Topology

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<u>Text</u>

Topology (2^{rd} edition)

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Chapter 1

Topological Spaces and Continuous Functions.

1.1 Topological Spaces.

Definition. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X such that:

(1) $\emptyset, X \in \mathcal{T}$.

(2) For any subcollection $\{U_\alpha\}$ of subsets of X, $\bigcup_\alpha U_\alpha \in \mathcal{T}$.

(3) For any finite subcollection $\{U_i\}_{i=1}^n$ of subsets of X, $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

We call the pair $(X, \mathcal{T})$ a **topological space**, and we call the elements of $\mathcal{T}$ **open sets**.

Example 1.1. (1) Let X be any set, the collection of all subsets of X, 2^X is a topology on X, which we call the **discrete topology**. We call the topology $\mathcal{T} = \{\emptyset, X\}$ the **indiscrete topology**.

(2) The set of three points $\{a, b, c\}$ has the 9 following topologies in figure 1.1.

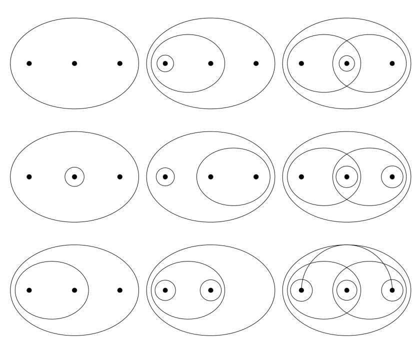

Figure 1.1: The Topologies on $\{a, b, c\}$.

(3) Let X be any set, and let $\mathcal{T}_f = \{U \subseteq X : X\backslash U$ is finite, or $X\backslash U = X\}$. Then $\mathcal{T}_f$ is a topology and called the **finite complement topology**.

(4) Let X be any set, and let $\mathcal{T}_c = \{U \subseteq X : X\backslash U$ is countable, or $X\backslash U = X\}$. Then $\mathcal{T}_c$ is a topology on X.

Definition. Let X be a set, and let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on X. We say that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$, and $\mathcal{T}'$ **finer** than $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{T}'$. If two topologies are either coarser, or finer than each other, we call them **comparable**.

Example 1.2. The topologies $\mathcal{T}_f$ and $\mathcal{T}_c$ are comparable, and we see that $\mathcal{T}_f \subseteq \mathcal{T}_f$, so $\mathcal{T}_f$ is coarser than $\mathcal{T}_c$, and $\mathcal{T}_c$ is finer than $\mathcal{T}_f$.

1.2 The Basis and Subbasis for a Topology.

Definition. If X is a set, the **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X, called **basis elements**, such that:

(1) For every $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$.

(2) For $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

We define the topology $\mathcal{T}$ **generated** by $\mathcal{B}$ to be collection of open sets: $\mathcal{T} = \{U \subseteq X : x \in B$ for some $B \in \mathcal{B}\}$.

Theorem 1.2.1. *Let X be a set, and $\mathcal{B}$ a basis of X, then the collection of subsets of X, $\mathcal{T} = \{U \subseteq X : x \in B$ for some $B \in \mathcal{B}\}$ is a topology on X.*

Proof. Let $\mathcal{B}$ be a basis for a topology in X, and consider $\mathcal{T}$ as defined above. Cleary, $\emptyset \in X$ and so is X.

Now let $\{U_\alpha\}$ be a subcollection of subsets of X, and let $U = \bigcup U_\alpha$. Then if $x \in U$ for some α, there is a B_α such that $x \in B_\alpha \subseteq U_\alpha$, thus $x \in B_\alpha \subseteq U$.

Now let $x \in U_1 \cap U_2$, and choose $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Then by definition, there is a B_3 for which $x \in B_3 \subseteq B_1 \cap B_2$. Now suppose for arbitrary n, that $U = \bigcap_{i=1}^n U_i \in \mathcal{T}$, for some finite subcollection $\{U_i\}$ of subsets of X. Then by let $B_n, B_{n+1} \in \mathcal{B}$ such that $x \in B_n \subseteq U$ and $x \in B_{n+1} \subseteq U_{n+1}$. Then by our hypothesis, there is a B for which $x \in B \subseteq B_n \cap B_{n+1}$, thus $U \cap U_{n+1} = \bigcap_{i=1}^{n+1} U_i \in \mathcal{T}$. This make $\mathcal{T}$ a topology on X. ∎

Example 1.3. (1) Let $\mathcal{B}$ be the set of all circular regions in the plane $\mathbb{R} \times \mathbb{R}$, then $\mathcal{B}$ satisfies the conditions needed for a basis.

(2) The collection $\mathcal{B}'$ in $\mathbb{R} \times \mathbb{R}$ of all rectangular region also forms a basis for a topology on $\mathbb{R} \times \mathbb{R}$.

(3) For any set X, the set of all 1-point elements of X forms a basis for a topology on X.

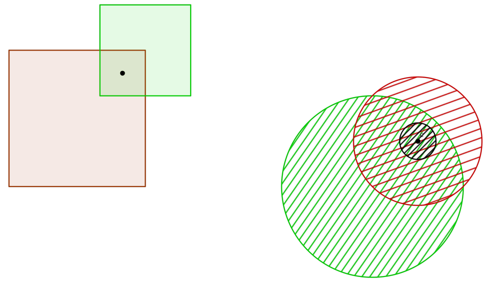

Figure 1.2: The basis for $\mathcal{B}$ and $\mathcal{B}'$ in $\mathbb{R} \times \mathbb{R}$ (see example (2)).

