Analysis

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<u>Text</u>

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Chapter 1

The Real and Complex Numbers

1.1 Ordered Sets

Definition. Let S be any set. An **order** on S is a relation $<$ such that:

(1) For $x, y \in S$, one and only one of the following hold:

$$x < y \qquad\qquad x = y \qquad\qquad y < x$$

We call this property the **trichotomy law**

(2) $<$ is transitive over S.

We denote the relations $>$ and $\leq$ to mean $x > y$ if and only if $y < x$, and $x \leq y$ if and only if $x < y$, or $x = y$. We call S together with $<$ an **ordered set**.

Example 1.1. Define $<$ on $\mathbb{Q}$ such that for $r, s \in \mathbb{Q}$, $r < s$ implies $< 0s - r$.

Definition. Let S be an ordered set, and let $E \subseteq S$. We say that E is **bounded above** there is some $\beta \in S$ for which $x \leq \beta$, for all $x \in E$. We say that E is **bounded below** if $\beta \leq x$, for call $x \in E$. We say an $\alpha \in S$ is a **least upperbound** of E, if α is an upperbound of E, and for all other upperbounds, γ, of E, $\alpha \leq \gamma$. Likewise, α is a **greatest lowerbound** of E if α is a lowerbound of E, and for all other lowerbounds γ of E, $\gamma \leq \alpha$. We denote the least upperbound, and greatest lowerbound by $\sup E$ and $\inf E$, respectively.

Lemma 1.1.1. *Let S be an ordered set, and let $E \subseteq S$. Then E has (if they exist) a unique least upperbound, and a unique greatest lowerbound.*

Proof. Let $\alpha, \beta \in S$ be least upperbounds of E. Then by definition, we have that $\alpha \leq \beta$, and $\beta \leq \alpha$; thus by the trichotomy law, $\alpha = \beta$. The proof is the same for greatest lowerbounds. ■

Example 1.2. (1) Let $A = \{p \in \mathbb{Q} : p^2 < 2\}$, and $B = \{p \in \mathbb{Q} : p^2 > 2\}$. Clearly, we have that every element of B is an upperbound of A, and every element of A is a lowerbound of B. Now take $p \in \mathbb{Q}$ a positive rational, and take $q \in \mathbb{Q}$ such that $q = p - \frac{p^2-2}{p+2}$. Then

$q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2}$. Now if $p \in A$, then $p^2 - 2 < 0$, which implies that $p < q$, and $q^2 < 2$; thus A has no largest element; similarly, if $p \in B$, then $p^2 - 2 > 0$, which implies that $q < p$ and $q^2 > 2$, which shows that B has no least element. Thus $\sup A$ and $\inf B$ do not exist in $\mathbb{Q}$.

(2) If $\alpha = \sup E \in S$, it may or may not be that $\alpha \in E$. Take $E_1 = \{r \in \mathbb{Q} : r < 0\}$, and $E_2 = \{r \in \mathbb{Q} : r \leq 0\}$. Then $\sup E_1 = \sup E_2 = 0$, but $0 \notin E_1$, where as $0 \in E_2$

(3) Consider the set $\frac{1}{\mathbb{Z}^+} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$. By the well ordering principle, 1 is the least element, and is also an upperbound of all $\frac{1}{n}$ for $n > 1$. Now also notice that as n gets arbitrarily large, then $\frac{1}{n}$ gets arbitratirly small; that is to say $\frac{1}{n}$ "tends" to 0, so $\sup \frac{1}{\mathbb{Z}^+} = 1 \in \frac{1}{\mathbb{Z}^+}$, and $\inf \frac{1}{\mathbb{Z}^+} = 0 \notin \frac{1}{\mathbb{Z}^+}$.

Definition. We say an ordered set S has the **least upperbound property**, if whenever $E \subseteq S$, nonempty, and bounded above, then $\sup E \in S$ exists; likewise, S has the **greatest lowerbound property** if whenever E is nonempty, bounded below then $\inf E \in S$ exists.

Example 1.3. The set of all rationals $\mathbb{Q}$ does not have the least upperbound property, nor the greatest lowerbound property, take A, B as in the previous example. Letting $E = \{1, \frac{1}{2}, \frac{1}{4}\} \subseteq \frac{1}{\mathbb{Z}^+}$, we see that $\frac{1}{\mathbb{Z}^+}$ satisfies both properties, with $\sup E = 1$, and $\inf E = \frac{1}{4}$.

