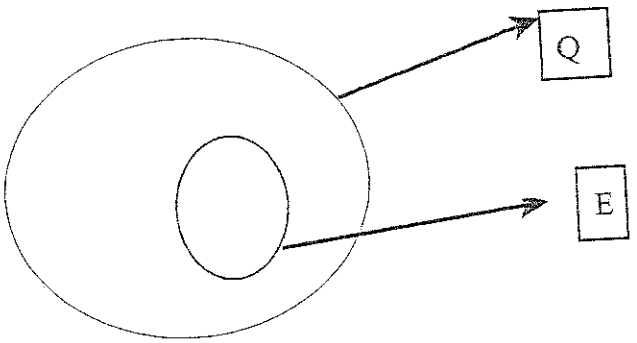
Chapter 03 Probability

Outcomes: You must be able to
* calculate simple theoretical probabilities
* calculate simple empirical probabilities
* apply combinatorics, theoretical probability, empirical probability, logic and problem solving skills to more difficult problems.

3.1 Theoretical Probability ("*a priori*" Probability).

3.1.1 Definition. Suppose there is a set Q of equally likely outcomes to a given action. Suppose further that a particular event E can be attained by the occurrence of any one of a certain subset of these outcomes.



The probability of E is defined by

$$P(E) = \frac{n(E)}{n(Q)}$$

3.1.2 Examples.

1. What is the probability of pulling a picture card out of a pack of cards?

2. A coin is tossed three times (or 3 coins are tossed once!). What is the probability of obtaining exactly 2 heads?

3. If two distinguishable dice are thrown, what is the probability that the two numbers on the upper faces will add up to 6?

4. Five beads are taken out one at a time, with no replacement, from a bowl containing 8 red beads and 6 blue beads. What is the probability that 2 are red and 3 are blue? (Consider Q to be the set of all possible combinations of size 5). NB beads of the same colour are indistinguishable from each other.

3.1.3 Notes.

1. $0 \leq P(E) \leq 1$

2. If $P(E) = 1$ then E must occur.

3. If $P(E) = 0$ then E cannot occur.

3.2 Dependent and Independent Events.

3.2.1 Definition.

If the probability of an event E_2 depends on whether an event E_1 has occurred then the event E_2 is said to be *dependent* on E_1. The probability of the event E_2 happening given that E_1 has happened is denoted by $P(E_2|E_1)$.

If the probability of an event E_2 does not depend on whether an event E_1 has occurred then the event E_2 is said to be *independent* of E_1. The expression $P(E_2|E_1)$ becomes simply $P(E_2)$.

3.3 The Laws of Probability.

3.3.1 Laws of Probability.

Law I $P(E) + P(\overline{E}) = 1$

Law IIa $P(E_1 \text{ and } E_2) = P(E_1) \times P(E_2|E_1) = P(E_2) \times P(E_1|E_2)$ if E_1 and E_2 are dependent events.

Law IIb $P(E_1 \text{ and } E_2) = P(E_1) \times P(E_2)$ if E_1 and E_2 are independent events.

Law IIIa $P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2)$ if E_1 and E_2 are not mutually exclusive events.

Law IIIb $P(E_1 \text{ or } E_2) = P(E_1) + P(E_2)$ if E_1 and E_2 are mutually exclusive events.

NB These laws can be generalised to three or more events.

3.3.2 Examples.

1. Two cards are taken one at a time, without replacement, from a well-shuffled pack of cards. What is the probability that they are both aces?

2. One card is taken form a well-shuffled pack of cards. What is the probability that it is a diamond picture card?

4. One card is taken form a well-shuffled pack of cards. What is the probability that it is a diamond or a picture card?

5. One card is taken form a well-shuffled pack of cards. What is the probability that it is an ace or a picture card?

6. What is the probability of obtaining 7 heads in 7 throws of a coin?

$$P(H\ H\ H\ H\ H\ H\ H) = \left(\frac{1}{2}\right)^7 = 0,00078125$$

7. A die is thrown 7 times. What is the probability of obtaining exactly three sixes in the 7 throws?

8. What is the probability of obtaining an ace or a diamond or a picture card when a single card is taken from a well-shuffled pack of cards?

9. Consider a box containing 8 red beads and 6 green beads. One bead is selected at random from the box. The probability that it is a red bead is $_{14} = \frac{4}{7}$. If the box is now partitioned into two identical sections so that there are 3 red and 2 green beads in one side and 5 red and 4 green beads in the other section, what is the probability of selecting a red bead from the box?

10. Consider a box containing 8 red, 3 white and 9 blue beads. Selection is done one at a time with no replacement.

a) If 3 beads are selected, what is the probability that they are all red?

b) If 3 beads are selected, what is the probability that 2 are red and 1 is white?

c) If 5 beads are selected, what is the probability that 2 are red and 3 are blue?

