APPM4058A & COMS7238A: Digital Image Processing

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Contents

1. Mathematical Morphology
2. Some basic morphological algorithms

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Outline

1. Mathematical Morphology
2. Some basic morphological algorithms

Mathematical Morphology

- Mathematical morphology is the study of shape and form of objects.
- Mathematical morphology in image processing is the processing and analysis of shapes within an image.
- The language of mathematical morphology is the language of set theory.
- We will look at the ideas on binary images, and indicate how they can be extended to grey scale (and thus colour) images also.
- For binary images, the shapes are defined by the pixel vectors.

Some basic concepts from set theory

- The elements of a set we are concerned here are coordinates of pixels representing objects or other features of interest in an image.
- For example, $C = \{\mathbf{w} | \mathbf{w} = -\mathbf{d}, \text{for } \mathbf{d} \in D\}$. C is the set of elements $\mathbf{w}$ formed by multiplying each of two coordinates of all elements of D by -1.
- $A \subset B$ – every element of A is also an element of B;
- $A \cup B$ – union, the set of elements in either A or B;
- $A \cap B$ – intersection, the set of elements in both A and B;
- A^c – complement of set A, the set of elements not contained in A;
- $A - B = \{\mathbf{w} | \mathbf{w} \in A, \mathbf{w} \notin B\} = A \cap B^c$;
- The reflection of B, $\hat{B} = \{\mathbf{w} | \mathbf{w} = -\mathbf{b}, \text{for } \mathbf{b} \in B\}$,
- The translation of A by point $\mathbf{z} = (z_1, z_2)$,
  $A_z = \{\mathbf{c} | \mathbf{c} = \mathbf{a} + \mathbf{z}, \text{for } \mathbf{a} \in A\}$

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Example

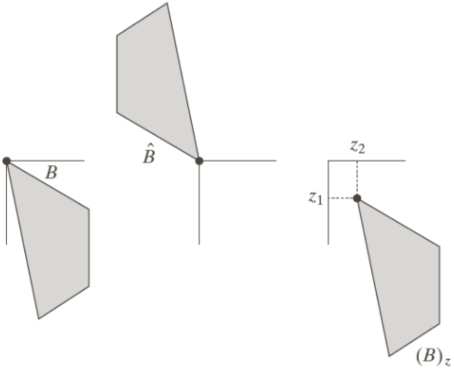

Figure: Reflection of B, and translation of set B by z

For example, $B = \{(0,1),(1,1),(2,1),(2,2),(3,0)\}$, $\mathbf{z} = (0,1)$; then

$$(B)_z = \{(0,2),(1,2),(2,2),(2,3),(3,1)\}.$$

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Morphology operators

- Dilation: Grows the shape, fills small gaps, smooths.
- Erosion: Shrinks the shape, breaks thin ties (artificial links), generates outlines.
- Opening and closing: Idempotent shape operations.
- Skeleton: The "kernel" of the shape.

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Dilation

- Dilation grows or thickens objects in an image.
- The specific manner and extent to this thickening are determined by the structuring element.
- A is a shape in a (binary) image.
- B is a structuring element.
- The dilation of A by B (written $A \oplus B$) is

$$A \oplus B = \{\mathbf{a} + \mathbf{b}; \mathbf{a} \in A, \mathbf{b} \in B.\} \tag{2}$$

- After dilation, A will be swelled by B.
- Commutative, $A \oplus B = B \oplus A$, and associative, $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.
- Require $(0, 0) \in B$, so that $A \subseteq A \oplus B$, that is, dilation is increasing.

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Structuring element

Structuring element (SE):

- A small set used to probe an image under study
- Usually, an origin for each SE is required. This origin allows the positioning of SE at a given pixel. An SE at a point or pixel **x** means the origin of the SE coincides with **x**.
- The shape and size of the structuring element must be adapted to the geometric properties of the image objects to be processed.

Dilation cont.