Lemma 1.2.2. *Let X be a set, and $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T} = \{\bigcup B : B \in \mathcal{B}\}$.*

Proof. Given a collection $\{B\}$ of basis elements in $\mathcal{B}$, since they are all in $\mathcal{T}$, their unions are also in $\mathcal{T}$. Conversely, given $U \in \mathcal{T}$, then for every point $x \in U$, choose a $B_x \in \mathbb{B}_x$ such that $x \in B_x \subseteq U$, then $U = \bigcup_{x \in U} B_x$. ■

Lemma 1.2.3. *Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{C} \subseteq \mathcal{T}$ be a collection of open sets of X such that for every $x \in U$, there is a $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is the basis for a $\mathcal{T}$ on X.*

Proof. Take any $x \in X$, then there is a $C \in \mathcal{C}$ such that $x \in C \subseteq U$, thus the first condition for a basis is satisfied. Now let $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C}$, since $C_1 \cap C_2$ is open in X, there is a $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. Therefore $\mathcal{C}$ is a basis for a topology on X.

Now let $\mathcal{T}_\mathcal{C}$ be the the topology generated by $\mathcal{C}$, now for $U \in \mathcal{T}$, we have by the hypothesis, that $U \in \mathcal{T}_\mathcal{C}$; and by lemma 1.2.2, $W \in \mathcal{T}_\mathcal{C}$ is the union of elements of $\mathcal{C}$, which is a subcollection of $\mathcal{T}$, thus $W \in \mathcal{T}$. Therefore $\mathcal{T}_\mathcal{C} = \mathcal{T}$. ■

Lemma 1.2.4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ on X. Then the $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for all $x \in X$, and all $B \in \mathcal{B}$, there is a $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Proof. Suppose first that $\mathcal{T} \subseteq \mathcal{T}'$, and let $x \in X$, and choose $B \in \mathcal{B}$ such that $x \in B$, then B is open in $\mathcal{T}$, thus it is open in $\mathcal{T}'$, thus there is a $B' \in \mathcal{B}'$ such that $x \in b' \subseteq B$. Conversely, suppose there is a $B' \in \mathcal{B}'$ for which $x \in B' \subseteq B$ for all $x \in X$, $B \in \mathcal{B}$. Take $x \in U \in \mathcal{T}$, since $\mathcal{B}$ generates $\mathcal{T}$, $x \in B \subseteq U$, since $B' \subseteq B$, this implies that $U \in \mathcal{T}'$ and $\mathcal{T} \subseteq \mathcal{T}'$. ■

Definition. If $\mathcal{B}$ is the collection of open intervals (a, b) in $\mathbb{R}$, we call the topology generated by $\mathcal{B}$ the **standard topology** on $\mathbb{R}$, and we denote it simply by $\mathbb{R}$.

Definition. If $\mathcal{B}$ is the collection of half open intervals $[a, b)$ in $\mathbb{R}$, we call the topology generated by $\mathcal{B}$ the **lower limit topology** on $\mathbb{R}$, and we denote it simply by $\mathbb{R}_l$. If $\mathcal{B}'$ is the collection of all half open intervals $(a, b]$ in $\mathbb{R}$, then we call the topology generated by $\mathcal{B}'$ the **upper limt topology** on $\mathbb{R}$, and denote it $\mathbb{R}_L$.

Definition. If $\mathcal{B}$ is the collection of all open intervals of the form $(a,b)\setminus \frac{1}{\mathbb{Z}^+}$, where $\frac{1}{\mathbb{Z}^+} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$, we call the topology generated by $\mathcal{B}$ the $\frac{1}{\mathbb{Z}^+}$**-topology** on $\mathbb{R}$, and we denote it $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$.

Lemma 1.2.5. *The topologies $\mathbb{R}_l$, $\mathbb{R}_L$, and $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ are all strictly finer than $\mathbb{R}$, but are not comparable with each other.*

Proof. Let (a,b) be a basis element for $\mathbb{R}$, and let $x \in (a,b)$, the basis element $[x,b) \in \mathbb{R}_l$ lies in (a,b) and contains x, however, there can be no interval (a,b) in $[x,b)$ as $x \leq a$, thus $\mathbb{R}\mathbb{R}_l$; a similar argument holds for $\mathbb{R}_L$.

Similarly, for $(a,b) \in \mathbb{R}$, the basis element $(a,b)\setminus\frac{1}{\mathbb{Z}^+}$ of $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ lies in (a,b), however, choose the basis $B = (-1,1)\setminus\frac{1}{\mathbb{Z}^+}$, and choose $0 \in B$, since $\mathbb{Z}^+$ is dense in $\mathbb{R}$, there is no interval (a,b) containing 0 and lying in B, thus $\mathbb{R} \subsetneq \mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$.

Now choose $[0,1)$ in $\mathbb{R}_l$, and choose $\frac{1}{k} \in [0,1)$ such that $k \in \mathbb{Z}^+$. Now $(0,1) \subsetneq [0,1)]$, so we cannot say that $[0,1)$ is a basis for $\mathbb{R}$, and moreover, $[0,1)\setminus\frac{1}{\mathbb{Z}^+}$ cannot be said to be a basis in $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$, thus $\mathbb{R}_l$ and $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ are incompararble, a similar argument holds for $\mathbb{R}_L$.

