Analysis

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Text

Principles of Mathematical Analysis (3^{rd} edition)

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Chapter 1

The Real and Complex Numbers

1.1 Ordered Sets

Definition. Let S be any set. An **order** on S is a relation $<$ such that:

(1) For $x, y \in S$, one and only one of the following hold:

$$x < y \qquad x = y \qquad y < x$$

We call this property the **trichotomy law**

(2) $<$ is transitive over S.

We denote the relations $>$ and $\leq$ to mean $x > y$ if and only if $y < x$, and $x \leq y$ if and only if $x < y$, or $x = y$. We call S together with $<$ an **ordered set**.

Example 1.1. Define $<$ on $\mathbb{Q}$ such that for $r, s \in \mathbb{Q}$, $r < s$ implies $< 0s - r$.

Definition. Let S be an ordered set, and let $E \subseteq S$. We say that E is **bounded above** there is some $\beta \in S$ for which $x \leq \beta$, for all $x \in E$. We say that E is **bounded below** if $\beta \leq x$, for call $x \in E$. We say an $\alpha \in S$ is a **least upperbound** of E, if α is an upperbound of E, and for all other upperbounds, γ, of E, $\alpha \leq \gamma$. Likewise, α is a **greatest lowerbound** of E if α is a lowerbound of E, and for all other lowerbounds γ of E, $\gamma \leq \alpha$. We denote the least upperbound, and greatest lowerbound by $\sup E$ and $\inf E$, respectively.

Lemma 1.1.1. *Let S be an ordered set, and let $E \subseteq S$. Then E has (if they exist) a unique least upperbound, and a unique greatest lowerbound.*

Proof. Let $\alpha, \beta \in S$ be least upperbounds of E. Then by definition, we have that $\alpha \leq \beta$, and $\beta \leq \alpha$; thus by the trichotomy law, $\alpha = \beta$. The proof is the same for greatest lowerbounds. ■

Example 1.2. (1) Let $A = \{p \in \mathbb{Q} : p^2 < 2\}$, and $B = \{p \in \mathbb{Q} : p^2 > 2\}$. Clearly, we have that every element of B is an upperbound of A, and every element of A is a lowerbound of B. Now take $p \in \mathbb{Q}$ a positive rational, and take $q \in \mathbb{Q}$ such that $q = p - \frac{p^2-2}{p+2}$. Then

$q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2}$. Now if $p \in A$, then $p^2 - 2 < 0$, which implies that $p < q$, and $q^2 < 2$; thus A has no largest element; similarly, if $p \in B$, then $p^2 - 2 > 0$, which implies that $q < p$ and $q^2 > 2$, which shows that B has no least element. Thus $\sup A$ and $\inf B$ do not exist in $\mathbb{Q}$.

(2) If $\alpha = \sup E \in S$, it may or may not be that $\alpha \in E$. Take $E_1 = \{r \in \mathbb{Q} : r < 0\}$, and $E_2 = \{r \in \mathbb{Q} : r \leq 0\}$. Then $\sup E_1 = \sup E_2 = 0$, but $0 \notin E_1$, where as $0 \in E_2$

(3) COnsider the set $\frac{1}{\mathbb{Z}^+} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$. By the well ordering principle, 1 is the least element, and is also an upperbound of all $\frac{1}{n}$ for $n > 1$. Now also notice that as n gets arbitrarily large, then $\frac{1}{n}$ gets arbitratirly small; that is to say $\frac{1}{n}$ "tends" to 0, so $\sup \frac{1}{\mathbb{Z}^+} = 1 \in \frac{1}{\mathbb{Z}^+}$, and $\inf \frac{1}{\mathbb{Z}^+} = 0 \notin \frac{1}{\mathbb{Z}^+}$.

