Problem 1: Decision Tree Basic Questions

Note: Using log base 2

Note: Used entropy.py to calculate entropies

1.

(a)

Root

For the root, we have that $S_1 = \{No, Yes, Yes, Yes\}$ with $p_{S_1} = .75$, which means we have an entropy of

$$L(S) = 4 \cdot 0.811278124459 = 3.245112497836$$

Depth-1

Splitting on Package type would give us
Bagged: $S_1 = \{Yes, Yes\}$ with $p_{S_1} = 1$
Canned: $S_2 = \{No, Yes\}$ with $p_{S_2} = 0.5$
This means we have an entropy of

$$L(S) = L(S_1) + L(S_2) = 2 \cdot 0 + 2 \cdot 1.0 = 2.0$$

Splitting on Unit price > \$5 would give us
Yes: $S_1 = \{No, Yes\}$ with $p_{S_1} = 0.5$
No: $S_2 = \{Yes, Yes\}$ with $p_{S_2} = 1$
This means we have an entropy of

$$L(S) = L(S_1) + L(S_2) = 2 \cdot 1.0 + 2 \cdot 0 = 2.0$$

Splitting on Contains > 5 grams of fat would give us
Yes: $S_1 = \{No, Yes\}$ with $p_{S_1} = 0.5$
No: $S_2 = \{Yes, Yes\}$ with $p_{S_2} = 1$
This means we have an entropy of

$$L(S) = L(S_1) + L(S_2) = 2 \cdot 1.0 + 2 \cdot 0 = 2.0$$

Since these all have the same entropy and we are using information gain as the splitting criteria, we can pick any of the splits to use. We will pick the second (Unit price). Note that we only need to split S_1 because S_2 is already completely pure.

Depth-2

Splitting on Package type would give us
Bagged: $S_1 = \{Yes\}$ with $p_{S_1} = 1$
Canned: $S_2 = \{No\}$ with $p_{S_2} = 0$
This means we have an entropy of

$$L(S) = L(S_1) + L(S_2) = 1 \cdot 0 + 1 \cdot 0 = 0$$

Splitting on Contains > 5 grams of fat would give us
Yes: $S_1 = \{No\}$ with $p_{S_1} = 0$
No: $S_2 = \{Yes\}$ with $p_{S_2} = 1$
This means we have an entropy of

$$L(S) = L(S_1) + L(S_2) = 1 \cdot 0 + 1 \cdot 0 = 0$$

(b)

Split-1

We can see from above that our information gain is as follows:

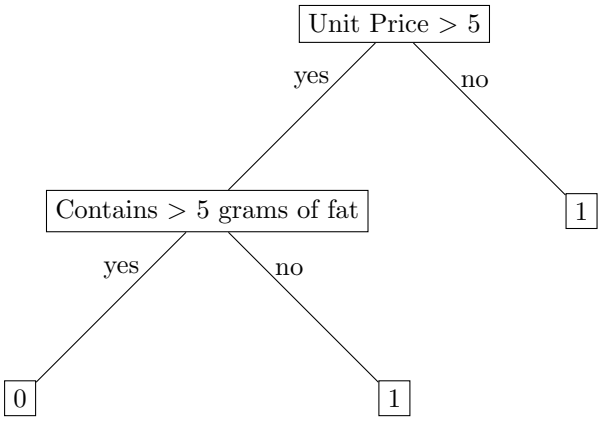
Information Gain $= L(S_{root}) - (L(S_{1_{depth-1}}) + L(S_{2_{depth-1}})) = 3.245112497836 - 2.0 = 1.2451124978360002$

Split-2

We can see from above that our information gain is as follows:

$$\text{Information Gain} = L(S_{depth-1}) - (L(S_{1_{depth-2}}) + L(S_{2_{depth-2}})) = 2.0 - 0 = 2.0$$

(c)



Step 1: $L(S) = 3.245112497836$
Step 2: $L(S) = 2.0$
Step 3: $L(S) = 0.0$

(d)

