Topology

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<u>Text</u>

Topology (2^{rd} edition)

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Chapter 1

Topological Spaces and Continuous Functions.

1.1 Topological Spaces.

Definition. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X such that:

(1) $\emptyset, X \in \mathcal{T}$.

(2) For any subcollection $\{U_\alpha\}$ of subsets of X, $\bigcup_\alpha U_\alpha \in \mathcal{T}$.

(3) For any finite subcollection $\{U_i\}_{i=1}^n$ of subsets of X, $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

We call the pair $(X, \mathcal{T})$ a **topological space**, and we call the elements of $\mathcal{T}$ **open sets**.

Example 1.1. (1) Let X be any set, the collection of all subsets of X, 2^X is a topology on X, which we call the **discrete topology**. We call the topology $\mathcal{T} = \{\emptyset, X\}$ the **indiscrete topology**.

(2) The set of three points $\{a, b, c\}$ has the 9 following topologies in figure 1.1.

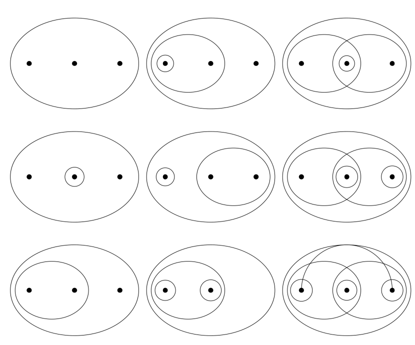

Figure 1.1: The Topologies on $\{a, b, c\}$.

(3) Let X be any set, and let $\mathcal{T}_f = \{U \subseteq X : X \backslash U \text{ is finite, or } X \backslash U = X\}$. Then $\mathcal{T}_f$ is a topology and called the **finite complement topology**.

(4) Let X be any set, and let $\mathcal{T}_c = \{U \subseteq X : X \backslash U \text{ is countable, or } X \backslash U = X\}$. Then $\mathcal{T}_c$ is a topology on X.

Definition. Let X be a set, and let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on X. We say that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$, and $\mathcal{T}'$ **finer** than $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{T}'$. If two topologies are either coarser, or finer than each other, we call them **comparable**.

Example 1.2. The topologies $\mathcal{T}_f$ and $\mathcal{T}_c$ are comparable, and we see that $\mathcal{T}_f \subseteq \mathcal{T}_f$, so $\mathcal{T}_f$ is coarser than $\mathcal{T}_c$, and $\mathcal{T}_c$ is finer than $\mathcal{T}_f$.

1.2 The Basis and Subbasis for a Topology.

Definition. If X is a set, the **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X, called **basis elements**, such that:

(1) For every $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$.

(2) For $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

We define the topology $\mathcal{T}$ **generated** by $\mathcal{B}$ to be collection of open sets: $\mathcal{T} = \{U \subseteq X : x \in B \text{ for some } B \in \mathcal{B}\}$.

Theorem 1.2.1. *Let X be a set, and $\mathcal{B}$ a basis of X, then the collection of subsets of X, $\mathcal{T} = \{U \subseteq X : x \in B \text{ for some } B \in \mathcal{B}\}$ is a topology on X.*

Proof. Let $\mathcal{B}$ be a basis for a topology in X, and consider $\mathcal{T}$ as defined above. Cleary, $\emptyset \in X$ and so is X.

Now let $\{U_\alpha\}$ be a subcollection of subsets of X, and let $U = \bigcup U_\alpha$. Then if $x \in U$ for some α, there is a B_α such that $x \in B_\alpha \subseteq U_\alpha$, thus $x \in B_\alpha \subseteq U$.

Now let $x \in U_1 \cap U_2$, and choose $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Then by definition, there is a B_3 for which $x \in B_3 \subseteq B_1 \cap B_2$. Now suppose for arbitrary n, that $U = \bigcap_{i=1}^{n} U_i \in \mathcal{T}$, for some finite subcollection $\{U_i\}$ of subsets of X. Then by let $B_n, B_{n+1} \in \mathcal{B}$ such that $x \in B_n \subseteq U$ and $x \in B_{n+1} \subseteq U_{n+1}$. Then by our hypothesis, there is a B for which $x \in B \subseteq B_n \cap B_{n+1}$, thus $U \cap U_{n+1} = \bigcap_{i=1}^{n+1} U_i \in \mathcal{T}$. This make $\mathcal{T}$ a topology on X. ■

Example 1.3. (1) Let $\mathcal{B}$ be the set of all circular regions in the plane $\mathbb{R} \times \mathbb{R}$, then $\mathcal{B}$ satisfies the conditions needed for a basis.