- There are other forms of definition of dilation, such as

$$A \oplus B = \{\mathbf{z} | ((\hat{B})_z \cap A) \neq \emptyset\}, \tag{3}$$

  where
  - $\hat{B}$ is a reflection of B, and defined as $\hat{B} = \{\mathbf{w} | \mathbf{w} = -\mathbf{b},\ \mathbf{b} \in B\}$;
  - The translation,$(B)_z$, of a set B by point $\mathbf{z} = (z_1, z_2)$ is defined as $(B)_z = \{\mathbf{c} | \mathbf{c} = \mathbf{b} + \mathbf{z}$ for $\mathbf{b} \in B\}$.
- If B is symmetric, then $\hat{B} = B$.
- By (3), the dilation of A by B is the set of all displacements, z, such that B and A overlap by at least one element.
- Compare it with convolution. The translation of the structuring element in dilation is similar to the mechanics of spatial convolution.

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Dilation cont.

The dilation of A by B can be expressed as a union of translations of A by the elements of B. That is,

$$A \oplus B = \bigcup_{\mathbf{b} \in B} A_b. \tag{4}$$

Dilation - example

	1	1	1	1	
	1		1	1	
	1		1		
		1	1	1	

(a)

1	1
1	1

(b)

Table: (a) A binary image; (b) A structuring element. (The pixel in red colour indicates the origin.)

Dilation - example

	1	1	1	1	
	1		1	1	
	1		1		
		1	1	1	

(a)

	1	1	1	1	
	1	1	1	1	
	1		1		
		1	1	1	

(b)

Dilation - example

	1	1	1	1	
	1		1	1	
	1		1		
		1	1	1	

(a)

	1	1	1	1	
	1	1	1	1	
	1		1		
		1	1	1	

(b)

	1	1	1	1	
	1	1	1	1	
	1		1		
		1	1	1	

(c)

Table: (a) Original image; (b) The structuring element aligned with the first nonzero pixel (in red) in (a); (c) The result after dilation.

Dilation - example

	1	1	1	1	
	1		1	1	
	1		1		
		1	1	1	

(a)

	1	1	1	1	
	1	1	1	1	
	1	1	1		
		1	1	1	

(b)

Dilation - example

	1	1	1	1	
	1		1	1	
	1		1		
		1	1	1	

(a)

	1	1	1	1	
	1	1	1	1	
	1	1	1		
		1	1	1	

(b)

	1	1	1	1	
	1	1	1	1	
	1	1	1		
		1	1	1	

(c)

Table: (a) The next pixel (in red) to be dilated; (b) The structuring element aligned with the pixel in (a); (c) The result after dilation.

Dilation - example

	1	1	1	1	
	1	1	1	1	
	1	1	1		
		1	1	1	

(a)

	1	1	1	1	
	1	1	1	1	
	1	1	1		
	1	1	1	1	

(b)

	1	1	1	1	
	1	1	1	1	
	1	1	1		
	1	1	1	1	

(c)

Table: (a) The next pixel to be dilated; (b) The structuring element aligned with the red pixel in (a); (c) The result after dilation.

Dilation - example

	1	1	1	1	
	1		1	1	
	1		1		
		1	1	1	

(a)

	1	1	1	1	1
	1	1	1	1	1
	1	1	1	1	1
	1	1	1	1	1
		1	1	1	1

(c)

	1	1	1	1	1
	1	1	1	1	1
	1	1	1	1	1
	1	1	1	1	1
		1	1	1	1

(c)

Table: (a) Original image; (b) All the pixels in the original image are processed; (c) The final result of dilation.

Dilation cont.

- Matlab function for dilation:

  `D=imdilate(A,B).`
- Matlab function `strel(shape, parameters)` constructs structuring elements with a variety of shapes and sizes.
- One of the simplest applications of dilation is bridging gaps.

Dilation cont.

- Using Python Scikit-image

```
from skimage import morphology
>>> B = morphology.diamond(1)
>>> B
array([[0, 1, 0],
[1, 1, 1],
[0, 1, 0]], dtype=uint8)
>>> A = np.zeros((6,6), dtype=np.uint8)
>>> A[2:4,3:4]=1
 C = morphology.binary_dilation(A, B.astype(np.uint
>>> C
array([[0, 0, 0, 0, 0, 0],
[0, 0, 0, 1, 0, 0],
[0, 0, 1, 1, 1, 0],
[0, 0, 1, 1, 1, 0],
[0, 0, 0, 1, 0, 0],
[0, 0, 0, 0, 0, 0]], dtype=uint8)
```