Lastly, let (a,b) be in $\mathbb{R}$ and choose $x \in (a,b)$. Then $(a,x]$ and $[x,b)$ are both in (a,b), however it is clear that $(a,x]$ and $[x,b)$ connot be contained in each other, thus $\mathbb{R}_l$ and $\mathbb{R}_L$ are incomparable. ∎

Definition. A **subbasis**, $\mathcal{S}$, for a topology on X is a collection of subsets of X whose union equals X. We call the **topology generated by** $\mathcal{S}$ to be the collection of all unions of finite intersections of elements of $\mathcal{S}$, that is:

$$\mathcal{T} = \{\bigcup\bigcap_{i=1}^{n} S_i : S_i \in \mathcal{S} \text{ for } 1 \leq i \leq n\}$$

Theorem 1.2.6. *Let $\mathcal{S}$ be a subbasis for a topology on X. Then the collection $\mathcal{T} = \{\bigcup\bigcap_{i=1}^{n} S_i : S_i \in \mathcal{S}$ for $1 \leq i \leq n\}$ is a topology on X.*

Proof. It is sufficient to show that the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ is a basis for a topology on X. By lemma 1.2.1, for $x \in X$, it belongs to an element S of $\mathcal{S}$, and therefore, to an element of $\mathcal{B}$. Now let $B_1 = \bigcap_{i=1}^{m} S_i$ and $B_2 = \bigcap_{j=1}^{n} S_j'$ be basis elements of $\mathcal{B}$. The intersection $\mathbb{B}_1 \cap B_2$ is a finite intersection of elements of $\mathcal{S}$, and hence also belongs in $\mathcal{B}$, and hence we can take another basis element B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$. ∎

1.3 The Order Topology.

Definition. Let X be a set with a simple order relation, and suppose that $|X| > 1$. Let $\mathcal{B}$ be the collection of sets of the following forms:

(1) All open intervalcs $(a,b) \in X$.

(2) All half open intervals $[a_0,b)$ where a_0 is the least element (if any) of X.

(3) All half open intervals of the form $(a,b_0]$ where b_0 is the greatest element (if any) of X.

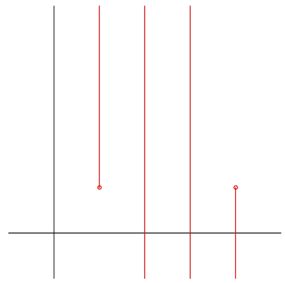

Figure 1.3: The order topology on $\mathbb{R} \times \mathbb{R}$.

Then $\mathcal{B}$ forms the basis for a topology on X called the **order topology**

Theorem 1.3.1. *The collection $\mathcal{B}$ forms a basis.*

Proof. Consider $x \in X$, if x is the least element of X, then it liess in all intervals of type (2), if it is the largest, then it lies in all intervals of type (3). If x is neither the least nor largest element, then $x \in (a_0, b_0)$ with a_0 and b_0 the least and largest elements (if any) of X. If no such elements exist, then $x \in (a, b)$, for some lowerbound a and upperbound b. Thus, in all three cases, there is a basis element containing x.

Now suppose $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$. If B_1 and B_2 are both of type (1), then let $B_1 = (a, b)$, $B_2 = (c, d)$, then $B_1 \cap B_2$ is an open interval of type (1), now fix B_1 to be of type one. If B_2 is of type (2), then letting $B_2 = [a_0, c)$, then $x \in [a_0, d)$ for some $d \in X$. Likewise, if $B_2 = (c, b_0]$, is of type (3), we get a similar result. Moreover, the results are analogous if we fix B_2 and let B_1 range between intervals of the three types. Thusm in all cases, there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. ■

Example 1.4. (1) The standard topology on $\mathbb{R}$ is the order topology on $\mathbb{R}$ induced by the usual order relation. We have that $\mathbb{R}$ under this topology has no intervals of type (2), nor (3), so all bases elements in the standard topology are open intervals in $\mathbb{R}$.

(2) Consider the dictionary order on $\mathbb{R} \times \mathbb{R}$. Since $\mathbb{R} \times \mathbb{R}$ has no intervals of type (2), nor (3), the bases of $\mathbb{R} \times \mathbb{R}$ under the dictionary order are the open intervals of the form $(a \times b, c \times d)$ Where $a \leq c$, and $b < d$.

(3) The positive integers $\mathbb{Z}^+$ with the least element 1 form an ordered set under the usual order. Taking $n > 1$, we see the bases of $\mathbb{Z}^+$ under the order topology are of the form $(n-1, n+1) = \{n\}$ and $[1, n) = \{1, \dots, n-1\}$. Thus the order topology on $\mathbb{Z}^+$ is the discrete topology.

(4) The set $X = \{1, 2\} \times \mathbb{Z}^+$ over the dictionary order is also an ordered set, with the least elelment 1×1. Denote $1 \times n$ as a_n and $2 \times n$ as b_n. Then X consist of the elements $a_1, a_2, \dots, b_1, b_2, \dots$.

Now take $\{b_1\}$, then any open set containing b_1 must have a basis about b_1, and also contains points a_i with $i \in \mathbb{Z}^+$; thus the order topology on X is not the discrete topology.

Definition. Let X be an ordered set, and let $a \in X$. There are two subsets in X, $(a, \infty) = \{x \in X : x > a\}$ and $(-\infty, a) = \{x \in X : x < a\}$ called **open rays** of X. There are also two sets $[a, \infty) = \{x \in X : x \geq a\}$ and $(-\infty, a] = \{x \in X : x \leq a\}$ called **closed rays** of X.