Impurities Using Gini Index

Root

For the root, we have that $S_1 = \{No, Yes, Yes, Yes\}$ with $p_{S_1} = .75$, which means we have an impurity of

$$L(S) = 4 \cdot 0.375 = 1.5$$

Depth-1

Splitting on Package type would give us
Bagged: $S_1 = \{Yes, Yes\}$ with $p_{S_1} = 1$
Canned: $S_2 = \{No, Yes\}$ with $p_{S_2} = 0.5$
This means we have an impurity of

$$L(S) = L(S_1) + L(S_2) = 2 \cdot 0 + 2 \cdot 0.5 = 1.0$$

Splitting on Unit price > \$5 would give us
Yes: $S_1 = \{No, Yes\}$ with $p_{S_1} = 0.5$
No: $S_2 = \{Yes, Yes\}$ with $p_{S_2} = 1$
This means we have an impurity of

$$L(S) = L(S_1) + L(S_2) = 2 \cdot 0.5 + 2 \cdot 0 = 1.0$$

Splitting on Contains > 5 grams of fat would give us
Yes: $S_1 = \{No, Yes\}$ with $p_{S_1} = 0.5$
No: $S_2 = \{Yes, Yes\}$ with $p_{S_2} = 1$
This means we have an impurity of

$$L(S) = L(S_1) + L(S_2) = 2 \cdot 0.5 + 2 \cdot 0 = 1.0$$

Since these all have the same impurity and we are using information gain as the splitting criteria, we can pick any of the splits to use. We will pick the second (Unit price). Note that we only need to split S_1 because S_2 is already completely pure.

Depth-2

Splitting on Package type would give us
Bagged: $S_1 = \{Yes\}$ with $p_{S_1} = 1$
Canned: $S_2 = \{No\}$ with $p_{S_2} = 0$
This means we have an impurity of

$$L(S) = L(S_1) + L(S_2) = 1 \cdot 0 + 1 \cdot 0 = 0$$

Splitting on Contains > 5 grams of fat would give us
Yes: $S_1 = \{No\}$ with $p_{S_1} = 0$
No: $S_2 = \{Yes\}$ with $p_{S_2} = 1$
This means we have an impurity of

$$L(S) = L(S_1) + L(S_2) = 1 \cdot 0 + 1 \cdot 0 = 0$$

Information Gain

Split-1

We can see from above that our information gain is as follows:

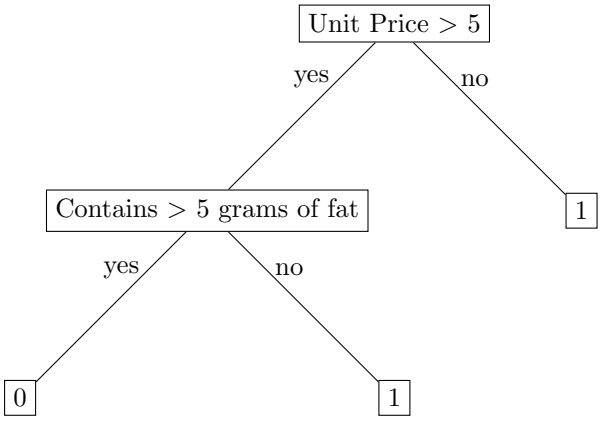
$$\text{Information Gain} = L(S_{root}) - (L(S_{1_{depth-1}}) + L(S_{2_{depth-1}})) = 1.5 - 1.0 = 0.5$$

Split-2

We can see from above that our information gain is as follows:

$$\text{Information Gain} = L(S_{depth-1}) - (L(S_{1_{depth-2}}) + L(S_{2_{depth-2}})) = 1.0 - 0.0 = 1.0$$

