CS21
Decidability and Tractability

Lecture 26

March 10, 2014

Outline

- "Challenges to the (extended) Church-Turing Thesis"
  - randomized computation
  - quantum computation

Challenges to the extended Church-Turing thesis

Extended Church-Turing Thesis

- the belief that TMs formalize our intuitive notion of an efficient algorithm is:

> The "extended" Church-Turing Thesis
>
> everything we can compute in time t(n) on a physical computer can be computed on a Turing Machine in time $t(n)^{O(1)}$ (polynomial slowdown)

- randomized computation challenges this belief

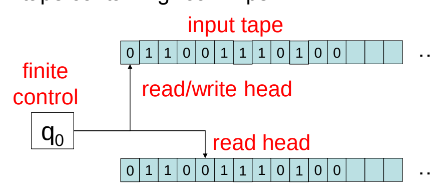
Extended Church-Turing Thesis

- Common to insert "probabilistic":

> The "extended" Church-Turing Thesis
>
> everything we can compute in time t(n) on a physical computer can be computed on a *probabilistic* Turing Machine in time $t(n)^{O(1)}$ (polynomial slowdown)

Randomized complexity classes

- model: probabilistic Turing Machine
  - deterministic TM with additional read-only tape containing "coin flips"



Randomized complexity classes

- **RP** (Random Polynomial-time)
  - L $\in$ **RP** if there is a p.p.t. TM M:

    $x \in L \Rightarrow \Pr_y[M(x,y) \text{ accepts}] \geq ½$

    $x \notin L \Rightarrow \Pr_y[M(x,y) \text{ rejects}] = 1$
- **coRP** (complement of Random Polynomial-time)
  - L $\in$ **coRP** if there is a p.p.t. TM M:

    $x \in L \Rightarrow \Pr_y[M(x,y) \text{ accepts}] = 1$

    $x \notin L \Rightarrow \Pr_y[M(x,y) \text{ rejects}] \geq ½$

“p.p.t” = probabilistic polynomial time

Randomized complexity classes

- **BPP** (Bounded-error Probabilistic Poly-time)
  - L $\in$ **BPP** if there is a p.p.t. TM M:

    $x \in L \Rightarrow \Pr_y[M(x,y) \text{ accepts}] \geq 2/3$

    $x \notin L \Rightarrow \Pr_y[M(x,y) \text{ rejects}] \geq 2/3$

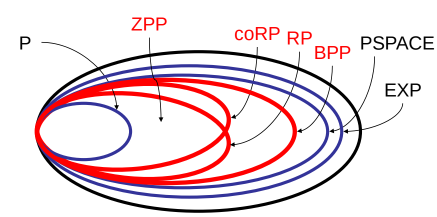
Randomized complexity classes

These classes may capture “efficiently computable” better than **P**.

One more important class:

- **ZPP** (Zero-error Probabilistic Poly-time)
  - **ZPP = RP $\cap$ coRP**
  - $\Pr_y[M(x,y)$ outputs “fail”$] \leq ½$
  - otherwise outputs correct answer

RP,coRP, BPP



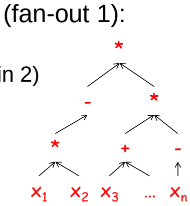
- from definitions: ZPP $\subset$ RP, coRP $\subset$ BPP

Relationship to other classes

- all these classes contain **P**
  - they can simply ignore the tape with coin flips
- all are in **PSPACE**
  - can exhaustively try all strings y
  - count accepts/rejects; compute probability
- **RP $\subset$ NP** (and **coRP $\subset$ coNP**)
  - multitude of accepting computations
  - **NP** requires only one

Polynomial identity testing

- Given: polynomial $p(x_1, x_2, \ldots, x_n)$ as arithmetic formula (fan-out 1):
  - multiplication (fan-in 2)
  - addition (fan-in 2)
  - negation (fan-in 1)



Polynomial identity testing

- Question: Is p identically zero?
  - i.e., is $p(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbf{F}^n$
  - (assume $|\mathbf{F}|$ larger than degree…)
- "polynomial identity testing" because given two polynomials p, q, we can check the identity $p \equiv q$ by checking if $(p - q) \equiv 0$

Polynomial identity testing

- try all $|\mathbf{F}|^n$ inputs?
  - may be exponentially many
- multiply out symbolically, check that all coefficients are zero?
  - may be exponentially many coefficients
- Best known deterministic algorithm places in EXP