Theorem 1.3.2. *Let X be an ordered set. Then the collection of all open rays in X form a subbasis for the order topology on X.*

Proof. Let $\mathcal{S}$ be the collection of all open rays of X, let (a, ∞) and $(-\infty, b) \in \mathcal{S}$, then $(a, b) = (a, \infty) \cap (-\infty, b)$. Now take:

$$S = \bigcup_{a,b \in X} (a, b)$$

then $S \subseteq X$, likewise, since S runs through all intersections of open rays of X, it contains all open intervals in X, hence $X \subseteq S$, and so $X = S$ as required. ∎

1.4 The Product Topology.

Definition. Let X and Y be topological spaces. We define the **product topology** on $X \times Y$ to be the topology having as basis the collection $\mathcal{B} = \{U \times V \subseteq X \times Y : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$

Theorem 1.4.1. *The collection $\mathcal{B} = \{U \times V \subseteq X \times Y : U$ is open in X and V is open in $Y\}$ forms a basis for the product topology on $X \times Y$.*

Proof. Clearly, we have that $X \times Y$ is a basis element of $\mathcal{B}$. Now take $U_1 \times V_1$ and $U_2 \times V_2$ in $\mathcal{B}$. Since $U_1 \times V_1 \cap U_2 \times V_2 = U_1 \cap U_2 \times V_1 \cap V_2$, since $U_1 \cap U_2$ and $V_1 \cap V_2$ are open in X and Y respectively, then we have that $U_1 \times V_1 \cap U_2 \times V_2$ is a basis element as well. ∎

Theorem 1.4.2. *If $\mathcal{B}$ is the basis for a topology on X, and $\mathcal{C}$ is the basis for a topology on Y, then the collection:*

$$\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

Is a basis for the topology on $X \times Y$.

Proof. By lemma 1.2.3, let W be an open set of $X \times Y$, and let $x \times y \in W$. Then there is a basis $U \times V$ such that $x \times y \in U \times V \subseteq W$. Since $\mathcal{B}$ and $\mathcal{C}$ are bases of X and Y respectively, choosing $B \in \mathcal{B}$ and $C \in \mathcal{C}$, we have that $x \in B \subseteq U$, and $y \in C \subseteq Y$, thus $x \times y \in B \times C \subseteq U \times V \subseteq W$. Therefore, $\mathcal{D}$ is the basis for a topology on $X \times Y$. ∎

Example 1.5. The product of the standard topology on $\mathbb{R}$ with itself is called the **standard topology on** $\mathbb{R} \times \mathbb{R}$, and has as basis the collection of all products of open sets in $\mathbb{R}$. By theorem 1.4.2, if we take the collection of all open intervals $(a, b) \times (a, b)$ in $\mathbb{R} \times \mathbb{R}$, we form a basis. Constructing this basis geometrically gives the interior of a rectangle, whose boundaries are the intervals (a, b) and (ac, d).

Definition. Let $\pi_1 : X \times Y \to X$ be defined such that $\pi_1(x, y) = x$, and define $\pi_2 : X \times Y \to Y$ such that $\pi_2(x, y) = y$. We call π_1 and π_2 **projections** of $X \times Y$ onto its first and secnd **factors**; that is onto X and Y, respectively.

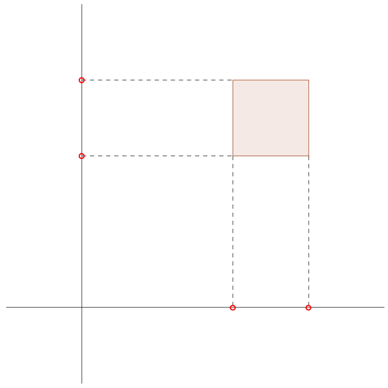

Figure 1.4: A basis element for $\mathbb{R} \times \mathbb{R}$

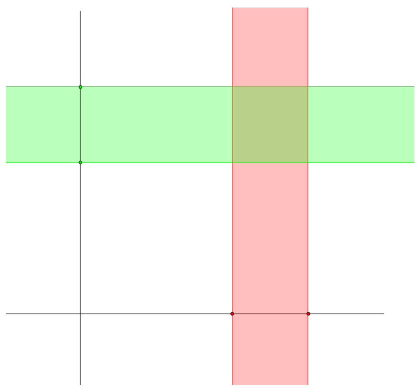

Figure 1.5: The inverse images, $\pi_1^{-1}(U)$ and $\pi_2^{-1}(V)$, of the projections π_1 and π_2 onto the $X \times Y$ plane.

Clearly, π_1 and π_2 are both onto. Now let U be open in X, then $\pi_1^{-1}(U) = U \times Y$ is open in $X \times Y$; similarly, $\pi_2^{-1}(V) = X \times V$ is also open in $X \times Y$, for V open in Y.

Theorem 1.4.3. *The collection* $\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ is open in } Y\}$ *is a subbasis for the product topology on* X*.*

Proof. Let $\mathcal{T}$ be the product topology on $X \times Y$, and let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$. Since every element of $\mathcal{S}$ is open in $\mathcal{T}$, $\mathcal{T} \subseteq \mathcal{T}'$. Conversely, consider the basis element $U \times V$ of $\mathcal{T}$, then $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times Y \cap X \times V = U \times V$, thus $\mathcal{T} \subseteq \mathcal{T}'$. Therefore, $\mathcal{S}$ is a subbasis for the product topology. ∎

1.5 The Subspace Topology.

Theorem 1.5.1. *Let* X *he a topological space with topology* $\mathcal{T}$*, and let* $Y \subseteq X$*. Then the collection:*

$$\mathcal{T}_y = \{Y \cap U : U \in \mathcal{T}\}$$

forms a topology on Y*.*

Proof. Cleary, $Y \cap \emptyset = \emptyset \in \mathcal{T}_Y$ and $Y \cap X = Y \in \mathcal{T}_Y$. Now consider the collection $\{U_{alpha}\}$. Then $\bigcup Y \cap U_\alpha = Y \cap \bigcup U_\alpha$, similarly, for $\{U_i\}_{i=1}^n$, $\bigcap Y \cap U_i = Y \cap \bigcap U_i$, hence $\mathcal{T}$ is a topology on Y. ■

Definition. Let X be a topological space, and let $Y \subseteq X$. We call the $\mathcal{T}$ defined in theorem 1.5.1 the **subspace topology** on Y. We say that $U \subseteq Y$ is **open in** Y if $U \in \mathcal{T}_Y$.

