

Topology

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Text

Topology (2rd edition)

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Chapter 1

Topological Spaces and Continuous Functions.

1.1 Topological Spaces.

Definition. A **topology** on a set X is a collection \mathcal{T} of subsets of X such that:

- (1) $\emptyset, X \in \mathcal{T}$.
- (2) For any subcollection $\{U_\alpha\}$ of subsets of X , $\bigcup_\alpha U_\alpha \in \mathcal{T}$.
- (3) For any finite subcollection $\{U_i\}_{i=1}^n$ of subsets of X , $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

We call the pair (X, \mathcal{T}) a **topological space**, and we call the elements of \mathcal{T} **open sets**.

Example 1.1. (1) Let X be any set, the collection of all subsets of X , 2^X is a topology on X , which we call the **discrete topology**. We call the topology $\mathcal{T} = \{\emptyset, X\}$ the **indiscrete topology**.

- (2) The set of three points $\{a, b, c\}$ has the 9 following topologies in figure 1.1.

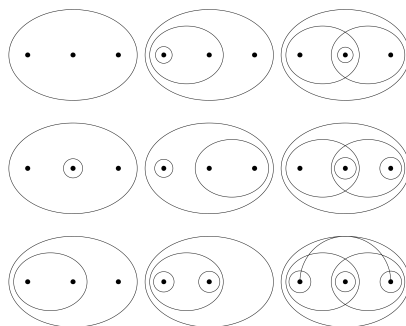


Figure 1.1: The Topologies on $\{a, b, c\}$.

- (3) Let X be any set, and let $\mathcal{T}_f = \{U \subseteq X : X \setminus U \text{ is finite, or } X \setminus U = X\}$. Then \mathcal{T}_f is a topology and called the **finite complement topology**.
- (4) Let X be any set, and let $\mathcal{T}_c = \{U \subseteq X : X \setminus U \text{ is countable, or } X \setminus U = X\}$. Then \mathcal{T}_c is a topology on X .

Definition. Let X be a set, and let \mathcal{T} and \mathcal{T}' be topologies on X . We say that \mathcal{T} is **coarser** than \mathcal{T}' , and \mathcal{T}' **finer** than \mathcal{T} if $\mathcal{T} \subseteq \mathcal{T}'$. If two topologies are either coarser, or finer than each other, we call them **comparable**.

Example 1.2. The topologies \mathcal{T}_f and \mathcal{T}_c are comparable, and we see that $\mathcal{T}_c \subseteq \mathcal{T}_f$, so \mathcal{T}_f is coarser than \mathcal{T}_c , and \mathcal{T}_c is finer than \mathcal{T}_f .

1.2 The Basis and Subbasis for a Topology.

Definition. If X is a set, the **basis** for a topology on X is a collection \mathcal{B} of subsets of X , called **basis elements**, such that:

- (1) For every $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$.
- (2) For $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

We define the topology \mathcal{T} **generated** by \mathcal{B} to be collection of open sets: $\mathcal{T} = \{U \subseteq X : x \in U \text{ for some } B \in \mathcal{B}\}$.

Theorem 1.2.1. Let X be a set, and \mathcal{B} a basis of X , then the collection of subsets of X , $\mathcal{T} = \{U \subseteq X : x \in U \text{ for some } B \in \mathcal{B}\}$ is a topology on X .

Proof. Let \mathcal{B} be a basis for a topology in X , and consider \mathcal{T} as defined above. Clearly, $\emptyset \in \mathcal{T}$ and so is X .

Now let $\{U_\alpha\}$ be a subcollection of subsets of X , and let $U = \bigcup U_\alpha$. Then if $x \in U$ for some α , there is a B_α such that $x \in B_\alpha \subseteq U_\alpha$, thus $x \in B_\alpha \subseteq U$.

