Collaborated with Kshitij Grover and Siddharth Murching

Problem 1

(a)

(i)

We know the following inequality involving random variables is true (it can be easily checked for the two indicator value values):

$$t\mathbb{1}_{X\geq t} \leq X$$

Then, since expectation satisfies monotonicity and linearity, we have that

$$tE[\mathbb{1}_{X\geq t}] \leq E[X]$$

Finally, given that $E[\mathbb{1}_A] = P(A)$, we have that

$$P(X \geq t) \leq \frac{E[X]}{t}$$

(ii)

We know the following inequality involving random variables is true (it can be easily checked for the two indicator value values):

$$t\mathbb{1}_{|X-E[X]|\geq t} \leq |X - E[X]|$$
$$t^2\mathbb{1}_{|X-E[X]|\geq t} \leq (X - E[X])^2$$

Then, since expectation satisfies monotonicity and linearity, we have that

$$t^2 E[\mathbb{1}_{X-E[X]\geq t}] \leq E[(X - E[X])^2]$$

Next, given that $E[\mathbb{1}_A] = P(A)$, we have that

$$P(X - E[X] \geq t) \leq \frac{E[(X - E[X])^2]}{t^2}$$

Finally, since we can write $Var(X) = E[(X - E[X])^2] = \sigma_X^2$, we have that

$$P(X - E[X] \geq t) \leq \frac{\sigma_X^2}{t^2}$$

(iii)

We will first show that $E[\frac{S_n}{n}] = E[X]$.

$$E[\frac{S_n}{n}] = E[\frac{\sum_{i=1}^{n} X_i}{n}]$$

By linearity of expectation, and since n is a constant, we have

$$E[\frac{S_n}{n}] = \frac{E[X_1] + E[X_2] + ... + E[X_n]}{n}$$

Then, since $X_1, X_2, ...$ are i.i.d, we have

$$E[\frac{S_n}{n}] = \frac{nE[X]}{n}$$

$$E[\frac{S_n}{n}] = E[X]$$

Now we will that $var(\frac{S_n}{n}) = \frac{\sigma_X^2}{n}$

$$var(\frac{S_n}{n}) = \frac{1}{n^2}var(S_n)$$

Since $X_1, X_2, ...$ are i.i.d (and S_n represents their sum), we have

$$var(\frac{S_n}{n}) = \frac{1}{n^2}n\sigma_X^2$$

$$var(\frac{S_n}{n}) = \frac{\sigma_X^2}{n}$$

Now let's plug this all into Chebyshev's inequality. We can see that we can substitute $\frac{S_n}{n}$ for X. We can also see that $E[X]$ can be kept in the equation, since we showed that $E[\frac{S_n}{n}] = E[X]$. Then, we can substitute in the variance we found above, and use all $\epsilon > 0$ instead of all $t > 0$. This gives us the following.

$$Pr(|\frac{S_n}{n} - E[X]| > \epsilon) \leq \frac{\sigma_X^2}{n\epsilon^2}$$

Then, clearly, as $n \to \infty$, both sides go to zero. So we have that, for all $\epsilon > 0$,

$$\lim_{n\to\infty} Pr(|\frac{S_n}{n} - E[X]| > \epsilon) = 0$$

(b)

Let the diameter of our graph be k. Let the complete "body" of our graph, circled in the diagram, have n nodes. Then the "arm" portion of our graph has k nodes. Let the node in the arm that is connected to the body be connected to every node in the body. Now let us calculate the sum of all the distances in this graph. We have that, between nodes in the arm, there are:

$k-1$ paths of length 1
$k-2$ paths of length 2
...
1 path of length $k-1$

Overall, this is equivalent to the sum $\sum_{i=1}^{k-1} i(k-i) = \frac{1}{6}k(k-1)(k+1)$.

