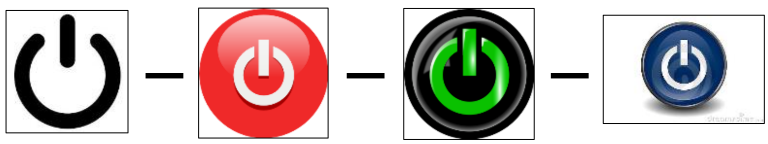
Ma/CS 6a

Class 22: Power Series



By Adam Sheffer

Power Series

- *Monomial*: ax^i.
- *Polynomial*:

  $$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n.$$

- *Power series*:

  $$A(x) = a_0 + a_1x + a_2x^2 + \cdots$$

- Also called *formal power series*, because we do not think about the meaning of x.

Sums and Products

- We define sums and products of power series as in the case of polynomials.
  - $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$
  - $B(x) = b_0 + b_1 x + b_2 x^2 + \cdots$

$$A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$$

$$A(x)B(x) = (a_0 b_0) + (a_1 b_0 + a_0 b_1)x + (a_2 b_0 + a_1 b_1 + a_0 b_2)x^2 + \cdots$$

More Sums and Products

- We define sums and products of power series as in the case of polynomials.
  - $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$
  - $B(x) = b_0 + b_1 x + b_2 x^2 + \cdots$
  - $C(x) = A(x) + B(x)$.
  - $D(x) = A(x)B(x)$.
  - $c_i = a_i + b_i$.
  - $d_i = \sum_{j=1}^{i} a_j b_{i-j}$.

Rings

- A *ring* is a set R together with two binary operations $+$ and $\cdot$ satisfying the following.
  - The set R is a commutative group under $+$.
  - The operation $\cdot$ satisfies the closure, associativity, and identity properties.
  - Distributive laws. For any $a, b, c \in R$, we have
$$a(b+c) = ab + ac,$$
$$(a+b)c = ac + bc.$$

Is this a Ring?

- The set $\mathbb{Z}_4 = \{0,1,2,3\}$ under addition and multiplication $mod\ 4$.
  - We already know that $\mathbb{Z}_4$ under addition $mod\ 4$ is a group.
  - For multiplication, we have associativity, and identity.
  - Standard addition and multiplication are distributive, so the same holds under $mod\ 4$.

Is this a Ring? #2

- 2×2 matrices with real entries, under matrix addition and multiplication.
  - 2×2 matrices with real entries under addition are a group.
  - For multiplication, we have closure, associativity, and identity.
  - Matrix addition and multiplication are distributive.

Polynomial Ring

- The *polynomial ring* $\mathbb{R}[x]$ is the set of polynomials in x with coefficients in $\mathbb{R}$ and standard addition and multiplication.
- The set of polynomials under addition is a group.
- Properties of $\cdot$
  - Closure. The product of two polynomials in $\mathbb{R}[x]$ is a polynomial in $\mathbb{R}[x]$.
  - Associativity. By the associativity of the standard multiplication.
  - Identity. We have $1 \in \mathbb{R}[x]$.
- Standard addition and multiplication are distributive.

Ring of Power Series

- The *power series ring* $\mathbb{R}[[x]]$ is the set of power series in x with coefficients in $\mathbb{R}$ under addition and multiplication.
- The set of power series under addition is a group.
- Properties of $\cdot$
  - Closure. By our definition, the product of two power series is a power series.
  - Associativity. Our definition of multiplication is associative.
  - Identity. We have $1 \in \mathbb{R}[[x]]$.
- Distributivity is not hard to verify.

What is Missing?

- In $\mathbb{R}[x]$ and $\mathbb{R}[[x]]$, why don't we have groups with respect to the operation $\cdot$?
  - The inverse does not always exist!
  - In $\mathbb{R}[x]$ only constant polynomials have an inverse.
  - What about $\mathbb{R}[[x]]$?
  - Is there an inverse of $1 - x$?

$$(1 - x)A(x) = 1.$$
$$A(x) = 1 + x + x^2 + x^3 + \cdots$$

Inverses in a Power Series

- **Theorem.** A power series
  $A(x) = a_0 + a_1x + a_2x^2 + \cdots \in \mathbb{R}[[x]]$
  has an inverse if and only if $a_0 \neq 0$.

- **Proof.** First assume that $A(x)$ has an inverse $B(x)$.
  $$(a_0 + a_1x + a_2x^2 + \cdots)(b_0 + b_1x + b_2x^2 + \cdots) = 1.$$
  - We have $a_0b_0 = 1$, so $a_0 \neq 0$.

Proof Cont.

- **Proof (cont.).** Assume that $a_0 \neq 0$.
  - We need to solve
  $$a_0b_0 = 1,$$
  $$a_1b_0 + a_0b_1 = 0,$$
  $$a_2b_0 + a_1b_1 + a_0b_2 = 0,$$
  $$\ldots$$
  - Since $a_0 \neq 0$, we obtain the solution
  $$b_0 = a_0^{-1},$$
  $$b_1 = -a_1b_0a_0^{-1},$$
  $$b_2 = (a_1b_1 + a_0b_2)a_0^{-1}.$$

More on Inverses

- Notation.
  - The inverse of $A(x)$ is sometimes written as $\boldsymbol{A(x)^{-1}}$ or as $\frac{\mathbf{1}}{\boldsymbol{A(x)}}$.