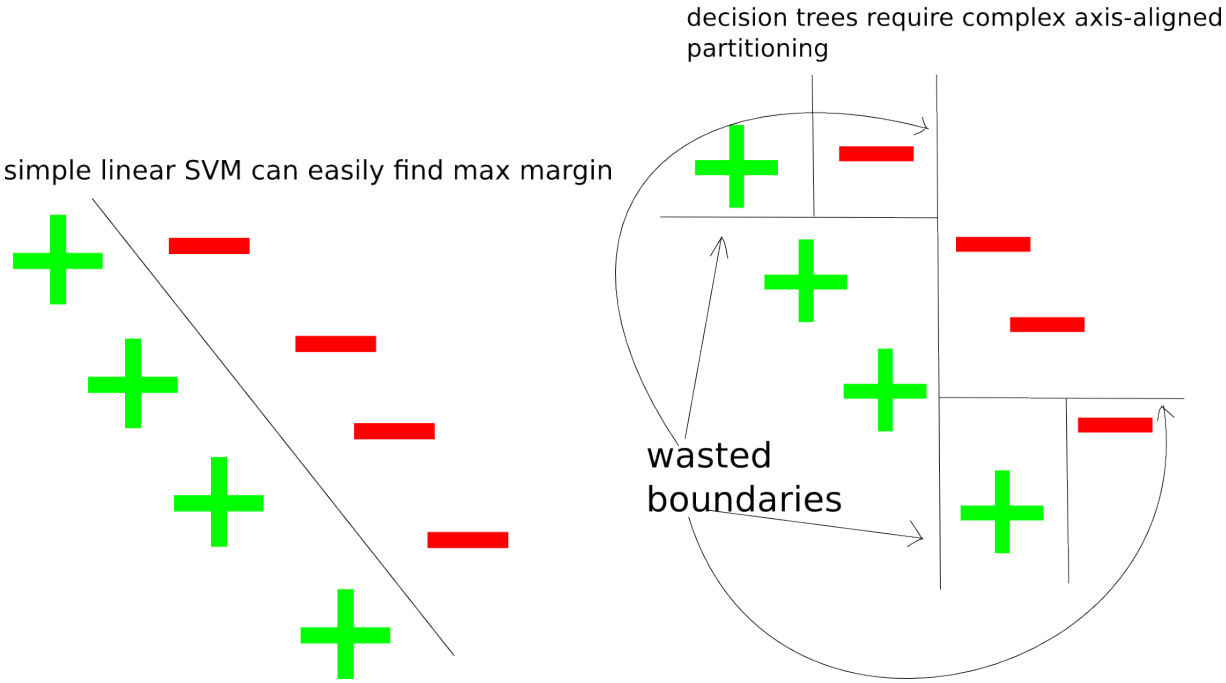
Draw the Tree



Step 1: $L(S) = 1.5$
Step 2: $L(S) = 1.0$
Step 3: $L(S) = 0.0$

2.

Compared to a linear classifier, the decision tree is not always preferred for classification problems. One example is the following.



3.

(a)

Impurities Using Gini Index
Root

For the root, we have that $p_{S_1} = 0.5$, which means we have an impurity of

$$L(S) = 4 \cdot 0.5 = 2.0$$

Depth-1

Splitting on X_1 would give us
$X_1 = 0$: $S_1 = \{Neg, Pos\}$ with $p_{S_1} = 0.5$
$X_1 = 1$: $S_2 = \{Pos, Neg\}$ with $p_{S_2} = 0.5$
This means we have an impurity of

$$L(S) = L(S_1) + L(S_2) = 2 \cdot 0.5 + 2 \cdot 0.5 = 2.0$$

Splitting on X_2 would give us
$X_2 = 0$: $S_1 = \{Neg, Pos\}$ with $p_{S_1} = 0.5$
$X_2 = 1$: $S_2 = \{Pos, Neg\}$ with $p_{S_2} = 0.5$
This means we have an impurity of

$$L(S) = L(S_1) + L(S_2) = 2 \cdot 0.5 + 2 \cdot 0.5 = 2.0$$

Since no split of the root results in any reduction in impurity, given our stopping condition we do not split the root at all, and end up with a tree that is just a root node. The resulting tree looks as follows, and has a classification error of 0.5.

1 (root)

(b)

Classification Error as Impurity
Root

For the root, we have an impurity of

$$L(S) = 4 \cdot 0.5 = 2.0$$

because the classification error is 0.5.
Depth-1

Splitting on X_1 would give us
$X_1 = 0$: $S_1 = \{Neg, Pos\}$, which means $E_{S_1} = 0.5$
$X_1 = 1$: $S_2 = \{Pos, Neg\}$, which means $E_{S_2} = 0.5$
This means we have an impurity of

$$L(S) = L(S_1) + L(S_2) = 2 \cdot 0.5 + 2 \cdot 0.5 = 2.0$$

because the classification errors are 0.5.