Definition. We say an ordered set S has the **least upperbound property**, if whenever $E \subseteq S$, nonempty, and bounded above, then $\sup E \in E$; likewise, S has the **greatest lowerbound property** if whenever E is nonempty, bounded below then $\inf E \in E$

Example 1.3. The set of all rationals $\mathbb{Q}$ does not have the least upperbound property, nor the greatest lowerbound property, take A, B as in the previous example. Letting $E = \{1, \frac{1}{2}, \frac{1}{4}\} \subseteq \frac{1}{\mathbb{Z}^+}$, we see that $\frac{1}{\mathbb{Z}^+}$ satisfies both properties, with $\sup E = 1$, and $\inf E = \frac{1}{4}$.

Theorem 1.1.2. *If S is an ordered set with the least upperbound property, then S also inherits the greatest lowerbound property.*

Proof. Let $B \subseteq S$, and let $L \subseteq S$ be the set of all lowerbounds of B. Then we have for any $y \in L$, $x \in B$, $y \leq x$. So every element of B is an upperbound of L, and L is nonempty, hence $\alpha = \sup L \in S$ exists. Now if $\gamma \leq \alpha$, then γ is not an upperbound of L, hence $\gamma \notin B$; thus $\alpha \leq x$ for all $x \in B$, so $\alpha \in L$, and by definition of the greatest lowerbound, we get $\alpha = \inf B$. ∎

1.2 Fields

Definition. A **field** is a set F, together with binary operations $+$ and $\cdot$ (called **addition** and **multiplication**, respectively) such that:

(1) F forms an abelian group under $+$.

(2) $F \backslash \{0\}$ forms an abelian group under $\cdot$ (where 0 is the additive identity of F).

(3) $\cdot$ distributes over $+$.

We now state the following propositions without proof.

Proposition 1.2.1. *For all $x, y, x \in F$:*

(1) $x + y = x + y$ implies $y = z$

(2) $x + y = x$ implies $y = 0$

(3) $x + y = 0$ *implies* $y = -x$

(4) $-(-x) = x$.

Proposition 1.2.2. *For all* $x, y, x \in F \backslash \{0\}$*:*

(1) $xy = xy$ *implies* $y = z$

(2) $xy = x$ *implies* $y = 1$

(3) $xy = 1$ *implies* $y = x^{-1}$

(4) $(x^{-1})^{-1} = x$.

Proposition 1.2.3. *For all* $x, y, x \in F$*:*

(1) $0x = 0$

(2) $x \neq 0$ *and* $y \neq 0$ *implies* $xy \neq 0$

(3) $(-x)y = -(xy) = x(-y)$

(4) $(-x)(-y) = xy$.

Definition. An **ordered field** is a field F that is also an ordered set, such that:

(1) $x + y < x + z$ whenever $y < z$, for $x, yz, z \in F$

(2) $xy > 0$ whenever $x > 0$ and $y > 0$, for $x, y \in F$.

Proposition 1.2.4. *Let* F *be an ordered field, then for any* $x, y, z \in F$*, the following hold:*

(1) $x > 0$ *implies* $-x < 0$.

(2) *If* $x > 0$ *and* $y < z$*, then* $xy < xz$.

(3) *If* $x < 0$ *and* $y < z$*, then* $xz < xy$.

(4) *If* $x \neq 0$*, then* $x^2 > 0$*, in particular,* $1 > 0$.

(5) $0 < x < y$ *implies that* $0 < y^{-1} < x^{-1}$.

Proof. (1) If $x > 0$, then $0 = x + (-x) > 0 + (-x)$, so $-x < 0$.

(2) We have $0 < z - y$, so $0 < x(z - y) = xz - xy$, so $xy < xz$.

(3) Do the same as (2),, multiplying $z - y$ by $-x$.

(4) If $x > 0$, we are done. Now suppose that $x < 0$, then $-x > 0$, so $(-x)(-x) = xx = x^2 > 0$; in particular, we also have that $1 \neq 0$, and $1 = 1^2$, so $1 > 0$.

(5) We have $0 < xy^{-1} < yy^{-1} = 1$, then $0 < x^{-1}xy^{-1} = y^{-1} < x^{-1}1 = x^{-1}$

■

1.3 The Field of Real Numbers