CS21
Decidability and Tractability

Lecture 20

February 24, 2014

Outline

- the class NP
  - 3-SAT is NP-complete (finishing up)
  - NP-complete problems: independent set, vertex cover, clique
  - NP-complete problems: Hamilton path and cycle,Traveling Salesperson Problem

CIRCUIT-SAT is NP-complete

Theorem: CIRCUIT-SAT is NP-complete

CIRCUIT-SAT = {C : C is a Boolean circuit for which there exists a satisfying truth assignment}

Proof:

- Part 1: need to show CIRCUIT-SAT $\in$ NP.
  - can express CIRCUIT-SAT as:

CIRCUIT-SAT = {C : C is a Boolean circuit for which $\exists$ x such that (C, x) $\in$ R}

R = {(C, x) : C is a Boolean circuit and C(x) = 1}

3SAT is NP-complete

Theorem: 3SAT is NP-complete

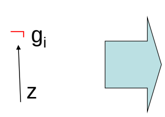
3SAT = {φ : φ is a 3-CNF formula for which there exists a satisfying truth assignment}

Proof:

- Part 1: need to show 3-SAT $\in$ NP
  - already done
- Part 2: need to show 3-SAT is NP-hard
  - we will give a poly-time reduction from CIRCUIT-SAT to 3-SAT

3SAT is NP-complete

- given a circuit C
  - variables $x_1, x_2, \ldots, x_n$
  - AND ($\wedge$), OR ($\vee$), NOT ($\neg$) gates $g_1, g_2, \ldots, g_m$
- reduction f(C) produces these clauses for φ on variables $x_1, x_2, \ldots, x_n, g_1, g_2, \ldots, g_m$:

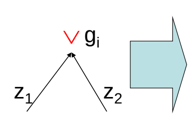


- $(g_i \vee z)$
- $(\neg z \vee \neg g_i)$

$\Big\}\ (z \Leftrightarrow \neg g_i)$

3SAT is NP-complete

- given a circuit C
  - variables $x_1, x_2, \ldots, x_n$
  - AND ($\wedge$), OR ($\vee$), NOT ($\neg$) gates $g_1, g_2, \ldots, g_m$
- reduction f(C) produces these clauses for φ on variables $x_1, x_2, \ldots, x_n, g_1, g_2, \ldots, g_m$:

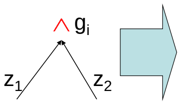


- $(\neg z_1 \vee g_i)$
- $(\neg z_2 \vee g_i)$
- $(\neg g_i \vee z_1 \vee z_2)$

$\Big\}\ (z_1 \vee z_2 \Leftrightarrow g_i)$

3SAT is NP-complete

- given a circuit C
  - variables $x_1, x_2, \ldots, x_n$
  - AND ($\wedge$), OR ($\vee$), NOT ($\neg$) gates $g_1, g_2, \ldots, g_m$
- reduction f(C) produces these clauses for φ on variables $x_1, x_2, \ldots, x_n, g_1, g_2, \ldots, g_m$:



- $(\neg g_i \vee z_1)$
- $(\neg g_i \vee z_2)$
- $(\neg z_1 \vee \neg z_2 \vee g_i)$

$\Big\}$ $(z_1 \wedge z_2 \Leftrightarrow g_i)$

3SAT is NP-complete

- finally, reduction f(C) produces single clause (g_m) where g_m is the output gate.
- f(C) computable in poly-time?
  - yes, simple transformation
- YES maps to YES?
  - if C(x) = 1, then assigning x-values to x-variables of φ and gate values of C when evaluating x to the g-variables of φ gives satisfying assignment.

3SAT is NP-complete

- NO maps to NO?
  - show that φ satisfiable implies C satisfiable
  - satisfying assignment to φ assigns values to x-variables and g-variables
  - output gate g_m must be assigned 1
  - every other gate must be assigned value it would take given values of its inputs.
  - the assignment to the x-variables must be a satisfying assignment for C.