(2) The collection $\mathcal{B}'$ in $\mathbb{R} \times \mathbb{R}$ of all rectangular region also forms a basis for a topology on $\mathbb{R} \times \mathbb{R}$.

(3) For any set X, the set of all 1-point elements of X forms a basis for a topology on X.

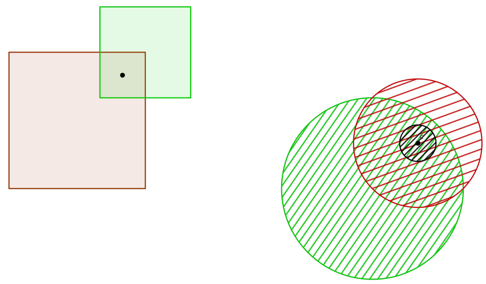

Figure 1.2: The basis for $\mathcal{B}$ and $\mathcal{B}'$ in $\mathbb{R} \times \mathbb{R}$ (see example (2)).

Lemma 1.2.2. *Let X be a set, and $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T} = \{\bigcup B : B \in \mathcal{B}\}$.*

Proof. Given a collection $\{B\}$ of basis elements in $\mathcal{B}$, since they are all in $\mathcal{T}$, their unions are also in $\mathcal{T}$. Conversely, given $U \in \mathcal{T}$, then for every point $x \in U$, choose a $B_x \in \mathbb{B}_x$ such that $x \in B_x \subseteq U$, then $U = \bigcup_{x \in U} B_x$. ■

Lemma 1.2.3. *Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{C} \subseteq \mathcal{T}$ be a collection of open sets of X such that for every $x \in U$, there is a $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is the basis for a $\mathcal{T}$ on X.*

Proof. Take any $x \in X$, then there is a $C \in \mathcal{C}$ such that $x \in C \subseteq U$, thus the first condition for a basis is satisfied. Now let $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C}$, since $C_1 \cap C_2$ is open in X, there is a $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. Therefore $\mathcal{C}$ is a basis for a topology on X.

Now let $\mathcal{T}_{\mathcal{C}}$ be the the topology generated by $\mathcal{C}$, now for $U \in \mathcal{T}$, we have by the hypothesis, that $U \in \mathcal{T}_{\mathcal{C}}$; and by lemma 1.2.2, $W \in \mathcal{T}_{\mathcal{C}}$ is the union of elements of $\mathcal{C}$, which is a subcollection of $\mathcal{T}$, thus $W \in \mathcal{T}$. Therefore $\mathcal{T}_{\mathcal{C}} = \mathcal{T}$. ■

Lemma 1.2.4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ on X. Then the $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for all $x \in X$, and all $B \in \mathcal{B}$, there is a $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Proof. Suppose first that $\mathcal{T} \subseteq \mathcal{T}'$, and let $x \in X$, and choose $B \in \mathcal{B}$ such that $x \in B$, then B is open in $\mathcal{T}$, thus it is open in $\mathcal{T}'$, thus there is a $B' \in \mathcal{B}'$ such that $x \in b' \subseteq B$. Conversely, suppose there is a $B' \in \mathcal{B}'$ for which $x \in B' \subseteq B$ for all $x \in X$, $B \in \mathcal{B}$. Take $x \in U \in \mathcal{T}$, since $\mathcal{B}$ generates $\mathcal{T}$, $x \in B \subseteq U$, since $B' \subseteq B$, this implies that $U \in \mathcal{T}'$ and $\mathcal{T} \subseteq \mathcal{T}'$. ■

Definition. If $\mathcal{B}$ is the collection of open intervals (a, b) in $\mathbb{R}$, we call the topology generated by $\mathcal{B}$ the **standard topology** on $\mathbb{R}$, and we denote it simply by $\mathbb{R}$.