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Dilation cont.

```
 B = morphology.disk(6)
>>> B
array([[0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0],
[0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0],
[0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0],
[0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0],
[0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0],
[0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0],
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1],
[0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0],
[0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0],
[0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0],
[0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0],
[0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0],
[0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]], dtype=uint8)
```

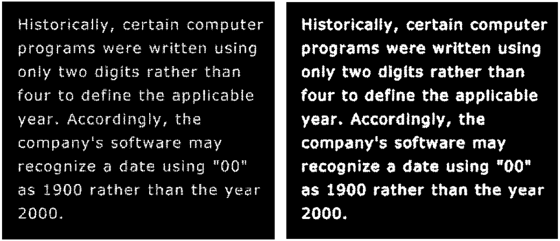
Check out the other structuring element you can use – `square`, `rectangle`, `diamond`, ...

Dilation cont.

```
>>> C = morphology.dilation(A,B)
```

Dilation - example

$$SE=\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & \color{red}{1} & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array}$$



(a) (b)

Figure: (a) Original image; (b) Result of dilation using the SE.

Erosion

- Erosion 'shrinks' or 'thins' objects in a binary image. The specific manner and extent of shrinking is controlled by a structuring element.
- The erosion of A by B is

$$A \ominus B = \{\mathbf{x}; \mathbf{x} + \mathbf{b} \in A \text{ for all } \mathbf{b} \in B\} \tag{5}$$

- An alternative definition

$$A \ominus B = \{\mathbf{z} | (B)_z \subseteq A\}, \tag{6}$$

- The above equation says that the erosion of A by B is the set of all points $\mathbf{z}$ such that B, translated by $\mathbf{z}$, is contained in A.
- The equation in (6) is equivalent to

$$A \ominus B = \{\mathbf{z} | (B)_z \cap A^c = \emptyset\}, \tag{7}$$

where A^c is the complement of A, and defined as $A^c = \{\mathbf{w} | \mathbf{w} \notin A\}$.

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Erosion cont.

Erosion of an image A by B is the intersection of all translations of A by the points $-b$, where $b \in B$. That is,

$$A \ominus B = \bigcap_{\mathbf{b} \in B} (A)_{-b}. \tag{8}$$

From the definition, if $\mathbf{x} \in A \ominus B$, then for every $\mathbf{b} \in B, \mathbf{x} + \mathbf{b} \in A$. But $\mathbf{x} + \mathbf{b} \in A$ implies $\mathbf{x} \in (A)_{-b}$. Hence for every $\mathbf{b} \in B$, $\mathbf{x} \in (A)_{-b}$. This implies $\mathbf{x} \in \bigcap_{\mathbf{b} \in B} (A)_{-b}$. Hence for every $\mathbf{b} \in B, \mathbf{x} + \mathbf{b} \in A$, by definition of erosion $\mathbf{x} \in A \ominus B$.

Erosion - example

	1	1	1	1	
	1		1	1	
	1		1		
		1	1	1	

(a)

1
1

(b)

Table: (a) A binary image; (b) A structuring element. (The pixel in red colour indicates origin of the structuring element.)

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Erosion - example

	1	1	1	1	
	1		1	1	
	1		1		
		1	1	1	

(a)

	1	1	1	1	
	1		1	1	
	1		1		
		1	1	1	

(b)

	1	1	1	1	
	1		1	1	
	1		1		
		1	1	1	

(c)

Table: (a) The pixel (in red) to be eroded; (b) Align the origin of the structuring element with the red pixel in (a); (c) The result of erosion.

Erosion - example

	1	1	1	1	
	1		1	1	
	1		1		
		1	1	1	

(a)

	1	1	1	1	
	1	1	1	1	
	1		1		
		1	1	1	

(b)

	1		1	1	
	1		1	1	
	1		1		
		1	1	1	

(c)

Erosion - example

	1	1	1	1	
	1		1	1	
	1		1		
		1	1	1	

(a)

	1	1	1	1	
	1		1	1	
	1		1		
		1	1	1	

(b)

	1		1	1	
	1		1		
			1		

(c)

Table: (a) Original image; (b) The red pixels are removed after erosion; (c) The final result of erosion.

Erosion cont.

- Matlab function for erosion: `imerode`
- Python Scikit-image functions for erosion: `binary_erosion` and `erosion`.
- One of the simple applications of erosion is to eliminate irrelevant details.