Polynomial identity testing

Lemma (Schwartz-Zippel): Let

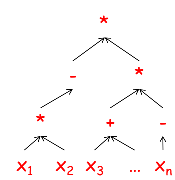
$$p(x_1, x_2, \ldots, x_n)$$

be a total degree d polynomial over a field **F** and let S be any subset of **F**. Then if p is not identically 0,

$$\Pr_{r_1, r_2, \ldots, r_n \in S}[\, p(r_1, r_2, \ldots, r_n) = 0] \le d/|S|.$$

Polynomial identity testing

- Given: polynomial $p(x_1, x_2, \ldots, x_n)$ over field **F**
- Is p identically zero?



- Note: degree d is at most the size of input

Polynomial identity testing

- randomized algorithm: pick a subset $S \subset \mathbf{F}$ of size 2d
  - pick $r_1, r_2, \ldots, r_n$ from S uniformly at random
  - if $p(r_1, r_2, \ldots, r_n) = 0$, answer "yes"
  - if $p(r_1, r_2, \ldots, r_n) \neq 0$, answer "no"
- if p identically zero, never wrong
- if not, Schwartz-Zippel ensures probability of error at most ½

Randomized complexity classes

- We have shown:
  - Polynomial Identity Testing is in coRP
  - note: no sub-exponential time deterministic algorithm know

Randomized complexity classes

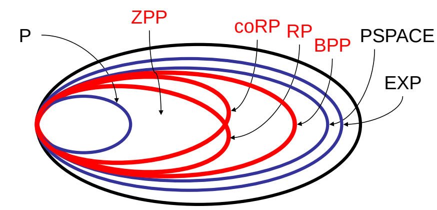
- How powerful is randomized computation?
- We have seen an example of a problem in

  BPP

  that we only know how to solve deterministically in **EXP**.

Is randomness a panacea for intractability?

Randomized complexity classes



- believed that P = ZPP = RP = coRP = BPP (!)

Review

- Highest level: 2 main points

1. Decidability
   - problem solvable by an algorithm = problem is decidable
   - some problems are not decidable (e.g. HALT)

Review

- Highest level: 2 main points

2. Tractability
   - problem solvable in polynomial time = problem is tractable
   - some problems are not tractable (EXP-complete problems)
   - huge number of problems are likely not to be tractable (NP-complete problems)

Review

- Important ideas
  - “problem” formalized as language
    - language = set of strings
  - “computation” formalized as simple machine
    - finite automata
    - pushdown automata
    - Turing Machine
  - “power” of machine formalized as the set of languages it recognizes

Review

- Important ideas (continued):
  - simulation used to show one model at least as powerful as another
  - diagonalization used to show one model strictly more powerful than another
    - also Pumping Lemma
  - reduction used to compare one problem to another

Review

- Important ideas (continued):
  - complexity theory investigates the resources required to solve problems
    - time, space, others…
  - complexity class = set of languages
  - language L is C-hard if every problem in C reduces to L
  - language L is C-complete if L is C-hard and L is in C.

Review

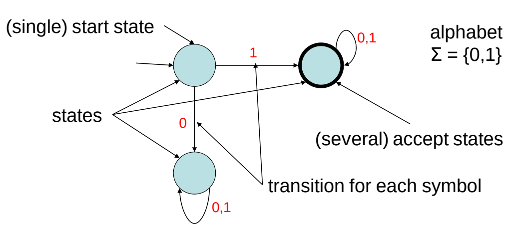
- Important ideas (continued):

A complete problem is a surrogate for the entire class.

Summary

Part I: automata

Finite Automata



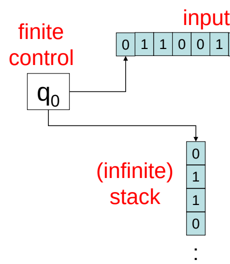
- read input one symbol at a time; follow arrows; accept if end in accept state

Finite Automata

- Non-deterministic variant: NFA
- Regular expressions built up from:
  - unions
  - concatenations
  - star operations

Main results: same set of languages recognized by FA, NFA and regular expressions ("regular languages").

Pushdown Automata



New capabilities:

- can push symbol onto stack
- can pop symbol off of stack

Context-Free Grammars

start symbol

terminal symbols

$A \rightarrow 0A1$

$A \rightarrow B$

$\underbrace{B \rightarrow \#}_{\text{production}}$

non-terminal symbols

Pushdown Automata

Main results: same set of languages recognized by NPDA, and context-free grammars ("context-free languages").

