CMS/CS/EE 144:
Networks: Structure & Economics

A probability refresher

Outline

- Probability space
- Random variables
- Expectation
- Indicator random variable
- Important discrete random variables
- Important continuous random variables

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$

- Sample Space Ω
  - The collection of all possible outcomes of an experiment
- Events
  - Collections of outcomes
  - We will only be concerned with the probability of events, rather than individual outcomes.
- A certain collection of events $\mathcal{F}$
  - Specifically, a $\sigma\text{-}algebra$
- Probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$

Example: Toss a fair coin twice

- Sample Space Ω
  - $\{HH,\ HT,\ TH,\ TT\}$
- A certain collection of events $\mathcal{F}$
  - $\{\emptyset\ ,\ \{HH\},\ldots,\ \{HH,\ HT\},\ldots,\ \{HH,\ HT,\ TH\},\ldots,$ $\{HH,\ HT,\ TH,\ TT\}\}$
- Probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$
  - E.g., $P(\{HH\}) = \frac{1}{4}$, $P(\{HH,\ HT\}) = \frac{1}{2}$

Useful properties

Inclusive/Exclusive Principle : For any pair of events A_1 and A_2,

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)$$

Union Bound: Given events $A_1, A_2, \ldots, A_n$,

$$\mathbb{P}(\cup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} \mathbb{P}(A_i)$$

Conditional Probability

Conditional Probability: Given two events A and B, assume we know that B occurs. The probability that A occurs given that it is known that B occurs is called the conditional probability of A given B and is defined by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Bayes Theorem : Given two events A and B,

$$\mathbb{P}(A|B) = \mathbb{P}(B|A)\frac{\mathbb{P}(A)}{\mathbb{P}(B)}$$

Conditional Probability

Law of total probability: Given any partition of events $E_1, E_2, \ldots, E_n$ of Ω, where $\mathbb{P}(E_i) > 0$,

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(E_i)\mathbb{P}(A|E_i)$$

Exercise

Independence

Independence : Two events A_1 and A_2 are independent if and only if,

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1) \times \mathbb{P}(A_2)$$

Question: If two events are mutual exclusive, are they independent?

Answer: No! Unless one of them has zero probability!

Independence

Independence : Two events A_1 and A_2 are independent if and only if,

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1) \times \mathbb{P}(A_2)$$

Example: Tossing a fair die and denote the outcome by O. Let A be the event {O is odd}, and B be the event {O is less than 3}. Then A and B are independent.

Random variables

A **random variable** X is a function mapping $\Omega \to \mathbb{R}$ (with some conditions).

The **distribution** of a random variable X is defined as

$$\mathbb{P}(X \in A) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\})$$

Example: X = number of heads in two coin tosses.

1) $X(HH) = 2,\ X(HT) = X(TH) = 1,\ X(TT) = 0$.
2) The event that there is at most one head: $\{X \leq 1\}$

Random variables

Cumulative Distribution Function:

$$F_X(x) = \mathbb{P}(X \in (\infty, x])$$

Probability Density Function: For continuous random variable X, If $F_X(x)$ is differentiable everywhere, its derivative $f_X(x)$ is called the p.d.f of X.

Probability Mass Function: For discrete random variable X, its probability mass function

$$p_X(\cdot) = \mathbb{P}(X = x_i)$$

Independence of random variables

Previously, we defined the independence of two events.

Independence: Two events A_1 and A_2 are independent if and only if,

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1) \times \mathbb{P}(A_2)$$

Independence of random variables

Previously, we defined the independence of two events.

Question: How to define the independence of random variables?

Idea: The inverse mapping of a random variable X^{-1} will map any set A in $\mathbb{R}$ to event $\{X \in A\}$... Then we have to check every sets in $\mathbb{R}$ and its corresponding events...

Too messy

Independence of random variables

Two random variables X and Y are **independent** if and only if, for every x and y,

$$\mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y)$$

Question: How to define the independence of random variables?

Idea: The inverse mapping of a random variable X^{-1} will map any set A in $\mathbb{R}$ to event $\{X \in A\}$... Then we have to check every sets in $\mathbb{R}$ and its corresponding events...

→ Too messy

Conditional density of random variables

Conditional Probability: Given two random variables X and Y,

$$\mathbb{P}(X \in A | Y \in B) = \frac{\mathbb{P}(X \in A, Y \in B)}{\mathbb{P}(Y \in B)}$$

Previously we defined the conditional density of two events.

Conditional Probability: Given two events A and B,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Expectation

Expectation of a random variable X can be interpreted as the average value of X over many repetitions of the probability experiment. Also called the "**mean**" or the "**first moment**."