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Erosion - example

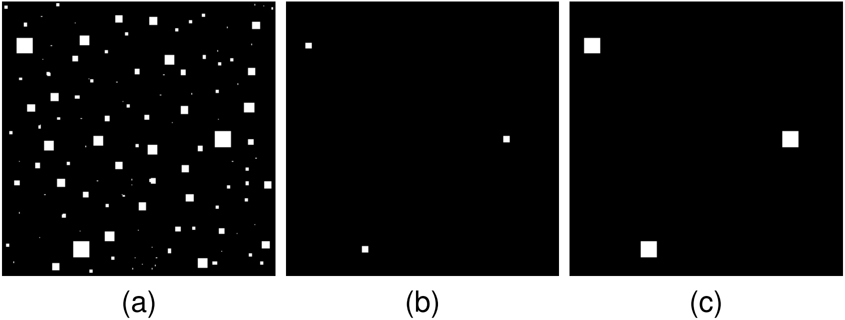

Figure: (a) Original image; (b) Result of erosion using an SE of 13 $\times$ 13 1's; (c) Result of dilation of (b) using the same SE.

Dilation and erosion

- Dilation and erosion are duals of each other with respect to set complementation
$$(A \ominus B)^c = A^c \oplus \hat{B}. \tag{9}$$

Proof - 1:

- $\mathbf{x} \in (A \ominus B)^c$ implies $\mathbf{x} \notin (A \ominus B)$;
- $\mathbf{x} \notin (A \ominus B)$ implies $\exists \mathbf{b} \in B$ such that $\mathbf{x} + \mathbf{b} \notin A$
- $\exists \mathbf{b} \in B$ such that $\mathbf{x} + \mathbf{b} \in A^c$, which is equivalent to $\exists \mathbf{b} \in B$ such that $\mathbf{x} \in (A^c)_{-\mathbf{b}}$.
- $\exists \mathbf{b} \in B$ such that $\mathbf{x} \in (A^c)_{-\mathbf{b}}$ implies $\mathbf{x} \in \cup_{\mathbf{b} \in B}(A^c)_{-\mathbf{b}}$.
- $\mathbf{x} \in \cup_{\mathbf{b} \in B}(A^c)_{-\mathbf{b}}$ implies $\mathbf{x} \in \cup_{\mathbf{b} \in \hat{B}}(A^c)_{\mathbf{b}}$.
- $\mathbf{x} \in \cup_{\mathbf{b} \in \hat{B}}(A^c)_b$ implies $\mathbf{x} \in A^c \oplus \hat{B}$.

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Dilation and erosion cont.

- The dual can also be put as

$$(A \ominus B)^c = \{\mathbf{z} | (B)_z \subseteq A\}^c. \tag{10}$$

Proof - 2:

- If set $(B)_z$ is contained in set A, then $(B)_z \cap A^c = \emptyset$, then (10) becomes

$$(A \ominus B)^c = \{\mathbf{z} | (B)_z \cap A^c = \emptyset\}^c. \tag{11}$$

- The complement of the set of z's that satisfy $(B)_z \cap A^c = \emptyset$ is the set of $\mathbf{z}$'s such that $(B)_z \cap A^c \neq \emptyset$. Thus

$$\begin{aligned} (A \ominus B)^c &= \{\mathbf{z} | (B)_z \cap A^c \neq \emptyset\} \\ &= A^c \oplus \hat{B}. \end{aligned} \tag{12}$$

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Opening and closing

- Opening
  - Remove sharp features
  - Break narrow links and joins
  - Remove noise
  - Idempotent
- Closing
  - Rounds sharp features
  - Fills narrow gaps and holes
  - Remove noise
  - Idempotent.

Opening and closing cont.

- The opening of an image A by structuring element B is defined as

$$A \circ B = (A \ominus B) \oplus B \tag{13}$$

- The closing of A by B is

$$A \bullet B = (A \oplus B) \ominus B \tag{14}$$

- Geometric interpretation
  - Opening: The boundary of $A \circ B$ is established by the points in B that reach the farthest into the boundary of A as B is shifted around the inside of this boundary. It can be expressed as a fitting process

$$A \circ B = \cup\{(B)_z | (B)_z \subseteq A\}. \tag{15}$$

  - Closing: a point $\mathbf{w}$ is a point of $A \bullet B$ if and only if $(B)_z \cap A \neq \emptyset$ for any translate of $(B)_z$ that contains $\mathbf{w}$.

