Ma/CS 6a

Class 24: More Generating Functions

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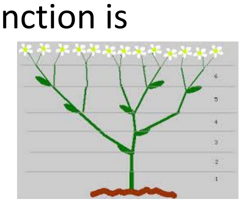
Reminder: Generating Functions

- Given an infinite sequence of numbers $a_0, a_1, a_2 \ldots$, the *generating function* of the sequence is the power series
$$a_0 + a_1 x + a_2 x^2 + \cdots$$
- **Example.**
  - Recall the *Fibonacci numbers*:
  $$F_0 = F_1 = 1 \qquad F_i = F_{i-1} + F_{i-2}.$$
  - The corresponding generating function is
  $$1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$$



Reminder: Using Generating Functions

- By rephrasing the solution to a problem as a generating function, we obtain expressions such as
  $$A(x) = x + 5xA(x) - 6x^2A(x).$$
- Solving the problem is equivalent to finding the power series $A(x)$.

Homogeneous Linear Recursion

- The generating function of the Fibonacci numbers $A(x) = a_0 + a_1x + a_2x + \cdots$ satisfies
  - $a_0 = a_1 = 1$.
  - $a_i = a_{i-1} + a_{i-2}$ (for $i \geq 2$).
- This is a special case of the ***homogeneous linear recursion*** (or ***HLR***) defined as
  - $a_0 = c_0, a_1 = c_1, \dots, a_{k-1} = c_{k-1}$.
  - $a_{n+k} + d_1a_{n+k-1} + \cdots + d_ka_n = 0$

HLR Property

- **Theorem.** Given a generating function $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ with an HLR, we have

$$A(x) = \frac{R(x)}{1 + d_1 x + \cdots + d_k x^k},$$

where $R(x)$ is a polynomial and $\deg R(x) < k$.

Example: Fibonacci Numbers

- The generating function of the Fibonacci numbers $A(x) = a_0 + a_1 x + a_2 x + \cdots$ satisfies
  - $a_0 = a_1 = 1$.
  - $a_i - a_{i-1} - a_{i-2} = 0$ (for $i \geq 2$).

$$\begin{aligned} A(x) &= 1 + x + x^2(a_0 + a_1) \\ &\quad + x^3(a_1 + a_2) + \cdots \\ &= 1 + x(a_0 + a_1 x + a_2 x^2 + \cdots) \\ &\quad + x^2(a_0 + a_1 x + a_2 x^2 + \cdots) \\ &= 1 + xA(x) + x^2 A(x). \end{aligned}$$

Fibonacci Numbers (cont.)

- We have
$$A(x) = 1 + xA(x) + x^2 A(x).$$
- That is,
$$A(x) = \frac{1}{1 - x - x^2}.$$

Proof of HLR Property

- We rewrite
$$A(x) = \frac{R(x)}{1 + d_1 x + \cdots + d_k x^k}$$
as
$$R(x) = (1 + d_1 x + \cdots + d_k x^k) A(x)$$
$$= (1 + d_1 x + \cdots + d_k x^k)(a_0 + a_1 x + a_2 x^2 + \cdots).$$
  - The coefficient of x^{n+k} (for $n \geq 0$) is
$$a_{n+k} + d_1 a_{n+k-1} + \cdots + d_k a_n.$$
  - By the HLR, this expression equals zero, so $\deg R(x) < k$.

The Auxiliary Equation

- The *auxiliary equation* of the HLR

$$a_0 = c_0, a_1 = c_1, \ldots, a_{k-1} = c_{k-1}.$$
$$a_{n+k} + d_1 a_{n+k-1} + \cdots + d_k a_n = 0$$

is $\boldsymbol{t^k + d_1 t^{k-1} + \cdots + d_k = 0.}$

  - If the auxiliary equation has k roots (not necessarily distinct), then we can rewrite it as

$$(t - \alpha_1)^{m_1}(t - \alpha_2)^{m_2} \cdots (t - \alpha_s)^{m_s} = 0,$$

where $m_1 + \cdots + m_s = k$.

