

# Analysis

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**Text**

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# Chapter 1

## The Real and Complex Numbers

### 1.1 Ordered Sets

**Definition.** Let  $S$  be any set. An **order** on  $S$  is a relation  $<$  such that:

- (1) For  $x, y \in S$ , one and only one of the following hold:

$$x < y$$

$$x = y$$

$$y < x$$

We call this property the **trichotomy law**

- (2)  $<$  is transitive over  $S$ .

We denote the relations  $>$  and  $\leq$  to mean  $x > y$  if and only if  $y < x$ , and  $x \leq y$  if and only if  $x < y$ , or  $x = y$ . We call  $S$  together with  $<$  an **ordered set**.

**Example 1.1.** Define  $<$  on  $\mathbb{Q}$  such that for  $r, s \in \mathbb{Q}$ ,  $r < s$  implies  $< 0s - r$ .

**Definition.** Let  $S$  be an ordered set, and let  $E \subseteq S$ . We say that  $E$  is **bounded above** if there is some  $\beta \in S$  for which  $x \leq \beta$ , for all  $x \in E$ . We say that  $E$  is **bounded below** if  $\beta \leq x$ , for all  $x \in E$ . We say an  $\alpha \in S$  is a **least upperbound** of  $E$ , if  $\alpha$  is an upperbound of  $E$ , and for all other upperbounds,  $\gamma$ , of  $E$ ,  $\alpha \leq \gamma$ . Likewise,  $\alpha$  is a **greatest lowerbound** of  $E$  if  $\alpha$  is a lowerbound of  $E$ , and for all other lowerbounds  $\gamma$  of  $E$ ,  $\gamma \leq \alpha$ . We denote the least upperbound, and greatest lowerbound by  $\sup E$  and  $\inf E$ , respectively.

**Lemma 1.1.1.** *Let  $S$  be an ordered set, and let  $E \subseteq S$ . Then  $E$  has (if they exist) a unique least upperbound, and a unique greatest lowerbound.*

*Proof.* Let  $\alpha, \beta \in S$  be least upperbounds of  $E$ . Then by definition, we have that  $\alpha \leq \beta$ , and  $\beta \leq \alpha$ ; thus by the trichotomy law,  $\alpha = \beta$ . The proof is the same for greatest lowerbounds. ■

**Example 1.2.** (1) Let  $A = \{p \in \mathbb{Q} : p^2 < 2\}$ , and  $B = \{p \in \mathbb{Q} : p^2 > 2\}$ . Clearly, we have that every element of  $B$  is an upperbound of  $A$ , and every element of  $A$  is a lowerbound of  $B$ . Now take  $p \in \mathbb{Q}$  a positive rational, and take  $q \in \mathbb{Q}$  such that  $q = p - \frac{p^2 - 2}{p + 2}$ . Then

$q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2}$ . Now if  $p \in A$ , then  $p^2 - 2 < 0$ , which implies that  $p < q$ , and  $q^2 < 2$ ; thus  $A$  has no largest element; similarly, if  $p \in B$ , then  $p^2 - 2 > 0$ , which implies that  $q < p$  and  $q^2 > 2$ , which shows that  $B$  has no least element. Thus  $\sup A$  and  $\inf B$  do not exist in  $\mathbb{Q}$ .

- (2) If  $\alpha = \sup E \in S$ , it may or may not be that  $\alpha \in E$ . Take  $E_1 = \{r \in \mathbb{Q} : r < 0\}$ , and  $E_2 = \{r \in \mathbb{Q} : r \leq 0\}$ . Then  $\sup E_1 = \sup E_2 = 0$ , but  $0 \notin E_1$ , where as  $0 \in E_2$
- (3) Consider the set  $\frac{1}{\mathbb{Z}^+} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ . By the well ordering principle, 1 is the least element, and is also an upperbound of all  $\frac{1}{n}$  for  $n > 1$ . Now also notice that as  $n$  gets arbitrarily large, then  $\frac{1}{n}$  gets arbitraritirly small; that is to say  $\frac{1}{n}$  “tends” to 0, so  $\sup \frac{1}{\mathbb{Z}^+} = 1 \in \frac{1}{\mathbb{Z}^+}$ , and  $\inf \frac{1}{\mathbb{Z}^+} = 0 \notin \frac{1}{\mathbb{Z}^+}$ .

**Definition.** We say an ordered set  $S$  has the **least upperbound property**, if whenever  $E \subseteq S$ , nonempty, and bounded above, then  $\sup E \in S$  exists; likewise,  $S$  has the **greatest lowerbound property** if whenever  $E$  is nonempty, bounded below then  $\inf E \in S$  exists.

**Example 1.3.** (1) The set of all rationals  $\mathbb{Q}$  does not have the least upperbound property, nor the greatest lowerbound property, take  $A, B$  as in the previous example. Letting  $E = \{1, \frac{1}{2}, \frac{1}{4}\} \subseteq \frac{1}{\mathbb{Z}^+}$ , we see that  $\frac{1}{\mathbb{Z}^+}$  satisfies both properties, with  $\sup E = 1$ , and  $\inf E = \frac{1}{4}$ .

