Analysis

Alec Zabel-Mena

Text

Principles of Mathematical Analysis (3^{rd} edition)

Walter Rudin

November 5, 2020

Chapter 1

The Real and Complex Numbers

1.1 Ordered Sets

Definition. Let S be any set. An **order** on S is a relation $<$ such that:

(1) For $x, y \in S$, one and only one of the following hold:

$$x < y \qquad\qquad x = y \qquad\qquad y < x$$

We call this property the **trichotomy law**

(2) $<$ is transitive over S.

We denote the relations $>$ and $\leq$ to mean $x > y$ if and only if $y < x$, and $x \leq y$ if and only if $x < y$, or $x = y$. We call S together with $<$ an **ordered set**.

Example 1.1. Define $<$ on $\mathbb{Q}$ such that for $r, s \in \mathbb{Q}$, $r < s$ implies $< 0s - r$.

Definition. Let S be an ordered set, and let $E \subseteq S$. We say that E is **bounded above** there is some $\beta \in S$ for which $x \leq \beta$, for all $x \in E$. We say that E is **bounded below** if $\beta \leq x$, for call $x \in E$. We say an $\alpha \in S$ is a **least upperbound** of E, if α is an upperbound of E, and for all other upperbounds, γ, of E, $\alpha \leq \gamma$. Likewise, α is a **greatest lowerbound** of E if α is a lowerbound of E, and for all other lowerbounds γ of E, $\gamma \leq \alpha$. We denote the least upperbound, and greatest lowerbound by $\sup E$ and $\inf E$, respectively.

Lemma 1.1.1. *Let S be an ordered set, and let $E \subseteq S$. Then E has (if they exist) a unique least upperbound, and a unique greatest lowerbound.*

Proof. Let $\alpha, \beta \in S$ be least upperbounds of E. Then by definition, we have that $\alpha \leq \beta$, and $\beta \leq \alpha$; thus by the trichotomy law, $\alpha = \beta$. The proof is the same for greatest lowerbounds. ■

Example 1.2. (1) Let $A = \{p \in \mathbb{Q} : p^2 < 2\}$, and $B = \{p \in \mathbb{Q} : p^2 > 2\}$. Clearly, we have that every element of B is an upperbound of A, and every element of A is a lowerbound of B. Now take $p \in \mathbb{Q}$ a positive rational, and take $q \in \mathbb{Q}$ such that $q = p - \frac{p^2-2}{p+2}$. Then

$q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2}$. Now if $p \in A$, then $p^2 - 2 < 0$, which implies that $p < q$, and $q^2 < 2$; thus A has no largest element; similarly, if $p \in B$, then $p^2 - 2 > 0$, which implies that $q < p$ and $q^2 > 2$, which shows that B has no least element. Thus $\sup A$ and $\inf B$ do not exist in $\mathbb{Q}$.

(2) If $\alpha = \sup E \in S$, it may or may not be that $\alpha \in E$. Take $E_1 = \{r \in \mathbb{Q} : r < 0\}$, and $E_2 = \{r \in \mathbb{Q} : r \leq 0\}$. Then $\sup E_1 = \sup E_2 = 0$, but $0 \notin E_1$, where as $0 \in E_2$

(3) Consider the set $\frac{1}{\mathbb{Z}^+} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$. By the well ordering principle, 1 is the least element, and is also an upperbound of all $\frac{1}{n}$ for $n > 1$. Now also notice that as n gets arbitrarily large, then $\frac{1}{n}$ gets arbitratirly small; that is to say $\frac{1}{n}$ "tends" to 0, so $\sup \frac{1}{\mathbb{Z}^+} = 1 \in \frac{1}{\mathbb{Z}^+}$, and $\inf \frac{1}{\mathbb{Z}^+} = 0 \notin \frac{1}{\mathbb{Z}^+}$.

Definition. We say an ordered set S has the **least upperbound property**, if whenever $E \subseteq S$, nonempty, and bounded above, then $\sup E \in S$ exists; likewise, S has the **greatest lowerbound property** if whenever E is nonempty, bounded below then $\inf E \in S$ exists.

Example 1.3. (1) The set of all rationals $\mathbb{Q}$ does not have the least upperbound property, nor the greatest lowerbound property, take A, B as in the previous example. Letting $E = \{1, \frac{1}{2}, \frac{1}{4}\} \subseteq \frac{1}{\mathbb{Z}^+}$, we see that $\frac{1}{\mathbb{Z}^+}$ satisfies both properties, with $\sup E = 1$, and $\inf E = \frac{1}{4}$.

(2) Let $A \subseteq \mathbb{R}$ be nonempty, and be bounded below. Then by the greatest lowerbound property, $\alpha = \inf A \in \mathbb{R}$ exists; Then for all $x \in A$, $\alpha \leq x$, and for all other lowerbounds γ, $\gamma \leq \alpha$. Then $-x \leq -\alpha$, and $-\alpha \leq -\gamma$, then we see that $-\gamma$ and $-\alpha$ are upperounds of $-A$, and that $-\alpha$ is the least upperbound of $-A$

Theorem 1.1.2. *If S is an ordered set with the least upperbound property, then S also inherits the greatest lowerbound property.*

Proof. Let $B \subseteq S$, and let $L \subseteq S$ be the set of all lowerbounds of B. Then we have for any $y \in L$, $x \in B$, $y \leq x$. So every element of B is an upperbound of L, and L is nonempty, hence $\alpha = \sup L \in S$ exists. Now if $\gamma \leq \alpha$, then γ is not an upperbound of L, hence $\gamma \notin B$; thus $\alpha \leq x$ for all $x \in B$, so $\alpha \in L$, and by definition of the greatest lowerbound, we get $\alpha = \inf B$. ■

1.2 Fields

Definition. A **field** is a set F, together with binary operations $+$ and $\cdot$ (called **addition** and **multiplication**, respectively) such that:

(1) F forms an abelian group under $+$.

(2) $F \backslash \{0\}$ forms an abelian group under $\cdot$ (where 0 is the additive identity of F).

(3) $\cdot$ distributes over $+$.

We now state the following propositions without proof.

Proposition 1.2.1. *For all $x, y, x \in F$:*

(1) $x + y = x + y$ *implies* $y = z$

(2) $x + y = x$ *implies* $y = 0$

(3) $x + y = 0$ *implies* $y = -x$

(4) $-(-x) = x$.

Proposition 1.2.2. *For all $x, y, x \in F\backslash\{0\}$:*

(1) $xy = xy$ *implies* $y = z$

(2) $xy = x$ *implies* $y = 1$

(3) $xy = 1$ *implies* $y = x^{-1}$

(4) $(x^{-1})^{-1} = x$.

Proposition 1.2.3. *For all $x, y, x \in F$:*

(1) $0x = 0$

(2) $x \neq 0$ *and* $y \neq 0$ *implies* $xy \neq 0$

(3) $(-x)y = -(xy) = x(-y)$

(4) $(-x)(-y) = xy$.

Definition. An **ordered field** is a field F that is also an ordered set, such that:

(1) $x + y < x + z$ whenever $y < z$, for $x, yz, z \in F$

(2) $xy > 0$ whenever $x > 0$ and $y > 0$, for $x, y \in F$.

Proposition 1.2.4. *Let F be an ordered field, then for any $x, y, z \in F$, the following hold:*

(1) $x > 0$ *implies* $-x < 0$.

(2) If $x > 0$ *and* $y < z$*, then* $xy < xz$.

(3) If $x < 0$ *and* $y < z$*, then* $xz < xy$.

(4) If $x \neq 0$*, then* $x^2 > 0$*, in particular,* $1 > 0$.

(5) $0 < x < y$ *implies that* $0 < y^{-1} < x^{-1}$.

Proof. (1) If $x > 0$, then $0 = x + (-x) > 0 + (-x)$, so $-x < 0$.

(2) We have $0 < z - y$, so $0 < x(z - y) = xz - xy$, so $xy < xz$.

(3) Do the same as (2),, multiplying $z - y$ by $-x$.

(4) If $x > 0$, we are done. Now suppose that $x < 0$, then $-x > 0$, so $(-x)(-x) = xx = x^2 > 0$; in particular, we also have that $1 \neq 0$, and $1 = 1^2$, so $1 > 0$.

(5) We have $0 < xy^{-1} < yy^{-1} = 1$, then $0 < x^{-1}xy^{-1} = y^{-1} < x^{-1}1 = x^{-1}$

■

1.3 The Field of Real Numbers

Theorem 1.3.1. *There exists an ordered field $\mathbb{R}$ with the least upperbound property, such that $\mathbb{Q} \subseteq \mathbb{R}$.*

Definition. We call the field $\mathbb{R}$ the **field of real numbers**,and we call the elements of $\mathbb{R}$ **real numbers**.

Definition. Let S be an ordered field, and let $E \subseteq S$. We say that E is **dense** in S, if for all $r, s \in S$, with $r < s$, there is an $\alpha \in E$ such that $r < \alpha < s$.