Theorem 1.1.2. *If S is an ordered set with the least upperbound property, then S also inherits the greatest lowerbound property.*

Proof. Let $B \subseteq S$, and let $L \subseteq S$ be the set of all lowerbounds of B. Then we have for any $y \in L$, $x \in B$, $y \leq x$. So every element of B is an upperbound of L, and L is nonempty, hence $\alpha = \sup L \in S$ exists. Now if $\gamma \leq \alpha$, then γ is not an upperbound of L, hence $\gamma \notin B$; thus $\alpha \leq x$ for all $x \in B$, so $\alpha \in L$, and by definition of the greatest lowerbound, we get $\alpha = \inf B$. ■

1.2 Fields

Definition. A **field** is a set F, together with binary operations $+$ and $\cdot$ (called **addition** and **multiplication**, respectively) such that:

(1) F forms an abelian group under $+$.

(2) $F \backslash \{0\}$ forms an abelian group under $\cdot$ (where 0 is the additive identity of F).

(3) $\cdot$ distributes over $+$.

We now state the following propositions without proof.

Proposition 1.2.1. *For all $x, y, x \in F$:*

(1) $x + y = x + y$ implies $y = z$

(2) $x + y = x$ implies $y = 0$

(3) $x + y = 0$ *implies* $y = -x$

(4) $-(-x) = x$.

Proposition 1.2.2. *For all* $x, y, x \in F \backslash \{0\}$*:*

(1) $xy = xy$ *implies* $y = z$

(2) $xy = x$ *implies* $y = 1$

(3) $xy = 1$ *implies* $y = x^{-1}$

(4) $(x^{-1})^{-1} = x$.

Proposition 1.2.3. *For all* $x, y, x \in F$*:*

(1) $0x = 0$

(2) $x \neq 0$ *and* $y \neq 0$ *implies* $xy \neq 0$

(3) $(-x)y = -(xy) = x(-y)$

(4) $(-x)(-y) = xy$.

Definition. An **ordered field** is a field F that is also an ordered set, such that:

(1) $x + y < x + z$ whenever $y < z$, for $x, yz, z \in F$

(2) $xy > 0$ whenever $x > 0$ and $y > 0$, for $x, y \in F$.

Proposition 1.2.4. *Let* F *be an ordered field, then for any* $x, y, z \in F$*, the following hold:*

(1) $x > 0$ *implies* $-x < 0$.

(2) If $x > 0$ *and* $y < z$*, then* $xy < xz$.

(3) If $x < 0$ *and* $y < z$*, then* $xz < xy$.

(4) If $x \neq 0$*, then* $x^2 > 0$*, in particular,* $1 > 0$.

(5) $0 < x < y$ *implies that* $0 < y^{-1} < x^{-1}$.

Proof. (1) If $x > 0$, then $0 = x + (-x) > 0 + (-x)$, so $-x < 0$.

(2) We have $0 < z - y$, so $0 < x(z - y) = xz - xy$, so $xy < xz$.

(3) Do the same as (2),, multiplying $z - y$ by $-x$.

(4) If $x > 0$, we are done. Now suppose that $x < 0$, then $-x > 0$, so $(-x)(-x) = xx = x^2 > 0$; in particular, we also have that $1 \neq 0$, and $1 = 1^2$, so $1 > 0$.

(5) We have $0 < xy^{-1} < yy^{-1} = 1$, then $0 < x^{-1}xy^{-1} = y^{-1} < x^{-1}1 = x^{-1}$

■

1.3 The Field of Real Numbers

Theorem 1.3.1. *There exists an ordered field $\mathbb{R}$ with the least upperbound property, such that $\mathbb{Q} \subseteq \mathbb{R}$.*

Definition. We call the field $\mathbb{R}$ the **field of real numbers**,and we call the elements of $\mathbb{R}$ **real numbers**.

Definition. Let S be an ordered field, and let $E \subseteq S$. We say that E is **dense** in S, if for all $r, s \in S$, with $r < s$, there is an $\alpha \in E$ such that $r < \alpha < s$.

