CS21 Decidability and Tractability

Lecture 18

February 19, 2014

Midterm

- 30 points
- Mean: 20.3 Median: 20

  (last year: 30 pts; mean 22.4; median: 24)
- Distribution:

30: 6 (8)	17-21: 29 (14)	< 9 : 3 (3)
26-30: 18 (26)	13-17: 20 (11)	
21-26: 32 (22)	9-13: 7 (5)	

Outline

- The complexity class EXP
- Time Hierarchy Theorem
- hardness and completeness
- an EXP-complete problem

Another puzzle

- Find an efficient algorithm to solve the following problem.
- Input: sequence of *triples* of symbols

  e.g. (A, b, C), (E, D, b), (d, A, C), (c, b, a)
- Goal: determine if it is possible to circle at least one symbol in each *triple* without circling upper and lower case of same symbol.

3SAT

- This is a disguised version of the language

  3SAT = {formulas in Conjunctive Normal Form with 3 literals per clause for which there exists a satisfying truth assignment}

  e.g. (A, b, C), (E, D, b), (d, A, C), (c, b, a)

  $(x_1 \vee \neg x_2 \vee x_3) \wedge (x_5 \vee x_4 \vee \neg x_2) \wedge (\neg x_4 \vee x_1 \vee x_3) \wedge (\neg x_3 \vee \neg x_2 \vee \neg x_1)$
- observe that this language is in TIME(2^n)

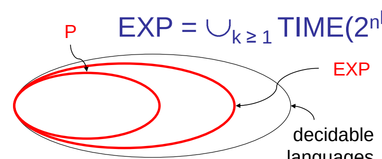
Time Complexity

Key definition: "P" or "polynomial-time" is

$$P = \cup_{k \geq 1} \mathrm{TIME}(n^k)$$

Definition: "EXP" or "exponential-time" is

$$\mathrm{EXP} = \cup_{k \geq 1} \mathrm{TIME}(2^{n^k})$$



EXP

$P = \cup_{k \geq 1} \text{TIME}(n^k)$

$EXP = \cup_{k \geq 1} \text{TIME}(2^{n^k})$

- Note: $P \subseteq$ EXP.
- We have seen 3SAT $\in$ EXP.
  - **does not rule out possibility that it is in P**
- Is P different from EXP?

Time Hierarchy Theorem

Theorem: For every *proper complexity function* $f(n) \geq n$:

$$\textbf{TIME(f(n))} \subsetneq \textbf{TIME(f(2n)}^3\textbf{)}.$$

- Note: $P \subseteq \text{TIME}(2^n) \subsetneq \text{TIME}(2^{(2n)^3}) \subseteq \text{EXP}$
- Most natural functions (and 2^n in particular) are proper complexity functions. We will ignore this detail in this class.

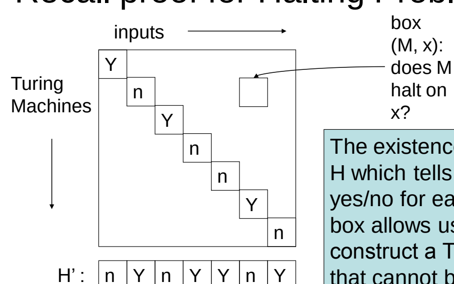
Time Hierarchy Theorem

Theorem: For every *proper complexity function* $f(n) \geq n$:

$$\textbf{TIME(f(n))} \subsetneq \textbf{TIME(f(2n)}^3\textbf{)}.$$

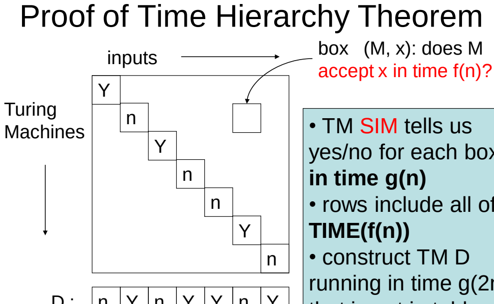
- Proof idea:
  - use diagonalization to construct a language that is not in TIME(f(n)).
  - constructed language comes with a TM that decides it and runs in time $f(2n)^3$.