Now let $x \in U_1 \cap U_2$, and choose $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Then by definition, there is a B_3 for which $x \in B_3 \subseteq B_1 \cap B_2$. Now suppose for arbitrary n , that $U = \bigcap_{i=1}^n U_i \in \mathcal{T}$, for some finite subcollection $\{U_i\}$ of subsets of X . Then by let $B_n, B_{n+1} \in \mathcal{B}$ such that $x \in B_n \subseteq U$ and $x \in B_{n+1} \subseteq U_{n+1}$. Then by our hypothesis, there is a B for which $x \in B \subseteq B_n \cap B_{n+1}$, thus $U \cap U_{n+1} = \bigcap_{i=1}^{n+1} U_i \in \mathcal{T}$. This make \mathcal{T} a topology on X . ■

Example 1.3. (1) Let \mathcal{B} be the set of all circular regions in the plane $\mathbb{R} \times \mathbb{R}$, then \mathcal{B} satisfies the conditions needed for a basis.

- (2) The collection \mathcal{B}' in $\mathbb{R} \times \mathbb{R}$ of all rectangular region also forms a basis for a topology on $\mathbb{R} \times \mathbb{R}$.
- (3) For any set X , the set of all 1-point elements of X forms a basis for a topology on X .

Figure 1.2: The basis for \mathcal{B} and \mathcal{B}' in $\mathbb{R} \times \mathbb{R}$ (see example (2)).

Lemma 1.2.2. *Let X be a set, and \mathcal{B} be a basis for a topology \mathcal{T} on X . Then $\mathcal{T} = \{\bigcup B : B \in \mathcal{B}\}$.*

Proof. Given a collection $\{B\}$ of basis elements in \mathcal{B} , since they are all in \mathcal{T} , their unions are also in \mathcal{T} . Conversely, given $U \in \mathcal{T}$, then for every point $x \in U$, choose a $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$, then $U = \bigcup_{x \in U} B_x$. ■

Lemma 1.2.3. *Let (X, \mathcal{T}) be a topological space, and let $\mathcal{C} \subseteq \mathcal{T}$ be a collection of open sets of X such that for every $x \in U$, there is a $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then \mathcal{C} is the basis for a \mathcal{T} on X .*

Proof. Take any $x \in X$, then there is a $C \in \mathcal{C}$ such that $x \in C \subseteq U$, thus the first condition for a basis is satisfied. Now let $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C}$, since $C_1 \cap C_2$ is open in X , there is a $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. Therefore \mathcal{C} is a basis for a topology on X .

Now let $\mathcal{T}_{\mathcal{C}}$ be the topology generated by \mathcal{C} , now for $U \in \mathcal{T}$, we have by the hypothesis, that $U \in \mathcal{T}_{\mathcal{C}}$; and by lemma 1.2.2, $W \in \mathcal{T}_{\mathcal{C}}$ is the union of elements of \mathcal{C} , which is a subcollection of \mathcal{T} , thus $W \in \mathcal{T}$. Therefore $\mathcal{T}_{\mathcal{C}} = \mathcal{T}$. ■

Lemma 1.2.4. *Let \mathcal{B} and \mathcal{B}' be bases for topologies \mathcal{T} and \mathcal{T}' on X . Then the $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for all $x \in X$, and all $B \in \mathcal{B}$, there is a $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Proof. Suppose first that $\mathcal{T} \subseteq \mathcal{T}'$, and let $x \in X$, and choose $B \in \mathcal{B}$ such that $x \in B$, then B is open in \mathcal{T} , thus it is open in \mathcal{T}' , thus there is a $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. Conversely, suppose there is a $B' \in \mathcal{B}'$ for which $x \in B' \subseteq B$ for all $x \in X$, $B \in \mathcal{B}$. Take $x \in U \in \mathcal{T}$, since \mathcal{B} generates \mathcal{T} , $x \in B \subseteq U$, since $B' \subseteq B$, this implies that $U \in \mathcal{T}'$ and $\mathcal{T} \subseteq \mathcal{T}'$. ■

Definition. If \mathcal{B} is the collection of open intervals (a, b) in \mathbb{R} , we call the topology generated by \mathcal{B} the **standard topology** on \mathbb{R} , and we denote it simply by \mathbb{R} .

Definition. If \mathcal{B} is the collection of half open intervals $[a, b)$ in \mathbb{R} , we call the topology generated by \mathcal{B} the **lower limit topology** on \mathbb{R} , and we denote it simply by \mathbb{R}_l . If \mathcal{B}' is the collection of all half open intervals $(a, b]$ in \mathbb{R} , then we call the topology generated by \mathcal{B}' the **upper limit topology** on \mathbb{R} , and denote it \mathbb{R}_L .