Theorem 1.3.2 (The Archimedean Principle)**.** *If $x, y \in \mathbb{R}$, and $x > 0$, then there is an $n \in \mathbb{Z}^+$ such that $nx > y$.*

Proof. Let $A = \{nx : n \in \mathbb{Z}^+\}$, and suppose that $nx \leq y$. Then y is an upperbound of A, abd since A is nonempty, $\alpha = \sup A \in \mathbb{R}$, since $x > 0$, we have $\alpha - x < \alpha$, so $\alpha - x$ is not an upperbound of A. Hence $\alpha - x < mx$ for some $m \in \mathbb{Z}^+$. Then $\alpha < (1 - m)x \in A$, contradicting that α is an upperbound of A. ■

Theorem 1.3.3 (The density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

Proof. Let $x < y$ be realnumbers, then $y - x > 0$, so by the Archimedean principle, there is an $n \in \mathbb{Z}^+$ fir which $n(y-x) > 1$. By the Archimedean principle again, we have $m_1, m_2 \in \mathbb{Z}^+$ for which $m_1 > nx$ and $m_2 > -nx$, thus $-m_2 < nx < m_1$, and we also have that there is an $m \in \mathbb{Z}^+$ for which $-m_2 < m < m_1$, and $m - 1 \leq nx < m$. Thus combining inequalities, we get $nx < m < ny$, thus $x < \frac{m}{n} < y$. ■

Theorem 1.3.4 (The existence of n^th roots of positive reals)**.** *For every real number $X > 0$, and for every $n \in \mathbb{Z}^+$, there is one, and only one positive real number y for which $y^n = x$.*

Proof. Let $y > 0$ be a real number; then $y^n > 0$, so there is atmost one such y for which $y^n = x$. Now let $E = \{t : \mathbb{R} : t^n < x\}$, choosing $t = \frac{x}{1+x}$, we see that $0 \leq t < 1$, hence $t^n < t < x$, so E is nonempty. Now if $1 + x < t$, then $t^n \geq x$, so $t \notin E$, and E has $1 + x$ as an upperbound. Therefore, $\alpha = \sup E \in \mathbb{R}$ exists.

Now suppose that $y^n < x$, choose $0 \leq h < 1$ such that $h < \frac{x-y^n}{n(y+1)^{n-1}}$, then $(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)n - 1 < x - y^n$, thus $(y+h)^n < x$, so $y + h \in E$, contraditing that y is an upperbound. On the other hand, if $y^n > x$, choosing $k = \frac{y^n - x}{ny^{n-1}}$, then $0 \leq k < y$, and letting $t \geq y - k$, we get that $y^n - t^n \leq y^n + (y-k)^n < kny_{n-1} = y^n - x^n$, so $t^n \geq x$, making $y - k$ an uppeerbound of E, which contradicts $y = \sup E$. ■

Remark. We denote y as $\sqrt[n]{x}$, or as $x^{\frac{1}{n}}$.

Corollary. *If $a, b \in \mathbb{R}$, with $a, b > 0$, and $n \in \mathbb{Z}^+$, then $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$.*

Proof. Let $\alpha = \sqrt[n]{a}$, and $\beta = \sqrt[n]{b}$. Then $\alpha^n = a$, and $\beta^n = b$, so $ab = \alpha^n\beta^n = (l\alpha\beta)^n$, we are done. ■

Definition. We define the **extended real number system** to be the field $\mathbb{R}$, together with symbols ∞, and $-\infty$, called **positive infinity** and **negative infinity**, such that $-\infty < x < \infty$ for all $x \in \mathbb{R}$. We call elements of the extended real numbers **infinite**, and every other element not in the extended real numbers **finite**.

Lemma 1.3.5. ∞ *is an upperbound for every subset* E*, of* $\mathbb{R}$*, and* $-\infty$ *is a lowerbound for every subset* E *of* $\mathbb{R}$*. Moreover, if* E *is not bounded above, then* $\sup E = \infty$*, and if* E *is not bounded below, then* $\inf E = -\infty$*.*

Remark. We make the following assumptions for extended real numbers:

(1) If $x \in \mathbb{R}$, then $x + \infty = \infty$, $x - \infty = -\infty$, and $\frac{x}{\infty} = \frac{x}{-\infty} = 0$.

