Chapter 2 Combinatorics

Outcomes: You must be able to

* evaluate factorials, combinations and permutations
* decide if a problem is a combination or a permutation
* evaluate the number of options in a combinatorial problem

2.1 The Symbol a^n

When the number denoted by the symbol a^n [pronounced "*a* to the *n*"] is first introduced to learners at school it is defined as

$$\boxed{a^n = a \times a \times a \,\ldots\ldots\ldots\ldots\, a \times a}$$

where a is repeated n times on the right hand side of the equation. [The number n is called an exponent or index, and a is called a base.] Clearly, according to this definition, the symbol a^n is only meaningful if n is a positive integer greater than zero, however it can be shown that n can be any real number.

Let m and n be real numbers. The laws of exponents are:

I $a^m \times a^n = a^{m+n}\ ;\ m \geq 1, n \geq 1$ **II** $\dfrac{a^m}{a^n} = a^{m-n},\ m > n \geq 1$

III $(a^m)^n = a^{mn},\ m \geq 1, n \geq 1$

Exercise: a) Show that the value of $a^0 = 1$.

b) Write $\sqrt[n]{a^m}$ as a raised to a power.

c) Write $\dfrac{1}{a^n}$ as a raised to a power.

2.2 The Symbol $n!$

2.2.1 Definition

The symbol $n!$ is read as "n factorial". It is defined as

$$n! = n(n - 1)(n - 2)\ldots\ldots\ldots\ldots 3\times2\times1 \quad \text{where } n \text{ is an integer, } n \geq 1.$$

2.2.2 Examples

$6! = 6\times5\times4\times3\times2\times1 = 720$
$5! = 5\times4\times3\times2\times1 = 120$
$1! = 1$

2.2.3 Theorem If $n \geq 2$ then $\frac{n!}{n} = (n-1)!$ for $n \geq 2$, n an integer.

So $4! = \frac{5!}{5}$; $3! = \frac{4!}{4}$; $2! = \frac{3!}{3}$; $1! = \frac{2!}{2}$

This suggests the following **definition :** $0! = 1$

2.3 Mutually Exclusive Events

2.3.1 Definition If in a single action the occurrence of one event implies the non-occurrence of a second event then the two events are said to be mutually exclusive.

2.3.2 Examples Consider the single action of taking one card out of a pack of 52 cards.

If an ace is taken out (event 1 = E_1) and a queen is taken out (event 2 = E_2) then the two events are mutually exclusive because an ace cannot be a queen.

If E_1 = taking a heart from the pack and E_2 = taking a queen from the pack then the two events are not mutually exclusive because a queen can be a heart.

2.4 The Fundamental Laws of Counting.

2.4.1 Law Ia If in a sequence of actions, the number of ways in which an event E_1 can occur is denoted by $n(E_1)$, and if following the occurrence of E_1 the number of ways in which a second event E_2 can occur is denoted by $n(E_2 \mid E_1)$ {said as "the number of ways of E_2 happening given that E_1 has happened"}, and if following the occurrence of E_1 and E_2 the number of ways in which a third event E_3 can occur is denoted by $n(E_3 \mid E_1 \text{ and } E_2)$ {said as "the number of ways of E_3 happening given E_1 and E_2 have happened"}, and so on, then the number of ways in which these events can occur in sequence (ie $n(E_1 \text{ and } E_2 \text{ and } E_3 \text{ and } ....)$) is:

$$n(E_1 \text{ and } E_2 \text{ and } E_3 \text{ and } ....) = n(E_1) \times n(E_2|E_1) \times n(E_3|E_1 \text{ and } E_2) \times ....$$

sequential "and"

2.4.2 Law Ib If the number of ways in which any particular event can occur is independent of any previous events having occurred then Law Ia reduces to

$$n(E_1 \text{ and } E_2 \text{ and } E_3 \text{ and } ....) = n(E_1) \times n(E_2) \times n(E_3) \times ....$$

Note. In Law Ia and Law Ib the concepts of dependent and independent events were introduced without defining the terms. The definitions will be left until the concept of probability has been dealt with. In the meantime these concepts will be assumed at an intuitive level.

2.4.3 Examples

1. In a race between 4 runners, in how many different ways can the first three places be filled?

2. If there are 5 different routes from place A to place B, 3 different routes from B to C, and 4 different routes from C to D, how many different routes are there from A to D passing through B and C?

3. In how many ways can 3 raffle tickets be drawn in sequence from a barrel containing 100 tickets if as each ticket is drawn it is (a) immediately replaced, and (b) not replaced?