- (2) Let  $A \subseteq \mathbb{R}$  be nonempty, and be bounded below. Then by the greatest lowerbound property,  $\alpha = \inf A \in \mathbb{R}$  exists; Then for all  $x \in A$ ,  $\alpha \leq x$ , and for all other lowerbounds  $\gamma$ ,  $\gamma \leq \alpha$ . Then  $-x \leq -\alpha$ , and  $-\alpha \leq -\gamma$ , then we see that  $-\gamma$  and  $-\alpha$  are upperbounds of  $-A$ , and that  $-\alpha$  is the least upperbound of  $-A$

**Theorem 1.1.2.** *If  $S$  is an ordered set with the least upperbound property, then  $S$  also inherits the greatest lowerbound property.*

*Proof.* Let  $B \subseteq S$ , and let  $L \subseteq S$  be the set of all lowerbounds of  $B$ . Then we have for any  $y \in L$ ,  $x \in B$ ,  $y \leq x$ . So every element of  $B$  is an upperbound of  $L$ , and  $L$  is nonempty, hence  $\alpha = \sup L \in S$  exists. Now if  $\gamma \leq \alpha$ , then  $\gamma$  is not an upperbound of  $L$ , hence  $\gamma \notin B$ ; thus  $\alpha \leq x$  for all  $x \in B$ , so  $\alpha \in L$ , and by definition of the greatest lowerbound, we get  $\alpha = \inf B$ . ■

## 1.2 Fields

**Definition.** A **field** is a set  $F$ , together with binary operations  $+$  and  $\cdot$  (called **addition** and **multiplication**, respectively) such that:

- (1)  $F$  forms an abelian group under  $+$ .
- (2)  $F \setminus \{0\}$  forms an abelian group under  $\cdot$  (where 0 is the additive identity of  $F$ ).
- (3)  $\cdot$  distributes over  $+$ .

We now state the following propositions without proof.

**Proposition 1.2.1.** *For all  $x, y, z \in F$ :*

- (1)  $x + y = x + z$  implies  $y = z$
- (2)  $x + y = x$  implies  $y = 0$
- (3)  $x + y = 0$  implies  $y = -x$
- (4)  $-(-x) = x$ .

**Proposition 1.2.2.** *For all  $x, y, z \in F \setminus \{0\}$ :*

- (1)  $xy = xz$  implies  $y = z$
- (2)  $xy = x$  implies  $y = 1$
- (3)  $xy = 1$  implies  $y = x^{-1}$
- (4)  $(x^{-1})^{-1} = x$ .

**Proposition 1.2.3.** *For all  $x, y, z \in F$ :*

- (1)  $0x = 0$
- (2)  $x \neq 0$  and  $y \neq 0$  implies  $xy \neq 0$
- (3)  $(-x)y = -(xy) = x(-y)$
- (4)  $(-x)(-y) = xy$ .

**Definition.** An **ordered field** is a field  $F$  that is also an ordered set, such that:

- (1)  $x + y < x + z$  whenever  $y < z$ , for  $x, y, z \in F$
- (2)  $xy > 0$  whenever  $x > 0$  and  $y > 0$ , for  $x, y \in F$ .

**Proposition 1.2.4.** *Let  $F$  be an ordered field, then for any  $x, y, z \in F$ , the following hold:*

- (1)  $x > 0$  implies  $-x < 0$ .
- (2) If  $x > 0$  and  $y < z$ , then  $xy < xz$ .
- (3) If  $x < 0$  and  $y < z$ , then  $xz < xy$ .
- (4) If  $x \neq 0$ , then  $x^2 > 0$ , in particular,  $1 > 0$ .
- (5)  $0 < x < y$  implies that  $0 < y^{-1} < x^{-1}$ .

*Proof.* (1) If  $x > 0$ , then  $0 = x + (-x) > 0 + (-x)$ , so  $-x < 0$ .

(2) We have  $0 < z - y$ , so  $0 < x(z - y) = xz - xy$ , so  $xy < xz$ .

(3) Do the same as (2), multiplying  $z - y$  by  $-x$ .

(4) If  $x > 0$ , we are done. Now suppose that  $x < 0$ , then  $-x > 0$ , so  $(-x)(-x) = xx = x^2 > 0$ ; in particular, we also have that  $1 \neq 0$ , and  $1 = 1^2$ , so  $1 > 0$ .

(5) We have  $0 < xy^{-1} < yy^{-1} = 1$ , then  $0 < x^{-1}xy^{-1} = y^{-1} < x^{-1}1 = x^{-1}$

■

### 1.3 The Field of Real Numbers

**Theorem 1.3.1.** *There exists an ordered field  $\mathbb{R}$  with the least upperbound property, such that  $\mathbb{Q} \subseteq \mathbb{R}$ .*

**Definition.** We call the field  $\mathbb{R}$  the **field of real numbers**, and we call the elements of  $\mathbb{R}$  **real numbers**.

**Definition.** Let  $S$  be an ordered field, and let  $E \subseteq S$ . We say that  $E$  is **dense** in  $S$ , if for all  $r, s \in S$ , with  $r < s$ , there is an  $\alpha \in E$  such that  $r < \alpha < s$ .