(2) If $x > 0$, then $x(\infty) = \infty$ and $x(-\infty) = -\infty$.

(3) If $x < 0$, then $x(\infty) = -\infty$ and $x(-\infty) = \infty$.

1.4 The Complex Field

Definition. We define a **complex number** to be a pair of real numbers (a, b). We denote the set of all comlex numbers by $\mathbb{C}$. We define the **addition** and **multiplication** of complex numbers to be the binary operations $+ : \mathbb{C} \to \mathbb{C}$ and $\cdot : \mathbb{C} \to \mathbb{C}$ such that

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d)\\ (a, b)(c, d) &= (ac - bd, ad + bc)\end{aligned}$$

Lastly, we define i to be the complex number such that $i = (0, 1)$.

Theorem 1.4.1. $\mathbb{C}$ *forms a field together with* $+$ *and* $\cdot$*.*

Theorem 1.4.2. *For* $(a, 0), (b, 0) \in C$*,* $(a, 0) + (b, 0) = (a + b, 0)$*, and* $(a, 0)(b, 0) = (ab, o)$*.*

Proof. This is a straightforward application of the addition and multiplication of complex numbers. ∎

Theorem 1.4.3. $i^2 = -1$*.*

Proof. $i^2 = (0, 1)(0, 1) = (0 - 1, 1 - 1) = (-1, 0) = -1$. ∎

Theorem 1.4.4. *Let* $(a, b) \in \mathbb{C}$*, then* $(a + b) = a + ib$*.*

Proof. $(a, b) = (a, 0) + (0, b) = (a, 0) + (0, 1)(b, 0) = a + ib$. ∎

Definition. Let $a, b \in \mathbb{R}$, and let $z \in \mathbb{C}$ such that $z = a + ib$. We define the **complex conjugate** of z to be the complex number $\bar{z} = a - ib$. Moreover, we define the **real part** of z to be a, and the **imaginary part** of z to be b, and we denote them $a = \operatorname{Re} z$, $b = \operatorname{Im} z$

Theorem 1.4.5. *Let* $z, w \in \mathbb{C}$*. Then*

(1) $\overline{z + w} = \bar{z} + \bar{w}$*.*

(2) $\overline{zw} = \bar{z}\bar{w}$*.*

(3) $z + \bar{z} = 2 \operatorname{Re} z$ *and* $z - \bar{z} = 2i \operatorname{Im} z$*.*

(4) $z\bar{z}$ *is a nonegative real number.*

Proof. Let $z = a + ib$, and let $w = c + id$. Then $z + w = (a + c) + i(b + d$, so $\overline{z + w} = (a+b) - i(b+d) = (a - ib) + (c - id) = \bar{z} + \bar{w}$; similarly, we get $\overline{zw} = \bar{z}\bar{w}$. Moreover, we have $(a+ib)+(a-ib) = 2a$, and $(a+ib)-(a-ib) = 2ib$, we also have that $z\bar{z} = (a+ib)(a-ib) = a^2 + b^2 \geq 0$, and $z\bar{z} = 0$ if and only if $a = b = 0$. ■

Definition. Let $z \in \mathbb{C}$. We define the **modulus** of z to be $|z| = \sqrt{z\bar{z}}$.

Remark. $|z|$ exists and is unique.

Theorem 1.4.6. *Let* $z, w \in \mathbb{C}$*, then:*

(1) $|z| \geq 0$ *and* $|z| = 0$ *if and only if* $z = 0$.

(2) $|\bar{z}| = |z|$.

(3) $|zw| = |z||w|$.

(4) $\operatorname{Re} z \leq |z|$.

(5) $|z + w + \leq |z| + |w|$.

Proof. Let $z = a + ib$, and $w = c + id$. Then $|z| = \sqrt{a^2 + b^2} \geq 0$, and $|z| = 0$ if and only if $a, b = 0$. Moreover, $|\bar{z}| = |a + i(-b)| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$. We also habe $|zw|^2 = (a^2 + b^2)(c^2 + d^) = |z|^2|w|^2$, likewise, $\|rez| = |a + i0| = \sqrt{a^2} \leq \sqrt{a^2 + b^2}$. Finally we prove (5).