Search vs. Decision

- Definition: given a graph G = (V, E), an independent set in G is a subset $V' \subseteq V$ such that for all $u,w \in V'$ $(u,w) \notin E$
- A problem:

  given G, find the largest independent set
- This is called a search problem
  - searching for *optimal* object of some type
  - comes up frequently

Search vs. Decision

- We want to talk about languages (or decision problems)
- Most search problems have a natural, related decision problem by adding a bound "k"; for example:
  - search problem: given G, find the largest independent set
  - decision problem: given (G, k), is there an independent set of size *at least* k

Ind. Set is NP-complete

Theorem: the following language is NP-complete:

IS = {(G, k) : G has an IS of size $\geq$ k}.

- Proof:
  - Part 1: IS $\in$ NP. Proof?
  - Part 2: IS is NP-hard.
    - reduce from 3-SAT

Ind. Set is NP-complete

- We are reducing from the language:

  3SAT = { φ : φ is a 3-CNF formula that has a satisfying assignment }

  to the language:

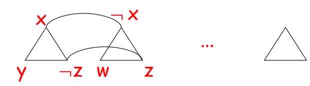
  IS = {(G, k) : G has an IS of size ≥ k}.

Ind. Set is NP-complete

The reduction f: given

$\varphi = (x \vee y \vee \neg z) \wedge (\neg x \vee w \vee z) \wedge \ldots \wedge (\ldots)$

we produce graph G_φ:

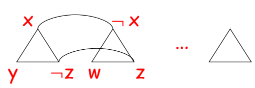


- one triangle for each of m clauses
- edge between every pair of contradictory literals
- set k = m

Ind. Set is NP-complete

$\varphi = (x \vee y \vee \neg z) \wedge (\neg x \vee w \vee z) \wedge \ldots \wedge (\ldots)$

f(φ) =
(G, # clauses)

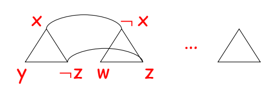


- Is f poly-time computable?
- YES maps to YES?
  - 1 true literal per clause in satisfying assign. A
  - choose corresponding vertices (1 per triangle)
  - IS, since no contradictory literals in A

Ind. Set is NP-complete

$\varphi = (x \vee y \vee \neg z) \wedge (\neg x \vee w \vee z) \wedge \ldots \wedge (\ldots)$

f(φ) =
(G, # clauses)



- NO maps to NO?
  - IS can have at most 1 vertex per triangle
  - IS of size ≥ # clauses must have exactly 1 per
  - since IS, no contradictory vertices
  - can produce satisfying assignment by setting these literals to true

Vertex cover

- Definition: given a graph G = (V, E), a vertex cover in G is a subset $V' \subseteq V$ such that for all $(u,w) \in E$, $u \in V'$ or $w \in V'$
- A search problem:

  given G, find the smallest vertex cover

- corresponding language (decision problem):

  VC = {(G, k) : G has a VC of size ≤ k}.

Vertex Cover is NP-complete

Theorem: the following language is NP-complete:

VC = {(G, k) : G has a VC of size ≤ k}.

- Proof:
  - Part 1: VC ∈ NP. Proof?
  - Part 2: VC is NP-hard.
    - reduce from?

Vertex Cover is NP-complete

- We are reducing from the language:

IS = {(G, k) : G has an IS of size $\geq$ k}

to the language:

VC = {(G, k) : G has a VC of size $\leq$ k}.

Vertex Cover is NP-complete

- How are IS, VC related?
- Given a graph G = (V, E) with n nodes
  - if V' $\subseteq$ V is an independent set of size k
  - then V-V' is a vertex cover of size n-k
- Proof:
  - suppose not. Then there is some edge with neither endpoint in V-V'. But then both endpoints are in V'. contradiction.

Vertex Cover is NP-complete

- How are IS, VC related?
- Given a graph G = (V, E) with n nodes
  - if V' $\subseteq$ V is a vertex cover of size k
  - then V-V' is an independent set of size n-k
- Proof:
  - suppose not. Then there is some edge with both endpoints in V-V'. But then neither endpoint is in V'. contradiction.

Vertex Cover is NP-complete

The reduction:

- given an instance of IS: (G, k) f produces the pair (G, n-k)

- f poly-time computable?
- YES maps to YES?
  - IS of size $\geq$ k in G $\Rightarrow$ VC of size $\leq$ n-k in G
- NO maps to NO?
  - VC of size $\leq$ n-k in G $\Rightarrow$ IS of size $\geq$ k in G

Clique

- Definition: given a graph G = (V, E), a clique in G is a subset V'$\subseteq$ V such that for all u,v $\in$ V', (u, v) $\in$ E
- A search problem:

given G, find the largest clique

- corresponding language (decision problem):

CLIQUE = {(G, k) : G has a clique of size $\geq$ k}.