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Opening and closing cont.

- Opening and closing are dual to each other with respect to set complementation and reflection. That is

$$(A \bullet B)^c = (A^c \circ \hat{B}). \tag{16}$$

- Properties of opening
  - $A \circ B$ is a subset (subimage) of A;
  - If C is a subset of D, then $C \circ B$ is also a subset of $D \circ B$.
  - Idempotence: $(A \circ B) \circ B = A \circ B$.
- Properties of closing
  - A is a subset (subimage) of $A \bullet B$;
  - If C is a subset of D, then $C \bullet B$ is also a subset of $D \bullet B$.
  - Idempotence: $(A \bullet B) \bullet B = A \bullet B$.
- Increasing/decreasing: $A \circ B \subseteq A \subseteq A \bullet B$.

Note that multiple openings of closings of a set have no effect after the operator has been done for once.

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Proof of duality of opening and closing

Proof of duality of opening and closing

$$\begin{aligned}(A \bullet B)^c =& ((A \oplus B) \ominus B)^c \\ =& (A \oplus B)^c \oplus \hat{B} \\ =& (A^c \ominus \hat{B}) \oplus \hat{B} \\ =& A^c \circ \hat{B}\end{aligned} \tag{17}$$

Opening - example

Using the same structuring element in the erosion example,

1
1

	1	1	1	1	
	1		1	1	
	1		1		
		1	1	1	

(a)

	1		1	1	
	1		1		
			1		

(b)

	1		1	1	
	1		1	1	
	1		1		
			1		

(c)

Table: (a) Original image; (b) The result of erosion of (a); (c) The result of dilation of (b).

Closing - example

Using the same SE in the dilation example

1	1
1	1

	1	1	1	1	
	1		1	1	
	1		1		
		1	1	1	

(a)

	1	1	1	1	1
	1	1	1	1	1
	1	1	1	1	1
	1	1	1	1	1
		1	1	1	1

(b)

	1	1	1	1	
	1	1	1	1	
	1	1	1	1	
		1	1	1	

(b)

Table: (a) Original image; (b) The result of dilation of (a); (c) The result of erosion of (b).

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Opening and closing cont.

Matlab functions

- `imopen(A,B)` for opening, and
- `imclose(A,B)` for closing.

Python Scikit-image functions

- `skimage.morphology.binary_opening()`
- `skimage.morphology.binary_closing()`
- `skimage.morphology.opening()`
- `skimage.morphology.closing()`

Opening - geometric interpretation example

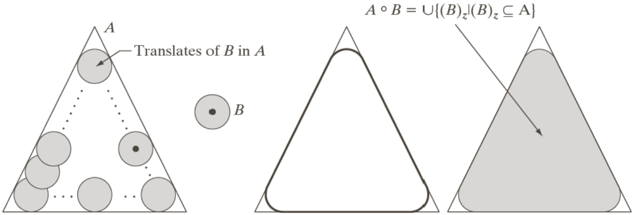


Figure: Structuring element B “rolling” along the inner boundary of set A (The dots indicate the origin of B). Bold line is the outer boundary of the opening. The shaded is the complete opening.

Closing - geometric interpretation example

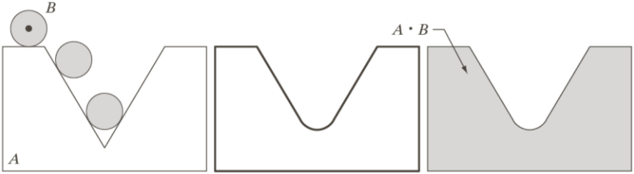


Figure: Structuring element B "rolling" on the outer boundary of set A. Bold line is the outer boundary of the closing. The shaded is the complete closing.

Opening and closing - geometric interpretation example

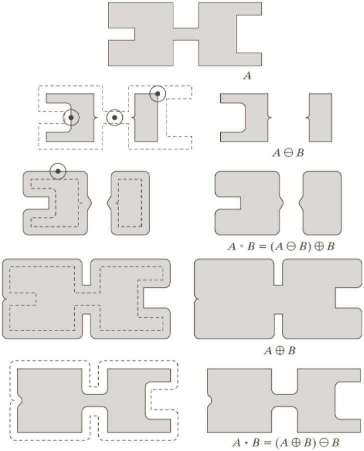


Figure: Morphological opening and closing. The structuring element is the small circle with a dark dot as its origin.