Theorem 1.3.2 (The Archimedean Principle)**.** *If $x, y \in \mathbb{R}$, and $x > 0$, then there is an $n \in \mathbb{Z}^+$ such that $nx > y$.*

Proof. Let $A = \{nx : n \in \mathbb{Z}^+\}$, and suppose that $nx \leq y$. Then y is an upperbound of A, abd since A is nonempty, $\alpha = \sup A \in \mathbb{R}$, since $x > 0$, we have $\alpha - x < \alpha$, so $\alpha - x$ is not an upperbound of A. Hence $\alpha - x < mx$ for some $m \in \mathbb{Z}^+$. Then $\alpha < (1-m)x \in A$, contradicting that α is an upperbound of A. ■

Theorem 1.3.3 (The density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

Proof. Let $x < y$ be realnumbers, then $y - x > 0$, so by the Archimedean principle, there is an $n \in \mathbb{Z}^+$ fir which $n(y-x) > 1$. By the Archimedean principle again, we have $m_1, m_2 \in \mathbb{Z}^+$ for which $m_1 > nx$ and $m_2 > -nx$, thus $-m_2 < nx < m_1$, and we also have that there is an $m \in \mathbb{Z}^+$ for which $-m_2 < m < m_1$, and $m - 1 \leq nx < m$. Thus combining inequalities, we get $nx < m < ny$, thus $x < \frac{m}{n} < y$. ■

Theorem 1.3.4 (The existence of n^th roots of positive reals)**.** *For every real number $X > 0$, and for every $n \in \mathbb{Z}^+$, there is one, and only one positive real number y for which $y^n = x$.*

Proof. Let $y > 0$ be a real number; then $y^n > 0$, so there is atmost one such y for which $y^n = x$. Now let $E = \{t : \mathbb{R} : t^n < x\}$, choosing $t = \frac{x}{1+x}$, we see that $0 \leq t < 1$, hence $t^n < t < x$, so E is nonempty. Now if $1 + x < t$, then $t^n \geq x$, so $t \notin E$, and E has $1 + x$ as an upperbound. Therefore, $\alpha = \sup E \in \mathbb{R}$ exists.

Now suppose that $y^n < x$, choose $0 \leq h < 1$ such that $h < \frac{x-y^n}{n(y+1)^{n-1}}$, then $(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)n - 1 < x - y^n$, thus $(y+h)^n < x$, so $y + h \in E$, contraditing that y is an upperbound. On the other hand, if $y^n > x$, choosing $k = \frac{y^n-x}{ny^{n-1}}$, then $0 \leq k < y$, and letting $t \geq y - k$, we get that $y^n - t^n \leq y^n + (y-k)^n < kny_{n-1} = y^n - x^n$, so $t^n \geq x$, making $y - k$ an uppeerbound of E, which contradicts $y = \sup E$. ■

Remark. We denote y as $\sqrt[n]{x}$, or as $x^{\frac{1}{n}}$.

Corollary. *If $a, b \in \mathbb{R}$, with $a, b > 0$, and $n \in \mathbb{Z}^+$, then $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$.*

Proof. Let $\alpha = \sqrt[n]{a}$, and $\beta = \sqrt[n]{b}$. Then $\alpha^n = a$, and $\beta^n = b$, so $ab = \alpha^n\beta^n = (l\alpha\beta)^n$, we are done. ■

Definition. We define the **extended real number system** to be the field $\mathbb{R}$, together with symbols ∞, and $-\infty$, called **positive infinity** and **negative infinity**, such that $-\infty < x < \infty$ for all $x \in \mathbb{R}$. We call elements of the extended real numbers **infinite**, and every other element not in the extended real numbers **finite**.

Lemma 1.3.5. *∞ is an upperbound for every subset E, of $\mathbb{R}$, and $-\infty$ is a lowerbound for every subset E of $\mathbb{R}$. Moreover, if E is not bounded above, then $\sup E = \infty$, and if E is not bounded below, then $\inf E = -\infty$.*

Remark. We make the following assumptions for extended real numbers:

(1) If $x \in \mathbb{R}$, then $x + \infty = \infty$, $x - \infty = -\infty$, and $\frac{x}{\infty} = \frac{x}{-\infty} = 0$.

(2) If $x > 0$, then $x(\infty) = \infty$ and $x(-\infty) = -\infty$.

(3) If $x < 0$, then $x(\infty) = -\infty$ and $x(-\infty) = \infty$.

1.4 The Complex Field

Definition. We define a **complex number** to be a pair of real numbers (a, b). We denote the set of all comlex numbers by $\mathbb{C}$. We define the **addition** and **multiplication** of complex numbers to be the binary operations $+ : \mathbb{C} \to \mathbb{C}$ and $\cdot : \mathbb{C} \to \mathbb{C}$ such that

$$\begin{aligned} (a, b) + (c, d) &= (a + c, b + d) \\ (a, b)(c, d) &= (ac - bd, ad + bc) \end{aligned}$$

Lastly, we define i to be the complex number such that $i = (0, 1)$.

Theorem 1.4.1. *$\mathbb{C}$ forms a field together with $+$ and $\cdot$.*

Theorem 1.4.2. *For $(a, 0), (b, 0) \in C$, $(a, 0) + (b, 0) = (a + b, 0)$, and $(a, 0)(b, 0) = (ab, o)$.*

Proof. This is a straightforward application of the addition and multiplication of complex numbers. ∎

Theorem 1.4.3. *$i^2 = -1$.*

Proof. $i^2 = (0, 1)(0, 1) = (0 - 1, 1 - 1) = (-1, 0) = -1$. ∎

Theorem 1.4.4. *Let $(a, b) \in \mathbb{C}$, then $(a + b) = a + ib$.*

Proof. $(a, b) = (a, 0) + (0, b) = (a, 0) + (0, 1)(b, 0) = a + ib$. ∎

Definition. Let $a, b \in \mathbb{R}$, and let $z \in \mathbb{C}$ such that $z = a + ib$. We define the **complex conjugate** of z to be the complex number $\overline{z} = a - ib$. Moreover, we define the **real part** of z to be a, and the **imaginary part** of z to be b, and we denote them $a = \operatorname{Re} z$, $b = \operatorname{Im} z$

Theorem 1.4.5. *Let $z, w \in \mathbb{C}$. Then*

(1) $\overline{z + w} = \overline{z} + \overline{w}$.

(2) $\overline{zw} = \overline{zw}$.

(3) $z + \overline{z} = 2 \operatorname{Re} z$ *and* $z - \overline{z} = 2i \operatorname{Im} z$.

(4) $z\overline{z}$ *is a nonegative real number.*

Proof. Let $z = a + ib$, and let $w = c + id$. Then $z + w = (a + c) + i(b + d$, so $\overline{z + w} = (a + b) - i(b + d) = (a - ib) + (c - id) = \overline{z} + \overline{w}$; similarly, we get $\overline{zw} = \overline{zw}$. Moreover, we have $(a + ib) + (a - ib) = 2a$, and $(a + ib) - (a - ib) = 2ib$, we also have that $z\overline{z} = (a + ib)(a - ib) = a^2 + b^2 \geq 0$, and $z\overline{z} = 0$ if and only if $a = b = 0$. ∎

Definition. Let $z \in \mathbb{C}$. We define the **modulus** of z to be $|z| = \sqrt{z\overline{z}}$.

Remark. $|z|$ exists and is unique.

Theorem 1.4.6. *Let* $z, w \in \mathbb{C}$*, then:*

(1) $|z| \geq 0$ *and* $|z| = 0$ *if and only if* $z = 0$.

(2) $|\overline{z}| = |z|$.

(3) $|zw| = |z||w|$.

(4) $\operatorname{Re} z \leq |z|$.

(5) $|z + w + \leq |z| + |w|$.

Proof. Let $z = a + ib$, and $w = c + id$. Then $|z| = \sqrt{a^2 + b^2} \geq 0$, and $|z| = 0$ if and only if $a, b = 0$. Moreover, $|\overline{z}| = |a + i(-b)| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$. We also habe $|zw|^2 = (a^2 + b^2)(c^2 + d^) = |z|^2|w|^2$, likewise, $\|rez| = |a + i0| = \sqrt{a^2} \leq \sqrt{a^2 + b^2}$. Finally we prove (5).

