CS21 Decidability and Tractability

Lecture 17

February 14, 2014

Outline

- The complexity class P
  - examples of problems in P

Time complexity

Definition: the running time ("time complexity") of a TM M is a function

$$f:\mathbf{N} \rightarrow \mathbf{N}$$

where f(n) is the maximum number of steps M uses on any input of length n.

- "M runs in time f(n)," "M is a f(n) time TM"

Time complexity

- We do not care about fine distinctions
  - e.g. how many additional steps M takes to check that it is at the left of tape
- We care about the behavior on large inputs
  - general-purpose algorithm should be "scalable"
  - overhead for e.g. initialization shouldn't matter in big picture

Time complexity

- Measure time complexity using asymptotic notation ("big-oh notation")
  - disregard lower-order terms in running time
  - disregard coefficient on highest order term
- example:

$$f(n) = 6n^3 + 2n^2 + 100n + 102781$$

  - "f(n) is order n^3"
  - write $f(n) = O(n^3)$

Time complexity

On input x:	
• scan tape left-to-right, reject if 0 to right of 1	O(n) steps
• repeat while 0's, 1's on tape:	≤ n repeats
• scan, crossing off one 0, one 1	O(n) steps
• if only 0's or only 1's remain, reject; if neither 0's nor 1's remain, accept	O(n) steps

- total = $O(n) + n\cdot O(n) + O(n) = O(n^2)$

Asymptotic notation facts

- "logarithmic": O(log n)
  - $\log_b n = (\log_2 n)/(\log_2 b)$
  - so $\log_b n = O(\log_2 n)$ for any constant b; therefore suppress base when write it
- "polynomial": $O(n^c) = n^{O(1)}$
  - also: $c^{O(\log n)} = O(n^{c'}) = n^{O(1)}$
- "exponential": $O(2^{n^\delta})$ for $\delta > 0$

each bound asymptotically less than next

Time complexity

- Recall:
  - language is a set of strings
  - a complexity class is a set of languages
  - complexity classes we've seen:
    - Regular Languages, Context-Free Languages, Decidable Languages, RE Languages, co-RE languages

Definition: TIME(t(n)) = {L : there exists a TM M that decides L in time O(t(n))}

Time complexity

- We saw that $L = \{0^k1^k : k \geq 0\}$ is in TIME(n^2).
- Book: it is also in TIME(n log n) by giving a more clever algorithm
- Can prove: O(n log n) time required on a single tape TM.
- How about on a multitape TM?

Time Complexity

- 2-tape TM M deciding $L = \{0^k1^k : k \geq 0\}$.

On input x:	
• scan tape left-to-right, reject if 0 to right of 1	O(n)
• scan 0's on tape 1, copying them to tape 2	O(n)
• scan 1's on tape 1, crossing off 0's on tape 2	O(n)
• if all 0's crossed off before done with 1's reject	
• if 0's remain after done with ones, reject; otherwise accept.	total: 3*O(n) = O(n)

Multitape TMs

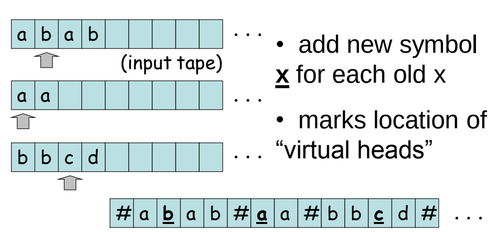
- Convenient to "program" multitape TMs rather than single ones
  - equivalent when talking about decidability
  - not equivalent when talking about time complexity

Theorem: Let t(n) satisfy t(n) ≥ n. Every multi-tape TM running in time t(n) has an equivalent TM running in time $O(t(n)^2)$.

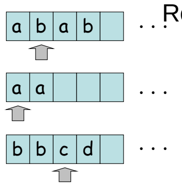
Multitape TMs

simulation of k-tape TM by single-tape TM:

- add new symbol **x** for each old x
- marks location of "virtual heads"



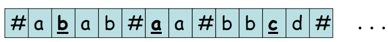
Multitape TMs



Repeat: O(t(n)) times

- scan tape, remembering the symbols under each virtual head in the state
  O(k t(n)) = O(t(n))
- make changes to reflect 1 step of M; if hit #, shift to right to make room.
  O(k t(n)) = O(t(n))

when M halts, erase all but 1st string
O(t(n))



Multitape TMs

- Moral: feel free to use k-tape TMs, but be aware of slowdown in conversion to TM
  - note: if $t(n) = O(n^c)$ then $t(n)^2 = O(n^{2c}) = O(n^{c'})$
  - note: if $t(n) = O(2^{n^\delta})$ for $\delta > 0$ then $t(n)^2 = O(2^{2n^\delta}) = O(2^{n^{\delta'}})$ for $\delta' > 0$
- high-level operations you are used to using can be simulated by TM with only polynomial slowdown
  - e.g., copying, moving, incrementing/decrementing, arithmetic operations +, -, *, /

Extended Church-Turing Thesis

- the belief that TMs formalize our intuitive notion of an efficient algorithm is:

<u>The "extended" Church-Turing Thesis</u>

everything we can compute in time t(n) on a physical computer can be computed on a Turing Machine in time $t(n)^{O(1)}$ (polynomial slowdown)

- quantum computers challenge this belief

Time Complexity

- interested in a coarse classification of problems. For this purpose,
  - treat any polynomial running time as "efficient" or "tractable"
  - treat any exponential running time as inefficient or "intractable"

<u>Key definition</u>: "P" or "polynomial-time" is

$$P = \cup_{k \geq 1} \mathrm{TIME}(n^k)$$

Time Complexity

- Why polynomial-time?
  - insensitive to particular deterministic model of computation chosen
  - closed under modular composition
  - empirically: qualitative breakthrough to achieve polynomial running time is followed by quantitative improvements from impractical (e.g. n^{100}) to practical (e.g. n^3 or n^2)

Examples of languages in P

- Recall: positive integers x, y are relatively prime if their Greatest Common Divisor (GCD) is 1.
- will show the following language is in P:

  RELPRIME = {<x, y> : x and y are relatively prime}
- what is the running time of the algorithm that tries all divisors up to max{x, y}?