- and DPDA's weaker than NPDA's…

Non-regular languages

Pumping Lemma: Let L be a regular language. There exists an integer p ("pumping length") for which every $w \in L$ with $|w| \geq p$ can be written as

$w = xyz$ such that

1. for every $i \geq 0$, $xy^iz \in L$, and
2. $|y| > 0$, and
3. $|xy| \leq p$.

Pumping Lemma for CFLs

CFL Pumping Lemma: Let L be a CFL. There exists an integer p ("pumping length") for which every $w \in L$ with $|w| \geq p$ can be written as

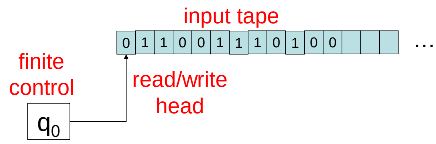
$w = uvxyz$ such that

1. for every $i \geq 0$, $uv^ixy^iz \in L$, and
2. $|vy| > 0$, and
3. $|vxy| \leq p$.

Summary

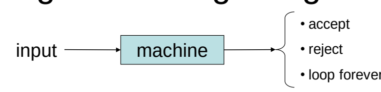
Part II: Turing Machines and decidability

Turing Machines



- New capabilities:
  - infinite tape
  - can read OR write to tape
  - read/write head can move left and right

Deciding and Recognizing



- TM M:
  - L(M) is the language it recognizes
  - if M rejects every x ∉ L(M) it decides L
  - set of languages recognized by some TM is called Turing-recognizable or recursively enumerable (RE)
  - set of languages decided by some TM is called Turing-decidable or decidable or recursive

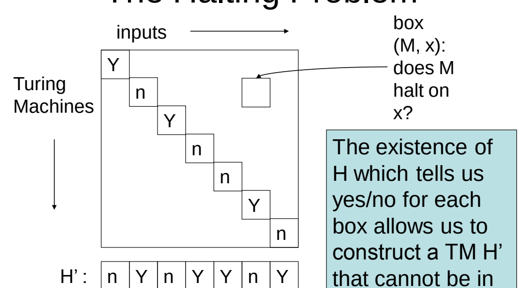
Church-Turing Thesis

- the belief that TMs formalize our intuitive notion of an algorithm is:

> The Church-Turing Thesis
>
> everything we can compute on a physical computer
>
> can be computed on a Turing Machine

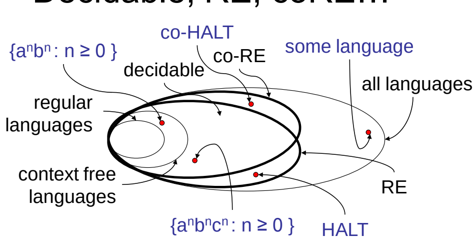
- Note: this is a belief, not a theorem.

The Halting Problem



The existence of H which tells us yes/no for each box allows us to construct a TM H' that cannot be in the table.

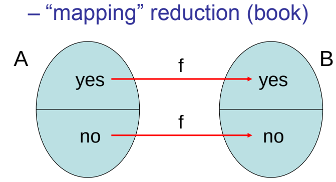
Decidable, RE, coRE…



some problems (e.g HALT) have no algorithms

Definition of reduction

- More refined notion of reduction:
  - "many-one" reduction (commonly)
  - "mapping" reduction (book)



reduction from language A to language B

Using reductions

- Used reductions to prove lots of problems were:
  - undecidable (reduce from undecidable)
  - non-RE (reduce from non-RE)
    - or show undecidable, and coRE
  - non-coRE (reduce from non-coRE)
    - or show undecidable, and RE

Rice's Theorem: Every nontrivial TM property is undecidable.

The Recursion Theorem

Theorem: Let T be a TM that computes fn:

$$t: \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$$

There is a TM R that computes the fn:

$$r: \Sigma^* \rightarrow \Sigma^*$$

defined as $r(w) = t(w, <R>)$.

- In the course of computation, a Turing Machine can output its own description.

Incompleteness Theorem

Theorem: Peano Arithmetic is not complete.