- For example, we have

$$\begin{aligned}\frac{1+x}{1-x} &= (1+x)(1-x)^{-1}\\ &= (1+x)(1+x+x^2+\cdots)\\ &= 1+2x+2x^2+2x^3+\cdots\end{aligned}$$

Recall: The Binomial Theorem

- By the *binomial theorem*, for $n \geq 1$ we have

$$\begin{aligned}(x+1)^n &= \sum_{\substack{0\le i,j\le n\\ i+j=n}} \binom{n}{i} x^i\, 1^j\\ &= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} + \cdots + \binom{n}{n}\end{aligned}$$

- What about the case where n is negative?

Negative Exponents

- What is $(x+1)^{-1}$?

$$(x+1)(a_0 + a_1 x + a_2 x^2 + \cdots) = 1,$$

$$(x+1)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

- What is $(x+1)^{-2}$?

$$(x+1)^{-2} = \left((x+1)^{-1}\right)^2$$
$$= (1 - x + x^2 - x^3 + \cdots)(1 - x + x^2 - x^3 + \cdots) = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

- What is $(x+1)^{-m}$?

Negative Exponents Formula

- **Theorem.** For any positive integer m,

$$(x+1)^{-m} = \sum_{n\geq 0} (-1)^n \binom{m+n-1}{n} x^n$$

- **Examples.**
  - $(x+1)^{-1} = \sum_{n\geq 0}(-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$
  - $(x+1)^{-2} = \sum_{n\geq 0}(-1)^n (n+1) x^n = 1 - 2x + 3x^2 - 4x^3 + \cdots$

Proof

- We look for the coefficient of x^k in

$$(x+1)^{-m} = \left((x+1)^{-1}\right)^m = \left(\sum_{n\geq 0}(-1)^n x^n\right)^m.$$

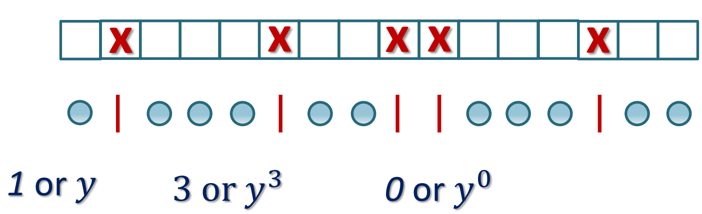
- To simplify, we replace $y = -x$ and have

$$(1-y)^{-m} = \left((1-y)^{-1}\right)^m = \left(\sum_{n\geq 0} y^n\right)^m.$$

- The problem turns to: *In how many ways can we write k as a sum of up to m positive integers.*

Proof (cont.)

- In how many ways can we write k as a sum of up to m positive integers?
  - This is equivalent to choosing $m-1$ cells in an array of $m+k-1$ cells.
  - There are $\binom{m+k-1}{m-1} = \binom{m+k-1}{k}$ ways.



Concluding the Proof

- There are $\binom{m+k-1}{k}$ ways to write k as a sum of at most m positive integers.
- The coefficient of y^k in

$$(1-y)^{-m} = \left((1-y)^{-1}\right)^m = \left(\sum_{n\geq 0} y^n\right)^m$$

is $\binom{m+k-1}{k}$. This implies

$$(x+1)^{-m} = \sum_{k\geq 0} (-1)^k \binom{m+k-1}{k} x^k .$$

Combining Both Cases?

- For integers $m \geq 1$ we have

$$(x+1)^m = \sum_{0\leq i\leq m} \binom{m}{i} x^i .$$

- For integers $m \leq -1$ we have

$$(x+1)^m = \sum_{n\geq 0} (-1)^n \binom{-m+n-1}{n} x^n .$$

- *Can we combine the two formulas into one?*

Generalizing the Binomial Numbers

- Given an integer m and a positive integer n, we define

$$\binom{m}{0} = 1 \text{ and } \binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}.$$

- This definition subsumes the standard binomial numbers $\binom{m}{n}$.
  - If $n \le m$, this is identical to $\frac{m!}{n!(m-n)!}$.
  - If $n > m$, we have $\binom{m}{n} = 0$.

Negative Binomial Numbers

- Given positive integers m and n, we have

$$\binom{-m}{n} = \frac{-m(-m-1)\cdots(-m-n+1)}{n!}$$
$$= (-1)^n \frac{m(m+1)\cdots(m+n-1)}{n!}$$
$$= (-1)^n \binom{m+n-1}{n}.$$

Combining Both Cases

- For integers $m \geq 1$ we have
$$(x+1)^m = \sum_{0 \leq i \leq m} \binom{m}{i} x^i.$$
- For integers $m \leq -1$ we have
$$(x+1)^m = \sum_{n \geq 0} (-1)^n \binom{-m+n-1}{n} x^n.$$
- Either way, we have
$$(x+1)^m = \sum_{n \geq 0} \binom{m}{n} x^n.$$

A Variant

- **Problem.** Find the value of $(1+ax)^m$ for any integer m and $a \in \mathbb{R}$.
- **Solution.**
  - Substitute $y = ax$. We have
$$(1+y)^m = \sum_{n \geq 0} \binom{m}{n} y^n.$$
  - By bringing x back, we have
$$(1+ax)^m = \sum_{n \geq 0} \binom{m}{n} a^n x^n.$$

An Example

- **Problem.** Write the power series of
$$\frac{2+x}{1-3x+3x^2-x^3}$$
- **Solution.**
  - First, $(1-3x+3x^2-x^3)=(1-x)^3$.
  - By the previous theorem, we have

$$(1-x)^{-3}=\sum_{n\geq 0}\binom{-3}{n}x^n=\sum_{n\geq 0}\binom{2+n}{n}(-x)^n.$$

$$(2+x)(1-x)^{-3}$$
$$=\sum_{n\geq 0}\left(2\binom{2+n}{n}-\binom{1+n}{n-1}\right)(-x)^n$$

The End