Lemma 1.5.2. *Let $\mathcal{B}$ be the basis for a topology on X. Then the collection $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$, where $Y \subseteq X$, is a basis for the subspace topology on Y.*

Proof. Let U be open in X, and let $y \in Y \cap U$, and choose $B \in \mathcal{B}$ such that $y \in B \subseteq U$, then $y \in B \cap Y \subseteq U \cap Y$, then by lemma 1.2.2, $\mathcal{B}_y$ is the basis fpr the subspace topology on Y. ■

Lemma 1.5.3. *Let Y be a subspace of X, If $U \subseteq Y$ is open in Y, then U is open in X.*

Proof. The proof is rather trivial, however, it is worth going through the motions. Let $U \in \mathcal{T}_Y$, then for some $V \subseteq X$, $U = Y \cap V$. Now since Y is open in X, and so is V, then it follows that U is also open in X. ■

Remark. What this lemma says is that given a topological space X, and a subspace Y of X, then the subspace topology of Y is courser than the topology on X, i.e. $\mathcal{T}_Y \subseteq \mathcal{T}$.

Theorem 1.5.4. *If A is a subspace of X, and B is a subspace of Y, then the product topology on $A \times B$ is the topology that $A \times B$ inherits as a subspace of $X \times Y$.*

Proof. We have that $U \times V$ is the basis element for $X \times Y$, with U open in X, and V open in Y. Thus $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$ is a basis element for the subspace topology on $X \times Y$. Since $U \cap A$ and $V \cap B$ are open in the subspace topologies of A and B respectively, then $(U \cap A) \times (v \cap B)$ is a basis for the product topology on $A \times B$. ■

Example 1.6. (1) Consider $[0,1] \subseteq \mathbb{R}$. In the subspace topology of $[0,1]$, we have as basis elements of the form $(a,b) \cap [0,1]$, with $(a,b) \subseteq \mathbb{R}$. If we have that $(a,b) \subseteq [0,1]$, then $(a,b) \cap [0,1] = (a,b)$. On the other hand, if $a \in [0,1]$ or $b \in [0,1]$, then we get $(a,b) \cap [0,1] = (a,1]$ or $(a,b) \cap [0,1] = [0,b)$, lastly if neither a nor b are in $[0,1]$, then we have $(a,b) \cap [0,1] = [0,1]$ only if $[0,1] \subseteq (a,b)$, and $(a,b) \cap [0,1] = \emptyset$ otherwise.

   Now each of these sets are open in $\mathbb{R}$, under the standard topology, except for $(a,1]$ and $[0,b)$.

(2) For $[0,1) \cup \{2\} \subseteq \mathbb{R}$, the singletomn $\{2\}$ is open in the subspace topology on $[0,1) \cup \{2\}$; for observe, that $(\frac{3}{5}, \frac{5}{2}) \cap ([0,1) \cup \{2\}) = \{2\}$, however, in the order topology, on that same set, $\{2\}$ is not open. Any basis element on $[0,1) \cup \{2\}$ containing 2 is of the form $(a,2]$, where $a \in [0,1) \cup \{2\}$.

(3) The dictionary order on $[0,1] \times [0,1]$ is a restriction of the dictionary order on $\mathbb{R} \times \mathbb{R}$. Now the set $\{\frac{1}{2}\} \times (\frac{1}{2}, 1]$ is open in the subspace topology on $[0,1] \times [0,1]$, but it is not open in the dictionary order on the same set.

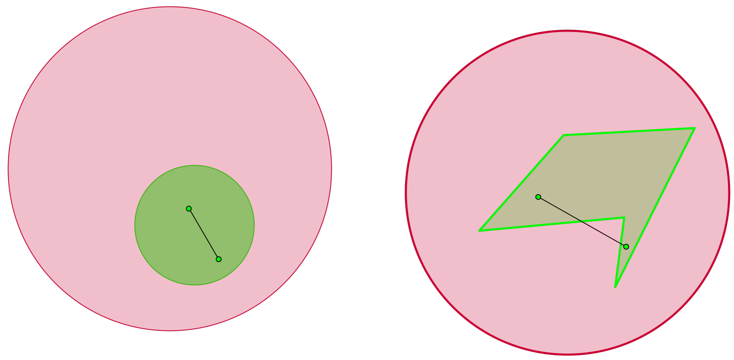

Figure 1.6: A convex set, and a con convex set.

Definition. We call the set $[0,1] \times [0,1]$ on the dictionary odere the **ordered square**, and we denote it by I_0^2.

Definition. Let X be an ordered set. We say that a nonempty subset $Y \subset X$ is **convex** in X if for each pair of points $a, b \in Y$, with $a < b$, then the open interval $(a, b) \subseteq X$ is also contained in Y.

Example 1.7. Let X be any ordered set. Then by definition, all open intervals and rays in X are convex in X.

Theorem 1.5.5. *Let X be an ordered set on the order toplogy, and let $Y \subseteq X$ be convex in X. Then the order topology on Y is the same as the subspace topology on Y.*

Proof. Consider $(a,) \subseteq X$. If $a \in Y$, then $(a, \infty) \cap Y = \{x \in Y : x > a\}$, which is by definition an open ray on Y. Now if $a \notin Y$, then a is either a lowerbound, or an upperbound. Then $(a, \infty) \cap Y = \emptyset$ and $(-\infty, a) \cap Y = Y$ if a is an upperbound, similarly, if a is a lowerbound we get $(a, \infty) \cap Y = Y$ and $(-\infty, a) \cap Y = \emptyset$.

Since $(a, \infty)Y$ and $(-\infty, a) \cap Y$ form a subbasis on the subspace topology on Y, and since they are also open in the order topology, then the order topology contains the subspace topology.