The Auxiliary Equation and $A(x)$

- We have $A(x) = \frac{R(x)}{1+d_1x+\cdots+d_kx^k}$.
- The denominator can be obtained by
  - Taking the expression $t^k + d_1 t^{k-1} + \cdots + d_k$ from the auxiliary equation.
  - Dividing by t^k.
  - Substituting $x = 1/t$.
- Thus, we can rewrite

$$A(x) = \frac{t^k \cdot R(x)}{(t - \alpha_1)^{m_1}(t - \alpha_2)^{m_2} \cdots (t - \alpha_s)^{m_s}}$$
$$= \frac{R(x)}{(1 - \alpha_1 x)^{m_1}(1 - \alpha_2 x)^{m_2} \cdots (1 - \alpha_s x)^{m_s}}.$$

Stronger HLR Property

- **Theorem.** Consider the sequence $a_0, a_1, \ldots$, that is defined by an HLR with auxiliary equation $(t-\alpha_1)^{m_1}(t-\alpha_2)^{m_2}\cdots(t-\alpha_s)^{m_s}=0$. Then

$$a_n = P_1(n)\alpha_1^n + P_2(n)\alpha_2^n + \cdots + P_s(n)\alpha_s^n,$$

where $P_i(n)$ is a polynomial of degree at most $m_i - 1$.

Proof of Stronger HLR Property

$$\bar{v}_{i-1} \le i\bar{v}_i + 4\sum_{j\ge i}\bar{v}_j + M_i\sum_{j\ge i}\bar{v}_j$$

$< D_i\bar{v}_i$ $\quad$ $< \gamma_{i-1,h}N/2$

Do not read this slide!!!

$$N\gamma_{i-1,h} \le \bar{v}_{i-1} \le A_i\bar{v}_i + B_i\sum_{j\ge i+1}\bar{v}_j = A_i\bar{v}_i + B_i\sum_{i+1\le j\le t}\bar{v}_j + B_i\sum_{t<j}\bar{v}_j$$

$\left(for\ t > 12B_i/\gamma_{i-1,h}\right)$

$$N\gamma_{i-1,h} \le A_i\bar{v}_i + D_i\bar{v}_i + N\gamma_{i-1,h}/2$$

$$\boxed{\frac{N\gamma_{i-1,h}}{2(A_i+D_i)} < \bar{v}_i}$$

Using the Stronger HLR Property

- **Problem.** Solve the following HLR.

$$u_0 = 0, \quad u_1 = -9, \quad u_2 = -1, \quad u_3 = 21.$$
$$u_{n+4} - 5u_{n+3} + 6u_{n+2} + 4u_{n+1} - 8u_n = 0.$$

- **Solution.**
  - The auxiliary equation is
$$t^4 - 5t^3 + 6t^2 + 4t - 8 = 0.$$
  - This can be rewritten as
$$(t-2)^3(t+1) = 0.$$
  - By the theorem, we have
$$\begin{aligned} u_n &= P_1(n)\alpha_1^n + P_2(n)\alpha_2^n \\ &= P_1(n)2^n + P_2(n)(-1)^n. \end{aligned}$$

Solution (cont.)

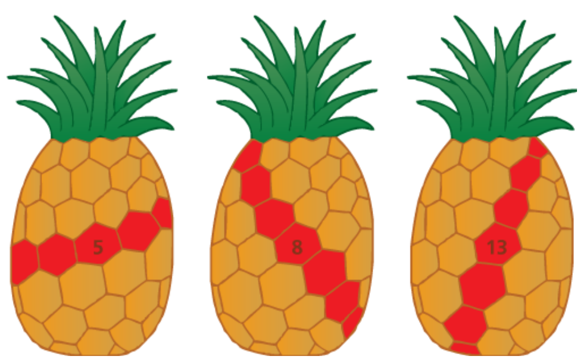
- From the auxiliary equation
$$(t-2)^3(t+1) = 0,$$
we know that
$$u_n = (An^2 + Bn + C)2^n + D(-1)^n.$$

- From the initial values
$$u_0 = 0, \quad u_1 = -9, \quad u_2 = -1, \quad u_3 = 21,$$
we get the system of equations
$$\begin{aligned} C + D &= 0, \\ 2A + 2B + 2C - D &= -9, \\ 16A + 8B + 4C + D &= -1, \\ 72A + 24B + 8C - D &= 21. \end{aligned}$$

Concluding the Solution

- We have
$$u_n = (An^2 + Bn + C)2^n + D(-1)^n.$$
- From the initial conditions, we obtain
$$C + D = 0,$$
$$2A + 2B + 2C - D = -9,$$
$$16A + 8B + 4C + D = -1,$$
$$72A + 24B + 8C - D = 21.$$
- Solving these equations yield $A = 1, B = -1, C = -3, D = 3$. Therefore
$$u_n = (n^2 - n - 3)2^n + 3(-1)^n.$$

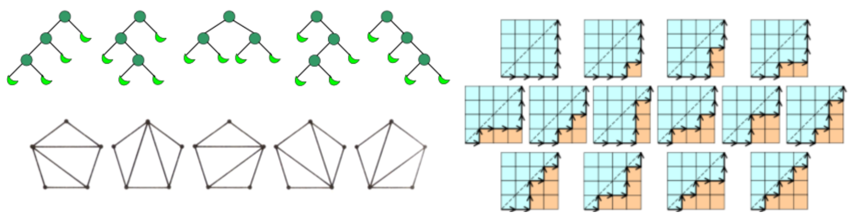
Pineapples Like Fibonacci Numbers!