Clique is NP-complete

Theorem: the following language is NP-complete:

CLIQUE = {(G, k) : G has a clique of size $\geq$ k}

- Proof:
  - Part 1: CLIQUE $\in$ NP. Proof?
  - Part 2: CLIQUE is NP-hard.
    - reduce from?

Clique is NP-complete

- We are reducing from the language:

IS = {(G, k) : G has an IS of size ≥ k}

to the language:

CLIQUE = {(G, k) : G has a CLIQUE of size ≥ k}.

Clique is NP-complete

- How are IS, CLIQUE related?
- Given a graph G = (V, E), define its complement G' = (V, E' = {(u,v) : (u,v) ∉ E})
  - if V' ⊆ V is an independent set in G of size k
  - then V' is a clique in G' of size k
- Proof:
  - *Every* pair of vertices u,v ∈ V' has no edge between them in G. Therefore they have an edge between them in G'.

Clique is NP-complete

- How are IS, CLIQUE related?
- Given a graph G = (V, E), define its complement G' = (V, E' = {(u,v) : (u,v) ∉ E})
  - if V' ⊆ V is a clique in G' of size k
  - then V' is an independent set in G of size k
- Proof:
  - *Every* pair of vertices u,v ∈ V' has an edge between them in G'. Therefore they have no edge between them in G.

Clique is NP-complete

The reduction:

- given an instance of IS: (G, k) f produces the pair (G', k)

- f poly-time computable?
- YES maps to YES?
  - IS of size ≥ k in G ⇒ CLIQUE of size ≥ k in G'
- NO maps to NO?
  - CLIQUE of size ≥ k in G' ⇒ IS of size ≥ k in G

Hamilton Path

- Definition: given a directed graph G = (V, E), a Hamilton path in G is a directed path that touches every node exactly once.

- A language (decision problem):

HAMPATH = {(G, s, t) : G has a Hamilton path from s to t}

HAMPATH is NP-complete

Theorem: the following language is NP-complete:

HAMPATH = {(G, s, t) : G has a Hamilton path from s to t}

- Proof:
  - Part 1: HAMPATH ∈ NP. Proof?
  - Part 2: HAMPATH is NP-hard.
    - reduce from?

HAMPATH is NP-complete

- We are reducing from the language:

3SAT = { φ : φ is a 3-CNF formula that has a satisfying assignment }

to the language:

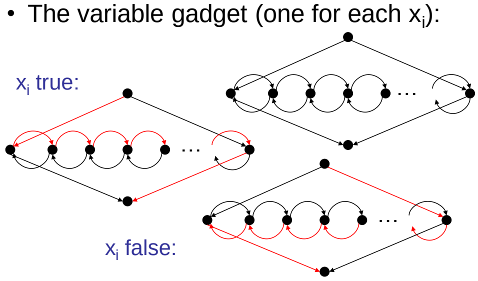
HAMPATH = {(G, s, t) : G has a Hamilton path from s to t}

HAMPATH is NP-complete

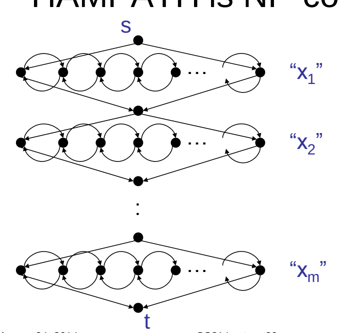
- We want to construct a graph from φ with the following properties:
  - a satisfying assignment to φ translates into a Hamilton Path from s to t
  - a Hamilton Path from s to t can be translated into a satisfying assignment for φ
- We will build the graph up from pieces called gadgets that "simulate" the clauses and variables of φ.

HAMPATH is NP-complete

- The variable gadget (one for each x_i):



HAMPATH is NP-complete



- path from s to t translates into a truth assignment to $x_1 \ldots x_m$
- why must the path be of this form?