Now, within the body, we have that there are $\binom{n}{2}$ paths of length 1, since the body is complete. Then, between the arm and the body, we have the following sum: $\sum_{i=1}^{k} in$. This is because the farthest node in the arm has n paths of length k, the next farthest has n paths of length $k-1$, etc. So our overall distance sum is as follows:

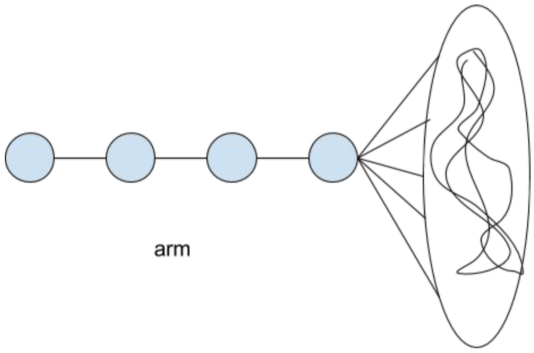
$$\text{distance sum} = \frac{1}{6}k(k-1)(k+1) + \binom{n}{2} + \sum_{i=1}^{k} in$$

$$\text{distance sum} = \frac{1}{6}k(k-1)(k+1) + \frac{1}{2}(n-1)n + \frac{1}{2}k(k+1)n$$

Then we have that, since our graph is connected, the total number of paths is $\binom{k+n}{2}$. So the average distance is

$$\text{avg distance} = \frac{\frac{1}{6}k(k-1)(k+1) + \frac{1}{2}(n-1)n + \frac{1}{2}k(k+1)n}{\binom{k+n}{2}}$$

We can observe that we can make this average go to 1 as $n \to \infty$. Intuitively, this makes sense; we add a bunch of paths that are length 1, and only a few longer ones. Mathematically, we can see that the number of longer paths scales linearly with k and n, and the number of paths of length one scales quadratically with n. So for a fixed k, if we just take n to be extremely large, we can get arbitrary diameter to average distance ratios (where the ratios are greater than 1). An example where the diameter is more than 3 times the average distance is if $n = 100000$ and $k = 4$.



Problem 2

(a)

Let H = heads and T = tails. To get an unbiased bit, we can do the following. We will segment our flips into rounds of two. That is, if we have the flips HHTTHT, we will consider the results HH, TT, and HT. Then, if we come across the result HT, we will call that 0. If we come across the result TH, we will call that 1. We will keep flipping the coin until we get one of these results. This is unbiased because the probability of getting HT equals the probability of getting TH, since each flip is independent.

Let X be the number of flips it takes to get one bit.
Let A be the event that we get either HT or TH.

Then, to calculate the expected number of flips, we can write the following.

$$E[X] = 2P(A) + (2 + E[X])P(A^c)$$
$$E[X] = 2(2p(1-p)) + (2 + E[X])(p^2 + (1-p^2))$$
$$E[X] = 4p(1-p) + (2 + E[X])(2p^2 - 2p + 1)$$
$$E[X] = \frac{1}{p - p^2}$$

The initial equation is explained as follows: when we get either HT or TH, then it takes us two flips to get our bit. Else, it takes us two flips plus the expected value of flips. The probabilities are fairly straightforward: $P(A)$ represents the chance of getting HT or TH, and $P(A^c)$ the chance of getting HH or TT.

(b)

Let K be the number of chunks we want to download.
Let $E[K]$ be the expected time to download k chunks.
Let A be the event that the randomly selected server has a chunk we have not already downloaded.

Then we can write the following recurrence relation:

$$E[K] = P(A)(t_1 + E[K-1]) + P(A^c)(t_2 + E[K])$$
$$E[K] = \frac{K}{n}(t_1 + E[K-1]) + \frac{n-K}{n}(t_2 + E[K])$$

Knowing that $E[0] = 0$, we can solve to get

$$E[K] = nt_2H_K + K(t_1 - t_2)$$

where H_K is the Kth harmonic number. The initial equation is explained as follows. When a server is selected, we know that there are two possibilities. We can either download a new chunk, which takes time t_1 and reduces the remaining number of chunks by 1; or, we can spend time t_2, which leaves us with the same number of chunks to download. The probabilities for these scenarios are K/n and $(n-K)/n$, respectively.

Then, since we want to download all n chunks, $E[n]$ is the following:

$$E[n] = nt_2 H_n + n(t_1 - t_2)$$

(c)

(i)

The probability of getting the first record at time n is the following:

$$\frac{1}{n-1}\frac{1}{n}$$

In words, this represents the following. $\frac{1}{n-1}$ is the probability that the first value is the second biggest and $\frac{1}{n}$ is the probability that the current value is the biggest. By doing this, we account for the fact that a record has not yet occurred (account for the previous $n-1$ values), while also factoring in the probability that the current variable will be the record (account for the current value). Intuitively, this makes sense. If only 1 value has passed by, the outlook is optimistic that we can beat the first value. If 1000 values have passed by and none of them have beaten the first one, the outlook is a bit more bleak. With this in mind, it becomes easy to write out the expected value.