We have $|z + w|^2 = (x + w)(\overline{z} + \overline{w}) = z\overline{z} + \overline{z}w + \overline{w}z + w\overline{w} = |z|^2 + w \operatorname{Re} z\overline{w} + |w|^2 \leq |z|^2 + 2|s\overline{w}| + |w|^2 = (|z| + |w|)^2$. ∎

Theorem 1.4.7 (The Cauchy Schwarz Inequality)**.** *Let* $a_i, b_i \in \mathbb{C}$*, for* $1 \leq i \leq n$*. Then:*

$$|\sum_{i=1}^{n} a_i\overline{b_i}| \leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_j|^2 \tag{1.1}$$

Proof. Let $A = \sum a_j|^2$, $B = \sum |b_i|^2$, and $C = \sum a_i\overline{b_i}$. If $B = 0$, then $b_i = 0$ for $1 \leq i \leq n$, and we are done; so suppose that $B > 0$. Then

$$\begin{aligned}
\sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(B\overline{a_j} - \overline{Cb_j}) \\
&= B \sum |a_j|^2 - B\overline{C} \sum a_j\overline{b_j} - BC \sum \overline{a_j}b_j + |C^2| \sum |b_j|^2 \\
&= (B^2A - B|C|^2) = B(AB - |C|^2) \geq 0
\end{aligned}$$

Since $B > 0$, we get $|C|^2 \leq AB$ as required. ∎

1.5 Euclidean Spaces

Definition. Let $k \in \mathbb{Z}^+$, and let $\mathbb{R}^k$ be the set of all ordered k-tuples $(x_1, x_2, \ldots, x_k)$, with $x_i \in \mathbb{R}$ for $1 \leq i \leq k$. We call $\mathbb{R}^k$ the **Euclidean space** of **dimension** k; more simply the **Euclidean k-space**. We call elements of $\mathbb{R}^k$ **vectors** or **points**; and we define **vector addition** and **scalar multiplication** to be:

$$(x_1, \ldots, x_k) + (y_1, \ldots, y_k) = (x_1 + y_1, \ldots, x_k + y_k)$$
$$\alpha(x_1, \ldots, x_k) = (\alpha x_1, \ldots, \alpha x_k)$$

for $(x_1, \ldots, x_k), (y_1, \ldots, y_k) \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$.

Theorem 1.5.1. $\mathbb{R}^k$ *forms a vector space together with vector addition and scalar multiplication.*

Definition. Let $x, y \in \mathbb{R}^k$. We define the **inner product** of x and y to be the binary operation $\langle , \rangle : \mathbb{R}^k\mathbb{R}^k \to \mathbb{R}$ such that

$$\langle x, y \rangle = \sum_{i=1}^{k} x_i y_i$$

We define the **norm** of x to be $||x|| = \sqrt{\langle x, x \rangle} = \sqrt{\sum x_i^2}$.

Theorem 1.5.2. *Let* $x, y \in \mathbb{R}^k$*, and* $\alpha \in \mathbb{R}$*. Then:*

(1) $||x|| \geq 0$ *and* $||x|| = 0$ *if and only if* $x_i = 0$ *for all* $1 \leq i \leq k$*.*

(2) $||\alpha x|| = |\alpha|||x||$*.*

(3) $||\langle x, y \rangle|| \leq ||x||||y||$*.*

(4) $||x + y|| \leq ||x|| + ||y||$*, and* $||x - z|| \leq ||x - y|| + ||y - z||$

Proof. (1) follows by definition of the norm. We also have that $||\alpha x|| = \sqrt{\sum \alpha^2 x_i^2} = \sqrt{\alpha^2}\sqrt{\sum x_i^2} = |\alpha|||x||$.

Now by the Cauchy Schwarz inequality, we have that $||\langle x, y \rangle||^2 = \sum x_i^2 y_i^2 \leq \sum x_i^2 \sum y_i^2 = ||x||||y||$. Finally we have that $||x + y|| = \langle x + y, x + y \rangle = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \leq ||x||^2 + 2||x||||y|| + ||y^2|| = (||x|| + ||y||)^2$, the last result follows immediately. ■

Chapter 2

Topological Foundations

2.1 Finite, Countable, and Uncountable Sets

Definition. Let A be a set, and let $E \subseteq \mathbb{N}$. We say that A is **finite** if there exists a 1-1 mapping of A ont E, we say A is **countable** if $E = \mathbb{N}$, and we say A is **atmost countable** if A is either finite or countable.

Example 2.1. The set of all integers $\mathbb{Z}$ is countable. Take $f : \mathbb{N} \to \mathbb{Z}$ such that $f(n) = 2$ if n is even, and $f(n) = -n$ if n is odd.

Definition. Let A be a set, and let $E \subseteq \mathbb{N}$. A **sequence** in A is a mapping $f : E \to A$ such that $f(n) = x_n$, for $x_n \in A$. We call the values of f **terms** of the sequence. We denote sequences by $\{x_n\}_{i=1}^n$, and when $E = \mathbb{N}$, we denote them simply by $\{x_n\}$.

Theorem 2.1.1. *Every infinite subset of a countable set is countable.*

Proof. Let A be countable, and let $E \subseteq A$ be infinite. Arrange the elements of A into a sequence $\{x_n\}$, and construct a sequence $\{n_k\}$ such that n_1 is the least term for which $\{x_{n_k}\} \in E$, and n_k is the least term greater than n_{k-1} for which $x_{n_k} \in E$. Let $f(k) = x_{n_k}$, and we get a 1-1 mapping of $\mathbb{N}$ onto E. ■

Theorem 2.1.2. *Let $\{E_n\}$ be a sequence of countable sets. Then $S = \bigcup E_n$ is also countable.*

Proof. Arrange every set E_n in a sequence $\{x_{nk}\}$, and consider the infinite array (x_{ij}), in which the elements of E_n form the n-th row. Then (x_{ij}) contains all the elements of S, and we can arrange them is a sequence

$$x_{11}, (x_{21}, x_{12}), (x_{31}, x_{22}, x_{13}), \dots$$

Moreover, if $E_j \cap E_j \neq \emptyset$, for $i \neq j$, then the elements of $E_i \cap E_j$ appear more than once in the sequence of S; so taking $T \subseteq \mathbb{N}$, we get a 1-1 mapping of T onto S, hence S is atmost countable, and since $E_i \subseteq S$ for $i \in \mathbb{N}$, is infinite, by theorem 2.1.1, S is infinite, thus S is countable. ■

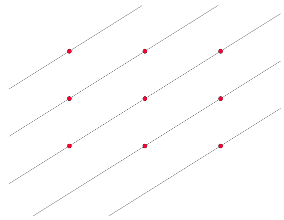

Figure 2.1: The infinite array (x_{ij})

Corollary. *Let A be atmost countable, and suppose for all $\alpha \in A$ that the sets B_α are atmost countable. Then*

$$T = \bigcup_{\alpha \in A} B_\alpha$$

is atmost countable.

Theorem 2.1.3. *Let A be countable, and let B_n be the set of all n-tuples $(a_1, \ldots, a_n)$ such that $a_i \in A$ for $1 \leq i \leq n$. Then B_n is countable.*

Proof. By induction on n, we have that $B_1 = A$, which is countable. Now suppose that B_n is countable, and consider B_{n+1} whose elements are of the form (b, a) where $b \in B_n$ and $a \in A$. Fixing b, we get a 1-1 correspondence between the elements of B_{n+1} and A; therefore B is countable. ■

Corollary. $\mathbb{Q}$ *is countable.*

Proof. For every rational $\frac{p}{q} \in \mathbb{Q}$, represent $\frac{p}{q}$ as (p, q). Then the countability of $\mathbb{Q}$ follows from theorem 2.1.3. ■

Theorem 2.1.4. *Let A be the set of all sequences of 0 and 1; then A is uncountable.*

Proof. Let EA be countable, and let E consist of all the sequences of 0 and 1, $s_1, s_2, s_3, \ldots$. Construct the sequence s such that if the n-th term of the sequence s_i is 0, then the n-th term of s is 1, and vice versa, for $i \in \mathbb{Z}^+$. Then the sequence s differs from the sequence s_i at atleast one place; thus $s \notin E$, but $s \in A$. Therefore $E \subset A$, which establishes the uncountablitiy of A. ■

2.2 Metric Spaces

Definition. A set X, whose elements we will call **points**, is said to be a **metric space** if there exists a mapping $d : X \times X \to \mathbb{R}$, called a **metric** (or **distance function**) such that for $x, y \in X$

(1) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$.

(3) $d(x, y) \leq d(x, z) + d(z, y)$ (The Triangle Inequality).

Example 2.2. The absolute value, $|\cdot|$ for real numbers, the modulus $|\cdot|$ for complex numbers, and the norm $||\cdot||$ for vectors are all metrics. They turn $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{R}^k$ into metric spaces respectively.

Definition. An **open interval** in $\mathbb{R}$ (or **segment**) is a set of the form $(a,b) = \{a, b \in \mathbb{R} : a < x < b\}$, a **closed interval** in $\mathbb{R}$ is a set of the form $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$; and **half open intervals** in $\mathbb{R}$are sets of the form $[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$ and $(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$.

If $a_i < b_i$, for $1 \leq i \leq k$, the set of all points $(x_1, \ldots, x_k) \in \mathbb{R}^k$ which satisfy the Inequalities $a_i \leq x_i \leq b_i$ is called a **k-cell** in $\mathbb{R}^k$. If $x \in \mathbb{R}^k$, and $r > 0$, we call the set $B_r(x) = \{y \in \mathbb{R}^k : ||x-y|| < r\}$ an **open ball** in $\mathbb{R}^k$, and we call the set $B_r[x] = \in \mathbb{R}^k : ||x-y|| \leq r\}$ a **closed ball** in $\mathbb{R}^k$.