We have $|z + w|^2 = (x + w)(\bar{z} + \bar{w}) = z\bar{z} + \bar{z}w + \bar{w}z + w\bar{w} = |z|^2 + w \operatorname{Re} z\bar{w} + |w|^2 \leq |z|^2 + 2|s\bar{w}| + |w|^2 = (|z| + |w|)^2$. ■

Theorem 1.4.7 (The Cauchy Schwarz Inequality)**.** *Let* $a_i, b_i \in \mathbb{C}$*, for* $1 \leq i \leq n$*. Then:*

$$|\sum_{i=1}^{n} a_i\bar{b_i}| \leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_j|^2 \tag{1.1}$$

Proof. Let $A = \sum a_j|^2$, $B = \sum |b_i|^2$, and $C = \sum a_i\bar{b_i}$. If $B = 0$, then $b_i = 0$ for $1 \leq i \leq n$, and we are done; so suppose that $B > 0$. Then

$$\begin{aligned}\sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(B\bar{a_j} - \bar{C}\bar{b_j}) \\ &= B \sum |a_j|^2 - B\bar{C} \sum a_j\bar{b_j} - BC \sum \bar{a_j}b_j + |C^2| \sum |b_j|^2 \\ &= (B^2A - B|C|^2) = B(AB - |C|^2) \geq 0\end{aligned}$$

Since $B > 0$, we get $|C|^2 \leq AB$ as required. ■

1.5 Euclidean Spaces

Definition. Let $k \in \mathbb{Z}^+$, and let $\mathbb{R}^k$ be the set of all ordered k-tuples $(x_1, x_2, \ldots, x_k)$, with $x_i \in \mathbb{R}$ for $1 \leq i \leq k$. We call $\mathbb{R}^k$ the **Euclidean space** of **dimension** k; more simply the **Euclidean k-space**. We call elements of $\mathbb{R}^k$ **vectors** or **points**; and we define **vector addition** and **scalar multiplication** to be:

$$(x_1, \ldots, x_k) + (y_1, \ldots, y_k) = (x_1 + y_1, \ldots, x_k + y_k)$$
$$\alpha(x_1, \ldots, x_k) = (\alpha x_1, \ldots, \alpha x_k)$$

for $(x_1, \ldots, x_k), (y_1, \ldots, y_k) \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$.

Theorem 1.5.1. $\mathbb{R}^k$ *forms a vector space together with vector addition and scalar multiplication.*

Definition. Let $x, y \in \mathbb{R}^k$. We define the **inner product** of x and y to be the binary operation $\langle , \rangle : \mathbb{R}^k \mathbb{R}^k \to \mathbb{R}$ such that

$$\langle x, y \rangle = \sum_{i=1}^{k} x_i y_i$$

We define the **norm** of x to be $||x|| = \sqrt{\langle x, x \rangle} = \sqrt{\sum x_i^2}$.

Theorem 1.5.2. *Let $x, y \in \mathbb{R}^k$, and $\alpha \in \mathbb{R}$. Then:*

(1) $||x|| \geq 0$ *and* $||x|| = 0$ *if and only if* $x_i = 0$ *for all* $1 \leq i \leq k$.

(2) $||\alpha x|| = |\alpha| ||x||$.

(3) $||\langle x, y \rangle|| \leq ||x|| ||y||$.

(4) $||x + y|| \leq ||x|| + ||y||$, *and* $||x - z|| \leq ||x - y|| + ||y - z||$

Proof. (1) follows by definition of the norm. We also have that $||\alpha x|| = \sqrt{\sum \alpha^2 x_i^2} = \sqrt{\alpha^2}\sqrt{\sum x_i^2} = |\alpha| ||x||$.

Now by the Cauchy Schwarz inequality, we have that $||\langle x, y \rangle||^2 = \sum x_i^2 y_i^2 \leq \sum x_i^2 \sum y_i^2 = ||x|| ||y||$. Finally we have that $||x + y|| = \langle x + y, x + y \rangle = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \leq ||x||^2 + 2||x|| ||y|| + ||y^2|| = (||x|| + ||y||)^2$, the last result follows immediately. ■

Chapter 2

Topological Foundations

2.1 Finite, Countable, and Uncountable Sets