$$E[N] = \sum_{i=2}^{\infty} i\frac{1}{i-1}\frac{1}{i} = \infty$$

(ii)

The probability of getting *a* record at time n is simply

$$\frac{1}{n}$$

Note that it no longer matters whether or not the record we get is the first record, which simplifies the probability expression. Then we can calculate the expected number of records that we get within the first M steps. We will call this $E[M]$.

$$E[M] = \sum_{i=2}^{M}\frac{1}{i} = \Big(\sum_{i=1}^{M}\frac{1}{i}\Big) - 1$$

We do this because the expected number of records we get equals the probability of getting a record at step 2 plus the probability of getting a record at step 3 plus... etc etc. Then, for large M, this becomes

$$E[M] \approx \ln M - 1$$

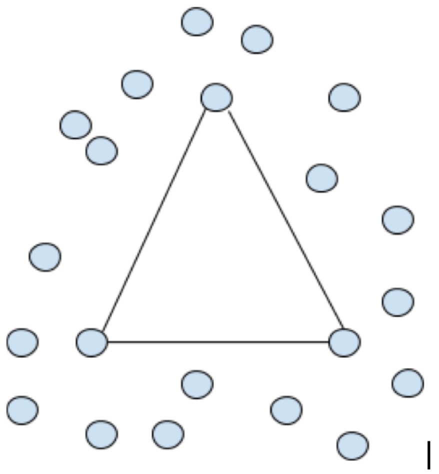
So to find out what M to choose, we can just solve the following (assuming x is sufficiently large):

$$\ln M - 1 = x$$
$$\ln M = x + 1$$
$$M = e^{(x+1)}$$

Problem 3

(a)

The example for this is fairly simple. First, we have a triangle. Then we have n vertices, all unconnected. As $n \to \infty$, Cl^{avg} goes to 0. It does this because $Cl_i(G) = 0$ for vertices with degree 0, and as we take $n \to \infty$, these dominate the average. $Cl(G)$ is always just 1, because there is 1 triangle and there are 3 connected triples. Here is an illustration:



Problem 4

(a)

First path (5 nodes):
Network congestion (click on **routers**) → Router (computing) (click on **core routers**) → Core router (click on **fiber [optic cable]**) → Optical fiber (click on **electromagnetic interference**) → Electromagnetic Interference (click on **capacitor**) → Capacitor

Shortest path (4 nodes):
Network congestion (click on **telephone circuits**) → Local loop (click on **electrical circuit**) → Electrical network (click on **capacitors**) → Capacitors

First path (5 nodes):
Paul Erdos (click on **Mathematics**) → Mathematics (click on **computer science**) → Computer science (click on **Distributed computing** → Distributed computing (click on **Chandy, Mani**) → K. Mani Chandy

Shortest path (4 nodes):
Paul Erdos (click on **John Selfridge**) → John Selfridge (click on **distributed computing**) → Distributed computing (click on **Chandy, Mani**) → K. Mani Chandy

(c)

Note: > indicates continuation of previous line.
This problem was done 1/09/16 at around 2:10 pm. My path uses 11 nodes.

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Star Wars: The Complete Saga (Episodes I-VI) [Blu-ray]
Leegoal Tri-wing Screwdriver for Nintendo Wii, Gamecube, Gameboy Advance
45 in 1 Professional Portable Opening Tool Compact Screwdriver Kit Set with
> Tweezers & Extension Shaft for Precise Repair or Maintenance Jk6089-A
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SE MH1047L Illuminated Multi-Power LED Head Magnifier
Pixnor 5-in-1 High-precision Stainless Steel Tweezers Repair Tools Set (Silver)
Organic Cocoa Butter By Sky Organics: Unrefined, 100% Pure Raw Cocoa Butter
> 16oz - Skin Nourishing, Moisturizing & Healing, for Dry Skin, Stretch
> Marks - For Skin Care, Hair Care & DIY Recipes
Premium Soft Replacement Toothbrush Heads Compatible with Oral B Toothbrush
> Handles, 4 Count
Brio SmartClean Sonic Electric Toothbrush
Brio Radius Nail Clippers - Toenail Clippers and Fingernail Clippers
Clorox Disinfecting Wipes Value Pack, Fresh Scent and Citrus Blend, 225 Count
Colgate Extra Clean Toothbrush, Full Head, Soft, 6 Count
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