Definition. We call a set $E \subseteq \mathbb{R}^k$ **convex**, if whenever $x, y \in E$, $\lambda x + (1-\lambda)y \in E$ for $0 < \lambda < 1$.

Lemma 2.2.1. *Open and closed balls, along with k-cells are convex.*

Proof. Let $B_r(x)$ be an open ball; let $y, x \in B_r(x)$, and $0 < \lambda < 1$. Then $||x-(\lambda y+(1-\lambda)z|| = ||\lambda(x-y)-(1-\lambda)(x-z)|| \leq \lambda||x-y|| + (1-\lambda)||x-z|| < \lambda r + (1-\lambda)r$. The proof is analogous for closed ball.

Now let K be a k-cell for $a_i < b_i$, for $1 \leq i \leq k$, let $x, y \in K$, then $a_i \leq x_i, y_i \leq b_i$, so $\lambda a_i \leq \lambda x_i \leq \lambda b_i$, and $(1-\lambda)a_i \leq (1-\lambda)y_i \leq (1-\lambda)b_i$, since $0 < \lambda < 1$, $a_i \leq a_i + (1-\lambda)a_i \leq \lambda x_i + (1-\lambda)y_i \leq \lambda b_i + (1-)b_i \leq b$. ■

Corollary. *Open and closed intervals, along with half open intervals are convex.*

Proof. We just notice that open and closed intervals are open and closed balls in $\mathbb{R}^1 = \mathbb{R}$, we also notice that half open intervals $[a,b)$ and $(a,b]$ are subsets of the closed interval $[a,b]$, and hence inherit convexity. ■

For the following definitions, let X be a metric space with metric d.

Definition. A **neighborhood** of a point $x \in X$ is the set $N_r(x) = \{y \in X : d(x,y) < r\}$ for some $r > 0$ called the **radius** of the neighborhood. We call x a **limit point** of a set $E \subseteq X$ if every neighborhood of x contains a point $y \neq x$ such that $y \in E$. If $y \in E$, and y is not a limit point, we call y an **isolated point**.

Definition. We call a set $E \subseteq X$ **closed** if every limit point of E is in E. A point $x \in X$ is an **interior point** of E if there is a neighborhood N of x such that $N \subseteq E$. We call E **open** if every point of E is an interior point of E.

Definition. $E \subseteq X$ is called **prefect** if E is closed, and every point of E is a limit point of E. We call E **dense** if every point of X is either a limit point of E, or a point of E, or both.

Definition. We call $E \subseteq X$ **bounded** if there is a real number $M > 0$, and a point $y \in X$ such that $d(x,y) < M$ for all $x \in E$.

Theorem 2.2.2. *Let X be a metric space and $x \in X$. Every neighborhood of x is open.*

Proof. Consider the neighborhood $N_r(x)$, and $y \in E$, there is a positive real number h such that $d(x, y) = r - h$, then for $z \in X$ such that $d(y, s) < h$, we have $d(x, s) \leq d(x, y) + d(y, s) < r - h + h = r$, thus $s \in E$, so y is an interior point of E. ■

Theorem 2.2.3. *If x is a limit point of a set E, then every neighborhood of x contains infinitely many points of E.*

Proof. Let N be a neighborhood of x contai ning only a finite number points of E. Let $y_1, \ldots, y_n$ be points of $N \cap E$ distinct from x and let $r = \min\{d(x, y_i)\}$ for $1 \leq i \leq n$, then $r > 0$, and the neighborhood $N_r(x)$ contains no point y of E for which $y \neq x$, so x is not a limit point; which is a contradiction. ■

Corollary. *A finite point set has no limit points.*

Proof. By theorem 2.3.3, if x is a limit point in the finite point set E, then evry neoghborhood of contains infinitely many points of E; contradicting its finiteness. ■

Example 2.3. (1) The set of all $z \in \mathbb{C}$ such that $|z| < 1$ is open, and bounded.

(2) The set of all $z \in \mathbb{C}$ for which $|z| \leq 1$ is closed, perfect, and bounded.

(3) Any nonempty finite set is closed, and bounded.

(4) $\mathbb{Z}$ is closed, but it is not open, perfect, or bounded.

(5) The set $\frac{1}{\mathbb{Z}^+}$ is neither closed, nor open, it is not perfect; but it is bounded..

(6) $\mathbb{C}$ is closed, open, and perfect, but it is not bounded.

(7) The open interval in (a, b) is open (only in $\mathbb{R}$), and bounded.

Theorem 2.2.4. *Let X be a metric space, a set $E \subseteq X$ is open if and only if $X \backslash E$ is closed.*

Proof. Suppose that $X \backslash E$ is closed, let $x \in E$, then $x \notin X \backslash E$, and x is not a limit point of $X \backslash E$. Thus there is a neighborhood N of x such that $N \cap E = \emptyset$, thus $N \subseteq E$, and so x is an interior point of E.

Conversely, suppose that E is open, and let x be a limit point of $X \backslash E$, then every neighborhood of of x contains a point of $X \backslash E$, so x is not an interior point of E, since E is open, it follows that $x \in X \backslash E$, thus $X \backslash E$ is closed. ■

Corollary. *E is closed if and only if $X \backslash E$ is open.*

Proof. This is the converse of theorem 2.3.4. ■

Theorem 2.2.5. *Let X be a metric space. The following are true:*

(1) If $\{G_\alpha\}$ is a collection of open sets, then $\bigcup G_\alpha$ is open.

(2) If $\{G_i\}_{i=1}^n$ is a finite collection of open sets, then $\bigcap_{i=1}^n G_i$ is open.

(3) if $\{G_\alpha\}$ is a collection of closed sets, then $\bigcap G_\alpha$ is closed.

(4) If $\{G_i\}_{i=1}^n$ is a finite collection of closed sets, then $\bigcup_{i=1}^n G_i$ is closed.

Proof. Let $G = \bigcup G_\alpha$, then if $x \in G$, $x \in G_\alpha$ for some α, then x is an interior point of G_α, hence an interior point of G, so G is open. Now let $G = \bigcap_{i=1}^n G_i$ For $x \in G$, there are neighborhoods N_i of x, with radii r_i such that $N_i \subseteq G_i$ for $1 \leq i \leq n$. Then let $r = \min\{r_1, \ldots, r_n\}$, and let N be the neighborhood of x with radius r, then $N \subseteq G_i$, hence $N \subseteq G$, so G is open.

The proofs of (3) and (4) are just the converse of the proofs of (1) and (2). ■

Definition. Let X be a metric space, and let $E \subseteq X$, and let E' be the set of all limit points of E. We define the **closure** of E to be the set $\overline{E} = E \cup E'$.

Theorem 2.2.6. *If X is a metric space, and $E \subseteq X$, then the following hold*

(1) $\overline{x}$ is closed.

(2) E is closed if and only if $E = \overline{E}$.

(3) If $F \subseteq X$ such that $E \subseteq F$, and F is closed, then $\overline{E} \subseteq F$.

Proof. If $x \in X$, and $x \notin \overline{E}$, then $x \notin E$, nor is it a limit point of E, thus there is a neighborhood of x that is disjoint from E, hence $X \backslash \overline{E}$ is open.

Now if E is closed, then $E' \subseteq E$, so $\overline{E} = E$, conversely, if $E = \overline{E}$, then clearly E is closed. Now if F is closed and $E \subseteq F$, then $F' \subseteq F$, and $E' \subseteq F$, therfore $\overline{E} \subseteq F$. ■

Theorem 2.2.7. *Let $E\mathbb{R}$ be nonempty and bnounded above, let y
$supE$, then $y \in \overline{E}$, hence $y \in E$ if E is closed.*

Proof. Suppose that $y \notin E$, then for every $h > 0$, there exists a point $x \in E$ such that $y - h < x < y$, then y is a limit point of E, thus $y \in \overline{E}$. ■

Theorem 2.2.8. *Let $Y \subseteq X$; a subset E of Y is open in Y if and only if $E = Y \cap G$ for some open subset G of X.*

Proof. Suppose E is open in Y, then for each $x \in E$, there is a $r_p > 0$ such that $d(x, y) < r_p$, if $y \in Y$, that implies that $y \in E$; hence let V_x be the set of all $y \in X$ such that $d(x, y) < r_p$, and define

$$G = \bigcup_{x \in E} V_p$$

Then by theorems 2.2.2 and 2.2.5, G is open in X, and $EG \cap Y$. Now we also have that $V_p \cap YE$, thus $G \cap YE$, thus $E = G \cap Y$. Conversely, if G is open in X, and $E = G \cap Y$, then every $x \in E$ has a neighborhood $v_p \in G$, thus $V_p \cap Y \subseteq E$, hence E is open in Y. ■

2.3 Compact Sets

Definition. Let X be a metric space, and let $E \subseteq X$. An **open cover** of E is a collection $\{G_\alpha\}$ of subsets of X such that $E \subseteq \bigcup G_\alpha$. We call a collection $\{E_\beta\}$ of subsets of X an **open subcover** of E if $\{E_\beta\}$ is a cover of E, and $\bigcup E_\beta \subseteq \bigcup G_\alpha$. We call E **compact** if every open cover of E contains a finite open subcover.