Now if (a, ∞) is an open ray in Y, then $(a, \infty) = (b, \infty) \cap Y$, with (b, ∞) some open ray in X, hence (a, ∞) is open in the subspace topology of Y, and since it also forms the subbasis for the order topology, we have that the ordertopology is contained within the subspace topology. Thus both topologies are equal. ■

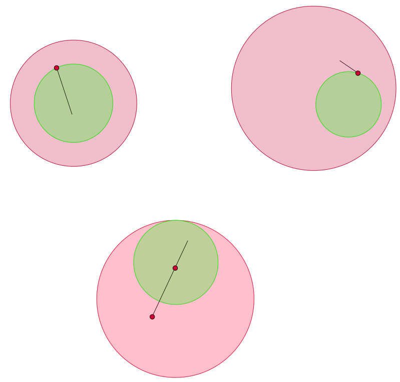

Figure 1.7: An illustration of theorem 1.5.5.

1.6 Closed Sets and Limit Points.

Definition. A subset A of a topological space X is said to be **closed** if $X\backslash A$ is open.

Example 1.8. (1) Consider $[a, b] \subseteq \mathbb{R}$, we have that $\mathbb{R}\backslash[a, b] = (-\infty, a) \cup (b, \infty)$ which is open in $\mathbb{R}$. So $[a, b]$ is closed.

(2) In $\mathbb{R} \times \mathbb{R}$, the set $A = \{x \times y : x, y \geq 0\}$ (i.e the first quadtant of the plane) is closed, for $\mathbb{R} \times \mathbb{R}\backslash A = (-\infty, 0) \times \mathbb{R} \cup \mathbb{R} \times (-\infty, 0)$, which is open in $\mathbb{R} \times \mathbb{R}$.

(3) Consider the finite complement topology $\mathcal{T}_C$ on a set X. We have that $X\backslash X = \emptyset \in \mathcal{T}$, so X is closed, similarly, $\emptyset$ is also closed. Likewise, if $A \subseteq X$ is a finite set, then $X\backslash A$ is also finite, and hence A is also closed. Thus, we have that all the closed sets of $\mathcal{T}_C$ are those finite subsets of X. As a consequence, this examle also illustrates that sets can be both closed and open.

(4) In the discrete topology 2^X, every open set is closed. This is another example where open sets are also closed sets.

(5) Consider $[0, 1] \cup (2, 3)$ in the subspace topology on $\mathbb{R}$. We have that $[0, 1]$ is open $([0, 1] = [0, 1] \cup (2, 3) \cap (-\frac{2}{3}), \frac{3}{2})$, similarly, $(2, 3)$ is also open. Now taking $[0, 1] \cup (2, 3)\backslash(2, 3) = [0, 1]$, which is open, so $[0, 1]$ is closed in the subspace topology on $\mathbb{R}$, bu the same reasoning, so is $(2, 3)$.

Theorem 1.6.1. *Let X be a topological space. Then:*

(1) $\emptyset$ and X are closed.

(2) Arbitrary intersections of closed sets are closed.

(3) Finite unions of closed sets are closed.

Proof. We have that $X\backslash\emptyset = X$ and $X\backslash X = \emptyset$, both of which are open in X, so they are also closed in X. Now let $\{U_\alpha\}$ be a collection of closed sets of X. We have that:

$$X\backslash\bigcap_{\alpha} U_\alpha = \bigcup_{\alpha} X\backslash U_\alpha.$$

Similarly, for $\{U_i\}_{i=1}^n$, we have

$$X \backslash \bigcup_{i=1}^{n} U_i = \bigcap_{i=1}^{n} X \backslash U_i.$$

Both of which are open in X. This completes the proof. ■

Definition. If Y is a subspace of X, we say that A is **closed in** Y if $A \subseteq Y$ and A is closed in the subspace topology of Y.

Theorem 1.6.2. *Let Y be a subspace of X. Then A is closed in Y if and only if A equals the intersection of a closed set of X with Y.*

Proof. Suppose that A is closed in Y, then $Y \backslash A$ is open in Y, hence we have that $Y \backslash A = U \cap Y$ for some open set U of X. Now $X \backslash U$ is closed in X, and with $A \subseteq Y$, we have that $A = Y \cap X \backslash U$.

Conversely, suppose that $A = C \cap Y$, with C closed in X. Then $X \backslash C$ is open in X, hence $X \backslash C \cap Y$ is open in Y, now since $X \backslash C \cap Y = Y \backslash A$, which is open, we have that A is closed in Y. ■

Theorem 1.6.3. *Let Y be a subspace of X. If A is closed in Y, and Y is closed in X, then A is closed in X; that is, closure is transitive.*

Proof. By theorem 1.6.2, if A is closed in Y, then $A = C \cap Y$ with C closed in X, now since Y is closed in X, then $Y = D \cap X$ with D closed in X. Thus $A = (C \cap D) \cap X$, therefore, A is closed in X. ■

We now go over the concepts of the closure, and the interior of a set.

Definition. Let $A \subseteq X$, with X a topological space. The **interior** of A is defined to be the union of all open sets in A. The **closure** of A is defined to be the intersection of all closed sets containing A. We denote the interior and the closure of A as $\operatorname{Int} A$ and $\overline{A}$ respectively

We have by the very definitions that $\operatorname{Int} A \subseteq A \subseteq \overline{A}$

Lemma 1.6.4. $\operatorname{Int} A = A$ *only when A is open, and $\overline{A} = A$ only when A is closed.*

Proof. Now, if A is open, then it is in the union of all open sets of A, hence $A \subseteq \operatorname{Int} A$, likewise, if A is closed, then since $\overline{A}$ is the intersection of all closed sets containing A, we get $\overline{A} \subseteq A$. ■

Corollary. *A is closed and open if and only if* $\operatorname{Int} A = \overline{A}$.