Definition. If \mathcal{B} is the collection of all open intervals of the form $(a, b) \setminus \frac{1}{\mathbb{Z}^+}$, where $\frac{1}{\mathbb{Z}^+} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$, we call the topology generated by \mathcal{B} the $\frac{1}{\mathbb{Z}^+}$ -**topology** on \mathbb{R} , and we denote it $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$.

Lemma 1.2.5. *The topologies \mathbb{R}_I , \mathbb{R}_L , and $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ are all strictly finer than \mathbb{R} , but are not comparable with each other.*

Proof. Let (a, b) be a basis element for \mathbb{R} , and let $x \in (a, b)$, the basis element $[x, b) \in \mathbb{R}_I$ lies in (a, b) and contains x , however, there can be no interval (a, b) in $[x, b)$ as $x \leq a$, thus \mathbb{R}_I ; a similar argument holds for \mathbb{R}_L .

Similarly, for $(a, b) \in \mathbb{R}$, the basis element $(a, b) \setminus \frac{1}{\mathbb{Z}^+}$ of $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ lies in (a, b) , however, choose the basis $B = (-1, 1) \setminus \frac{1}{\mathbb{Z}^+}$, and choose $0 \in B$, since \mathbb{Z}^+ is dense in \mathbb{R} , there is no interval (a, b) containing 0 and lying in B , thus $\mathbb{R} \subseteq \mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$.

Now choose $[0, 1)$ in \mathbb{R}_I , and choose $\frac{1}{k} \in [0, 1)$ such that $k \in \mathbb{Z}^+$. Now $(0, 1) \subseteq [0, 1)$, so we cannot say that $[0, 1)$ is a basis for \mathbb{R} , and moreover, $[0, 1) \setminus \frac{1}{\mathbb{Z}^+}$ cannot be said to be a basis in $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$, thus \mathbb{R}_I and $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ are incomparable, a similar argument holds for \mathbb{R}_L .

Lastly, let (a, b) be in \mathbb{R} and choose $x \in (a, b)$. Then $(a, x]$ and $[x, b)$ are both in (a, b) , however it is clear that $(a, x]$ and $[x, b)$ cannot be contained in each other, thus \mathbb{R}_I and \mathbb{R}_L are incomparable. ■

Definition. A **subbasis**, \mathcal{S} , for a topology on X is a collection of subsets of X whose union equals X . We call the **topology generated by \mathcal{S}** to be the collection of all unions of finite intersections of elements of \mathcal{S} , that is:

$$\mathcal{T} = \left\{ \bigcup_{i=1}^n S_i : S_i \in \mathcal{S} \text{ for } 1 \leq i \leq n \right\}$$

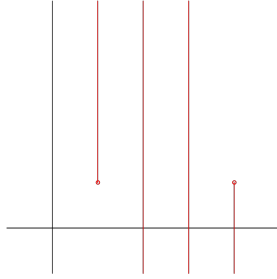
Theorem 1.2.6. *Let \mathcal{S} be a subbasis for a topology on X . Then the collection $\mathcal{T} = \left\{ \bigcup_{i=1}^n S_i : S_i \in \mathcal{S} \text{ for } 1 \leq i \leq n \right\}$ is a topology on X .*

Proof. It is sufficient to show that the collection \mathcal{B} of all finite intersections of elements of \mathcal{S} is a basis for a topology on X . By lemma 1.2.1, for $x \in X$, it belongs to an element S of \mathcal{S} , and therefore, to an element of \mathcal{B} . Now let $B_1 = \bigcap_{i=1}^m S_i$ and $B_2 = \bigcap_{j=1}^n S'_j$ be basis elements of \mathcal{B} . The intersection $B_1 \cap B_2$ is a finite intersection of elements of \mathcal{S} , and hence also belongs in \mathcal{B} , and hence we can take another basis element B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$. ■

1.3 The Order Topology.

Definition. Let X be a set with a simple order relation, and suppose that $|X| > 1$. Let \mathcal{B} be the collection of sets of the following forms:

- (1) All open intervals $(a, b) \in X$.
- (2) All half open intervals $[a_0, b)$ where a_0 is the least element (if any) of X .
- (3) All half open intervals of the form $(a, b_0]$ where b_0 is the greatest element (if any) of X .