Lemma 2.3.1. *Every finite set is compact.*

Proof. Let K be finite, and let $\{G_\alpha\}$ be an open subcover of K. Since K is finite, there is a 1-1 mapping of K onto the set $\{1, \dots, n\}$. Let $\{E_i\}_{i=1}^n$ be the finite collection of all subsets of K, clearly, $\{E_i\}$ is an open cover of K. Moreover, if $\bigcup E_i \subseteq \bigcup G_\alpha$, we are done, and if $\bigcup G_\alpha \subseteq \bigcup E_i$, then $\{G_i\}$ is a finite subcollection that covers K, so in either case, K is compact. ■

Theorem 2.3.2. *Let X be a metric space, and let $K \subseteq Y \subseteq X$. Then Y is compact in X if and only if K is compact in Y.*

Proof. Suppose K is compact in Y, and let $\{G_\alpha\}$ be a collection of subsets of Y X that cover K, and let $V_\alpha = Y \cap G_\alpha$, then $\{V_\alpha\}$ is a collection of subsets of X covering K, in which $V_\alpha \subseteq G_\alpha$ for all α, therefore K is compact in Y

conversely, suppose that K is compact in X, and let $\{V_\alpha\}$ be a collection of open sets in Y sucht that $K \subseteq \bigcup V_\alpha$, by theorem 2.2.8, there is a collection $\{G_\alpha\}$ of open sets in Y such that $V_\alpha = Y \cap G_\alpha$, for all α. Then $K \subseteq \bigcup_{i=1}^n G_{\alpha_i}$; therefore, K is compact in Y. ■

Theorem 2.3.3. *Compact subsets of metric spaces are closed.*

Proof. Let X be a metric space, and let K be compact in X and let $x \in X \backslash K$, if $y \in K$, let U and V be neighborhoods of x and y respectively, each of radius $r < \frac{1}{2}d(x, y)$. Since K is compact, there are finitely many points $y_1, \dots y_n$ such that $K \bigcup_{i=1}^n V_i = V$, where V_i is a neighborhood of y_i for $1 \leq i \leq n$. Let $U = \bigcap_{i=1}^n U_i$, then $V \cap W$ is empty, hence $UX \backslash V$, therefore, $x \in X \backslash K$, therefore K is closed. ■

Theorem 2.3.4. *Closed subsets of compact sets are compact.*

Proof. Let X be a metric space with $F \subseteq KX$, with F closed in X, and K compact. Let $\{V_\alpha\}$ be an open cover of F. If we append $X \backslash F$ to $\{V_\alpha\}$, we get an open cover Θ of K, and since K is compact, there is a finite subcollection Φ which covers K, so Φ is an open cover of F, $X \backslash F \Phi$, then $\Phi \backslash (X \backslash F)$ still covers F, therfore F is compact. ■

Theorem 2.3.5. *Let $\{K_\alpha\}$ be a collection of compact sets of a metric space X, such that every finite subcollection of $\{K_\alpha\}$ is nonempty. Then $\bigcap K_\alpha$ is nonempty.*

Proof. Fix $K_1 \subseteq \{K_\alpha\}$, and let $G_\alpha = X \backslash K_\alpha$. Suppose no point of K_1 is in $\bigcap K_\alpha$, then $\{G_\alpha\}$ covers K_1, and since K is compact, we have $K_1 \bigcup_{i=1}^n G_{\alpha_i}$, for $1 \leq i \leq n$, which implies that $\bigcap K_\alpha$ is empty, a contradiction. ■

Corollary. *If $\{K_\alpha\}$ is a sequence of nonempty compact sets, such that $K_{n+1} \subseteq K_n$, then $\bigcap_{i=1}^\infty K_n$ is nonempty.*

Theorem 2.3.6. *If E is a infinite subset of a compact set K, then E has a limit point in K.*

Proof. Suppose no point of K is a limit point of E, then for all $x \in K$, the neighborhood U_x contains atmost one point in E. Then no finite subcollection of $\{U_x\}$ covers E, which contradicts the compactness on K. ■

Theorem 2.3.7 (The Nested Interval Theorem)**.** *if $\{I_n\}$ is a sequence of intervals in $\mathbb{R}$ such that $I_{n+1} \subseteq I_n$, then $\bigcap_{j=1}^{\infty} I_n$ is nonempty.*

Proof. We let $I_n = [a_n, b_n]$. Letting E be the set of all a_n, E is nonempty and bounded above by b_1. Letting $x = \sup E$, and $m \geq n$, we have $[a_m, b_m] \subseteq [a_n, b_n]$, thus $a_m \leq x \leq b_m$ for all m, thus $x \in I_m = \bigcap_{j=i}^{n} I_j$ ■

Theorem 2.3.8. *Let $k \in \mathbb{Z}^+$, and $\{I_n\}$ be a nonempty sequence of k-cells of $\mathbb{R}^k$ such that $I_{n+1} I_n$. Then $\bigcap_{j=1}^{\infty} I_n$ is nonempty.*

Proof. Let I_n be the set of all points $x \in \mathbb{R}^k$ such that $a_{n,j} \leq x_j \leq b_{n,j}$, and let $I_{n,j} = [a_{n,j}, b_{n,j}]$. Then for each $1 \leq j \leq k$, by the nested interval theorem, $\bigcap_{l=1}^{\infty} I_{l,j}$ is nonempty, hence there are real numbers x'_j such that $a_{n,j} \leq x'_j \leq b_{n,j}$. Letting $x' = (x'_1, \ldots, x'_k)$, we get that $x' \in I \bigcap_{l=1}^{\infty} I_l$ ■

Theorem 2.3.9. *Every k-cell is compact.*

Proof. Let I be a k-cell, and let $\delta = \sqrt{\sum_{j=1}^{k} a(b_j - a_j)^2}$ we get for $x, y \in I$, $||x - y|| \leq \delta$. Now suppose there is an open cover $\{G_\alpha\}$ of I for which no finite subcover is contained. Let $c_j = \frac{a_j + b_j}{2}$, then the closed intervals $[a_j, c_j]$, $[c_j, b_j]$ determine the 2^k k-cells Q_i such that $\bigcup Q_i = I$. Then atleast one Q_i cannot be covered by any finite subcollectio of $\{G_\alpha\}$. Subdividing Q_1, we get a sequence $\{Q_n\}$ such that $Q_{n+1} \subseteq Q_n$, Q_n is not covered by any finite subcollection of $\{G_\alpha\}$, and $||x - y|| \leq \frac{\delta}{2^n}$ for $x, y \in Q_n$. Then by theorem 2.3.8, there is a point $x' \in Q_n$, and for some α, $x' \in G_\alpha$; since G_α is open, there is an $r > 0$ for which $||x - || < r$ implies $y \in G_\alpha$. THen for n sufficiently large, we have that $\frac{\delta}{2^n} < r$, then we get that $Q_n \in G_\alpha$, which is a contradiction. ■

Theorem 2.3.10 (The Heine-Borel Theorem)**.** *If E is a subset of $\mathbb{R}^k$, then the following are equivalent:*

(1) E is closed and bounded.

(2) E is compact.

(3) Every infinite subset of E has a limit point in E.

Proof. Suppose that E is closed and bounded, then $E \subseteq I$ for some k-cell I in $\mathbb{R}^k$, and hence it is compact. By theorem **??**, E is compact. Now suppose that E is compact, then by theorem 2.3.6, every infinite subset of E has a limit point in E.

Now suppose that every infinite subset of E has a limit point in E. If E is not bounded, then $||x_n|| > n$ for some $x_n \in E$ and $n \in \mathbb{Z}^+$. Then the set of all such x_n is infinite, and has no limit point in E, a contradiction; moreover suppose that E is not closed. Then there is a point $x_0 \in \mathbb{R}^k \backslash E$, which is a limit point of E. Then there are points $x_n \in E$ for which $||x_n - x_0|| < \frac{1}{n}$, let S be the set of all such points. Then S is infinite and has x_0 as its only limit point; for if $y \neq x_0 \in \mathbb{R}^k$, then $\frac{1}{2}||x_0 - y|| \leq ||x_0 - y|| - \frac{1}{n} \leq ||x_0 - y|| - ||x_n - x_0|| \leq ||x_n - y||$ for only some n. Thus by theorem 2.2.3, y is not a limit point of S Therefore, if every infinite subset of E has a limit point in E, E must be closed. ■

Theorem 2.3.11 (The Bolzano-Weierstrass Theorem). *Every bounded infinite subset E of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.*

Proof. We have that $E \subseteq I$, for some k-cell I in $\mathbb{R}^k$. Since k-cells are compact, by the Heine-Borel theorem, E is also compact and has a limit point in I. ■

2.4 Perfect Sets

Theorem 2.4.1. *If $P \subseteq \mathbb{R}^k$ is a nonempty perfect set, then P is uncountable.*