Recall proof for Halting Problem



The existence of H which tells us yes/no for each box allows us to construct a TM H' that cannot be in the table.

Proof of Time Hierarchy Theorem



- TM SIM tells us yes/no for each box **in time g(n)**
- rows include all of **TIME(f(n))**
- construct TM D running in time g(2n) that is not in table

Proof of Time Hierarchy Theorem

- Proof:
  - SIM is TM deciding language
    - { <M, x> : M accepts x in $\leq f(|x|)$ steps }
  - Claim: SIM runs in time $g(n) = f(n)^3$.
  - define new TM D: on input <M>
    - if SIM accepts <M, <M>>, reject
    - if SIM rejects <M, <M>>, accept
  - D runs in time g(2n)

Proof of Time Hierarchy Theorem

- Proof (continued):
  - suppose M in **TIME(f(n))** decides L(D)
    - M(<M>) = SIM(<M, <M>>) ≠ D(<M>)
    - but M(<M>) = D(<M>)
  - contradiction.

Proof of Time Hierarchy Theorem

- Claim: there is a TM SIM that decides

  {<M, x> : M accepts x in ≤ f(|x|) steps}

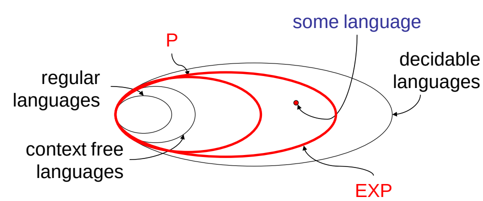
  and runs in time $g(n) = f(n)^3$.
- Proof sketch: SIM has 4 work tapes
  - contents and "virtual head" positions for M's tapes
  - M's transition function and state
  - f(|x|) "+"s used as a clock
  - scratch space

Proof of Time Hierarchy Theorem

- Proof sketch (continued): 4 work tapes
  - contents and "virtual head" positions for M's tapes
  - M's transition function and state
  - f(|x|) "+"s used as a clock
  - scratch space
  - initialize tapes
  - simulate step of M, advance head on tape 3; repeat.
  - can check running time is as claimed.

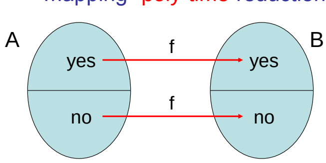
So far…

- We have defined the complexity classes P (polynomial time), EXP (exponential time)



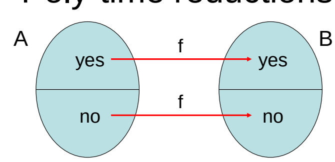
Poly-time reductions

- Type of reduction we will use:
  - "many-one" poly-time reduction (commonly)
  - "mapping" poly-time reduction (book)



reduction from language A to language B

Poly-time reductions



- function f should be poly-time computable

Definition: f : Σ*→ Σ* is poly-time computable if for some $g(n) = n^{O(1)}$ there exists a g(n)-time TM M_f such that on every w∈Σ*, M_f halts with f(w) on its tape.

Poly-time reductions

Definition: $A \leq_P B$ ("A reduces to B") if there is a poly-time computable function f such that for all w

$w \in A \Leftrightarrow f(w) \in B$

- as before, condition equivalent to:
  - YES maps to YES *and* NO maps to NO
- as before, meaning is:
  - B is at least as "hard" (or expressive) as A

Poly-time reductions

Theorem: if $A \leq_P B$ and $B \in P$ then $A \in P$.