For example, for discrete random variable,

$$\mathbb{E}[X] = \sum_{x \in \Omega} x p_X(x)$$

For continuous random variable,

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x$$

Expectation

Linearity: Given two random variables X and Y,

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$$

➡ **Very important**

Independence: Given two independent random variables X and Y,

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

Conditional expectation

Conditional expectation can be interpreted as the average value taken by a random variable over many repetitions of the probability experiment **conditioned on** the occurrence of a particular event.

For example, for discrete random variable,

$$\mathbb{E}[X|A] = \sum_{x \in \Omega} x p_X(x|A)$$

For continuous random variable,

$$\mathbb{E}[X|A] = \int_{-\infty}^{\infty} x f_X(x|A)\,\mathrm{d}x$$

Conditional expectation

Conditional expectation can be interpreted as the average value taken by a random variable over many repetitions of the probability experiment **conditioned on** the occurrence of a particular event.

Law of total probability: Given any partition of events $E_1, E_2, \ldots, E_n$ of Ω, where $\mathbb{P}(E_i) > 0$,

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{P}(E_i)\mathbb{E}(X|E_i)$$

Indicator random variable

The **indicator random variable** for an event E is given by

$$\mathcal{I}_E(w) = \begin{cases} 1, & \text{if } w \in E \\ 0, & \text{else} \end{cases}$$

Important fact:

$$\mathbb{E}[\mathcal{I}_E] = 1 \cdot \mathbb{P}(E) + 0 \cdot \mathbb{P}(E^C) = \mathbb{P}(E)$$

Indicators are tremendously useful for **counting expected number of occurrences**!

Exercise

Important probability distributions

- Bernoulli
- Binomial
- Geometric
- Poisson
- Uniform
- Exponential
- Normal/Gaussian

– When they arise
– Distribution functions
– Mean, variance, etc.
– How they relate to each other

Bernoulli random variable

- A biased coin has a probability p of showing heads when tossed. Let X take the value 1 if the outcome is heads and 0 if the outcome is tails.
- Often, the events corresponding to $X=1$ and $X=0$ are referred to as "success" and "failure".

$$p_X(x) = \begin{cases} 1-p, & \text{if } x = 0 \\ p, & \text{if } x = 1 \end{cases}$$

$$\mathbb{E}[X] = p$$

$$\mathrm{Var}(X) = p(1-p)$$

Binomial random variable

- A biased coin has a probability p of showing heads when tossed. Let X denote the total number of heads obtained over n tosses.
- Perform n independent Bernoulli trials $X_1,\ldots, X_n$ and let $X=X_1+\ldots+X_n$ denote the total number of successes.

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, k = 0, 1, \ldots, n$$

$$\mathbb{E}[X] = np$$

$$\mathrm{Var}(X) = np(1-p)$$

Geometric random variable

- A biased coin has a probability p of showing heads when tossed. Suppose the coin is tossed repeatedly until a head occurs. Let X denote the number of tosses required to obtain a head.
- X is also called a waiting time till the first success.

$$p_X(k) = (1-p)^{k-1}p,\ k = 1, 2, 3 \ldots$$

$$\mathbb{E}[X] = \frac{1}{p}$$

$$\mathrm{Var}(X) = \frac{1-p}{p^2}$$

Poisson random variable

- Number of phone calls arriving at a call center per minute.
- Number of jumps in a stock price in a given time interval.
- Number of misprints on the front page of a newspaper.

$$p_X(k) = \frac{e^{-\lambda}\lambda^k}{k!},\ k = 0, 1, 2\ldots$$

$$\mathbb{E}[X] = \lambda$$

$$\mathrm{Var}(X) = \lambda$$

Uniform random variable

- Let X denote a random variable that takes any value inside $[a,b]$ with equal probability.

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0, & \text{if } x < a \\ \frac{x-a}{b-a}, & \text{if } a \leq x \leq b \\ 1, & \text{if } x > b \end{cases}$$

$$\mathbb{E}[X] = \frac{a+b}{2} \qquad \mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

Exponential random variable

- Time it takes for a server to complete a request.
- Time it takes before your next telephone call.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-\lambda x}, & \text{if } x \geq 0 \end{cases}$$

$$\mathbb{E}[X] = \frac{1}{\lambda} \qquad \mathrm{Var}(X) = \frac{1}{\lambda^2}$$

Normal/Gaussian random variable

- Arises as a limit of the sample average of independent and identically distributed (i.i.d.) random variables, e.g. $Binomial(n,p)$ as n goes to infinity. This is known as the Central-Limit Theorem, which will be presented in a subsequent lecture.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\mathbb{E}[X] = \mu$$

$$\mathrm{Var}(X) = \sigma^2$$

Exercise

Questions?