Figure 1.3: The order topology on $\mathbb{R} \times \mathbb{R}$.

Then \mathcal{B} forms the basis for a topology on X called the **order topology**

Theorem 1.3.1. *The collection \mathcal{B} forms a basis.*

Proof. Consider $x \in X$, if x is the least element of X , then it lies in all intervals of type (2), if it is the largest, then it lies in all intervals of type (3). If x is neither the least nor largest element, then $x \in (a_0, b_0)$ with a_0 and b_0 the least and largest elements (if any) of X . If no such elements exist, then $x \in (a, b)$, for some lowerbound a and upperbound b . Thus, in all three cases, there is a basis element containing x .

Now suppose $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$. If B_1 and B_2 are both of type (1), then let $B_1 = (a, b)$, $B_2 = (c, d)$, then $B_1 \cap B_2$ is an open interval of type (1), now fix B_1 to be of type one. If B_2 is of type (2), then letting $B_2 = [a_0, c)$, then $x \in [a_0, d)$ for some $d \in X$. Likewise, if $B_2 = (c, b_0]$, is of type (3), we get a similar result. Moreover, the results are analogous if we fix B_2 and let B_1 range between intervals of the three types. Thus in all cases, there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. ■

Example 1.4. (1) The standard topology on \mathbb{R} is the order topology on \mathbb{R} induced by the usual order relation. We have that \mathbb{R} under this topology has no intervals of type (2), nor (3), so all bases elements in the standard topology are open intervals in \mathbb{R} .

- (2) Consider the dictionary order on $\mathbb{R} \times \mathbb{R}$. Since $\mathbb{R} \times \mathbb{R}$ has no intervals of type (2), nor (3), the bases of $\mathbb{R} \times \mathbb{R}$ under the dictionary order are the open intervals of the form $(a \times b, c \times d)$ Where $a \leq c$, and $b < d$.
- (3) The positive integers \mathbb{Z}^+ with the least element 1 form an ordered set under the usual order. Taking $n > 1$, we see the bases of \mathbb{Z}^+ under the order topology are of the form $(n-1, n+1) = \{n\}$ and $[1, n) = \{1, \dots, n-1\}$. Thus the order topology on \mathbb{Z}^+ is the discrete topology.
- (4) The set $X = \{1, 2\} \times \mathbb{Z}^+$ over the dictionary order is also an ordered set, with the least element 1×1 . Denote $1 \times n$ as a_n and $2 \times n$ as b_n . Then X consist of the elements $a_1, a_2, \dots, b_1, b_2, \dots$.

Now take $\{b_1\}$, then any open set containing b_1 must have a basis about b_1 , and also contains points a_i with $i \in \mathbb{Z}^+$; thus the order topology on X is not the discrete topology.

Definition. Let X be an ordered set, and let $a \in X$. There are two subsets in X , $(a, \infty) = \{x \in X : x > a\}$ and $(-\infty, a) = \{x \in X : x < a\}$ called **open rays** of X . There are also two sets $[a, \infty) = \{x \in X : x \geq a\}$ and $(-\infty, a] = \{x \in X : x \leq a\}$ called **closed rays** of X .

Theorem 1.3.2. *Let X be an ordered set. Then the collection of all open rays in X form a subbasis for the order topology on X .*

Proof. Let \mathcal{S} be the collection of all open rays of X , let (a, ∞) and $(-\infty, b) \in \mathcal{S}$, then $(a, b) = (a, \infty) \cap (-\infty, b)$. Now take:

$$S = \bigcup_{a, b \in X} (a, b)$$

then $S \subseteq X$, likewise, since S runs through all intersections of open rays of X , it contains all open intervals in X , hence $X \subseteq S$, and so $X = S$ as required. ■

1.4 The Product Topology.

Definition. Let X and Y be topological spaces. We define the **product topology** on $X \times Y$ to be the topology having as basis the collection $\mathcal{B} = \{U \times V \subseteq X \times Y : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$

Theorem 1.4.1. *The collection $\mathcal{B} = \{U \times V \subseteq X \times Y : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$ forms a basis for the product topology on $X \times Y$.*

