Ma/CS 6a

Class 16: Permutations

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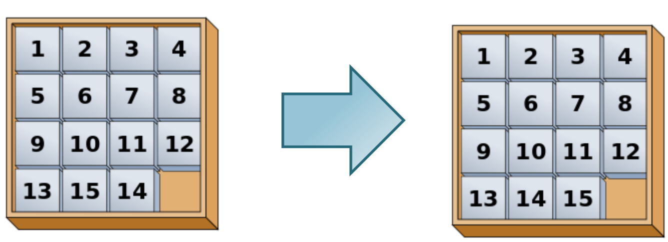
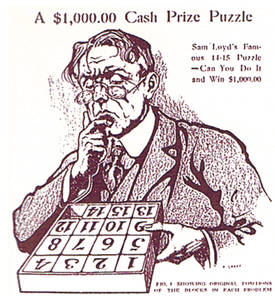
The 15 Puzzle

- **Problem.** Start with the configuration on the left and move the tiles to obtain the configuration on the right.



The 15 Puzzle (cont.)

- The game became a craze in the U.S. in 1880.
- Sam Loyd, a famous chess player and puzzle composer, offered a $1,000 prize for anyone who could provide a solution.



Reminder: Permutations

- **Problem.** Given a set $\{1,2,\ldots,n\}$, in how many ways can we order it?
- **The case $\boldsymbol{n=3}$.** Six distinct orders / permutations: 123, 132, 213, 231, 312, 321.
- **The general case.**

$$n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1$$

Options for placing 1

Options for placing 2

Options for placing n

The 15 Puzzle and Permutations

- How a configuration of the puzzle can be described as a permutation?
  - Denote the missing tile as 16.
  - The board below corresponds to the permutation

$$1\ 16\ 3\ 4\ 6\ 2\ 11\ 10\ 5\ 8\ 7\ 9\ 14\ 12\ 15\ 13$$

Permutations as Functions

- We can consider a ***permutation*** as a ***bijection*** from the set $\{1,2,\ldots,n\}$ to itself.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 1 & 3 & 2 & 6 & 4 \end{array}$$

- Denote the bijection as α.
  - $\alpha(1) = 5$.
  - $\alpha(3) = 3$.

The Permutation Set S_n

- S_n – The set of permutations of $\mathbb{N}_n = \{1,2,3, \ldots, n\}$.
- We have $|S_n| = n!$
- The set S_3:

$$\begin{array}{ccc} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{array} \qquad \begin{array}{ccc} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 \end{array} \qquad \begin{array}{ccc} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{array}$$

$$\begin{array}{ccc} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{array} \qquad \begin{array}{ccc} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{array} \qquad \begin{array}{ccc} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \end{array}$$

Combining Two Permutations

- $\alpha(1) = 2, \quad \alpha(2) = 4, \quad \alpha(3) = 5,$ $\alpha(4) = 1, \quad \alpha(5) = 3.$
- $\beta(1) = 3, \quad \beta(2) = 5, \quad \beta(3) = 1,$ $\beta(4) = 4, \quad \beta(5) = 2.$
- What is the function $\beta\alpha$.

First apply α and then β

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ \alpha: & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 2 & 4 & 5 & 1 & 3 \\ \beta: & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 5 & 4 & 2 & 3 & 1 \end{array}$$

Closure of S_n

- **Claim.** If α and β are in S_n, so does $\alpha\beta$.
- By definition, $\alpha\beta$ is a function from $\mathbb{N}_n$ to itself.
- It remains to show that for every $i \in \mathbb{N}_n$ there is a unique $j \in \mathbb{N}_n$ such that $i = \alpha\beta(j)$.
  - Since $\alpha \in S_n$, there is a unique k such that $i = \alpha(k)$.
  - Since $\beta \in S_n$, there is a unique j such that $k = \beta(j)$.

Symmetry

- Is it true that for every $\alpha, \beta \in S_n$, we have
  $$\alpha\beta = \beta\alpha?$$
- No!
  - Consider S_3 and

$$\alpha = \begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 \end{matrix} \qquad \beta = \begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow. \\ 2 & 1 & 3 \end{matrix}$$

$$\alpha\beta = \begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{matrix} \qquad \beta\alpha = \begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow. \\ 2 & 3 & 1 \end{matrix}$$

Associativity

- Is it true that for every $\alpha, \beta, \gamma \in S_n$, we have
$$(\alpha\beta)\gamma = \alpha(\beta\gamma)?$$

- Yes. In both cases the product looks like:

$$\begin{array}{cccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\alpha: & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & 2 & 4 & 5 & 1 & 3 \\
\beta: & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & 2 & 1 & 3 & 4 & 5 \\
\gamma: & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & 5 & 4 & 3 & 1 & 2
\end{array}$$