2.4.4 Law IIa If a set of mutually exclusive events E_1, E_2, E_3, ... can respectively occur in $n(E_1)$, $n(E_2)$, $n(E_3)$, ... number of ways, then the total number of ways in which any one of these events can occur is

$$n(E_1 \text{ or } E_2 \text{ or } E_3 \text{ or } \ldots) = n(E_1) + n(E_2) + n(E_3) + \ldots..$$

2.4.5 Example In how many ways can a diamond or a heart be taken from a pack of cards?

2.4.6 Law IIb For two non-mutually exclusive events:

$$n(E_1 \text{ or } E_2) = n(E_1) + n(E_2) - n(E_1 \text{ and } E_2)$$

Inclusive "or" Instantaneous "and"

2.4.7 Example In how many ways can a picture card or a spade be taken from a pack of cards in a single action?

2.4.8 Law IIc For three non-mutually exclusive events:

$$n(E_1 \text{ or } E_2 \text{ or } E_3) = n(E_1) + n(E_2) + n(E_3) - n(E_1 \text{ and } E_2) - n(E_1 \text{ and } E_3) - n(E_2 \text{ and } E_3) + n(E_1 \text{ and } E_2 \text{ and } E_3)$$

2.4.9 Example In how many ways can a diamond or a spade or a picture card be taken from a pack of cards in a single action?

2.4.10 Challenge Write down the law for 4 non-mutually exclusive events! There should be 15 terms in your answer.

2.5 Permutations and Combinations

2.5.1 Definitions

1. A **Permutation** is an *ordered* arrangement of any subset of a set of distinguishable elements in which any given element cannot appear more than once in the arrangement.

2. A **Combination** is an unordered arrangement of any subset of a set of distinguishable elements in which any given element cannot appear more than once in the arrangement.

2.5.2 Glossary

In mathematics the words permutation and combination have very specific meanings. The table below stresses the definitions of these terms.

Arrangements

Ordered Arrangements

Arrangements in which the order of the elements is a factor used to distinguish one arrangement from another.

- Permutations as defined above
- Ordered arrangements that are not permutations.

Unordered Arrangements

Arrangements in which the same elements in a different order do not constitute a different arrangement.

- Combinations as defined above.
- Unordered arrangements that are not combinations.

2.5.3 Examples

1. People sitting in a straight row : Permutation

2. The registration numbers of cars in South Africa : an ordered arrangement that is *not* a permutation.

3. The cards dealt to people playing a game of cards eg poker : Combination

4. A purse containing R10 in change (coins) : an unordered arrangement that is not a combination.

2.6 Precise Definitions of Permutation and Combination

2.6.1 Permutation Consider n distinguishable objects. Select any r of them ($r \leq n$) in an ordered sequence. Each of the ordered arrangements obtained is called a permutation. The total number of distinct permutations that is possible when r objects are taken from n objects is denoted by ${}^nP_r$. [Alternative notations are P_r^n or $P(n,r)$]

$$ {}^nP_r = \begin{bmatrix} n \\ r \end{bmatrix} = \frac{n!}{(n-r)!} $$

2.6.2 Examples

1. $$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 5 \times 4 = 20$$

2. $$\begin{bmatrix} 100 \\ 3 \end{bmatrix} = \frac{100!}{97!} = 100 \times 99 \times 98 = 970200$$

3. $$\begin{bmatrix} 7 \\ 7 \end{bmatrix} =$$

4. $$\begin{bmatrix} 7 \\ 0 \end{bmatrix} =$$

5. $$\begin{bmatrix} 0 \\ 0 \end{bmatrix} =$$

2.6.3 Examples

1. In how many ways can three chairs in a straight line be filled if there are 20 people to choose from?

2. In how many ways can the cards in a pack be permuted?

2.6.4 Combination Consider n distinguishable objects. Select any r of them ($r \leq n$) without giving consideration to the order in which they were chosen. The unordered arrangement obtained is called a combination. The total number of distinct combinations that is possible from n objects chosen in groups of r is denoted by ${}^nC_r$.
[Alternative notations : C_r^n or C(n, r)]

$$ {}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} $$

2.6.5 Examples

(1) $$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(2 \times 1)(3 \times 2 \times 1)} = \frac{5 \times 4}{2 \times 1} = 10$$