Proof. Since every point of P is a limit point of P, we gave that P must be infinite. Then suppose that P is countable. For points $x_n \in P$, construct the sequence $\{U_n\}$ of neighborhoods of x_n, for $n \in \mathbb{Z}^+$; now by induction, if U_1 is a neighborhood of x_1, then for $y \in \hat{U_1}$, $||x_1 - y|| \leq r$ for some $r > 0$. Now suppos the neighborhood U_n of x_n has been constructed such that $U_n \cap P$ is nonempty. Then there is a neighborhood U_{n+1} fo x_{n+1} such that $\hat{U_{n+1}} \subseteq U_n$, $x_n \notin \hat{U_{n+1}}$, and $U_{n+1} \cap P$ is nonempty. Therefore there is a nonempty $K_n = U_n \cap P$. Since $\hat{U_n}$ is close and bounded, $\hat{U}$ is compact, and since $x_n \notin K_{n+1}$, $x_n \notin \bigcap_{i=1}^{\infty} K_i$, and since $K_n \subseteq P$, $\bigcap K_i$ is empty, a contradiction. ■

Corollary. *Let $a < b$ be real numbers. Then the closed interval $[a, b]$ is uncountable. Moreover, $\mathbb{R}$ is uncountable.*

Proof. We have $[a, b]$ is closed, and perfect (since $(a, b)[a, b] isperfect$), thus $[a, b]$ is uncountable. Moreover, take $f : \mathbb{R} \to [a, b]$, by $f(x) = \frac{a+b}{2}x$; then f is a 1-1 mapping of $\mathbb{R}$ onto $[a, b]$, which makes $\mathbb{R}$ uncountable. ■

Theorem 2.4.2 (The construction of the Cantor set). *There exists a perfect set in $\mathbb{R}$ which contains no open interval.*

Proof. Let $E_0 = [0, 1]$, and remove $(\frac{1}{3}, \frac{2}{3})$, and let $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Now remove the open intervals $(\frac{1}{9}, \frac{2}{9})$ $(\frac{3}{9}, \frac{6}{9})$, $(\frac{7}{9}, \frac{8}{9})$, and let $E_2 = [0, \frac{2}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{7}{8}, \frac{8}{9}]$. Continuig the remove the middle third of each interval, we obtain the sequence of compact sets $\{E_n\}$, such that $E_{n+1}E_n$, and E_n is the union of 2^n closed intervals of length $\frac{1}{3^n}$. Then let:

$$P = \bigcap_{i=1}^{\infty} E_i \tag{2.1}$$

Then P is nonempty, and compact.

Now let I be the open interval of the form $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})$, with $k, m \in \mathbb{Z}^+$. Then by the construction of P, I has no point in P, we also see that every other open interval contains a subinterval of the form of I; them P contains no open interval.

Now let $x \in P$, and let S be any open interval for which $x \in S$. LEt I_n be the closed interval of E_n such that $x \in I_n$. Choose n sufficiently large such that $I_n S$. If $x_n \neq x$ is an endpoint of I_n, then $x_n \in P$, and so x is a limit point of P. Therefore P is perfect. ■

Definition. The we call the set P constructed in the proof of theorem 2.4.2 the **Cantor set**.

2.5 Connected Sets

Definition. Two subsets A and B of a metric space X are **seperated** if $A \cap \hat{B}$ and $\hat{A} \cap B$ are both empty. We say a subset E of X is **connected**, if E is not the union of two nonepmty speperated sets.

Theorem 2.5.1. *A subset E of $\mathbb{R}$ is connected if and only if $x, y \in E$ and $x < z < y$ imply $z \in E$.*

Proof. Let $x, y \in E$ such that for some $z \in (x, y))$, $z \notin E$. Then $E = A \cup B$, with $A = E \cup (-\infty, z)$ and $B = E \cup (z, \infty)$. Then A and B are seperated, which contradicts the connectedness of E.

Conversely suppose for $x, y \in E$, that $z \in E$ for $z \in (x, y)$. Then there are nonempty seperated sets A and B such that $A \cup B = E$. Choose $x \in A$, $y \in B$ such that $x < y$, and let $z = \sup(A \cap [x, y])$. Then by theorem 2.2.7, $z \in \hat{A}$, so $z notinB$. In particular, $x \leq x < y$. Now if $z \notin A$, then $x < z < y$, with $z \notin E$. Now if $z \in A$, then $z \notin \hat{B}$, hence there is a z' such that $z < z' < y$, and $z' \notin B$. Then $x < z' < y$ and $z' \not inE$. ∎

Chapter 3

Sequences

3.1 Convergent Sequences

Definition. A sequence $\{x_n\}$ in a metric space X is said to **converge** if there is a point $x \in X$ such that for every $\epsilon > 0$, there is an $N \in \mathbb{Z}^+$ such that $d(x_n, x) < \epsilon$ whenever $n \geq N$. We say $\{x_n\}$ **converges** to x, and we call x the **limit** of $\{x_n\}$ as n approaches ∞. We write $x_n \to x$ as $n \to \infty$, and $\lim_{n\to\infty} x_n = x$ (or $\lim x_n = x$). If $\{x_n\}$ does not converge, we say the $\{x_n\}$ **diverges**, or **is divergent**.

Example 3.1. Consider the following sequences in $\mathbb{C}$.

(1) $\{\frac{1}{n}\}$ is bounded, and $\lim_{n\to\infty} \frac{1}{n} = 0$.

(2) The sequence $\{n^2\}$ us unbounded and diverges.

(3) $1 + \frac{(-1)^n}{n} \to 1$ as $n \to \infty$, and $\{1 + \frac{(-1)^n}{n}\}$ is bounded.

(4) $\{i^n\}$ is bounded and divergent.

(5) $\{1\}$ is bounded and converges to 1.

Theorem 3.1.1. *Let $\{x_n\}$ be a sequence in a metric space, then:*

(1) $\{x_n\}$ converges to $x \in X$ if and only if every every neighborhood of x contains x_n for all but finitely many n.

(2) If $\{x_n\}$ converges to x, and x', then $x = x'$.

(3) If $\{x_n\}$ converges, then x_n is bounded.

(4) If $E \subseteq X$, and x is a limit point of E, then there is a sequence in E that converges to x.

Proof. Suppose $x_n \to x$, and let U be a neighborhood of x. For some $\epsilon > 0$, there is an $N \in \mathbb{Z}^+$ for which $d(x_n, x) < \epsilon$, whenever $n \geq N$, thus $x_n \in U$ for finitely many n. Conversely, suppose that $x_n \in U$ for some $n \geq N$, then letting $\epsilon > 0$, we havae $d(x, x_n) < \epsilon$, hence $x_n \to x$.

Let > 0, then there are $N_1, N_2 \in Z^+$ such that $d(x_n, x) < \frac{\epsilon}{2}$, and $d(x_n, x') < \frac{\epsilon}{2}$. Then choosing $N = \max\{N_1, N_2\}$, and letting ϵ be arbitrarily small, we have $d(x, x') \leq d(x, x_n) + d(x_n, x') < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$; and so we get that $x = x'$.

Let $x_n \to x$, then there is an $N \in \mathbb{Z}^+$ for which $d(x_n, x) < 1$ whenever $n \geq N$. Letting $r = \max\{1, d(x_N, x)\}$, then $d(x_n, x) \leq r$.

Finally, let x be a limit point of E, then for each $n \in Z^+$, there is an $x_n \in E$ such that $d(x, x_n) < \frac{1}{n}$, choose $N > \frac{1}{\epsilon}$, then whenever $n\ geqN$, $d(x, x_n) < \epsilon$; hence $x_n \to x$. ∎

Theorem 3.1.2. *Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathbb{C}$, and that $limx_n = x$, $\lim y_n = y$ as $n \to \infty$. Then the following hold as $n \to \infty$:*

(1) $\lim{(x_n + y_n)} = \lim x_n + \lim y_n = x + y$.

(2) $limx_n y_n = \lim x_n \lim y_n = xy$.

(3) $lim\frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n} = \frac{x}{y}$*; given that* $y_n, y \neq 0$.

Proof. (1) Let > 0, then for $N_1, N_2 \in \mathbb{Z}^+$, $|x_n - x| < \frac{\epsilon}{2}$ and $|y_n - y| < \frac{\epsilon}{2}$. Then choose $N = \max\{N_1, N_2\}$, then whenever $n \geq N$, we have $|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \epsilon$.

(2) Notice that $x_n y_n - xy = (x_n - x)(y_n - y) + x(y_n - y) + y(x_x - x)$, then for $N_1, N_2 \in \mathbb{Z}^+$, $|x_n - x| < \sqrt{\epsilon}$, and $|y_n - y| < \sqrt{\epsilon}$. Then choosing $N = \max\{N_1, N_2\}$, then $|(x_n - x)(y_n - y)| < \epsilon$, thus we have $|x_n y_n - xy| \leq |(x_n - x)(y_n - y)| + |x(y_n - y)| + |y(x_x - x)| < \epsilon$.