**Theorem 1.3.2** (The Archimedean Principle). *If  $x, y \in \mathbb{R}$ , and  $x > 0$ , then there is an  $n \in \mathbb{Z}^+$  such that  $nx > y$ .*

*Proof.* Let  $A = \{nx : n \in \mathbb{Z}^+\}$ , and suppose that  $nx \leq y$ . Then  $y$  is an upperbound of  $A$ , and since  $A$  is nonempty,  $\alpha = \sup A \in \mathbb{R}$ , since  $x > 0$ , we have  $\alpha - x < \alpha$ , so  $\alpha - x$  is not an upperbound of  $A$ . Hence  $\alpha - x < mx$  for some  $m \in \mathbb{Z}^+$ . Then  $\alpha < (1 + m)x \in A$ , contradicting that  $\alpha$  is an upperbound of  $A$ . ■

**Theorem 1.3.3** (The density of  $\mathbb{Q}$  in  $\mathbb{R}$ ).  *$\mathbb{Q}$  is dense in  $\mathbb{R}$ .*

*Proof.* Let  $x < y$  be real numbers, then  $y - x > 0$ , so by the Archimedean principle, there is an  $n \in \mathbb{Z}^+$  for which  $n(y - x) > 1$ . By the Archimedean principle again, we have  $m_1, m_2 \in \mathbb{Z}^+$  for which  $m_1 > nx$  and  $m_2 > -nx$ , thus  $-m_2 < nx < m_1$ , and we also have that there is an  $m \in \mathbb{Z}^+$  for which  $-m_2 < m < m_1$ , and  $m - 1 \leq nx < m$ . Thus combining inequalities, we get  $nx < m < ny$ , thus  $x < \frac{m}{n} < y$ . ■

**Theorem 1.3.4** (The existence of  $n^{\text{th}}$  roots of positive reals). *For every real number  $X > 0$ , and for every  $n \in \mathbb{Z}^+$ , there is one, and only one positive real number  $y$  for which  $y^n = x$ .*

*Proof.* Let  $y > 0$  be a real number; then  $y^n > 0$ , so there is at most one such  $y$  for which  $y^n = x$ . Now let  $E = \{t \in \mathbb{R} : t^n < x\}$ , choosing  $t = \frac{x}{1+x}$ , we see that  $0 \leq t < 1$ , hence  $t^n < t < x$ , so  $E$  is nonempty. Now if  $1 + x < t$ , then  $t^n \geq x$ , so  $t \notin E$ , and  $E$  has  $1 + x$  as an upperbound. Therefore,  $\alpha = \sup E \in \mathbb{R}$  exists.

Now suppose that  $y^n < x$ , choose  $0 \leq h < 1$  such that  $h < \frac{x - y^n}{n(y+1)^{n-1}}$ , then  $(y + h)^n - y^n < hn(y + h)^{n-1} < hn(y + 1)^{n-1} < x - y^n$ , thus  $(y + h)^n < x$ , so  $y + h \in E$ , contradicting that  $y$  is an upperbound. On the other hand, if  $y^n > x$ , choosing  $k = \frac{y^n - x}{ny^{n-1}}$ , then  $0 \leq k < y$ , and letting  $t \geq y - k$ , we get that  $y^n - t^n \leq y^n + (y - k)^n < kny_{n-1} = y^n - x^n$ , so  $t^n \geq x$ , making  $y - k$  an upperbound of  $E$ , which contradicts  $y = \sup E$ . ■

*Remark.* We denote  $y$  as  $\sqrt[n]{x}$ , or as  $x^{\frac{1}{n}}$ .

**Corollary.** *If  $a, b \in \mathbb{R}$ , with  $a, b > 0$ , and  $n \in \mathbb{Z}^+$ , then  $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ .*

*Proof.* Let  $\alpha = \sqrt[n]{a}$ , and  $\beta = \sqrt[n]{b}$ . Then  $\alpha^n = a$ , and  $\beta^n = b$ , so  $ab = \alpha^n \beta^n = (\alpha\beta)^n$ , we are done. ■

**Definition.** We define the **extended real number system** to be the field  $\mathbb{R}$ , together with symbols  $\infty$ , and  $-\infty$ , called **positive infinity** and **negative infinity**, such that  $-\infty < x < \infty$  for all  $x \in \mathbb{R}$ . We call elements of the extended real numbers **infinite**, and every other element not in the extended real numbers **finite**.

**Lemma 1.3.5.**  $\infty$  is an upperbound for every subset  $E$ , of  $\mathbb{R}$ , and  $-\infty$  is a lowerbound for every subset  $E$  of  $\mathbb{R}$ . Moreover, if  $E$  is not bounded above, then  $\sup E = \infty$ , and if  $E$  is not bounded below, then  $\inf E = -\infty$ .

*Remark.* We make the following assumptions for extended real numbers:

- (1) If  $x \in \mathbb{R}$ , then  $x + \infty = \infty$ ,  $x - \infty = -\infty$ , and  $\frac{x}{\infty} = \frac{x}{-\infty} = 0$ .
- (2) If  $x > 0$ , then  $x(\infty) = \infty$  and  $x(-\infty) = -\infty$ .
- (3) If  $x < 0$ , then  $x(\infty) = -\infty$  and  $x(-\infty) = \infty$ .