Definition. If $\mathcal{B}$ is the collection of half open intervals $[a, b)$ in $\mathbb{R}$, we call the topology generated by $\mathcal{B}$ the **lower limit topology** on $\mathbb{R}$, and we denote it simply by $\mathbb{R}_l$. If $\mathcal{B}'$ is the collection of all half open intervals $(a, b]$ in $\mathbb{R}$, then we call the topology generated by $\mathcal{B}'$ the **upper limt topology** on $\mathbb{R}$, and denote it $\mathbb{R}_L$.

Definition. If $\mathcal{B}$ is the collection of all open intervals of the form $(a,b)\setminus \frac{1}{\mathbb{Z}^+}$, where $\frac{1}{\mathbb{Z}^+} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$, we call the topology generated by $\mathcal{B}$ the $\frac{1}{\mathbb{Z}^+}$**-topology** on $\mathbb{R}$, and we denote it $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$.

Lemma 1.2.5. *The topologies $\mathbb{R}_l$, $\mathbb{R}_L$, and $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ are all strictly finer than $\mathbb{R}$, but are not comparable with each other.*

Proof. Let (a,b) be a basis element for $\mathbb{R}$, and let $x \in (a,b)$, the basis element $[x,b) \in \mathbb{R}_l$ lies in (a,b) and contains x, however, there can be no interval (a,b) in $[x,b)$ as $x \leq a$, thus $\mathbb{R}\mathbb{R}_l$; a similar argument holds for $\mathbb{R}_L$.

Similarly, for $(a,b) \in \mathbb{R}$, the basis element $(a,b)\setminus\frac{1}{\mathbb{Z}^+}$ of $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ lies in (a,b), however, choose the basis $B = (-1,1)\setminus\frac{1}{\mathbb{Z}^+}$, and choose $0 \in B$, since $\mathbb{Z}^+$ is dense in $\mathbb{R}$, there is no interval (a,b) containing 0 and lying in B, thus $\mathbb{R} \subseteq \mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$.

Now choose $[0,1)$ in $\mathbb{R}_l$, and choose $\frac{1}{k} \in [0,1)$ such that $k \in \mathbb{Z}^+$. Now $(0,1) \subseteq [0,1)]$, so we cannot say that $[0,1)$ is a basis for $\mathbb{R}$, and moreover, $[0,1)\setminus\frac{1}{\mathbb{Z}^+}$ cannot be said to be a basis in $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$, thus $\mathbb{R}_l$ and $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ are incompararble, a similar argument holds for $\mathbb{R}_L$.

Lastly, let (a,b) be in $\mathbb{R}$ and choose $x \in (a,b)$. Then $(a,x]$ and $[x,b)$ are both in (a,b), however it is clear that $(a,x]$ and $[x,b)$ connot be contained in each other, thus $\mathbb{R}_l$ and $\mathbb{R}_L$ are incomparable. ■

Definition. A **subbasis**, $\mathcal{S}$, for a topology on X is a collection of subsets of X whose union equals X. We call the **topology generated by** $\mathcal{S}$ to be the collection of all unions of finite intersections of elements of $\mathcal{S}$, that is:

$$\mathcal{T} = \{\bigcup\bigcap_{i=1}^{n} S_i : S_i \in \mathcal{S} \text{ for } 1 \leq i \leq n\}$$

Theorem 1.2.6. *Let $\mathcal{S}$ be a subbasis for a topology on X. Then the collection $\mathcal{T} = \{\bigcup\bigcap_{i=1}^{n} S_i : S_i \in \mathcal{S}$ for $1 \leq i \leq n\}$ is a topology on X.*

Proof. It is sufficient to show that the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ is a basis for a topology on X. By lemma 1.2.1, for $x \in X$, it belongs to an element S of $\mathcal{S}$, and therefore, to an element of $\mathcal{B}$. Now let $B_1 = \bigcap_{i=1}^{m} S_i$ and $B_2 = \bigcap_{j=1}^{n} S'_j$ be basis elements of $\mathcal{B}$. The intersection $\mathbb{B}_1 \cap B_2$ is a finite intersection of elements of $\mathcal{S}$, and hence also belongs in $\mathcal{B}$, and hence we can take another basis element B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$. ■

1.3 The Order Topology.