Proof. Clearly, we have that $X \times Y$ is a basis element of \mathcal{B} . Now take $U_1 \times V_1$ and $U_2 \times V_2$ in \mathcal{B} . Since $U_1 \times V_1 \cap U_2 \times V_2 = U_1 \cap U_2 \times V_1 \cap V_2$, since $U_1 \cap U_2$ and $V_1 \cap V_2$ are open in X and Y respectively, then we have that $U_1 \times V_1 \cap U_2 \times V_2$ is a basis element as well. ■

Theorem 1.4.2. *If \mathcal{B} is the basis for a topology on X , and \mathcal{C} is the basis for a topology on Y , then the collection:*

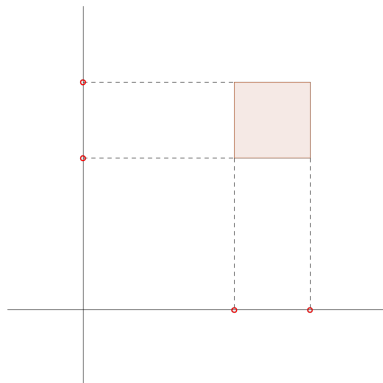
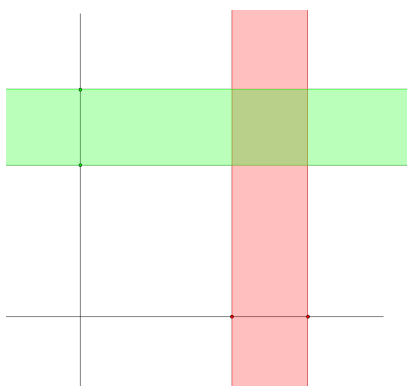
$$\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

Is a basis for the topology on $X \times Y$.

Proof. By lemma 1.2.3, let W be an open set of $X \times Y$, and let $x \times y \in W$. Then there is a basis $U \times V$ such that $x \times y \in U \times V \subseteq W$. Since \mathcal{B} and \mathcal{C} are bases of X and Y respectively, choosing $B \in \mathcal{B}$ and $C \in \mathcal{C}$, we have that $x \in B \subseteq U$, and $y \in C \subseteq V$, thus $x \times y \in B \times C \subseteq U \times V \subseteq W$. Therefore, \mathcal{D} is the basis for a topology on $X \times Y$. ■

Example 1.5. The product of the standard topology on \mathbb{R} with itself is called the **standard topology on $\mathbb{R} \times \mathbb{R}$** , and has as basis the collection of all products of open sets in \mathbb{R} . By theorem 1.4.2, if we take the collection of all open intervals $(a, b) \times (c, d)$ in $\mathbb{R} \times \mathbb{R}$, we form a basis. Constructing this basis geometrically gives the interior of a rectangle, whose boundaries are the intervals (a, b) and (c, d) .

Definition. Let $\pi_1 : X \times Y \rightarrow X$ be defined such that $\pi_1(x, y) = x$, and define $\pi_2 : X \times Y \rightarrow Y$ such that $\pi_2(x, y) = y$. We call π_1 and π_2 **projections** of $X \times Y$ onto its first and second **factors**; that is onto X and Y , respectively.

Figure 1.4: A basis element for $\mathbb{R} \times \mathbb{R}$ Figure 1.5: The inverse images, $\pi_1^{-1}(U)$ and $\pi_2^{-1}(V)$, of the projections π_1 and π_2 onto the $X \times Y$ plane.

Clearly, π_1 and π_2 are both onto. Now let U be open in X , then $\pi_1^{-1}(U) = U \times Y$ is open in $X \times Y$; similarly, $\pi_2^{-1}(V) = X \times V$ is also open in $X \times Y$, for V open in Y .

Theorem 1.4.3. *The collection $\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ is open in } Y\}$ is a subbasis for the product topology on $X \times Y$.*

Proof. Let \mathcal{T} be the product topology on $X \times Y$, and let \mathcal{T}' be the topology generated by \mathcal{S} . Since every element of \mathcal{S} is open in \mathcal{T} , $\mathcal{T} \subseteq \mathcal{T}'$. Conversely, consider the basis element $U \times V$ of \mathcal{T} , then $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times Y \cap X \times V = U \times V$, thus $\mathcal{T} \subseteq \mathcal{T}'$. Therefore, \mathcal{S} is a subbasis for the product topology. ■

1.5 The Subspace Topology.