CS21
Decidability and Tractability

Lecture 13

February 5, 2014

Outline

- undecidable problems
  - computation histories
  - surprising contrasts between decidable/undecidable
- Rice's Theorem
- Post Correspondence problem
- a non-RE and non-co-RE language
- the Recursion Theorem

Dec. and undec. problems

- two problems regarding Context-Free Grammars:
  - does a CFG generate all strings:
    $ALL_{CFG} = \{<G> : G \text{ is a CFG and } L(G) = \Sigma^*\}$
  - CFG emptiness:
    $E_{CFG} = \{<G> : G \text{ is a CFG and } L(G) = \varnothing\}$
- Both decidable? both undecidable? one decidable?

Dec. and undec. problems

Theorem: E_{CFG} is decidable.

Proof:

- observation: for each nonterminal A, the set
  $S_A = \{w : A \Rightarrow^* w\}$
  is non-empty iff there is some rule:
  $A \rightarrow \mathbf{x}$
  and $\forall$ non-terminals B in string x, $S_B \neq \varnothing$

Dec. and undec. problems

Proof:

- on input <G>
- mark all terminals in G
- repeat until no new non-terminals get marked:
  - if there is a production $A \rightarrow x_1x_2x_3\ldots x_k$
  - and each symbol $x_1, x_2, \ldots, x_k$ has been marked
  - then mark A
- if S marked, reject ($G \notin E_{CFG}$), else accept ($G \in E_{CFG}$).
- terminates? correct?

Dec. and undec. problems

Theorem: ALL_{CFG} is undecidable.

Proof:

- reduce from co-A_{TM} (i.e. show co-$A_{TM} \leq_m ALL_{CFG}$)
- what should f(<M, w>) produce?
- Idea:
  - produce CFG G that generates all strings that are ***not*** accepting computation histories of M on w

Dec. and undec. problems

Proof:

- build a NPDA, then convert to CFG
- want to accept strings not of this form,

$$\#C_1\#C_2\#C_3\#...\#C_k\#$$

plus strings of this form but where

- C_1 is not the start config. of M on input w, or
- C_k is not an accept. config. of M on input w, or
- C_i does not yield in one step C_{i+1} for some i

Dec. and undec. problems

Proof:

- our NPDA nondeterministically checks one of:
  - C_1 is not the start config. of M on input w, or
  - C_k is not an accept. config. of M on input w, or
  - C_i does not yield in one step C_{i+1} for some i
  - input has fewer than two #'s
- details of first two?
- to check third condition:
  - nondeterministically guess C_i starting position
  - how to check that C_i doesn't yield in 1 step C_{i+1} ?

Dec. and undec. problems

Proof:

- checking:
  - C_i does not yield in one step C_{i+1} for some i
- push C_i onto stack
- at #, start popping C_i and compare to C_{i+1}
  - accept if mismatch away from head location, or
  - symbols around head changed in a way inconsistent with M's transition function.
- is everything described possible with NPDA?

Dec. and undec. problems

Proof:

- Problem: cannot compare C_i to C_{i+1}
- could prove in same way that proved

$$\{ww: w \in \Sigma^*\} \text{ not context-free}$$

- recall that

$$\{ww^R: w \in \Sigma^*\} \text{ is context-free}$$

- free to tweak construction of G in the reduction
- solution: write computation history:

$$\#C_1\#C_2^R\#C_3\#C_4^R...\#C_k\#$$

Dec. and undec. problems

Proof:

- f(<M, w>) = <G> equiv. to NPDA below:

on input x, accept if not of form:

$$\#C_1\#C_2^R\#C_3\#C_4^R...\#C_k\#$$

- accept if C_1 is the not the start configuration for M on input w
- accept if check that C_i does not yield in one step C_{i+1}
- accept if C_k is not an accepting configuration for M

- is f computable?
- YES maps to YES?
  - $\langle M, w\rangle \in \text{co-}A_{TM} \Rightarrow f(M, w) \in ALL_{CFG}$
- NO maps to NO?
  - $\langle M, w\rangle \notin \text{co-}A_{TM} \Rightarrow f(M, w) \notin ALL_{CFG}$

Rice's Theorem

- We have seen that the following properties of TM's are undecidable:
  - TM accepts string w
  - TM halts on input w
  - TM accepts the empty language
  - TM accepts a regular language
- Can we describe a single generic reduction for all these proofs?
- Yes. *Every* property of TMs undecidable!

Rice's Theorem

- A TM property is a language P for which
  - if $L(M_1) = L(M_2)$ then $<M_1> \in P$ iff $<M_2> \in P$
- TM property P is nontrivial if
  - there exists a TM M_1 for which $<M_1> \in P$, and
  - there exists a TM M_2 for which $<M_2> \notin P$.

Rice's Theorem: Every nontrivial TM property is undecidable.