## 1.4 The Complex Field

**Definition.** We define a **complex number** to be a pair of real numbers  $(a, b)$ . We denote the set of all complex numbers by  $\mathbb{C}$ . We define the **addition** and **multiplication** of complex numbers to be the binary operations  $+: \mathbb{C} \rightarrow \mathbb{C}$  and  $\cdot: \mathbb{C} \rightarrow \mathbb{C}$  such that

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d) \\ (a, b)(c, d) &= (ac - bd, ad + bc)\end{aligned}$$

Lastly, we define  $i$  to be the complex number such that  $i = (0, 1)$ .

**Theorem 1.4.1.**  $\mathbb{C}$  forms a field together with  $+$  and  $\cdot$ .

**Theorem 1.4.2.** For  $(a, 0), (b, 0) \in \mathbb{C}$ ,  $(a, 0) + (b, 0) = (a + b, 0)$ , and  $(a, 0)(b, 0) = (ab, 0)$ .

*Proof.* This is a straightforward application of the addition and multiplication of complex numbers. ■

**Theorem 1.4.3.**  $i^2 = -1$ .

*Proof.*  $i^2 = (0, 1)(0, 1) = (0 - 1, 1 - 1) = (-1, 0) = -1$ . ■

**Theorem 1.4.4.** Let  $(a, b) \in \mathbb{C}$ , then  $(a + b) = a + ib$ .

*Proof.*  $(a, b) = (a, 0) + (0, b) = (a, 0) + (0, 1)(b, 0) = a + ib$ . ■

**Definition.** Let  $a, b \in \mathbb{R}$ , and let  $z \in \mathbb{C}$  such that  $z = a + ib$ . We define the **complex conjugate** of  $z$  to be the complex number  $\bar{z} = a - ib$ . Moreover, we define the **real part** of  $z$  to be  $a$ , and the **imaginary part** of  $z$  to be  $b$ , and we denote them  $a = \operatorname{Re} z$ ,  $b = \operatorname{Im} z$

**Theorem 1.4.5.** Let  $z, w \in \mathbb{C}$ . Then

- (1)  $\overline{z + w} = \bar{z} + \bar{w}$ .
- (2)  $\overline{zw} = \bar{z}\bar{w}$ .
- (3)  $z + \bar{z} = 2 \operatorname{Re} z$  and  $z - \bar{z} = 2i \operatorname{Im} z$ .

(4)  $z\bar{z}$  is a nonnegative real number.

*Proof.* Let  $z = a + ib$ , and let  $w = c + id$ . Then  $z + w = (a + c) + i(b + d)$ , so  $\overline{z + w} = (a + c) - i(b + d) = (a - ib) + (c - id) = \bar{z} + \bar{w}$ ; similarly, we get  $\overline{zw} = \bar{z}\bar{w}$ . Moreover, we have  $(a + ib) + (a - ib) = 2a$ , and  $(a + ib) - (a - ib) = 2ib$ , we also have that  $z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 \geq 0$ , and  $z\bar{z} = 0$  if and only if  $a = b = 0$ . ■

**Definition.** Let  $z \in \mathbb{C}$ . We define the **modulus** of  $z$  to be  $|z| = \sqrt{z\bar{z}}$ .

*Remark.*  $|z|$  exists and is unique.

**Theorem 1.4.6.** Let  $z, w \in \mathbb{C}$ , then:

(1)  $|z| \geq 0$  and  $|z| = 0$  if and only if  $z = 0$ .

(2)  $|\bar{z}| = |z|$ .

(3)  $|zw| = |z||w|$ .

(4)  $\operatorname{Re} z \leq |z|$ .

(5)  $|z + w| \leq |z| + |w|$ .

*Proof.* Let  $z = a + ib$ , and  $w = c + id$ . Then  $|z| = \sqrt{a^2 + b^2} \geq 0$ , and  $|z| = 0$  if and only if  $a, b = 0$ . Moreover,  $|\bar{z}| = |a + i(-b)| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$ . We also have  $|zw|^2 = (a^2 + b^2)(c^2 + d^2) = |z|^2|w|^2$ , likewise,  $|\operatorname{Re} z| = |a + i0| = \sqrt{a^2} \leq \sqrt{a^2 + b^2}$ . Finally we prove (5).

We have  $|z + w|^2 = (z + w)(\bar{z} + \bar{w}) = z\bar{z} + \bar{z}w + \bar{w}z + w\bar{w} = |z|^2 + w\operatorname{Re} z\bar{w} + |w|^2 \leq |z|^2 + 2|s\bar{w}| + |w|^2 = (|z| + |w|)^2$ . ■

**Theorem 1.4.7** (The Cauchy Schwarz Inequality). Let  $a_i, b_i \in \mathbb{C}$ , for  $1 \leq i \leq n$ . Then:

$$\left| \sum_{i=1}^n a_i \bar{b}_i \right| \leq \sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2 \quad (1.1)$$

