infinite set, and B is a finite set, then $C \backslash B$ is an infinite set.

Proof. (1) Let f be a bijection from $\mathbb{N}_m = \{1, \ldots, m\}$ onto A, and let g be a bijection from $\mathbb{N}_n = \{1, \ldots, n\}$ onto B.Define h from $\mathbb{N}_{m+n}$ into $A \cup B$ (without loss of generality) by:

$$h(i) = \begin{cases} f(i), \text{ for } i = 1, \ldots, m \\ g(i - m) \text{ for } i = m + 1, \ldots, m + n \end{cases} \tag{1.1}$$

We wish to show that h is a bijection. Notice that since A has m elements and B has n elements, then the set $A \cup B$ will have atmost $m + n$ elements; so all that is needed is to show that h is $1 - 1$. Let $i, j \in \mathbb{N}_m$, then $h(i) = h(j)$ implies that $f(i) = f(j)$, which is injective, thus $i = j$; now if $i, j = m + 1, \ldots m + +n$, then $h(i) = h(j)$ implies that $g(m - i) = m - j$, since g is also injective, $m - i = m - j$, so $i = j$. Thus h is injective and we are done.

(2) Let f be a bijection from $_m$ onto A, since CA, f is also a bijection from $\mathbb{N}_1$ onto C. Consider now h defined from $\mathbb{N}_{n-1}$ onto $A\backslash C$ by $h(i) = f(i)$ for $i = 2, \ldots m$. Then by the argument of (1), h is a bijecction.

(3) Let f be a bijection from $\mathbb{N}_m$ onto B, and consider h defined from $\mathbb{N}$ into $C\backslash B$ by $h(i) = g(i)$ for $i = m + 1, \ldots$. Clearly h is injective, but is it surjective? If $B = \emptyset$, then $C\backslash B = C$, and h is not surjective. Now suppose that B has only 1 element and suppose that h is indeed surjective. Then for every $j \in C\backslash B$, there exists and $i \in \mathbb{N}$ such that $h(i) = j$, Then clearly $j \notin B$, hence h is surjective still if we restrict it just to C, which would make C finite, a contradiction. Hence, $C\backslash B$ cannot be finite.

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Theorem 1.1.4. *Suppose that T and S are sets, and that TS. Then the following are true:*

(1) If S is finite, then T is also finite.

(2) If S is infinite, then T is also infinite.

Proof. (1) Suppose S is finite, then there is a bijection from onto S, for some $m \in T$. Now since TS, restricting f to N_n for some $n \leq m$, f is a bijection onto T; therefore T is finite.

(2) This is the contrapositive of the first argument.

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Definition. A set S is **denumerable** if there is a bijecction from $\mathbb{N}$ onto S. S is **countable** if it is either finite or denumerable. S is uncountable if it is not countable.

(1) The set $E = \{2n : n \in \mathbb{N}\}$ and $O = \{2n+1 : n \in \mathbb{N}\}$ are denumerable, take $f : \mathbb{N} \to E$ by $f(n) = 2n$ and take $g : \mathbb{N} \to O$ by $g(n) = 2n + 1$.

(2) The integers $\mathbb{Z}$ are denumerable, take $f(0) = 0$ and $f(n) = 2n$ for $n > 0$ and $f(n) = 2n + 1$ for $n < 0$.

(3) Then union of denumberable sets is also denumerable.

Theorem 1.1.5. *The set $\mathbb{N} \times \mathbb{N}$ is denumerable.*

Theorem 1.1.6. *Let S and T be sets with $T \subseteq S$; then:*

(1) If S is countable then T is countable.

(2) If S is uncountable then T is uhncountable.

Theorem 1.1.7. *The following statements are true:*

(1) S is denumerable.

(2) There exists a surjection of $\mathbb{N}$ onto S.

(3) There exists an injection of $\mathbb{N}$ onto S.

Proof. (1) If S is finite, then clearly it is countable. Now suppose that S is denumerable, then there is a bijection from $\mathbb{N}$ onto S, which by definition is surjective.

(2) If H is a surjection from $\mathbb{N}$ onto S, define $H_1 : S \to \mathbb{N}$ by taking $H_1(s)$ to be the least element in the set $H^{-1} = \{n \in \mathbb{N} : H(s) = n\}$, note that if $s, t \in S$ and $n_{st} = H_1(s) = H_1(t)$, then by definition of H_1, $s = H(n_{st}) = t$, so H_1 is injective.

(3) If $H_1 : S \to \mathbb{N}$ is injective, then it is bijective from S onto $H_1(S)\mathbb{N}$, by theorem 1.1.6 S is countable.

