Convexity

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<u>**Text**</u>

A Course in Convexity

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November 1, 2020

Chapter 1

Convex Sets.

1.1 Definitions and Examples

Definition. We call the d-dimensional vector space $\mathbb{R}^d$ the **Euclidean space**, and it is the set of all vectors (also called points) $(x_1, \ldots, x_d)$, where $x_i \in \mathbb{R}$ for $1 \leq i \leq d$. We define the **Euclidean norm** to be the function $|| \cdot || : \mathbb{R}^d \to \mathbb{R}$ such that for $x = (x_1, \ldots, x_m) \in \mathbb{R}^d$, $||x|| = \sqrt{x_1^2 + \cdots + x_m^2}$; and we define the distance of of two points $x, y \in \mathbb{R}^d$ to be the function $\Delta : \mathbb{R}^d\mathbb{R}^d \to \mathbb{R}$ such that $\Delta(x, y) = ||x - y||$.

Definition. Let $x_1, x_2, \ldots, x_m$ be points in $\mathbb{R}^d$. We call a point $x \in \mathbb{R}^d$, of the form $x = \sum_{i=1}^m \alpha_i x_i$, with $\sum \alpha_i = 1$, a **convex combination** of x. We call the set of all convex combinations of a subset $A \subseteq \mathbb{R}^d$ the **convex hull** of A, and denote it $\hat{A}$.

Definition. Let $x, y \in \mathbb{R}^d$. We call the set $[x, y] = \{\alpha x + (1 - \alpha)y : 0 \leq \alpha \leq 1\}$ of all convex combinations of x and y an **interval** with **endpoints** x, y.

We call a set $A \subseteq \mathbb{R}^d$ **convex** if whenever $x, y \in A$, $[x, y] \in A$.

Example 1.1. The empty set, regular polyhedra, and open balls in $\mathbb{R}^d$ are all convex.

Lemma 1.1.1. *The convex hull of a convex set is convex.*

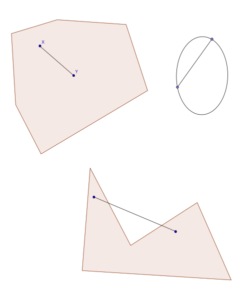

Figure 1.1: Two convex set, and a non-convex set

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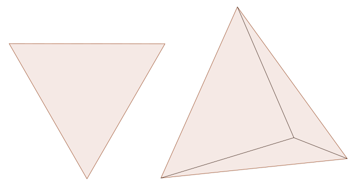

Figure 1.2: Regular polyhedra in $\mathbb{R}^2$ and $\mathbb{R}^3$
fig1.2

Proof. Let $A \subseteq \mathbb{R}^d$, and let $x, y \in \hat{A}$. Then by definition the set of all convex combinations of x and y is in $\hat{A}$, thus $[x, y] \in \hat{A}$ ∎

Definition. Let $c_1, \ldots, c_m \in \mathbb{R}^d$ and let $\beta_1, \ldots, \beta_m \in \mathbb{R}$; we call the set $A = \{x \in \mathbb{R}^d : \langle c_i, x\rangle \leq \beta_i, \text{ for } 1 \leq i \leq m\}$ a **regular polyhedron**.

Lemma 1.1.2. *Regular polyhedra in $\mathbb{R}^d$ are convex.*

Proof. Let $A \subseteq \mathbb{R}^d$ be a regular polyhedron and let $x, y \in A$. Then $\langle c_i, x\rangle, \langle c_i, y\rangle \leq \beta_i$ for $c_i \in \mathbb{R}^d$, $\beta_i \in \mathbb{R}$ for $1 \leq i \leq m$.Then by the scalar linearity of the innerproduct, $\langle c_i, \alpha x + (1-\alpha)y\rangle \leq \beta_i$. Thus $[x, y] \in$A. ∎