*Proof.* Let  $A = \sum |a_j|^2$ ,  $B = \sum |b_i|^2$ , and  $C = \sum a_i \bar{b}_i$ . If  $B = 0$ , then  $b_i = 0$  for  $1 \leq i \leq n$ , and we are done; so suppose that  $B > 0$ . Then

$$\begin{aligned} \sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(\overline{Ba_j - Cb_j}) \\ &= B \sum |a_j|^2 - B\bar{C} \sum a_j \bar{b}_j - BC \sum \bar{a}_j b_j + |C|^2 \sum |b_j|^2 \\ &= (B^2 A - B|C|^2) = B(AB - |C|^2) \geq 0 \end{aligned}$$

Since  $B > 0$ , we get  $|C|^2 \leq AB$  as required. ■



## 1.5 Euclidean Spaces

**Definition.** Let  $k \in \mathbb{Z}^+$ , and let  $\mathbb{R}^k$  be the set of all ordered  $k$ -tuples  $(x_1, x_2, \dots, x_k)$ , with  $x_i \in \mathbb{R}$  for  $1 \leq i \leq k$ . We call  $\mathbb{R}^k$  the **Euclidean space** of **dimension**  $k$ ; more simply the **Euclidean  $k$ -space**. We call elements of  $\mathbb{R}^k$  **vectors** or **points**; and we define **vector addition** and **scalar multiplication** to be:

$$\begin{aligned}(x_1, \dots, x_k) + (y_1, \dots, y_k) &= (x_1 + y_1, \dots, x_k + y_k) \\ \alpha(x_1, \dots, x_k) &= (\alpha x_1, \dots, \alpha x_k)\end{aligned}$$

for  $(x_1, \dots, x_k), (y_1, \dots, y_k) \in \mathbb{R}^k$  and  $\alpha \in \mathbb{R}$ .

**Theorem 1.5.1.**  $\mathbb{R}^k$  forms a vector space together with vector addition and scalar multiplication.

**Definition.** Let  $x, y \in \mathbb{R}^k$ . We define the **inner product** of  $x$  and  $y$  to be the binary operation  $\langle \cdot, \cdot \rangle : \mathbb{R}^k \mathbb{R}^k \rightarrow \mathbb{R}$  such that

$$\langle x, y \rangle = \sum_{i=1}^k x_i y_i$$

We define the **norm** of  $x$  to be  $\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\sum x_i^2}$ .

**Theorem 1.5.2.** Let  $x, y \in \mathbb{R}^k$ , and  $\alpha \in \mathbb{R}$ . Then:

- (1)  $\|x\| \geq 0$  and  $\|x\| = 0$  if and only if  $x_i = 0$  for all  $1 \leq i \leq k$ .
- (2)  $\|\alpha x\| = |\alpha| \|x\|$ .
- (3)  $|\langle x, y \rangle| \leq \|x\| \|y\|$ .
- (4)  $\|x + y\| \leq \|x\| + \|y\|$ , and  $\|x - z\| \leq \|x - y\| + \|y - z\|$

*Proof.* (1) follows by definition of the norm. We also have that  $\|\alpha x\| = \sqrt{\sum \alpha^2 x_i^2} = \sqrt{\alpha^2} \sqrt{\sum x_i^2} = |\alpha| \|x\|$ .

Now by the Cauchy Schwarz inequality, we have that  $|\langle x, y \rangle|^2 = \sum x_i^2 y_i^2 \leq \sum x_i^2 \sum y_i^2 = \|x\| \|y\|$ . Finally we have that  $\|x + y\|^2 = \langle x + y, x + y \rangle = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \leq \|x\|^2 + 2\|x\| \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$ , the last result follows immediately. ■



# Chapter 2

## Topological Foundations

### 2.1 Finite, Countable, and Uncountable Sets

**Definition.** Let  $A$  be a set, and let  $E \subseteq \mathbb{N}$ . We say that  $A$  is **finite** if there exists a 1-1 mapping of  $A$  onto  $E$ , we say  $A$  is **countable** if  $E = \mathbb{N}$ , and we say  $A$  is **atmost countable** if  $A$  is either finite or countable.

**Example 2.1.** The set of all integers  $\mathbb{Z}$  is countable. Take  $f : \mathbb{N} \rightarrow \mathbb{Z}$  such that  $f(n) = 2$  if  $n$  is even, and  $f(n) = -n$  if  $n$  is odd.

**Definition.** Let  $A$  be a set, and let  $E \subseteq \mathbb{N}$ . A **sequence** in  $A$  is a mapping  $f : E \rightarrow A$  such that  $f(n) = x_n$ , for  $x_n \in A$ . We call the values of  $f$  **terms** of the sequence. We denote sequences by  $\{x_n\}_{n=1}^{\infty}$ , and when  $E = \mathbb{N}$ , we denote them simply by  $\{x_n\}$ .