*(the **number of strips** of each of the three types)*

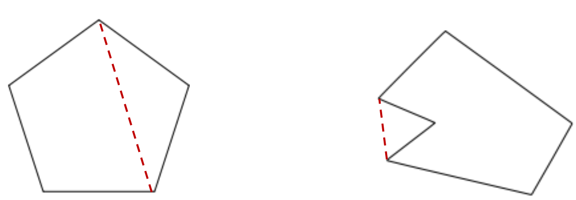
The Catalan Numers

- ***The Catalan numbers***. A sequence of numbers that solves a HUGE number of problems.
- In the exercises of the book "Enumerative Combinatorics" by Stanley, there are over 150 problems whose solution is the Catalan numbers.
- Obtained by **Euler** and **Lamé**.



Convex Polygons

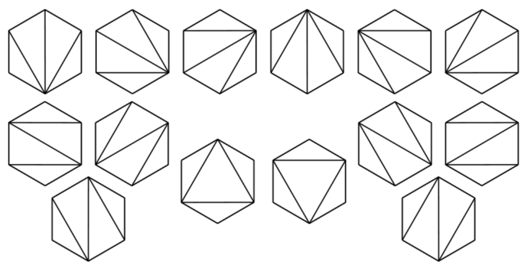
- A polygon is ***convex*** if no line segment between two of its vertices intersects the outside of the polygon.
- Equivalently, every interior angle of a convex polygon is smaller than $180°$.



Convex *Not Convex*

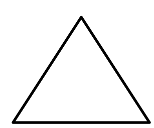
Triangulating of a convex Polygon

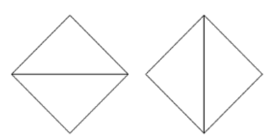
- A *triangulation* of a convex polygon P is the addition of non-crossing diagonals of P, partitioning the interior of P into triangles.

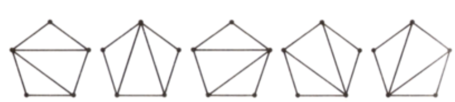


Number of Triangulations

- Let c_n denote the number of different triangulations of a convex polygon with $n+2$ vertices.
  - We set $c_0 = 1$.
  - $c_1 = 1$.
  - $c_2 = 2$.
  - $c_3 = 5$.
  - $c_4 = 14$.

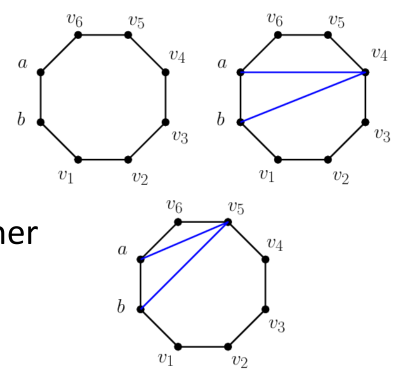






A Recursive Relation

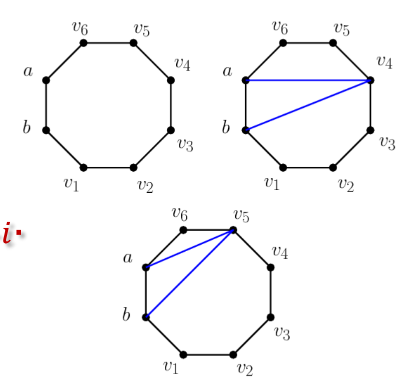
- We have initial values for c_n. Now we need a recursive relation.
  - Consider a side ab of an n-sided convex polygon P.
  - In every triangulation, ab belongs to exactly one triangle Δ.
  - The third vertex of Δ can be each of the other $n-2$ vertices of P.



A Recursive Relation (cont.)

- The number of triangulations that contain the triangle abv_4 is $c_2 c_3$.
- The number of triangulations that contain the triangle abv_5 is $c_1 c_4 = c_4$.
- Recursive relation:

$$c_{n-2} = \sum_{i=0}^{n-3} c_i c_{n-3-i}.$$



Are the Catalans an HLR?