Euclid's Algorithm

- possibly earliest recorded algorithm

on input <x, y>:
- repeat until y = 0
  - set x = x mod y
  - swap x, y
- x is the GCD(x, y). If x = 1, accept; otherwise reject

Example run on input <10, 22>:

x, y = 10, 22

x, y = 22, 10

x, y = 10, 2

x, y = 2, 0

reject

Euclid's Algorithm

- possibly earliest recorded algorithm

on input <x, y>:
- repeat until y = 0
  - set x = x mod y
  - swap x, y
- x is the GCD(x, y). If x = 1, accept; otherwise reject

Example run on input <24, 5>:

x, y = 24, 5

x, y = 5, 4

x, y = 4, 1

x, y = 1, 0

accept

Euclid's Algorithm

on input <x, y>:
- (1) repeat until y = 0
  - (2) set x = x mod y
  - (3) swap x, y
- x is the GCD(x, y). If x = 1, accept; otherwise reject

- every 2 times through loop, (x, y) each reduced by ½
- loops ≤ $2 \cdot \max\{\log_2 x, \log_2 y\}$ = O(n = |<x, y>|); poly time for each loop

Claim: value of x reduced by ½ at every execution of (2) except possibly first one.

Proof:
- after (2) x < y
- after (3) x > y
- if x/2 ≥ y, then x mod y < y ≤ x/2
- if x/2 < y, then x mod y = x – y < x/2

A puzzle

- Find an efficient algorithm to solve the following problem:
- Input: sequence of pairs of symbols

  e.g. (A, b), (E, D), (d, C), (B, a)
- Goal: determine if it is possible to circle at least one symbol in each pair without circling upper and lower case of same symbol.

A puzzle

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2SAT

- This is a disguised version of the language

  2SAT = {formulas in Conjunctive Normal Form with 2 literals per clause for which there exists a satisfying truth assignment}
  – CNF = "AND of ORs"

  (A, b), (E, D), (d, C), (b, a)

  $(x_1 \vee \neg x_2) \wedge (x_5 \vee x_4) \wedge (\neg x_4 \vee x_3) \wedge (\neg x_2 \vee \neg x_1)$
  – satisfying truth assignment = assignment of TRUE/FALSE to each variable so that whole formula is TRUE

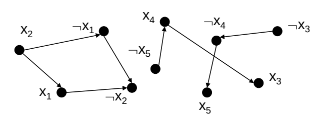
2SAT

Theorem: There is a polynomial-time algorithm deciding 2SAT ("2SAT $\in$ P").

Proof: algorithm described on next slides.

Algorithm for 2SAT

- Build a graph with separate nodes for each literal.
  - add directed edge (x, y) iff formula includes clause ($\neg x \vee y$) or ($y \vee \neg x$) (equiv. to $x \Rightarrow y$)



e.g. $(x_1 \vee \neg x_2)\wedge(x_5 \vee x_4)\wedge(\neg x_4 \vee x_3)\wedge(\neg x_2 \vee \neg x_1)$

Algorithm for 2SAT

Claim: formula is unsatisfiable iff there is some variable x with a path from x to $\neg x$ and a path from $\neg x$ to x in derived graph.

- Proof ($\Leftarrow$)
  - edges represent implication $\Rightarrow$. By transitivity of $\Rightarrow$, a path from x to $\neg x$ means $x \Rightarrow \neg x$, and a path from $\neg x$ to x means $\neg x \Rightarrow x$.

Algorithm for 2SAT

- Proof ($\Rightarrow$)
  - to construct a satisfying assign. (if no x with a path from x to $\neg x$ and a path from $\neg x$ to x):
    - pick unassigned literal s with no path from s to $\neg s$
    - assign it TRUE, as well as all nodes reachable from it; assign negations of these literals FALSE
    - note: path from s to t and s to $\neg t$ implies path from $\neg t$ to $\neg s$ and t to $\neg s$, implies path from s to $\neg s$
    - note: path s to t (assigned FALSE) implies path from $\neg t$ (assigned TRUE) to $\neg s$, so s already assigned at that point.

Algorithm for 2SAT

- Algorithm:
  - build derived graph
  - for every pair x, $\neg x$ check if there is a path from x to $\neg x$ and from $\neg x$ to x in the graph
- Running time of algorithm (input length n):
  - O(n) to build graph
  - O(n) to perform each check
  - O(n) checks
  - running time $O(n^2)$. 2SAT $\in$ P.

Another puzzle

- Find an efficient algorithm to solve the following problem.
- Input: sequence of *triples* of symbols

  e.g. (A, b, C), (E, D, b), (d, A, C), (c, b, a)
- Goal: determine if it is possible to circle at least one symbol in each *triple* without circling upper and lower case of same symbol.