**Theorem 2.1.1.** *Every infinite subset of a countable set is countable.*

*Proof.* Let  $A$  be countable, and let  $E \subseteq A$  be infinite. Arrange the elements of  $A$  into a sequence  $\{x_n\}$ , and construct a sequence  $\{n_k\}$  such that  $n_1$  is the least term for which  $\{x_{n_k}\} \in E$ , and  $n_k$  is the least term greater than  $n_{k-1}$  for which  $x_{n_k} \in E$ . Let  $f(k) = x_{n_k}$ , and we get a 1-1 mapping of  $\mathbb{N}$  onto  $E$ . ■

**Theorem 2.1.2.** *Let  $\{E_n\}$  be a sequence of countable sets. Then  $S = \bigcup E_n$  is also countable.*

*Proof.* Arrange every set  $E_n$  in a sequence  $\{x_{nk}\}$ , and consider the infinite array  $(x_{ij})$ , in which the elements of  $E_n$  form the  $n$ -th row. Then  $(x_{ij})$  contains all the elements of  $S$ , and we can arrange them in a sequence

$$x_{11}, (x_{21}, x_{12}), (x_{31}, x_{22}, x_{13}), \dots$$

Moreover, if  $E_i \cap E_j \neq \emptyset$ , for  $i \neq j$ , then the elements of  $E_i \cap E_j$  appear more than once in the sequence of  $S$ ; so taking  $T \subseteq \mathbb{N}$ , we get a 1-1 mapping of  $T$  onto  $S$ , hence  $S$  is atmost countable, and since  $E_i \subseteq S$  for  $i \in \mathbb{N}$ , is infinite, by theorem 2.1.1,  $S$  is infinite, thus  $S$  is countable. ■

Figure 2.1: The infinite array  $(x_{ij})$ 

**Corollary.** Let  $A$  be atmost countable, and suppose for all  $\alpha \in A$  that the sets  $B_\alpha$  are atmost countable. Then

$$T = \bigcup_{\alpha \in A} B_\alpha$$

is atmost countable.

**Theorem 2.1.3.** Let  $A$  be countable, and let  $B_n$  be the set of all  $n$ -tuples  $(a_1, \dots, a_n)$  such that  $a_i \in A$  for  $1 \leq i \leq n$ . Then  $B_n$  is countable.

*Proof.* By induction on  $n$ , we have that  $B_1 = A$ , which is countable. Now suppose that  $B_n$  is countable, and consider  $B_{n+1}$  whose elements are of the form  $(b, a)$  where  $b \in B_n$  and  $a \in A$ . Fixing  $b$ , we get a 1-1 correspondence between the elements of  $B_{n+1}$  and  $A$ ; therefore  $B$  is countable. ■

**Corollary.**  $\mathbb{Q}$  is countable.

*Proof.* For every rational  $\frac{p}{q} \in \mathbb{Q}$ , represent  $\frac{p}{q}$  as  $(p, q)$ . Then the countability of  $\mathbb{Q}$  follows from theorem 2.1.3. ■

**Theorem 2.1.4.** Let  $A$  be the set of all sequences of 0 and 1; then  $A$  is uncountable.

*Proof.* Let  $EA$  be countable, and let  $E$  consist of all the sequences of 0 and 1,  $s_1, s_2, s_3, \dots$ . Construct the sequence  $s$  such that if the  $n$ -th term of the sequence  $s_i$  is 0, then the  $n$ -th term of  $s$  is 1, and vice versa, for  $i \in \mathbb{Z}^+$ . Then the sequence  $s$  differs from the sequence  $s_i$  at atleast one place; thus  $s \notin E$ , but  $s \in A$ . Therefore  $E \subset A$ , which establishes the uncountablitiy of  $A$ . ■

## 2.2 Metric Spaces

**Definition.** A set  $X$ , whose elements we will call **points**, is said to be a **metric space** if there exists a mapping  $d : X \times X \rightarrow \mathbb{R}$ , called a **metric** (or **distance function**) such that for  $x, y \in X$

- (1)  $d(x, y) \geq 0$ , and  $d(x, y) = 0$  if and only if  $x = y$ .
- (2)  $d(x, y) = d(y, x)$ .
- (3)  $d(x, y) \leq d(x, z) + d(z, y)$  (The Triangle Inequality).

**Example 2.2.** The absolute value,  $|\cdot|$  for real numbers, the modulus  $|\cdot|$  for complex numbers, and the norm  $\|\cdot\|$  for vectors are all metrics. They turn  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{R}^k$  into metric spaces respectively.

**Definition.** An **open interval** in  $\mathbb{R}$  (or **segment**) is a set of the form  $(a, b) = \{a, b \in \mathbb{R} : a < x < b\}$ , a **closed interval** in  $\mathbb{R}$  is a set of the form  $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ ; and **half open intervals** in  $\mathbb{R}$  are sets of the form  $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$  and  $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ .

If  $a_i < b_i$ , for  $1 \leq i \leq k$ , the set of all points  $(x_1, \dots, x_k) \in \mathbb{R}^k$  which satisfy the Inequalities  $a_i \leq x_i \leq b_i$  is called a **k-cell** in  $\mathbb{R}^k$ . If  $x \in \mathbb{R}^k$ , and  $r > 0$ , we call the set  $B_r(x) = \{y \in \mathbb{R}^k : \|x - y\| < r\}$  an **open ball** in  $\mathbb{R}^k$ , and we call the set  $B_r[x] = \{y \in \mathbb{R}^k : \|x - y\| \leq r\}$  a **closed ball** in  $\mathbb{R}^k$ .

**Definition.** We call a set  $E \subseteq \mathbb{R}^k$  **convex**, if whenever  $x, y \in E$ ,  $\lambda x + (1 - \lambda)y \in E$  for  $0 < \lambda < 1$ .