3.4 Empirical Probability ("a posteriori" probability).

3.4.1 Definition.

If an event E has occurred M times out of a total of N times, where N is "sufficiently large", then the empirical probability of E is defined by

$$P(E) = \frac{M}{N}$$

3.4.2 Example.

Given the following statistics for men above 60:

gout : 10% hypertension: 40% gout and hypertension: 2%

a) If one man is randomly selected from the population, what is the probability that he has hypertension or gout?

b) If two men are selected without replacement, what is the probability that both have gout?

c) What is the probability that a man with gout has hypertension?

Tutorial 3 Probability

1. One ball is taken from a bag containing 4 white and 6 black balls. Find the probability that it is (a) white, (b) black.

2. Three balls are taken without replacement from a bag containing 8 white and 12 black balls. Find the probability that (a) all are white, (b) just 2 are white, (c) just 1 is white, (d) all are black.

3. Ten students are seated in a row. Find the probability that two particular students are *not* seated side by side.

4. If a die is cast three times, find the probability that (a) and even number will be thrown each time, (b) an odd number will appear just once, (c) the sum of the three numbers will be even.

5. From a box containing ten cards numbered 1, 2, 3, ..., 10, four cards are taken out. Find the probability that their sum will be even if (a) the cards are not replaced, (b) each card is replaced before the next card is taken out.

6. A and B, having equal skill, are playing a game where the winner is the first one to get three points. A has got 2 points and B has got 1 point. What is the probability that A will win the game?

7. One bag contains 3 white balls and 2 black balls, and another bag contains 2 white and 3 black balls. A ball is drawn from the second bag and placed in the first; then a ball is drawn from the first bag and placed in the second. When the pair of operations is repeated, what is the probability that the first bag will contain 5 whits balls?

8. Three bags contain respectively 3 white and 1 black ball, 3 white and 3 black balls, 6 white and 3 black balls. Two bags are selected and a ball is drawn from each. Find the probability that (a) both balls are white, (b) both balls are the same colour.

9. If four trials are made in Problem 8, find the probability that (a) the first two trials will result in pairs of white balls and the other two trials will result in pairs of black balls, (b) a pair of black balls will be obtained at least three times.

10. Five cards numbered 1, 2, 3, 4, 5 respectively are placed in a revolving box. If the cards are drawn one at a time from the box without replacement, what is the probability that they will be drawn in ascending numerical order?

11. Brown, Jones and Smith shoot at a target in alphabetical order with probabilities ¼, ⅓, ½ respectively of hitting the bull's-eye. Find the probability that (a) each on his first shot will be the first to hit the bull's-eye, (b) the bull's-eye is not hit in the first round of shooting, (c) the first to hit the bull's-eye is Jones on his second shot.

12. The probability that X will win a game of chess against Y is $\frac{2}{5}$. In a five game match, what is the probability that (a) X will win the first, third and fifth games, and lose the others? (b) X will win exactly three games? (c) that X will win at least three games?

13. A point is selected at random inside a circle. Find the probability that the point is closer to the centre of the circle than it is to its circumference.

14. The following data are known concerning the distribution of the four blood groups in any population and for either gender:

Group	O	A	B	AB
%	46	42	9	3

At a certain large university 40% of the students are women.

(a) A student is selected at random from this university. Find the probability that the student is a male with blood belonging to group B.

(b) If 4 students are selected at random find the probability that each student's blood group is different.

(c) If 4 women are selected at random find the probability that each student's blood group is different.

(d) If 4 students are selected at random find the probability that they are all women each with a different blood group.

15. In a certain multiple-choice examination there are 4 questions, each question has 6 listed answers, of which one and only one is correct. Five marks are given for a correct answer, and 0 for an incorrect answer. If 50% is the pass mark, find the probability that a student will pass the examination by guessing, assuming he attempts all the questions. Repeat the problem for a multiple-choice examination consisting of 8 questions.

16. In a survey conducted on the men above 50 years of age in a certain community it was empirically determined that 1 in 4 of this group smoked, that 3in 4 of those who had lung cancer were also smokers, and that 1 in 300 had lung cancer.

Determine the probability that (a) a smoker of this age group has lung cancer, (b) a man above the age of 50 is either a smoker or has lung cancer.

17. Tests used in the detection of a particular disease are 90% effective; they fail to detect it in 10% of the cases. In persons free of the disease, the tests indicate 1% to be affected and 99% not to be affected. From a large population, in which only 0.2% have the disease, one person id selected at random, is given the test, and the presence of the disease is indicated. What is the probability that the person really is affected?

Answers

1: 0.4; 0.6
2: 0.0491...; 0.2947; 0.4632; 0.1930
3: 0.8
4: 0.125; 0.375; 0.5
5: 0.5238; 0.5
6: 0.75
7: 0.0044..
8: 0.4027..; 0.5277..
9: 0.00253...; 0.007080..
10: 0.008333..
11: 0.25; 0.25; 0.25; 0.25; 0.0625
12: 0.02304; 0.2304; 0.31744
13: 0.25
14: 0.054; 0.0125; 0.0125; 0.00032
15: 0.1319444; 0.0306564
16: 0.01; 0.250833..
17: 0.1528...

Supplementary Problems

Spiegel: 6.40 - 6.51; 6.83 - 6.88