(3) We first show that $\frac{1}{y_n} \to \frac{1}{y}$, given that $y_n, y \neq 0$. Choose m such that $|y_n - y| < \frac{1}{2}|y|$ whenever $n \geq m$, then $|y_n| > \frac{1}{2}|y|$. Then for $\epsilon > 0$, there is an $N > m$ such that whenever $n \geq N$, $|y_n - y| < \frac{1}{2}|y|^2\epsilon$. Then $|\frac{1}{y_n} - \frac{1}{y}| \leq \frac{|y_n - y|}{|y_n y|} < \frac{2}{|y|^2}|y_n - y| < \epsilon$. Then choosing the sequences $\{x_n\}$ and $\{\frac{1}{y_n}\}$, the rest follows.

∎

Corollary. *(1) For any $c \in \mathbb{C}$, and a sequene $x_n \to x$, we have $\lim cx_n = c \lim x_n = cx$ and $\lim{(c + x_n)} = c + \lim x_n = c + x$ as $n \to \infty$.*

(2) Provided that $x, x_n \neq 0$, we have $\lim \frac{1}{x_n} = \frac{1}{\lim x_n} = \frac{1}{x}$, as $n \to \infty$.

Proof. We choose $\{x_n\}$ and $\{y_n\} = \{c\}$ for all n, then the results follow. ∎

Theorem 3.1.3. *(1) Let $x_n = (\alpha_{1n}, \ldots \alpha_{kn}) \in \mathbb{R}^k$. Then $\{x_n\}$ converges to x if and only if $\lim \alpha_{jn} = \alpha_j$ for $1 \leq j \leq k$, as $n \to \infty$.*

(2) Let $\{x_n\}, \{y_n\}$ be sequences in $\mathbb{R}^k$, and let $\{\beta_n\}$ be a sequence in $\mathbb{R}$ such that $x_n \to x$, $y_n \to y$, and $\beta_n \to \beta$. Then $\lim{(x_n + y_n)} = x + y$, $\lim x_n y_n = xy$, and $lim\beta_n x_n = \beta x$.

Proof. If $x_n \to x$, then $|\alpha_{jn} - \alpha_j| \leq ||x_n - x|| < \epsilon$, thus $\lim \alpha_{jn} = \alpha_j$. Conversely, suppose that $\alpha_{jn} \to \alpha_j$. Then for $\epsilon > 0$ there is an $N \in \mathbb{Z}^+$ such that $n \geq N$ implies $|\alpha_{jn} - \alpha_j| < \frac{\epsilon}{\sqrt{k}}$. Then for $n \geq N$,

$$||x_n - x|| = \sqrt{\sum |\alpha_{jn} - \alpha_j|^2} < epsilon$$

To prove (2), we appy part (1) of this theorem together with theorem 3.1.2. ∎

Theorem 3.1.4 (The Sandwhich Theorem)**.** *Let $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ be sequences in $\mathbb{R}$, and Suppose that $\lim x_n = \lim y_n = a$ and that there is an $N \in \mathbb{Z}^+$ such hat $x_n \leq w_n \leq y_n$ for all $n \geq N$. Then $\lim_{n\to\infty} w_n = a$.*

Proof. Let $\epsilon > 0$ and let $\{x_n\}$ and $\{y_n\}$ both converge to a. Then by definition there are $N_1, N_2 \in \mathbb{Z}^+$ such that $|x_n - a| < \epsilon$ and $|y_n - a| < \epsilon$ for $n \geq N_1, N_2$. Now choose $N = \max\{N_0, N_1, N_2\}$, if $n \geq N$, we have $-\epsilon < x_n - a < \epsilon$, and we also have $x_n - a < w_n - a < y_n - a$, thus we have that:

$$-\epsilon < x_n - a < w_n - a < y_n - a < \epsilon$$

Thus we have that $|w_n - a| < \epsilon$. ■

Corollary. *If $x_n \to \infty$ as $n \to \infty$, and $\{y_n\}$ is bounded, then $x_n y_n \to 0$ as $n \to \infty$.*

Proof. We have that $\{y_n\}$ is bounded, hence, there is $M > 0$ such that $|y_n| < M$ for all $n \in \mathbb{Z}^+$. And since $\{x_n\}$ converges to 0 we have that for any ϵ there is an $N \in \mathbb{Z}^+$ such that for $n \geq N$, $|x_n - 0| < \frac{\epsilon}{M}$. For $|x_n y_n - 0| = |x_n y_n| < M|x_n| < M\frac{\epsilon}{M} = \epsilon$. Therefore, $x_n y_n \to 0$ as $n \to \infty$. ■

Corollary. *Let $\{x_n\}$, $\{y_n\}$ be sequences such that $0 \leq x_n \leq y_n$ for $n \geq N > 0$. Then if $y_n \to 0$, then $x_n \to 0$ as $n\infty$.*

Proof. This is a special case of the sandwhich theorem. ■

3.2 Subsequences

Definition. Let $\{x_n\}$ be a sequence, and let $\{n_k\}\mathbb{Z}^+$ such that $n_k < n_{k+1}$. We call the sequence $\{x_{n_k}\}$ a **Subsequence** of $\{x_n\}$. If $\{x_{n_k}\}$ converges, we call its limit the **subsequential limit** of $\{x_n\}$.

Theorem 3.2.1. *A sequence $\{x_n\}$ converges to a point x if and only if every subsequence $\{x_{n_k}\}$ converges to x.*

Proof. Clearly if $x_n \to x$, then every subsequence $x_{n_k} \to x$, (since subsequences can be thought of as subsets of thier parent sequences). On the other hand, let $x_{n_k} \to x$ for $\{k\} \subseteq \mathbb{Z}^+$. Then for $\epsilon > 0$, there is a $K \in \mathbb{Z}^+$ for which $d(x_{n_k}, x) < \frac{\epsilon}{2}$ for $k \geq K$. Let $N \in \mathbb{Z}^+$, and choose $n \geq max\{N, K\}$, then $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, d) < \epsilon$. ■

Theorem 3.2.2. *If $\{x_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{x_n\}$ converges to a point x.*

Proof. If $\{x_n\}$ is finite, then thre is an $x \in \{x_n\}$ and a sequence $\{n_k\}$ with $n_k < n_{k+1}$ such that $x_{n_i} = x$ for $1 \leq i \leq k$, then the subsequence converges to x.

Now if $\{x_n\}$ is infinite, there is a limit point $x \in X$ of $\{x_n\}$, then choose n_i such that $d(x, x_i) < \frac{1}{i}$ for $1 \leq i \leq k$. Obtaining $\{n_k\}$ from this, we see that $n_k < n_{k+1}$, and so we get that $\{x_{n_k}\}$ converges to x. ■

Corollary. *Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.*

Theorem 3.2.3. *The subsequential limits of $\{x_n\}$ is a metric space X form a closed subset of X.*

Proof. Let E be the set of all subsequential limits of $\{x_n\}$, and let x be a limit point of E. Choose n_i such that $x_{n_i} \neq x$ and let $\delta = d(x, x_{n_i})$, for $1 \leq i \leq k$. Then consier the sequence $\{n_k\}$, since x is a limit point of E, there is an $x' \in E$ for which $d(x, x') < \frac{\delta}{2^i}$. Thus there is an $N_I > n_i$ such that $d(x', x_{n_i}) < \frac{\delta}{2^i}$, thus $d(x, x_{n_i}) < \frac{\delta}{2^i}$. So $\{x_n\}$ converges to x and $x \in E$. ∎

3.3 Cauchy Sequences

Definition. We call a sequence $\{x_n\}$ in a metric space X a **Cauchy sequence** in X, or more simply, **Cauchy** in X if for all $\epsilon > 0$, there is an $N \in \mathbb{Z}^+$ such that $d(x_n, x_m) < \epsilon$ whenever $m, n \geq N$.

Definition. Let E be a nonempty subset of a metrix space X, and lelt $S \subseteq \mathbb{R}$ be the all real numbers $d(x, y)$, with $x, y \in E$. We call $\sup S$ the **diameter** of E, and denote it $\operatorname{diam} E$.

Theorem 3.3.1. *Let $\{x_n\}$ be a sequence, and let E_N be the set of all points p_N such that ${}_N < p_{n+1}$. Then $\{x_n\}$ is Cauchy if and only if $\lim \operatorname{diam} E_N = 0$ as $N \to \infty$.*

Proof. Let $\{x_n\}$ be Cauchy, Let $x_{N_1}, x_{N_2} \in E$ such that $d(x_n, x_{N_1}) < \frac{\epsilon}{2}$, and $d(x_{N_2}, x_m) < \frac{\epsilon}{2}$. Then we see that $d(x_{N_1}, x_{N_2}) \leq d(x_{N_1}, x_n) + d(x_m, x_{N_2}) < \epsilon$, so $\{x_{N_k}\}$ is Cauchy and we see that $\lim \operatorname{diam} E_N = 0$. Now suppose that $\lim \operatorname{diam} E = 0$, then for any $x_n, x_m \in S$, $d(x_n, 0) < \frac{\epsilon}{2}$ and $d(0, x_m) < \frac{\epsilon}{2}$ implies that $d(x_n, x_m) \leq d(x_n, 0) + d(0, x_m) < \epsilon$, whenever $n, m > N$, for $\epsilon > 0$. ∎

Theorem 3.3.2. *(1) If $E \subseteq X$, then $\operatorname{diam} \hat{E} = \operatorname{diam} E$.*

(2) If $\{K_n\}$ is a sequence of compact sets in X, such that $K_{n+1} \subseteq K_n$, and if $\lim \operatorname{diam} K_n = 0$ as $n \to \infty$, then $\bigcap_{i=1}^{infty} K_i$ contains exactly one point.