Splitting on X_2 would give us
$X_2 = 0$: $S_1 = \{Neg, Pos\}$, which means $E_{S_1} = 0.5$
$X_2 = 1$: $S_2 = \{Pos, Neg\}$, which means $E_{S_2} = 0.5$
This means we have an impurity of

$$L(S) = L(S_1) + L(S_2) = 2 \cdot 0.5 + 2 \cdot 0.5 = 2.0$$

because the classification errors are 0.5.

Since no split of the root results in any reduction in the impurity, given our stopping condition we do not split the root at all, and end up with a tree that is just a root node. The resulting tree looks as follows, and has a classification error of 0.5.

1 (root)

(c)

In order to achieve zero classification training error, we need 99 unique thresholds (internal nodes) in the worst case.

One way to see this is to see that in the worst case, the decision tree must have 100 leaf nodes in order to achieve zero classification training error. That is, in the worst case, we must split each training point into its own leaf to achieve zero classification training error. And by definition, a binary tree with l leaf nodes has $n = 2l - 1$ nodes, which means it has $l - 1 = 99$ internal nodes.

Another way to see this is that adding one additional training point requires us to add only one additional internal node. For example, say we have two training points. We can classify these two with just one internal node and two leaf nodes. Then consider what happens when we add another training point. This new training point will fall into one of the existing leaf nodes (one of the existing splits). So all we need to do in order to maintain zero classification training error is split that leaf node into two new leaf nodes (the new training point will fall into one, the old will fall into the other), which gives us one more internal node. This logic continues, and we can see that for 100 training points, we need 99 internal nodes (since we start with one internal node for two training points and add one internal node for every additional training point).

4.

The worst-case complexity of the number of possible splits we must consider is

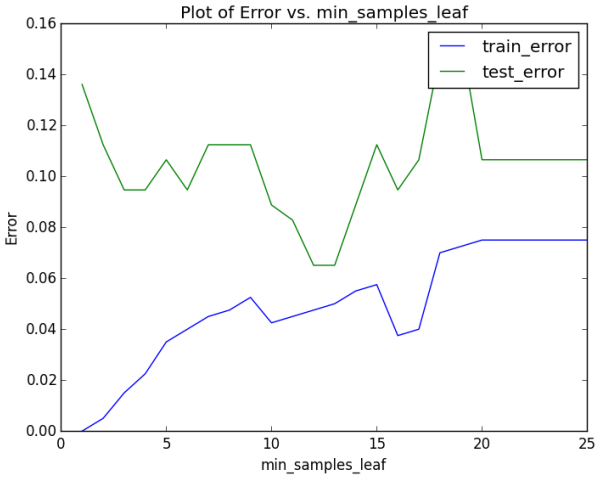
$$O(DN)$$

This is because given N data points, there are $N - 1$ possible positions at which we can split them, and at each position, we can split using one of D features/attributes (because all of them are continuous).

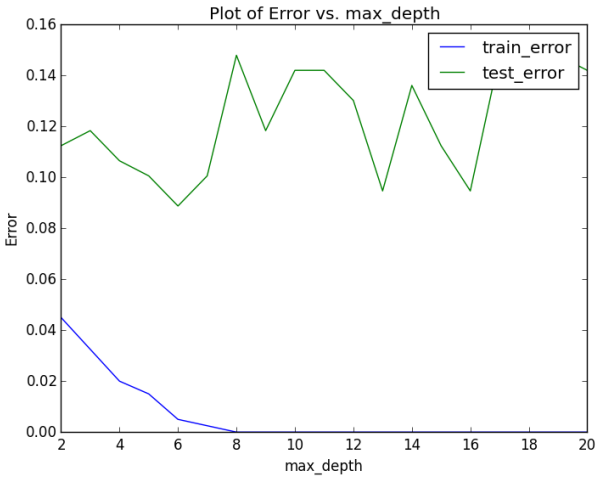
Problem 2: Implementation of Decision Tree

Note: Used decision_tree.py for this section

1.



2.



3.

Note that the graphs are not perfect and fluctuate quite a bit. This was most likely caused by the relatively small sizes of our training/test data sets.

First Plot

For the first plot, early stopping can be seen as we move right along the x-axis, because a higher min_samples_leaf value means that the decision trees we end up with have leaves with more points which in turn means the algorithm needed to use less splits and thus stopped earlier.