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Theorem 1.1.8. *The set of all rational numebers $\mathbb{Q}$ is denumerable.*

Proof. Since $\mathbb{N} \times \mathbb{N}$ is denumerable, it follows from theorem 1.1.7 that there is a surjection $f : \mathbb{N} rightarrow \mathbb{N} \times \mathbb{N}. Nowif$g: $\mathbb{N}\times\mathbb{N} \to \mathbb{Q}^+$, sends $(m, n) \to \frac{m}{n}$, then g is surjective. Then composing g with f, we get the surjection $g \circ f : \mathbb{B}\mathbb{Q}^+$, and so $\mathbb{Q}^+$ is countable. Similarly, $\mathbb{Q}^-$ is countable, hence $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$ is countable. ■

Theorem 1.1.9. *If A_m is countable for each $m \in \mathbb{N}$, then the union, $A = \bigcup_{m=1}^{\infty} A_m$ is countable.*

Proof. For each $m \in \mathbb{N}$, let $\phi_m : \mathbb{N} \to A$ be a surjection. Define $\beta : \mathbb{N} \times \mathbb{N}] rightarrow A$ by $\beta(m, n) = \phi_m(n)$. Then we claim that β is a surjection. If $a \in A$, then there exists an $m \in \mathbb{N}$ such that $a \in A_m$. Hence there exists an $n \in \mathbb{N}$ such that $a = \phi_m(n) = \beta(m, n)$. Therefore β is surjective, and A is countable. ■

Theorem 1.1.10 (Cantor's Theorem.)**.** *If A is any set, then there is no surjection from A onto 2^A.*

Proof. Suppose that $\phi : A \to 2^A$ is a surjection. Since $\phi(a) \subseteq A$, either $a \in A$ or $a \notin A$. Now let $D = \{a \in A : a \notin \phi(a)\}$. Then $D \subseteq A$; hence $D = \phi(a_0)$ for some $a_0 \in A$. Then either $a_0 \in D$ or $a_0 \notin D$. If $a \in D$, then $a = \phi(a_0)$, a contradiction of the definition of D. If $a \notin D$ then $a \notin \phi(a_0)$, another contradiction. Therefore, ϕ cannot be surjective. ■

Chapter 2

The Real Numbers.

2.1 The Algebraic and Order Properties of $\mathbb{R}$.

We begin first by examining the algebraic struture of a the real numbers, which we denote by $\mathbb{R}$. The following are called the **field axioms** for $\mathbb{R}$.

Axiom 2.1.1. *On the set $\mathbb{R}$ of real numbers, there are two binary operations $+$ and $\cdot$ called* ***addition*** *and* ***multiplication*** *respectively, such that:*

(1) $(\mathbb{R}, +)$ *is an abelian group.*

(2) $(\mathbb{R}\backslash\{0\}, \cdot)$ *is an abelian group.*

(3) $\cdot$ *distributes over* $+$*; that is for all* $a, b, c \in \mathbb{R}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

Theorem 2.1.1. *(1) If* $z, a \in \mathbb{R}$, *with* $z + a = a$, *then* $z = 0$.

(2) If $u, b \in \mathbb{R}$ *with* $b \neq 0$, *and* $ub = b$, *then* $u = 1$.

(3) For $a \in \mathbb{R}$, $a_0 = 0$.

The proof of this theorem is elementary and easy to reproduce.

Theorem 2.1.2. *(1) If* $a, b \in \mathbb{R}$ *with* $a \neq 0$, *and* $ab = 1$, *then* $b = a^{-1}$.

(2) If $ab = 0$, *then either* $a = 0$ *or* $b = 0$.

Another elementary proof.

Remark. For the real numbers $a, b \in \mathbb{R}$, we define **subtraction** of a and b, to be $a - b = a + (-b)$, similarly, we define **division** to be $\frac{a}{b} = ab^{-1}$. We also define the **exponent** recursively to be $a^0 = 1$, $a^1 = a$ and $a^{n+1} = a^n a$ for $n \geq 1$.We also denote $\frac{1}{a} = a^{-1}$ and we leave 0^0 and $\frac{1}{0}$ undefined.

We call the **rational numbers** tobe numbers of the form $\frac{a}{b}$ with $a, b \in \mathbb{Z}$ and denote them as $\mathbb{Q}$. There are however elements of $\mathbb{R}$ which are not in $\mathbb{Q}$; we calll the set of such elements the **irrational numbers** and denonte them by $\mathbb{R}\backslash\mathbb{Q}$, or $\mathbb{Q}^*$.

Theorem 2.1.3. *There is no rational number r such that $r^2 = 2$.*

Proof. Suppose there were. let $\frac{p}{q}$ be such a number where p and q have no common factors. Then $(\frac{p}{q})^2 = 2$, hence we have that $p^2 = 2q^2$, which makes p^2 even, hence p is even. Then $p = 2k$ for some $k \in \mathbb{N}$. Then $2q^2 = 4k^2$, hence $q^2 = 2k^2$ which is even, therefore q^2 and consequently q is even; contradicting the fact that p and q have no common factors. Therefore no such rational number exist. ■

2.2 The Order Properties of $\mathbb{R}$

We begin with the following definition.

Definition. There is a nonempty subset $\mathbb{R}^+$ of $\mathbb{R}$ called the set of **positive real numbers** satisfying the following properties:

(1) If $ab \in \mathbb{R}^+$, then $a + b \in R^+$.

(2) If $a, b \in \mathbb{R}^+$, then $ab \in \mathbb{R}^+$.

(3) One, and only one of the following hold for all $a \in \mathbb{R}^+$: either $a \in \mathbb{R}^+$, $a = 0$, or $-a \in \mathbb{R}^+$.

If $a \in \mathbb{R}^+$, then we say that $a > 0$, if $a \in \mathbb{R}^+ \cup \{0\}$, then we say that $a \geq 0$.