Theorem 1.6.5. *Let Y be a subspace of X, and let $A \subseteq Y$, and let $\overline{A}$ be the closure of A. Then $\overline{A} \cap Y$ is the closure of A in Y.*

Proof. Let $\hat{A}$ be the closure of A in Y. Since $\overline{A}$ is closed in X, by theorem 1.6.2, $\overline{A} \cap Y$ is closed in Y, now we have that $A \subseteq \overline{A} \cap Y$, and since $\hat{A} = \bigcap U$, then $\hat{A} \subseteq \overline{A} \cap Y$.

Conversely, suppose that $\hat{A}$ is closed in Y, again by theorem 1.6.2, we have that $\hat{A} = C \cap Y$, where C is closed in X, since $A \subseteq \hat{A}$, then $A \subseteq C$, and since C is closed, then $\overline{A} \subseteq C$, thus $\overline{A} \cap Y \subseteq \hat{A}$. ■

Definition. Let X be a topological space, and let $x \in X$. We call an open set U of X a **neighborhood** of x if $x \in U$.

Theorem 1.6.6. *If $A \subseteq X$, with X a topological space, then $\overline{A}$ is a neighborhood of $x \in X$ if and only if for every neighborhood U of x, $A \cap U \neq \emptyset$.*

Proof. We prove the contrapositve. If $x \notin \overline{A}$, then $U = X \backslash \overline{A}$ is an open set containing A, disjoint from A. Conversely, suppose there is a neighborhood U of x, with U disjoint from A, then $X \backslash U$ is closed, and therefore contains the closure of A, thus $x \notin \overline{A}$ ■

Corollary. *$\overline{A}$ is a neighborhood of x if and only if for every basis element B of X, containing x, intersects A. endcorollary*

Proof. This is a direct application of theorem 1.6.6, since basis elements are open sets. ■

Example 1.9. (1) We have the closure of $(0,1]$ in $\mathbb{R}$ is the closed interval $[0,1]$, since every neighborhood of 0 intersects $(0,1]$. Now every point outside of $[0,1]$ has a neighborhood disjoint from $[0,1]$ (take the neighborhood $(2,3)$ of 2).

(2) $\overline{\frac{1}{\mathbb{Z}^+}} = \{0\} \cup \frac{1}{\mathbb{Z}^+}$ and $\overline{\{0\} \cup (1,2))} = \{0\} \cup [1,2]$.

(3) $\overline{\mathbb{Q}} = \mathbb{R}$, $\overline{\mathbb{Z}^+} = \mathbb{Z}^+$, $\overline{\mathbb{R}^+} = \mathbb{R}^+ \cup \{0\}$. This first follows from the density of $\mathbb{Q}$ in $\mathbb{R}$. Every neighborhood $n \in \mathbb{Z}^+$ intersects $\mathbb{Z}^+$, so $\overline{Z^+} \subseteq \mathbb{Z}^+$, and we have that the neighborhood $(0,1)$ of 0 intersects $\mathbb{R}^+$, so $\overline{\mathbb{R}^+} \subseteq \mathbb{R}^+ \cup \{0\}$.

Definition. If $A \subseteq X$, with X a topological space, and if $x \in X$, we say that x is a **limit point** of A if every neighborhood of x intersects A at some distinct point. That is: $x \in \overline{X \backslash \{x\}}$.

Example 1.10. (1) Consider $(0,1]$, we have that $0 \in [0,1] = \overline{(0,1]} =$

$\{0\}, so 0 is a limit point of (0,1], the same can be said for any \mathrm{x} \in (0,1]$.

(2) For $\frac{1}{\mathbb{Z}^+}$, 0 is once again a limit point. Let $x \in \mathbb{R}$ be nonzero, and let $[x,b)$ be the neighborhood of x in the lower limit topology. Then $[x,b) \cap \frac{1}{\mathbb{Z}^+} = \emptyset$ or $\{x\}$, hence, 0 is the only limit point of $\frac{1}{\mathbb{Z}^+}$.

(3) $\overline{\{0\} \cup (1,2)} = \{0\} \cup [1,2]$ has all of its limit points in $[1,2]$. Likewise, every point in $\mathbb{R}$ is a limit point of $\mathbb{Q}$. $\mathbb{Z}^+$ has no limit points in $\mathbb{R}$, and the limit points of $\mathbb{R}^+$ are all the points of $\overline{\mathbb{R}^+}$.

Theorem 1.6.7. *Let $A \subseteq X$, X a topological space, and let A' be the set of all limit points in A. Then $\overline{A} = A \cup A'$.*

Proof. Let $x \in A'$, then every neighborhood of x intersects A at some distinct point x', by definition, so by theorem 1.6.6, $x \in \overline{A}$, hence $A' \subseteq \overline{A}$, so $A \cup A' \subseteq \overline{A}$. Now, let $x \in \overline{A}$. If $x \in A$, we are done. Otherwise, since every neighborhood of x intersects A, we have that they intersect at distinct points, thus $x \in A'$, therefore $\overline{A} \subseteq A \cup A'$. ■

Corollary. *$A \subseteq X$ is closed if and only if $A' \subseteq A$.*

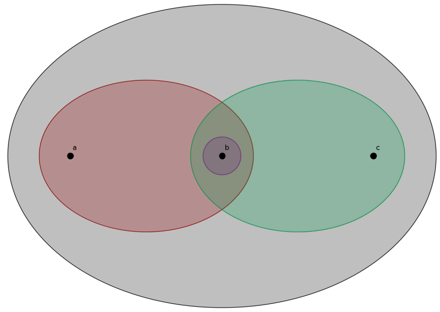

Figure 1.8: A topology on $\{a, b, c\}$, which turns out to be a Hausdorff space.