Opening and closing - examples

(a) (b) (c)

Figure: (a) Original image; (b) Opening by a disk of radius 5; (c) Closing by a disk of radius 5.

Outline

1. Mathematical Morphology

2. Some basic morphological algorithms

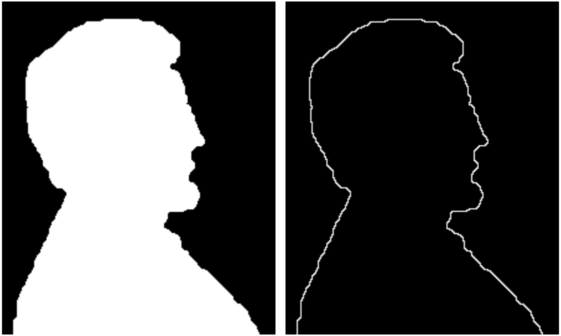
Boundary extraction

- The boundary of a set A, denoted by $\beta(A)$, can be obtained by first eroding A by B, and then performing the set difference between A and its erosion. That is

$$\beta(A) = A - (A \ominus B), \tag{18}$$

where B is a suitable structuring element.

Boundary extraction example



(a) (b)

Figure: (a) Original binary image; (b) Result of using $A - (A \ominus B)$ with B a 3×3 SE of 1's.

Skeleton

- The idea of a Skeleton for a shape is to distil the essence of the shape.
- An informal definition would be:
  - one-pixel thick,
  - through the ‘middle’ of the object, and,
  - preserves the topology of the object.
- Unfortunately no practical definition will work with these requirements. Why? It does not always realizable. For example

			1	1			
			1	1			
			1	1			
			1	1			
			1	1			
			1	1			
			1	1			

			1			
		1		1		
	1		1		1	
		1	1	1		
	1		1		1	
		1		1		
			1			

Skeleton cont.

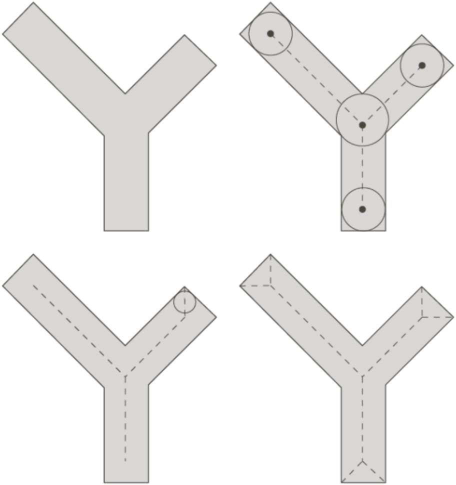


Figure: Various positions of maximum disks with centers on the skeleton of A. Bottom right: complete skeleton.

Skeleton using heuristic method

A quite practical construction of a skeleton can be implemented using the following heuristic algorithm:

- Set up a 3×3 moving window;
- Repeatedly scan the image with the window until it is stable using the rules
  - At each step we set the centre pixel to zero UNLESS one of the following conditions occurs:
    - an isolated pixel is found;
    - removing a pixel would change the connectivity;
    - removing a pixel would shorten a line.

	1	

(1)

		1
	1	
1	1	

(2)

	1	1

(3)

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Alternative idea of skeletons

- Construction of skeleton sets.
- Idea is to construct a finite sequence of sets which together encapsulate the concept of the shape. Let B be a small disc.
- The kth skeleton set ($k = 0, 1, \dots, K$) is defined by:

$$S_k(A) = (A \ominus kB) \cap ((A \ominus kB) \circ B)^c, \tag{19}$$

  where K is chosen as the last iterative step before A erodes to an empty set, that is

$$(A \ominus KB) \circ B = \emptyset. \tag{20}$$

- Note that

$$S_0(A) = A \cap (A \circ B)^c, \tag{21}$$

  so the skeleton sets builds inwards from the boundaries.
- The skeleton set itself will then be the disjoint union:

$$S(A) = \cup S_k(A), k = 0, 1, \dots, K. \tag{22}$$

Skeleton sets

Consider the following table.