- We have the initial values
  - $c_0 = 1, c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 14.$
- We have the relation

$$c_{n-2} = \sum_{i=0}^{n-3} c_i c_{n-3-i}.$$

- Is this an HLR?
  - No! This is not linear and number of elements in the recursion changes.

Solving the Recursion

- We have the generating series

$$C(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

  - We consider

$$\begin{aligned} C(x)^2 & \\ &= c_0 + (c_0 c_1 + c_1 c_0)x \\ &+ (c_0 c_2 + c_1 c_1 + c_2 c_0)x^2 + \cdots \end{aligned}$$

  - Since

$$c_n = \sum_{i=0}^{n-1} c_i c_{n-1-i},$$

  we have

$$C(x)^2 = 1 + c_2 x + c_3 x^2 + \cdots.$$

Solving the Recursion (cont.)

- We have
$$C(x)^2 = 1 + c_2 x + c_3 x^2 + \cdots.$$

- That is,
$$C(x) = 1 + xC(x)^2.$$

Solving the Recursion (cont.)

- We have $C(x) = 1 + xC(x)^2$.
- Setting $y = C(x)$, we obtain the quadratic equation
$$xy^2 - y - 1 = 0,$$
or
$$C(x) = y = \frac{1 \pm \sqrt{1-4x}}{2x}.$$
- How can we handle $\sqrt{1-4x}$?

More Binomial Formulas

- Recall that
$$(x+1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.$$
- By defining $\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}$ also for negative m's, we generalized the formula also to negative powers.
- *We can also consider values of m that are not integers!* In this case i goes to infinity (we do not prove this).

Fractional Powers

- Using the fractional formula, we have
$$(1-4x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n$$
$$= 1 + \frac{1/2}{1!} \cdot (-4x) + \frac{\frac{1}{2} \cdot \frac{-1}{2}}{2!} \cdot 16x^2 + \cdots$$
- This implies
$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x} = \frac{1 \pm \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n}{2x}.$$

Two Solutions

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x} = \frac{1 \pm \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n}{2x}.$$

- Considering the plus and minus cases separately, we have

$$C_-(x) = \frac{-\sum_{n=1}^{\infty} \binom{1/2}{n} (-4x)^n}{2x}$$
$$= \frac{-1}{2} \sum_{n=1}^{\infty} \binom{1/2}{n} (-4)^n x^{n-1}.$$
$$C_+(x) = \frac{1}{x} + \frac{-1}{2} \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} (-4)^n x^n.$$

The Correct Solution

$$C_-(x) = \frac{-1}{2} \sum_{n=1}^{\infty} \binom{1/2}{n} (-4)^n x^{n-1}.$$
$$\xcancel{C_+(x) = \frac{1}{x} + \frac{-1}{2} \sum_{n=1}^{\infty} \binom{1/2}{n} (-4)^n x^n.}$$

- Which one is the correct solution?
  - We saw that x^{-1} is not well defined!

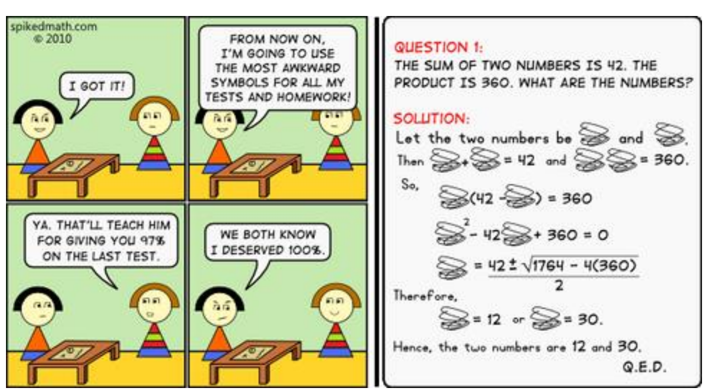
$$c_n = \frac{-1}{2} \binom{1/2}{n+1} (-4)^{n+1}$$
$$= -\frac{(-4)^{n+1}}{2(n+1)!} \cdot \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdots \frac{-(2n-1)}{2}.$$

Tidying Up

$$c_n = -\frac{(-4)^{n+1}}{2(n+1)!}\cdot\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdots\frac{-(2n-1)}{2}$$
$$= \frac{2^n}{(n+1)!}\cdot 3\cdot 5\cdot 7\cdots(2n-1)$$
$$= \frac{2^n}{(n+1)!}\cdot\frac{2n!}{n!\cdot 2^n}$$
$$= \frac{1}{n+1}\binom{2n}{n}.$$

This is the n'th Catalan number!

The End: A Request from the TAs



Please don't!