Proof. If A is closed, then $\overline{A} = A = A \cup A'$, thus $A' \subseteq A$. The converse is obvious. ∎

Definition. Let X be a topological space. A sequence $\{x_n\}$ is said to **converge** to a point $x \in X$ if for every neighborhood U of x, there is an $N \in \mathbb{Z}^+$ such that $x_n \in U$ for all $n \geq N$.

Example 1.11. Consider the following topological space on $\{a, b, c\}$ in figure 1.8, and define the sequence $\{x_n\}$ by $x_n = b$ for all $n \in \mathbb{Z}^+$. The neighborhoods of a, b, and c are $U_a = \{a, b\}$, $U_b = \{b\}$, and $U_c = \{b, c\}$. Now let $N > 0$, then we see that for all $n \geq N$, that $b \in U_b, U_a, U_c$, thus b converges to a and to c, and itself,

Definition. A topological space X is called a **Hausdorff space** if for each pair of distinct points x_1 , and x_2, there are neighborhoods U_1 and U_2 of x_1 and x_2 respectively such that U_1 and u_2 are disjoint.

Example 1.12. The topology of the previous example in figure **??** is not a Hausdorff space.

Theorem 1.6.8. *Every finite point set in a Hausdorff space is closed.*

Proof. Let X be a Hausdorff space, and let $x_0 \in X$. We have that $\overline{\{x_0\}} = \bigcap_{\{x_0\} \in U} U$. Now let $x \neq x_0 \in X$. Since $x \in \overline{\{x_0\}}$, and X is Hausdorff, the inters of the neighborhoods of x and x_0 is empty, thus $x \notin \overline{\{x_0\}}$, therefore $\overline{\{x_0\}} = \{x_0\}$. ∎

Remark. We can extend this proof to finite point sets of size n by induction.

Now the condition that finite point sets be closed need not depend on whether or not X is a Hausdorff space. In fact, we can assume the following for some topoltopological spaces.

Axiom 1.6.1 (The T_1 Axiom). *In any topological space, every finite point set of X is closed.*

Theorem 1.6.9. *Let X be a topological space satisfying the T_1 axiom, and let AX. Then a point x is a limit point of A if and only if every neighborhood of x contains infinitely manu points of A.*

Proof. Let U_x be a neighborhood of x. If U_x intersects A at infinitely many points of A, then it intersects A at a point distinct from x, thus x is a limit point of A.

Conversely suppose that x is a limit point of A, and let $U_x \cap A$ be finite, then $U_x \cap A \backslash \{x\}$. Now let $U_x \cap A \backslash \{x\} = \{x_1, \ldots x_m\}$. By the T_1 axiom, $\{x_1, \ldots, x_m\}$ is closed, so $X \backslash \{x_1, \ldots, x_m\}$ is open, thus $U_x \cap X \backslash \{x_1, \ldots, x_m\}$ is a neighborhood of x that does not intersect $A \backslash \{x\}$, which contradicts that x is a limit point. ■

Theorem 1.6.10. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point in X.*

Proof. Let $\{x_n\}$ be a sequence of points converging to x, and let $y \neq x$ and let U_x and U_y be neighborhoods of x and y respectively. Then $U_x \cap U_y = \emptyset$. Now since $\{x_n\}$ converges to x, we have that for $N > 0$, $x_n \in U_x$ whenever $n \geq N$. Then $x_n \notin U_y$, and so $\{x_n\}$ cannot converge to y. ■

Definition. Let $\{x_n\}$ be a sequence in a Hausdorff space X. If $\{x_n\}$ converges to a point $x \in X$, we call x the **limit** of $\{x_n\}$ and we write $\lim x_n = x$ or $\{x_n\} \to x$.

Theorem 1.6.11. *The following are true:*

(1) Every simply oredered set under the order topology is Hausdorff.

(2) The product of two Hausdorff spaces is Hausdorff.

(3) The subspace of a Hausdorff space is Hausdorff.

Proof. (1) Let X be an ordered set under the order topology. Take $x, y \in X$ distinct, and suppose without loss of generality that $x < y$. Then consider the neighborhoods $(-\infty, x]$ and $[y, \infty)$ of x and y respectively. Then $(-\infty, x] \cap [y, \infty) = \emptyset$.

(2) Let X and Y be Hausdorff, and consider $X \times Y$ in the product topology. Let $x_1 \times y_1$ and $x_2 \times y_2$ be distinct points, and let U_{x_1}, U_{x_2}, V_{y_1} and V_{y_2} be basis elements of x_1, x_2, y_1, and y_2 respectively. Then they are neighborhoods of those elements respectively.

Now we have that $U_{x_1} \times V_{y_1}$ and $U_{x_2} \times V_{y_2}$ are basis elements of $x_1 \times y_1$ and $x_2 \times y_2$, respectively, and hence neighborhoods of those elements respectively.Then we have $(U_{x_1} \times V_{y_1}) \cap (U_{x_2} \times V_{y_2}) = (U_{x_1} \cap U_{x_2}) \times (V_{y_1} \cap V_{y_2}) = \emptyset \times \emptyset = \emptyset)$.

(3) Let X be Hausdorff, and let Y be a subspace of X. Let x_1 and x_2 be distinct points, and let U_{x_1} and U_{x_2} be their neighborhoods. Since Y is open in X, then so are $Y \cap U_{x_1}$ and $Y \cap U_{x_2}$, so they are also neighborhoods of x_1 and x_2 respectively. Then $Y \cap U_{x_1} \cap Y \cap U_{x_2} = Y \cap (U_{x_1} \cap U_{x_2}) = \emptyset$.

■

1.7 Continuous Functions.