Erosions	Openings	Set differences
A	$A \circ B$	$A - (A \circ B)$
$A \ominus B$	$(A \ominus B) \circ B$	$(A \ominus B) - ((A \ominus B) \circ B)$
$A \ominus 2B$	$(A \ominus 2B) \circ B$	$(A \ominus 2B) - ((A \ominus 2B) \circ B)$
$A \ominus 3B$	$(A \ominus 3B) \circ B$	$(A \ominus 3B) - ((A \ominus 3B) \circ B)$
$\vdots$	$\vdots$	$\vdots$
$A \ominus kB$	$(A \ominus kB) \circ B$	$(A \ominus kB) - ((A \ominus kB) \circ B)$

Table: Construction of skeleton sets

We continue the set construction until $(A \ominus kB) \circ B$ is empty. The skeleton is then obtained by taking the union of all the set differences.

Skeleton sets - example (using a 3×3 cross SE)

1	1	1	1			
1	1	1	1			
1	1	1	1			
1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1

A

	1	1				
1	1	1	1			
1	1	1	1			
1	1	1	1	1	1	
1	1	1	1	1	1	1
1	1	1	1	1	1	1
	1	1	1	1	1	

$A \circ B$

1			1			
						1
1						1

$A - (A \circ B)$

	1	1				
	1	1				
	1	1	1			
	1	1	1	1	1	
	1	1	1	1	1	

$A \ominus B$

		1				
	1	1	1			
	1	1	1	1		
		1	1			

$(A \ominus B) \circ B$

	1	1				
	1					
					1	
	1			1	1	

$(A \ominus B) - ((A \ominus B) \circ B)$

Skeleton sets - example cont.

		1				
		1	1			

$A \ominus 2B$

$(A \ominus 2B) \circ 2B$

		1				
		1	1			

$(A \ominus 2B) - ((A \ominus 2B) \circ B)$

1			1			
	1	1				
	1					
		1				1
		1	1		1	
	1			1	1	
1						1

The final skeleton

Skeleton cont.

- The skeleton set can be used to completely reconstruct the shape A:

$$A = \cup(S_k(A) \oplus kB), k = 0, 1, \ldots, K, \quad (23)$$

where $S_k(A) \oplus kB = (\ldots(((S_k(A) \oplus B) \oplus B) \oplus B) \ldots) \oplus B$.

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Skeleton set theorem (23) proof.

- For any sets x and Y, $X = \{X \cap Y^c\} \cup \{X \cap Y\}$. Thus,

$$\begin{aligned} A \ominus qB =& ((A \ominus qB) \cap ((A \ominus qB) \circ B)^c) \cup ((A \ominus qB) \circ B) \\ =& S_q(A) \cup ((A \ominus qB) \circ B)) \end{aligned} \tag{24}$$

- Prove by induction that

$$A \ominus qB = \cup_{k=q}^{K} \left(S_k(A) \oplus (k-q)B \right). \tag{25}$$

Note that

- $q = 0$ gives the result;
- $q = K$ is true by definition of K;
- suppose the result holds for some $q + 1$. We deduce it is true for $q, q < K$.

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- Assume

$$A \ominus (q+1)B = \cup_{k=q+1}^{K}(S_k(A) \oplus (K-q-1)B). \tag{26}$$

Then

$$\begin{aligned}
\cup_{k=q}^{K} S_k(A) \oplus (K-q)B & \\
&= ((\cup_{k=q+1}^{K} S_k(A) \oplus (K-q-1)B) \oplus B) \cup \\
&\quad (S_q(A) \oplus (q-q)B) \\
&= ((\cup_{k=q+1}^{K} S_k(A) \oplus (K-q-1)B) \oplus B) \cup S_q(A) \\
&= ((A \ominus (q+1)B) \oplus B) \cup S_q(A) \\
&= (((A \ominus qB) \ominus B) \oplus B) \cup S_q(A) \\
&= ((A \ominus qB) \circ B) \cup S_q(A) \\
&= A \ominus qB
\end{aligned} \tag{27}$$

Skeleton - example

Using Matlab function

```
bwmorph(f, 'skel', Inf)
```

Using Python Scikit-image

```
from skimage.morphology import skeletonize
im_skeleton = skeletonize(image)
```