CS 21 Decidability and Tractability	**Winter 2014**

Problem Set 2

Out: January 22 | *Due:* **January 29**

Reminder: you are encouraged to work in groups of two or three; however you must turn in your own write-up and note with whom you worked. You may consult the course notes and the text (Sipser). The full honor code guidelines can be found in the course syllabus.

Please attempt all problems. **To facilitate grading, please turn in each problem on a separate sheet of paper and put your name on each sheet. Do not staple the separate sheets.**

1. For a positive integer i, let $N(i)$ denote its decimal representation (the usual string we write when writing the number i, with no leading zeros). Let $N'(i)$ denote the string $N(i)$ written in reverse order (least-significant digit first). Show that the language

$$L = \{N(i)\#N'(i+2) : i \geq 1\}$$

over the alphabet $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \#\}$ is context-free.

2. Show that CFLs are *not* closed under intersection. To do this you should come up with two CFLs A and B with the property that $C = A \cap B = \{x : x \in A \text{ and } x \in B\}$ is *not* a CFL. State the languages you are using and then:

   (a) Prove that A and B are CFLs.

   (b) Prove that $C = A \cap B$ is not a CFL.

3. Ogden's Lemma and non-context-free languages.

   (a) Consider the language $L = \{a^i b^j c^k d^\ell : i = 0 \text{ or } j = k = \ell\}$. Prove that L satisfies the conditions of the CFL Pumping Lemma. In other words, the CFL Pumping Lemma is *incapable* of proving that L is not a CFL (since we cannot derive a contradiction from the assumption that L is a CFL).

   (b) The following is a strengthening of the CFL Pumping Lemma:

   Lemma 2.1 (Ogden's Lemma) *If L is a CFL, then there exists a pumping length p such that for every $w \in L$ of length at least p* and every way of "marking" p or more positions in w, *w can be written $w = uvxyz$ such that:*

   - *for all $i \geq 0$, $uv^ixy^iz \in L$, and*
   - *vy contains at least 1 marked position of w, and*
   - *vxy contains at most p marked positions of w.*

Prove Ogden's Lemma. Hint: assume L is given by a CFG in Chomsky Normal Form, and consider a parse tree for w in this grammar. Pick a path from the root to a marked descendant by always travelling to the child with the greater number of marked descendants, and find a repeated nonterminal on that path. It may be useful in the course of the proof to single out "branch nodes," which are internal nodes whose left and right children both have marked descendants.

(c) Use Ogden's Lemma to prove that the language L from part (a) is not a CFL.

4. Deterministic vs. nondeterministic pushdown automata.

   (a) Show that CFL's are *not* closed under complement. (The complement of a language $L \subseteq \Sigma^*$ is $\overline{L} = \Sigma^* - L$.)

   (b) We saw in class that *deterministic* PDAs are closed under complement. Carefully explain why this implies that NPDAs are "more powerful" than deterministic PDAs, in the process describing the precise meaning of "more powerful."