**Lemma 2.2.1.** *Open and closed balls, along with k-cells are convex.*

*Proof.* Let  $B_r(x)$  be an open ball; let  $y, z \in B_r(x)$ , and  $0 < \lambda < 1$ . Then  $\|x - (\lambda y + (1 - \lambda)z)\| = \|\lambda(x - y) + (1 - \lambda)(x - z)\| \leq \lambda\|x - y\| + (1 - \lambda)\|x - z\| < \lambda r + (1 - \lambda)r$ . The proof is analogous for closed ball.

Now let  $K$  be a  $k$ -cell for  $a_i < b_i$ , for  $1 \leq i \leq k$ , let  $x, y \in K$ , then  $a_i \leq x_i, y_i \leq b_i$ , so  $\lambda a_i \leq \lambda x_i \leq \lambda b_i$ , and  $(1 - \lambda)a_i \leq (1 - \lambda)y_i \leq (1 - \lambda)b_i$ , since  $0 < \lambda < 1$ ,  $a_i \leq \lambda a_i + (1 - \lambda)a_i \leq \lambda x_i + (1 - \lambda)y_i \leq \lambda b_i + (1 - \lambda)b_i \leq b_i$ . ■

**Corollary.** *Open and closed intervals, along with half open intervals are convex.*

*Proof.* We just notice that open and closed intervals are open and closed balls in  $\mathbb{R}^1 = \mathbb{R}$ , we also notice that half open intervals  $[a, b)$  and  $(a, b]$  are subsets of the closed interval  $[a, b]$ , and hence inherit convexity. ■

For the following definitions, let  $X$  be a metric space with metric  $d$ .

**Definition.** A **neighborhood** of a point  $x \in X$  is the set  $N_r(x) = \{y \in X : d(x, y) < r\}$  for some  $r > 0$  called the **radius** of the neighborhood. We call  $x$  a **limit point** of a set  $E \subseteq X$  if every neighborhood of  $x$  contains a point  $y \neq x$  such that  $y \in E$ . If  $y \in E$ , and  $y$  is not a limit point, we call  $y$  an **isolated point**.

**Definition.** We call a set  $E \subseteq X$  **closed** if every limit point of  $E$  is in  $E$ . A point  $x \in X$  is an **interior point** of  $E$  if there is a neighborhood  $N$  of  $x$  such that  $N \subseteq E$ . We call  $E$  **open** if every point of  $E$  is an interior point of  $E$ .

**Definition.**  $E \subseteq X$  is called **perfect** if  $E$  is closed, and every point of  $E$  is a limit point of  $E$ . We call  $E$  **dense** if every point of  $X$  is either a limit point of  $E$ , or a point of  $E$ , or both.

**Definition.** We call  $E \subseteq X$  **bounded** if there is a real number  $M > 0$ , and a point  $y \in X$  such that  $d(x, y) < M$  for all  $x \in E$ .

**Theorem 2.2.2.** *Let  $X$  be a metric space and  $x \in X$ . Every neighborhood of  $x$  is open.*

*Proof.* Consider the neighborhood  $N_r(x)$ , and  $y \in E$ , there is a positive real number  $h$  such that  $d(x, y) = r - h$ , then for  $z \in X$  such that  $d(y, z) < h$ , we have  $d(x, z) \leq d(x, y) + d(y, z) < r - h + h = r$ , thus  $z \in E$ , so  $y$  is an interior point of  $E$ . ■

**Theorem 2.2.3.** *If  $x$  is a limit point of a set  $E$ , then every neighborhood of  $x$  contains infinitely many points of  $E$ .*

*Proof.* Let  $N$  be a neighborhood of  $x$  containing only a finite number of points of  $E$ . Let  $y_1, \dots, y_n$  be points of  $N \cap E$  distinct from  $x$  and let  $r = \min\{d(x, y_i)\}$  for  $1 \leq i \leq n$ , then  $r > 0$ , and the neighborhood  $N_r(x)$  contains no point  $y$  of  $E$  for which  $y \neq x$ , so  $x$  is not a limit point; which is a contradiction. ■

**Corollary.** *A finite point set has no limit points.*

*Proof.* By theorem 2.2.3, if  $x$  is a limit point in the finite point set  $E$ , then every neighborhood of  $x$  contains infinitely many points of  $E$ ; contradicting its finiteness. ■

**Example 2.3.** (1) The set of all  $z \in \mathbb{C}$  such that  $|z| < 1$  is open, and bounded.

(2) The set of all  $z \in \mathbb{C}$  for which  $|z| \leq 1$  is closed, perfect, and bounded.

(3) Any nonempty finite set is closed, and bounded.

(4)  $\mathbb{Z}$  is closed, but it is not open, perfect, or bounded.

(5) The set  $\frac{1}{\mathbb{Z}_+}$  is neither closed, nor open, it is not perfect; but it is bounded.

(6)  $\mathbb{C}$  is closed, open, and perfect, but it is not bounded.

(7) The open interval in  $(a, b)$  is open (only in  $\mathbb{R}$ ), and bounded.