Proof:
- a poly-time algorithm for deciding A:
- on input w, compute f(w) in poly-time.
- run poly-time algorithm to decide if $f(w) \in B$
- if it says "yes", output "yes"
- if it says "no", output "no"

Example

- 2SAT = {CNF formulas with 2 literals per clause for which there exists a satisfying truth assignment}
- L = {directed graph G, and list of pairs of vertices $(u_1, v_1), (u_2, v_2),\ldots, (u_k, v_k)$, such that there is no i for which [u_i is reachable from v_i in G and v_i is reachable from u_i in G]}
- We gave a poly-time reduction from 2SAT to L.
- determined that $2SAT \in P$ from fact that $L \in P$

Hardness and completeness

- Reasonable that can efficiently transform one problem into another.
- Surprising:
  - can often find a special language L so that **every** language in a given complexity class reduces to L!
  - powerful tool

Hardness and completeness

- Recall:
  - a language L is a set of strings
  - a complexity class C is a set of languages

Definition: a language L is C-hard if for every language $A \in C$, A poly-time reduces to L; i.e., $A \leq_P L$.

meaning: L is at least as "hard" as anything in C

Hardness and completeness

- Recall:
  - a language L is a set of strings
  - a complexity class C is a set of languages

Definition: a language L is C-complete if L is C-hard and $L \in C$

meaning: L is a "hardest" problem in C

An EXP-complete problem

- Version of A_{TM} with a time bound:

  ATM_B = {<M, x, m> : M is a TM that accepts x within at most m steps}

Theorem: ATM_B is EXP-complete.

Proof:
- what do we need to show?

An EXP-complete problem

- ATM_B = {<M, x, m> : M is a TM that accepts x within at most m steps}
- Proof that ATM_B is EXP-complete:
  - Part 1. Need to show $ATM_B \in$ EXP.
    - simulate M on x for m steps; accept if simulation accepts; reject if simulation doesn't accept.
    - running time $m^{O(1)}$.
    - n = length of input $\geq \log_2 m$
    - running time $\leq m^k = 2^{(\log m)k} \leq 2^{(kn)}$

An EXP-complete problem

- ATM_B = {<M, x, m> : M is a TM that accepts x within at most m steps}
- Proof that ATM_B is EXP-complete:
  - Part 2. For each language $A \in$ EXP, need to give poly-time reduction from A to ATM_B.
  - for a given language $A \in$ EXP, we know there is a TM M_A that decides A in time $g(n) \leq 2^{n^k}$ for some k.
  - what should reduction f(w) produce?

An EXP-complete problem

- ATM_B = {<M, x, m> : M is a TM that accepts x within at most m steps}
- Proof that ATM_B is EXP-complete:
  - f(w) = <M_A, w, m> where $m = 2^{|w|^k}$
  - is f(w) poly-time computable?
    - hardcode M_A and k…
  - YES maps to YES?
    - $w \in A \Rightarrow$ <M_A, w, m> $\in ATM_B$
  - NO maps to NO?
    - $w \notin A \Rightarrow$ <M_A, w, m> $\notin ATM_B$

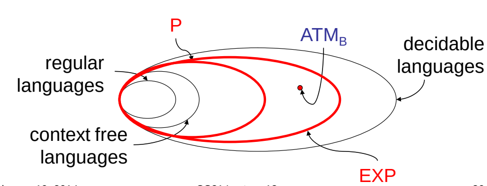
An EXP-complete problem

- A C-complete problem is a surrogate for the entire class C.
- For example: if you can find a poly-time algorithm for ATM_B then there is automatically a poly-time algorithm for every problem in EXP (i.e., EXP = P).

- Can you find a poly-time alg for ATM_B?

An EXP-complete problem

- Can you find a poly-time alg for ATM_B?
- NO! we showed that $P \subsetneq$ EXP.
- ATM_B is not tractable (intractable).



Back to 3SAT

- Remember 3SAT $\in$ EXP

  3SAT = {formulas in CNF with 3 literals per clause for which there exists a satisfying truth assignment}

- It seems hard. Can we show it is intractable?
  - formally, can we show 3SAT is EXP-complete?

Back to 3SAT

- can we show 3SAT is EXP-complete?
- Don't know how to. Believed unlikely.
- One reason: there is an important positive feature of 3SAT that doesn't seem to hold for problems in EXP (e.g. ATM_B):

3SAT is decidable in polynomial time by a nondeterministic TM