(2) $$\binom{100}{4} = \frac{100!}{4!96!} = 3921225$$

(3) $$\binom{5}{3} = \frac{5!}{3!2!} = 10 = \binom{5}{2}$$

2.6.6 Examples

1. In how many ways can a committee of three people be chosen from 50 potential candidates?

2. In how many ways can a pack of cards be combined?

2.6.7 Permuting a Group of Objects Some of Which Are Identical

Given n objects with n_1 of them identical, n_2 of them identical but different from the first group n_1, n_3 of them identical but different from the first group and the second group and so on. The number of distinguishable ways in which the n objects can be permuted is

$$N_D = \frac{n!}{(n_1!)(n_2!)(n_3!)\dots(n_k!)} \qquad \text{where } n_1 + n_2 + n_3 + \dots + n_k = n$$

2.6.8 Example

In how many distinguishable ways can the symbols α, α, β, β, β, γ be permuted?

2.7 Miscellaneous Examples

1. In how many ways can the symbols α, β, γ, δ be (a) permuted and (b) combined in groups of 2?

2. In how many different ways can the letters of the alphabet be arranged in sequences of 3 letters (a) if repetition of letters is not permitted, and (b) if repetition of letters is permitted?

3. In how many different ways can n objects be (a) permuted and (b) combined?

4. Suppose that in a test there are two sections to the question paper - Section A in which there are 6 questions and Section B in which there are 7 questions. Students must answer 10 questions, and at least 4 questions must be from Section A.
   How many different (a) combinations and (b) permutations of questions are possible assuming a student answers 10 questions?

5. In how many distinguishable ways can the symbols α, α, α, β, γ, δ, λ, μ be permuted?

6. In how many distinguishable ways can the symbols α, α, α, β, γ, δ, λ, μ be permuted in groups of 4?

2.8 Pascal's Triangle - for interest!

Compare the numbers in Pascal's triangle on the left with the corresponding values of $\binom{n}{r}$ on the right:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

Pascal's triangle suggests that $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$.

Tutorial 2 Combinatorics

1. Evaluate: (i) 10! (ii) $\begin{bmatrix}10\\3\end{bmatrix}$ (iii) $\begin{bmatrix}10\\7\end{bmatrix}$ (iv) $\binom{21}{19}$

2. Simplify: (i) $\frac{n!}{(n-1)!}$ (ii) $\frac{(n+2)!}{n!}$

3. Two jobs are advertised; there are 63 applications for the one and another 34 for the other. In how many different ways can people be allocated to the two jobs?

4. Using the digits 3,4,5,6,7 and 9, and if repetition of digits is not permitted, find

   a) how many three digit numbers can be formed,
   b) how many of the three-digit numbers are less than 400,
   c) how many of the three-digit numbers are even,
   d) how many of the three-digit numbers are multiples of 5.

5. a) In how many ways can 4 boys and 3 girls sit in a row?
   b) In how many ways can the children sit in a row if they must sit so that the girls are in a group and the boys are in a group?
   c) In how many ways can the children sit in a row if the girls must sit as a group but the boys can be scattered?

6. How many distinct permutations can be formed from all the letters of each of the following words : (a) THEM (b) UNUSUAL (c) SOCIOLOGICAL.

7. How many different signals, each consisting of 6 flags hung in a vertical, line can be formed from 4 identical red flags and 2 identical blue flags?
   (Solve this problem (i) using permutations and (ii) using combinations).

8. A box contains 8 balls. Find the number of ordered samples of size 3 that can be taken from the box if (i) replacement is allowed; (ii) replacement is not allowed.

9. A delegation of 4 students is to be selected to represent the university at a meeting of a certain national student association. There are 12 eligible students.

   a) In how many ways can the delegation be chosen if there are no restrictions on who can be chosen from the 12?

   b) In how many ways can the delegation be chosen if two of the students refuse to attend the meeting together?

   c) In how many ways can the delegation be chosen if two of the students are married and will only attend the meeting if they can go together?

10. A student must answer 8 out of 10 questions in an exam.

    a) In how many ways can this be done if s/he is free to choose any 8 of the 10?

    b) In how many ways can this be done if s/he must answer at least 4 of the first 5 questions?

11. In how many ways can a teacher choose one or more students from 6 eligible students?

Answers.

1.	3628800; 720; 604800; 210	2.	n; $n^2 + 3n + 2$	3.	2142
4.	120; 20; 40; 20	5.	5040; 288; 720	6.	24; 840; 9979200
7.	15	8.	512; 336	9.	495; 450; 255
10.	45; 35	11.	63		

Supplementary Problems

Wegner (3rd Ed): Ch 4 - 15 to 21