The Identity Element of S_n

- **Identity.** The *identity permutation* is defined as $\mathrm{id}(r) = r$ for every $r \in \mathbb{N}_n$. For any $\alpha \in S_n$, we have
$$\mathrm{id} \cdot \alpha = \alpha \cdot \mathrm{id} = \alpha.$$

Inverse

- Is it true that for every $\alpha \in S_n$, there exists an ***inverse permutation*** $\alpha^{-1} \in S_n$ satisfying
$$\alpha\alpha^{-1} = \alpha^{-1}\alpha = \mathrm{id}.$$

- Yes:

$$\begin{array}{rccccc} & 1 & 2 & 3 & 4 & 5 \\ \alpha: & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 2 & 4 & 5 & 1 & 3 \\ \alpha^{-1}: & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 1 & 2 & 3 & 4 & 5 \end{array}$$

Cycle Notation

- We can consider a permutation as a set of ***cycles***.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 6 & 3 & 2 & 1 & 4 \end{array}$$

$$1 \to 5 \qquad 2 \to 6 \to 4 \qquad 3$$

- We write this permutation as
$$(1\ 5)(2\ 6\ 4)(3).$$

Converting to Cycle Notation

- $\alpha(1) = 2, \quad \alpha(2) = 4, \quad \alpha(3) = 5,$
  $\alpha(4) = 1, \quad \alpha(5) = 3.$
- We start with 1 and construct its cycle: $1 \to 2 \to 4 \to 1.$
- We then choose a number that was not considered yet: $3 \to 5 \to 3.$
- We've dealt with every number of $\mathbb{N}_5$, so the cycle notation is $(1\ 2\ 4)(3\ 5)$.

Counting Cycles

- **Problem.** How many distinct cycles of length k exist in S_n?
- **Solution.**
  - There are $\binom{n}{k}$ ways of choosing k elements for the cycle.
  - There are $k!$ ways to order this elements.
  - Each cycle has k different representations.

$$\binom{n}{k} k! \frac{1}{k} = \frac{n!}{k \cdot (n-k)!}.$$

Card Shuffling

- **Problem.** Cards numbered 1 to 12 are picked up in row order and re-dealt in column order:

1	2	3		1	5	9
4	5	6		2	6	10
7	8	9		3	7	11
10	11	12		4	8	12

  How many times do we need to repeat this procedure until the cards return to their original positions?

Finding a Permutation

1	2	3		1	5	9
4	5	6		2	6	10
7	8	9		3	7	11
10	11	12		4	8	12

- A reshuffling corresponds to a permutation.
- For example, after each reshuffling 6 will move to the previous position of 5.

Solution

1	2	3		1	5	9
4	5	6		2	6	10
7	8	9		3	7	11
10	11	12		4	8	12

- The cycle structure of the permutation: $\alpha = (1)(2\ 5\ 6\ 10\ 4)(3\ 9\ 11\ 8\ 7)(12)$.
- Every cycle has length 1 or 5, so after five steps we return to the original position.

Classification of Permutations

- The ***type of a permutation*** of S_n is the number of cycles of each length in its cycle structure.
- Both $(1\ 2\ 4)(3\ 5)$ and $(1\ 2\ 3)(4\ 5)$ are of the same type: one cycle of length 3 and one of length 2.
  - We denote this type as $[2\ 3]$
- In general, we write a type as $[1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} 4^{\alpha_4} \dots]$.

Counting Permutations of a Given Type

- **Problem.** How many permutations of S_{14} are of the type $[2^2 3^2 4]$?
- We need to insert the numbers $1,2,\ldots,14$ into the cycle pattern
  $$(\cdot\ \cdot)(\cdot\ \cdot)(\cdot\ \cdot\ \cdot)(\cdot\ \cdot\ \cdot)(\cdot\ \cdot\ \cdot\ \cdot).$$
- We can place every permutation of $\mathbb{N}_{14}$ into this pattern.
  - $(12\ 1)(3\ 5)(2\ 6\ 4)(13\ 14\ 3)(7\ 8\ 9\ 10)$
  - Is the solution $14!$?

Fixing the Solution

- The following permutations are identical:
  - $(12\ 1)(3\ 5)(2\ 6\ 4)(13\ 14\ 3)(7\ 8\ 9\ 10)$
  - $(3\ 5)(12\ 1)(2\ 6\ 4)(13\ 14\ 3)(7\ 8\ 9\ 10)$
  - So is the answer $\frac{14!}{2!2!}$?
- Another identical permutation:
  - $(1\ 12)(3\ 5)(2\ 6\ 4)(13\ 14\ 3)(7\ 8\ 9\ 10)$
  - So is the answer $\frac{14!}{2!2!2\cdot 2\cdot 3\cdot 3\cdot 4}$?
  - Yes!