We can see that when there is no early stopping (i.e. when min_samples_leaf = 1) overfitting occurs, because training error goes down and test error goes up. When there is early stopping (min_samples_leaf around 10-15), we can see that training error rises but test error falls. Thus we can conclude that early stopping helps prevent overfitting and improves generalization. However, when we stop too early, we can see that both training error and test error rise, and that we begin to underfit. So we need to make sure not to stop too early.

Second Plot

For the first plot, early stopping can be seen as we move left along the x-axis, because a lower max_depth value means that the decision trees we end up with are limited to less levels, which in turn means the algorithm needed to use less splits and thus stopped earlier.

We can see that when there is no early stopping (i.e. when max_depth > 8) overfitting occurs, because training error goes down and test error goes up. When there is early stopping (max_depth around 6-8), we can see that training error rises but test error falls. Thus we can conclude that early stopping helps prevent overfitting and improves generalization. However, when we stop too early, we can see that both training error and test error rise, and that we begin to underfit. So we need to make sure not to stop too early.

Problem 3: The AdaBoost Algorithm

(a)

We want to show that

$$E = \frac{1}{m}\sum_{i=1}^{m}\exp(-y_i f(x_i)) \geq \frac{1}{m}\sum_{i=1}^{m}\mathbb{1}(H(x_i) \neq y_i)$$

To show this, it suffices to show that for each pair x_i, y_i,

$$\exp(-y_i f(x_i)) \geq \mathbb{1}(H(x_i) \neq y_i)$$

To show this, we just need to consider two cases. The first case is if y_i and $f(x_i)$ disagree in sign. In this case,

$$\mathbb{1}(H(x_i) \neq y_i) = 1$$

$$\exp(-y_i f(x_i)) = \exp(p)$$

where p is a nonnegative number. And we know that

$$\exp(x) \geq 1, x \geq 0$$

So overall, we have that

$$\exp(-y_i f(x_i)) \geq \mathbb{1}(H(x_i) \neq y_i)$$

The next case is if y_i and $f(x_i)$ agree in sign. In this case,

$$\mathbb{1}(H(x_i) \neq y_i) = 0$$

$$\exp(-y_i f(x_i)) = \exp(n)$$

where n is a negative number. And we know that

$$\exp(x) \geq 0, x \in \mathbb{R}$$

So overall, we have that

$$\exp(-y_i f(x_i)) \geq \mathbb{1}(H(x_i) \neq y_i)$$

Then we have that in both/all cases,

$$\exp(-y_i f(x_i)) \geq \mathbb{1}(H(x_i) \neq y_i)$$

Thus we can conclude that

$$E = \frac{1}{m}\sum_{i=1}^{m}\exp(-y_i f(x_i)) \geq \frac{1}{m}\sum_{i=1}^{m}\mathbb{1}(H(x_i) \neq y_i)$$

(b)

We have from lecture that

$$D_{t+1}(i) = \frac{D_t(i)\exp(-\alpha_t y_i h_t(x_i))}{Z_t}$$

So we can write any $D_t(i)$ as follows:

$$D_t(i) = \Big(\prod_{j=1}^{t-1}\frac{\exp(-\alpha_j y_i h_j(x_i))}{Z_j}\Big)\cdot D_1(i)$$

Then we can write Z_T as follows:

$$Z_T = \sum_{i=1}^{m}\Big(\prod_{j=1}^{T-1}\frac{\exp(-\alpha_j y_i h_j(x_i))}{Z_j}\Big)\cdot D_1(i)\cdot\exp(-\alpha_T y_i h_T(x_i))$$

$$Z_T = \sum_{i=1}^{m}\Big(\prod_{j=1}^{T-1}\frac{1}{Z_j}\Big)\cdot\Big(\prod_{j=1}^{T-1}\exp(-\alpha_j y_i h_j(x_i))\Big)\cdot D_1(i)\cdot\exp(-\alpha_T y_i h_T(x_i))$$

$$Z_T = \sum_{i=1}^{m} \Big(\prod_{j=1}^{T-1} \frac{1}{Z_j} \Big) \cdot \Big(\prod_{j=1}^{T} \exp(-\alpha_j y_i h_j(x_i)) \Big) \cdot D_1(i)$$