Proof. Clearly $\operatorname{diam} E \leq \operatorname{diam} \hat{E}$. Now let $\epsilon > 0$, and choose $x, y \in \hat{E}$, then there are points $x', y' \in \hat{E}$ such that $d(x, x') < \frac{\epsilon}{2}$ and $d(y, y') < \frac{\epsilon}{2}$. Hence, $d(x, y) \leq d(x, x') + d(x', y') + d(y'y) < \epsilon \operatorname{diam} E$, then choosing ϵ arbitrarily small, $\operatorname{diam} \hat{E} \leq \operatorname{diam} E$.

Now, we also have that by the nested interval theorem that $K = \bigcap K_i$ is nonempty. Now suppose that K contains more that one point. then $\operatorname{diam} K > 0$, and since $K \subseteq K_n$ for all n, $diam K \leq \operatorname{diam} K_n$, a contradiction. Thus K contains exactly one element. ∎

Theorem 3.3.3. *(1) In any metric space X, every convergent sequence is a Cauchy sequence.*

(2) If X is compact, and $\{x_n\}$ is Cauchy in X, then $\{x_n\}$ converges to a point in X.

Proof. (1) If $x_n \to x$, and $\epsilon > 0$ such that there is an $N \in \mathbb{Z}^+$ such that $d(x_n, x) < \frac{\epsilon}{2}$ for all $n \geq N$, then for $m \geq N$, we have $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \epsilon$. Thus $\{x_n\}$ is Cauchy.

(2) Let $\{x_n\}$ be Cauchy, and let E_N be the set of all points x_N for which $x_N < x_{N+1}$. Then $\lim \operatorname{diam} \hat{E} = 0$, then being closed in X, each $\hat{E_N}$ is compact in X, and $\hat{E_{N+1}} \subseteq \hat{E_N}$, so by theorem 3.3.2, there is a unique $x \in X$ in all of $\hat{E_N}$. Now for $\epsilon > 0$, there is an $N_0 \in \mathbb{Z}^+$ for which $\operatorname{diam} \hat{E} < \epsilon$. Then for all $x_n \in \hat{E}$, $d(x_n, x) < \epsilon$ whenever $n \geq N_0$. ∎

Corollary (The Cauchcy Criterion)**.** *Every Cauchy sequence in $\mathbb{R}^k$ converges to a point in $\mathbb{R}^k$.*

Proof. Let $\{x_n\}$ be Cauchy in $\mathbb{R}^k$, define E_N as in (2), then for some $N \in \mathbb{Z}^+$, $\operatorname{diam} E < 1$, and so $\{x_n\}$ us the union of all E_n, and ther set of points $\{x_1, \dots, x_{N-1}\}$, so $\{x_n\}$ is bounded, and thus has a compact closure, it follows then that $x_n \to x$ for some $x \in \mathbb{R}^k$. ∎

Definition. We call a metric space **complete** if every Cauchy sequence in the space converges.

Theorem 3.3.4. *All compact metric spaces, and all Euclidean spaces are complete.*

Example 3.2. Consider $\mathbb{Q}$ together with the metric $|x - y|$. The metric space induced on $\mathbb{Q}$ by $|\cdot|$ is not complete.

Definition. A sequence $\{x_n\}$ in $\mathbb{R}$ is said to be **monotonically increasing** if $x_n \leq x_{n+1}$, $\{x_n\}$ is said to be **monotonically decreasing** if $x_{n+1} < x_n$. We call $\{x_n\}$ **monotonic** if it is either monotonically increasing or monotonically decreasing.

Theorem 3.3.5. *A monotonic sequence converges if and only if it is bounded.*

Proof. Suppose, without loss of generality, that $\{x_n\}$ is monotonically increasing. If $\{x_n\}$ is bounded, then $x_n \leq x$, then for all $\epsilon > 0$, there is an $N \in \mathbb{Z}^+$ for which $x - \epsilon < x_N \leq x$. Then for $n \geq N$, $x_n \to x$. The converse follows from theorem 3.1.2. ∎

3.4 Upper and Loweer Limits.

Let $\{_n\}$ be a sequence in $\mathbb{R}$ such that for all $M > 0$, there is an $N \in \mathbb{Z}^+$ for which $n \geq N$ implies that either $x_n \geq M$, or $x_n \leq M$. Then we write $x_n \to \infty$ and $x_n \to -\infty$, respectively.

Definition. Let $\{x_n\}$ be a sequence in $\mathbb{R}$, and let E tbe the set of all extended real numbers x such that $x_{n_k} \to x$ for some subseqence $\{x_{n_k}\}$. Then E contains all subsequential limits of $\{x_n\}$, and possible $\pm\infty$. We then call $\sup E$ the **upper limit** of E, and $\inf E$ the **lower limit** of E.

Theorem 3.4.1. *Let $\{x_n\}$ be a sequence in $\mathbb{R}$, and let E be the set of all extended real numbers x, let $s = \sup E$ and $s' = \inf E$. Then the following hold:*

(1) s,s' $\in E$.

(2) If $x > s$, and $x' > s'$, there is an $N \in \mathbb{Z}^+$ such that $n \geq N$ implies that $x' < x_n < x$.

Proof. We prove the theorem for the case of s, since it is analogous for s'.

(1) If $s = \infty$, then E is not bounded above, so neither is $\{x_n\}$, and there is a subsequence for which $x_n \to \infty$. Now if $s \in \mathbb{R}$, then E is bounded above, and has at least one subsequential limit. Then $s \in E$. Now if $s = -\infty$, then E contains only $-\infty$, and so by definition $x_n \to -\infty$.

(2) Suppose there is an $x > s$, such that $x_n \geq x$ for all n. Then there is a $y \in E$ such that $y \geq x \geq s$, a contradiction of the definition of s.

■

Example 3.3. (1) Let $\{x_n\}$ be a sequence in $\mathbb{Q}$, then every real number is a subsequential limit, and $\limsup x_n = \infty$ and $\liminf x_n = -\infty$.

(2) Let $\{x_n\} = \{\frac{(-1)^n}{1+\frac{1}{n}}\}$; then $\limsup x_n = 1$ and $\liminf X_n = -1$ as $n \to \infty$.

(3) For a sequence $\{x_n\}$ in $\mathbb{R}$, $\lim x_n = x$ if and only if $\limsup x_n = \liminf x_n = x$ as $n \to \infty$.

Theorem 3.4.2. *If $x_n \leq y_n$, for $n \geq N > 0$, then $\liminf x_n \leq \liminf y_n$ and $\limsup x_n \leq \limsup y_n$ as $n \to \infty$.*

3.5 Special Sequences

Theorem 3.5.1. *Let $n, p \in \mathbb{Z}^+$. Then the following hold as $n \to \infty$.*

(1) $\lim \frac{1}{n^p} = 0$.

(2) $\lim \sqrt[p]{n} = 1$.

(3) $\lim \sqrt[n]{n} = 1$.

(4) If $\alpha \in \mathbb{R}$, then $\lim \frac{n^\alpha}{(1+p)^n} = 0$.

(5) If $|x| < 1$, then $\lim x^n = 0$.

Proof. (1) Let $n > [p]\frac{1}{\epsilon}$; then $|\frac{1}{n^p}| < \epsilon$.

(2) If $p = 1$, we are done. If $p > 1$, let $x_n = \sqrt[n]{p} - 1$, then $x_n > 0$. By the binomial theorem, $1 + nx_n \leq (1 + x_n)^p = p$, hence $0 \leq x_n \leq \frac{p-1}{p}$. Now if $1 > p > 0$, then $\frac{1}{p} > 0$, so we notice that $0 \leq \frac{1}{x_n} \leq \frac{1}{\frac{p-1}{n}}$.

(3) Let $x_n = \sqrt[n]{n} - 1$, then $x_n \geq 0$, then by the binomial theorem again, $n = (1 + x_n)^n \geq \frac{n(n-1)}{2}x_n^2$, then $0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$.

(4) Let $k \in \mathbb{Z}^+$ such that $k > \alpha$. Then $n > 2k$,let $(1+p)^n > \binom{n}{k}p^k > \frac{n^k p^k}{2^k k!}$. So $0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k}n^{-k}$, since $\alpha - k < 0$, $n^{\alpha-k} \to 0$ and we are done.

(5) Take $\alpha = 0$, and let $x = \frac{1}{1+p}$, then the result follow.

■

Chapter 4

Continuity

4.1 Limits of Functions.