**Theorem 2.2.4.** *Let  $X$  be a metric space, a set  $E \subseteq X$  is open if and only if  $X \setminus E$  is closed.*

*Proof.* Suppose that  $X \setminus E$  is closed, let  $x \in E$ , then  $x \notin X \setminus E$ , and  $x$  is not a limit point of  $X \setminus E$ . Thus there is a neighborhood  $N$  of  $x$  such that  $N \cap (X \setminus E) = \emptyset$ , thus  $N \subseteq E$ , and so  $x$  is an interior point of  $E$ .

Conversely, suppose that  $E$  is open, and let  $x$  be a limit point of  $X \setminus E$ , then every neighborhood of  $x$  contains a point of  $X \setminus E$ , so  $x$  is not an interior point of  $E$ , since  $E$  is open, it follows that  $x \in X \setminus E$ , thus  $X \setminus E$  is closed. ■

**Corollary.**  *$E$  is closed if and only if  $X \setminus E$  is open.*

*Proof.* This is the converse of theorem 2.2.4. ■

**Theorem 2.2.5.** *Let  $X$  be a metric space. The following are true:*

(1) *If  $\{G_\alpha\}$  is a collection of open sets, then  $\bigcup G_\alpha$  is open.*

(2) *If  $\{G_i\}_{i=1}^n$  is a finite collection of open sets, then  $\bigcap_{i=1}^n G_i$  is open.*

(3) *if  $\{G_\alpha\}$  is a collection of closed sets, then  $\bigcap G_\alpha$  is closed.*

(4) If  $\{G_i\}_{i=1}^n$  is a finite collection of closed sets, then  $\bigcup_{i=1}^n G_i$  is closed.

*Proof.* Let  $G = \bigcup G_\alpha$ , then if  $x \in G$ ,  $x \in G_\alpha$  for some  $\alpha$ , then  $x$  is an interior point of  $G_\alpha$ , hence an interior point of  $G$ , so  $G$  is open. Now let  $G = \bigcap_{i=1}^n G_i$ . For  $x \in G$ , there are neighborhoods  $N_i$  of  $x$ , with radii  $r_i$  such that  $N_i \subseteq G_i$  for  $1 \leq i \leq n$ . Then let  $r = \min\{r_1, \dots, r_n\}$ , and let  $N$  be the neighborhood of  $x$  with radius  $r$ , then  $N \subseteq G_i$ , hence  $N \subseteq G$ , so  $G$  is open.

The proofs of (3) and (4) are just the converse of the proofs of (1) and (2). ■

**Definition.** Let  $X$  be a metric space, and let  $E \subseteq X$ , and let  $E'$  be the set of all limit points of  $E$ . We define the **closure** of  $E$  to be the set  $\overline{E} = E \cup E'$ .

**Theorem 2.2.6.** If  $X$  is a metric space, and  $E \subseteq X$ , then the following hold

(1)  $\overline{x}$  is closed.

(2)  $E$  is closed if and only if  $E = \overline{E}$ .

(3) If  $F \subseteq X$  such that  $E \subseteq F$ , and  $F$  is closed, then  $\overline{E} \subseteq F$ .

*Proof.* If  $x \in X$ , and  $x \notin \overline{E}$ , then  $x \notin E$ , nor is it a limit point of  $E$ , thus there is a neighborhood of  $x$  that is disjoint from  $E$ , hence  $X \setminus \overline{E}$  is open.

Now if  $E$  is closed, then  $E' \subseteq E$ , so  $\overline{E} = E$ , conversely, if  $E = \overline{E}$ , then clearly  $E$  is closed. Now if  $F$  is closed and  $E \subseteq F$ , then  $F' \subseteq F$ , and  $E' \subseteq F$ , therefore  $\overline{E} \subseteq F$ . ■

**Theorem 2.2.7.** Let  $E \subseteq \mathbb{R}$  be nonempty and bounded above, let  $y = \sup E$ , then  $y \in \overline{E}$ , hence  $y \in E$  if  $E$  is closed.

*Proof.* Suppose that  $y \notin E$ , then for every  $h > 0$ , there exists a point  $x \in E$  such that  $y - h < x < y$ , then  $y$  is a limit point of  $E$ , thus  $y \in \overline{E}$ . ■

**Theorem 2.2.8.** Let  $Y \subseteq X$ ; a subset  $E$  of  $Y$  is open in  $Y$  if and only if  $E = Y \cap G$  for some open subset  $G$  of  $X$ .

*Proof.* Suppose  $E$  is open in  $Y$ , then for each  $x \in E$ , there is a  $r_p > 0$  such that  $d(x, y) < r_p$ , if  $y \in Y$ , that implies that  $y \in E$ ; hence let  $V_x$  be the set of all  $y \in X$  such that  $d(x, y) < r_p$ , and define

$$G = \bigcup_{x \in E} V_x$$

Then by theorems 2.2.2 and 2.2.5,  $G$  is open in  $X$ , and  $E \subseteq G \cap Y$ . Now we also have that  $V_p \cap Y \subseteq E$ , thus  $G \cap Y \subseteq E$ , thus  $E = G \cap Y$ . Conversely, if  $G$  is open in  $X$ , and  $E = G \cap Y$ , then every  $x \in E$  has a neighborhood  $V_p \subseteq G$ , thus  $V_p \cap Y \subseteq E$ , hence  $E$  is open in  $Y$ . ■

## 2.3 Compact Sets