Counting Instances of a Type

- In general, the number of permutations of S_n of type $[1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} 4^{\alpha_4} \ldots]$ is

$$\frac{n!}{\alpha_1!\,\alpha_2!\,\alpha_3!\,\alpha_4! \cdots 1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} 4^{\alpha_4} \cdots}$$

Types of S_5

Type	Example	Number
$[1^5]$	id	1
$[1^3 2]$	$(1\ 2)(3)(4)(5)$	10
$[1^2 3]$	$(1\ 2\ 3)(4)(5)$	20
$[1 2^2]$	$(1\ 2)(3\ 4)(5)$	15
$[14]$	$(1\ 2\ 3\ 4)(5)$	30
$[23]$	$(1\ 2\ 3)(4\ 5)$	20
$[5]$	$(1\ 2\ 3\ 4\ 5)$	24

Conjugate Permutations

- Permutations $\alpha, \beta \in S_n$ are said to be *conjugate* if there exists $\sigma \in S_n$ such that
$$\sigma\alpha\sigma^{-1} = \beta.$$
- Let $\alpha = (1\ 2)(3)$ and $\beta = (1)(3\ 2)$. The two permutations are conjugate, since we can take $\sigma = (1\ 2\ 3)$ and $\sigma^{-1} = (3\ 2\ 1)$.

$$\begin{array}{cccc} & 1 & 2 & 3 \\ \sigma^{-1} & \downarrow & \downarrow & \downarrow \\ & 3 & 1 & 2 \\ \alpha & \downarrow & \downarrow & \downarrow \\ & 3 & 2 & 1 \\ \sigma & \downarrow & \downarrow & \downarrow \\ & 1 & 3 & 2 \end{array}$$

Conjugation and Types

- **Theorem.** Two permutations of S_n are conjugate iff they are of the same type.

$$\alpha = (1\ 2)(3), \quad \beta = (1)(3\ 2), \quad \sigma = (1\ 2\ 3).$$

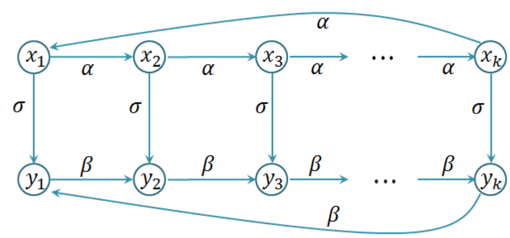
$$\sigma\alpha\sigma^{-1} = \beta$$

Proof: One Direction

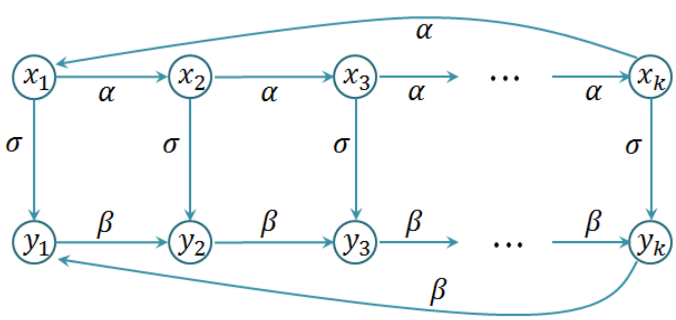
- Suppose that α, β are conjugate, so that $\sigma\alpha\sigma^{-1} = \beta$.
- Consider a cycle

$\alpha(x_1) = x_2, \alpha(x_2) = x_3, \ldots, \alpha(x_k) = x_1$.

- Set $y_i = \sigma(x_i)$. Then $\beta(y_i) = \sigma\alpha\sigma^{-1}\big(\sigma(x_i)\big) = \sigma(x_{i+1}) = y_{i+1}$.



Proof: One Direction (cont.)



- σ is a bijection between cycles of α and cycles of β.
- That is, α and β are of the same type.

Proof: The Other Direction

- Suppose α and β have the same type.
  - To prove conjugation, we need to find σ.
  - Set up a bijection between the cycles of α and β, so that corresponding cycles have the same length.
  - For every two such cycles $(x_1\ x_2\ \ldots x_k)$ and $(y_1\ y_2\ \ldots\ y_k)$, we set $\sigma(x_i) = y_i$. Then

$$\sigma\alpha\sigma^{-1}(y_i) = \sigma\alpha(x_i) = \sigma(x_{i+1}) = y_{i+1} = \beta(y_i)$$

  - That is, $\sigma\alpha\sigma^{-1} = \beta$.

The End

So how can we solve this?
In the next class, but you can try at home!