$$Z_T = \sum_{i=1}^{m} \Big(\prod_{j=1}^{T-1} \frac{1}{Z_j} \Big) \cdot \exp(-y_i \sum_{j=1}^{T} \alpha_j h_j(x_i)) \cdot D_1(i)$$

$$Z_T = \sum_{i=1}^{m} \Big(\prod_{j=1}^{T-1} \frac{1}{Z_j} \Big) \cdot \exp(-y_i f(x_i)) \cdot D_1(i)$$

$$\Big(\prod_{j=1}^{T-1} Z_j \Big) \cdot Z_T = \sum_{i=1}^{m} \exp(-y_i f(x_i)) \cdot D_1(i)$$

$$\prod_{j=1}^{T} Z_j = \sum_{i=1}^{m} \exp(-y_i f(x_i)) \cdot D_1(i)$$

Finally, we can use the fact that we initialize $D_1(x) = \frac{1}{m}$ to get the desired result.

$$\prod_{j=1}^{T} Z_j = \frac{1}{m} \sum_{i=1}^{m} \exp(-y_i f(x_i))$$

(c)

Let ϵ_t, the training set error of a weak classifier h_t for a weighted dataset, be given as follows:

$$\epsilon_t = \sum_{i=1}^{m} D_t(i) \mathbb{1}(h_t(x_i) \neq y_i)$$

We will consider a special class of weak classifiers $h_t(x)$ that return exactly $+1$ if h_t classifies example x as positive, and -1 if h_t if classifies x as negative. We will show that for this class of classifiers the normalizer Z_t can be written as

$$Z_t = (1 - \epsilon_t) \exp(-\alpha_t) + \epsilon_t \exp(\alpha_t)$$

We can see this is true by considering our only two cases.
First, we will consider when $h_t(x_i) \neq y_i \implies h_t(x_i) y_i = -1$. In this case, we have that (since we always normalize our weightings such that they sum to 1)

$$\epsilon_t = \sum_{i=1}^{m} D_t(i) = 1$$

which means that, under our new definition,

$$Z_t = \exp(\alpha_t)$$

We can see that under our old definition,

$$Z_t = \sum_{i=1}^{m} D_t(i) \exp(-\alpha_t y_i h_t(x_i))$$

$$Z_t = \sum_{i=1}^{m} D_t(i) \exp(\alpha_t)$$

$$Z_t = \exp(\alpha_t)$$

So in this case, the normalizer Z_t can be written as desired.

Next, we will consider when $h_t(x_i) = y_i \implies h_t(x_i) y_i = 1$. In this case, we have that

$$\epsilon_t = 0$$

which means that, under our new definition,

$$Z_t = \exp(-\alpha_t)$$

We can see that under our old definition,

$$Z_t = \sum_{i=1}^{m} D_t(i) \exp(-\alpha_t y_i h_t(x_i))$$

$$Z_t = \sum_{i=1}^{m} D_t(i) \exp(-\alpha_t)$$

Since we always normalize our weightings such that they sum to 1, this turns into

$$Z_t = \exp(-\alpha_t)$$

So in this case, the normalizer Z_t can be written as desired.

So in both/all cases, the normalizer can be written as desired. Now we will minimize Z_t with respect to α_t.

$$\frac{\delta Z_t}{\delta \alpha_t} = (1-\epsilon_t)(-\alpha_t)\exp(-\alpha_t) + \epsilon_t \alpha_t \exp(\alpha_t) = 0$$

$$\epsilon_t \alpha_t \exp(\alpha_t) = \alpha_t (1-\epsilon_t)\exp(-\alpha_t)$$

$$\exp(2\alpha_t) = \frac{1-\epsilon_t}{\epsilon_t}$$

$$\alpha_t = \frac{1}{2}\ln\left(\frac{1-\epsilon_t}{\epsilon_t}\right)$$

$$\implies \alpha_t^* = \frac{1}{2}\ln\left(\frac{1-\epsilon_t}{\epsilon_t}\right)$$

Thus, we can see that choosing α_t to greedily minimize Z_t leads to the choices in AdaBoost. We also want to choose h_t to greedily minimize Z_t. Since the only variable in our expression for Z_t that involves h_t is ϵ_t, this means we just want to choose the model h_t such that ϵ_t minimizes Z_t.