Rice's Theorem

- The setup:
  - let $T_\varnothing$ be a TM for which $L(T_\varnothing) = \varnothing$
    - technicality: if $<T_\varnothing> \in P$ then work with property co-P instead of P.
    - conclude co-P undecidable; therefore P undec. due to closure under complement
  - so, WLOG, assume $<T_\varnothing> \notin P$
  - non-triviality ensures existence of TM M_1 such that $<M_1> \in P$

Rice's Theorem

Proof:
- reduce from A_{TM} (i.e. show $A_{TM} \leq_m P$)
- what should f(<M, w>) produce?
- f(<M, w>) = <M'> described below:

on input x,
- accept iff M accepts w and M_1 accepts x

(intersection of two RE languages)

- f computable?
- YES maps to YES?

$<M, w> \in A_{TM} \Rightarrow$
$L(f(M, w)) = L(M_1) \Rightarrow$
$f(M, w) \in P$

Rice's Theorem

Proof:
- reduce from A_{TM} (i.e. show $A_{TM} \leq_m P$)
- what should f(<M, w>) produce?
- f(<M, w>) = <M'> described below:

on input x,
- accept iff M accepts w and M_1 accepts x

(intersection of two RE languages)

- NO maps to NO?

$<M, w> \notin A_{TM} \Rightarrow$
$L(f(M, w)) = L(T_\varnothing) \Rightarrow$
$f(M, w) \notin P$

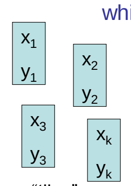
Post Correspondence Problem

- many undecidable problems unrelated to TMs and automata
- classic example: Post Correspondence Problem

$PCP = \{<(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)> :$
$x_i, y_i \in \Sigma^*$ and there exists $(a_1, a_2, \ldots, a_n)$ for which $x_{a_1}x_{a_2}\ldots x_{a_n} = y_{a_1}y_{a_2}\ldots y_{a_n}\}$

Post Correspondence Problem

$PCP = \{<(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)> :$
$x_i, y_i \in \Sigma^*$ and there exists $(a_1, a_2, \ldots, a_n)$ for which $x_{a_1}x_{a_2}\ldots x_{a_n} = y_{a_1}y_{a_2}\ldots y_{a_n}\}$



"tiles"

x_2	x_1	x_5	x_2	x_1	x_3	x_4	x_4
y_2	y_1	y_5	y_2	y_1	y_3	y_4	y_4

$x_2x_1x_5x_2x_1x_3x_4x_4 = y_2y_1y_5y_2y_1y_3y_4y_4$

"match"

Post Correspondence Problem

Theorem: PCP is undecidable.

Proof:

- reduce from A_{TM} (i.e. show $A_{TM} \leq_m$ PCP)
- two step reduction makes it easier
- first, show $A_{TM} \leq_m$ MPCP
  (MPCP = "modified PCP")
- next, show MPCP $\leq_m$ PCP

Post Correspondence Problem

MPCP = $\{<(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)> :$
$x_i, y_i \in \Sigma^*$ and there exists $(a_1, a_2, \ldots, a_n)$ for which $x_1 x_{a_1} x_{a_2} \ldots x_{a_n} = y_1 y_{a_1} y_{a_2} \ldots y_{a_n}\}$

Proof of MPCP $\leq_m$ PCP:

- notation: for a string $u = u_1 u_2 u_3 \ldots u_m$
  - $*u$ means the string $*u_1*u_2*u_3*u_4\ldots*u_m$
  - $u*$ means the string $u_1*u_2*u_3*u_4\ldots*u_m*$
  - $*u*$ means the string $*u_1*u_2*u_3*u_4\ldots*u_m*$

Post Correspondence Problem

Proof of MPCP $\leq_m$ PCP:

- given an instance $(x_1, y_1), \ldots, (x_k, y_k)$ of MPCP
- produce an instance of PCP:
  $(*x_1, *y_1*)$, $(*x_1, y_1*)$, $(*x_2, y_2*)$, …, $(*x_k, y_k*)$, $(*\blacklozenge, \blacklozenge)$
- YES maps to YES?
  - given a match in original MPCP instance, can produce a match in the new PCP instance
- NO maps to NO?
  - given a match in the new PCP instance, can produce a match in the original MPCP instance

Post Correspondence Problem

- YES maps to YES?
  - given a match in original MPCP instance, can produce a match in the new PCP instance

x_1	x_4	x_5	x_2	x_1	x_3	x_4	x_4
y_1	y_4	y_5	y_2	y_1	y_3	y_4	y_4

$*x_1$	$*x_4$	$*x_5$	$*x_2$	$*x_1$	$*x_3$	$*x_4$	$*x_4$	$*\blacklozenge$
$*y_1*$	y_4*	y_5*	y_2*	y_1*	y_3*	y_4*	y_4*	$\blacklozenge$

Post Correspondence Problem

can't match unless start with this tile

- NO maps to NO?
  - given a match in the new PCP instance, can produce a match in the original MPCP instance

$*x_1$	$*x_4$	$*x_5$	$*x_2$	$*x_1$	$*x_3$	$*x_4$	$*x_4$	$*\blacklozenge$
$*y_1*$	y_4*	y_5*	y_2*	y_1*	y_3*	y_4*	y_4*	$\blacklozenge$

x_1	x_4	x_5	x_2	x_1	x_3	x_4	x_4
y_1	y_4	y_5	y_2	y_1	y_3	y_4	y_4

"$*$" symbols must align

can only appear at the end