(same holds for any reasonable proof system for number theory)

Proof outline:
- the set of theorems of PA is RE
- the set of true sentences (= Th(**N**)) is not RE

Summary

Part III: Complexity

Complexity

- Complexity Theory = study of what is computationally feasible (or tractable) with limited resources:
  - running *time* — main focus
  - storage *space*
  - number of *random bits*
  - degree of *parallelism*
  - rounds of *interaction*
  - *others…*

  not in this course

Time and Space Complexity

Definition: the time complexity of a TM M is a function $f:\mathbf{N} \rightarrow \mathbf{N}$, where f(n) is the maximum number of steps M uses on any input of length n.

Definition: the space complexity of a TM M is a function $f:\mathbf{N} \rightarrow \mathbf{N}$, where f(n) is the maximum number of tape cells M scans on any input of length n.

Complexity Classes

Definition: TIME(t(n)) = {L : there exists a TM M that decides L in space O(t(n))}

$$P = \cup_{k \geq 1} TIME(n^k)$$

$$EXP = \cup_{k \geq 1} TIME(2^{n^k})$$

Definition: SPACE(t(n)) = {L : there exists a TM M that decides L in space O(t(n))}

$$PSPACE = \cup_{k \geq 1} SPACE(n^k)$$

Complexity Classes

Definition: NTIME(t(n)) = {L : there exists a NTM M that decides L in time O(t(n))}

$$NP = \cup_{k \geq 1} NTIME(n^k)$$

- Theorem: P $\neq$ EXP
- P $\subset$ NP $\subset$ PSPACE $\subset$ EXP
- Don't know if any of the containments are proper.

Alternate definition of NP

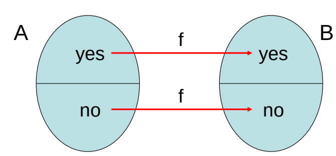
Theorem: language L is in NP if and only if it is expressible as:

$$L = \{ x \mid \exists\, y, |y| \leq |x|^k, (x, y) \in R \}$$

where R is a language in P.

Poly-time reductions

- Type of reduction we will use:
  - "many-one" poly-time reduction (commonly)
  - "mapping" poly-time reduction (book)



1. f poly-time computable
2. YES maps to YES
3. NO maps to NO

Hardness and completeness

Definition: a language L is C-hard if for every language A $\in$ C, A poly-time reduces to L; i.e., A $\leq_P$ L.

can show L is C-hard by reducing from a known C-hard problem

Definition: a language L is C-complete if L is C-hard and L $\in$ C

Complete problems

- EXP-complete: ATM_B = {<M, x, m> : M is a TM that accepts x within at most m steps}
- PSPACE-complete: QSAT = {φ : φ is a 3-CNF, and $\exists x_1 \forall x_2 \exists x_3 \ldots \forall x_n\ \varphi(x_1, x_2, \ldots x_n)$}
- NP-complete: 3SAT = {φ : φ is a satisfiable 3-CNF formula}

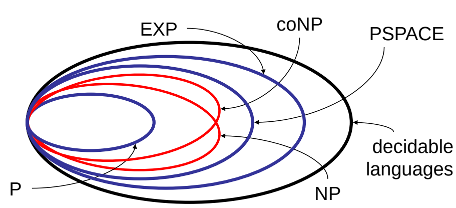
Lots of NP-complete problems

- Indendent Set
- Vertex Cover
- Clique
- Hamilton Path (directed and undirected)
- Hamilton Cycle and TSP
- Subset Sum
- NAE3SAT
- Max Cut
- Problem sets: max/min Bisection, 3-coloring, subgraph isomorphism, subset sum, (3,3)-SAT, Partition, Knapsack, Max2SAT…

Other complexity classes

- coNP – complement of NP
  - complete problems: UNSAT, DNF TAUTOLOGY
- NP intersect coNP
  - contains (decision version of) FACTORING
- PSPACE
  - complete problems: QSAT, GEOGRAPHY

Complexity classes



all containments believed to be proper

Extended Church-Turing Thesis

- the belief that TMs formalize our intuitive notion of an efficient algorithm is:

<u>The "extended" Church-Turing Thesis</u>

everything we can compute in time t(n) on a physical computer can be computed on a Turing Machine in time $t(n)^{O(1)}$ (polynomial slowdown)

Challenges to the Extended Church-Turing Thesis

- Randomized computation – BPP
  - POLYNOMIAL IDENTITY TESTING example of problem in BPP, not known to be in P
- Quantum computation
  - FACTORING example of problem solvable in quantum polynomial time, not believed to be in P