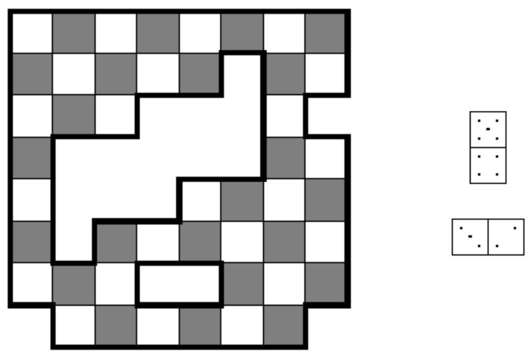
Ma/CS 6a

Class 11: Matchings



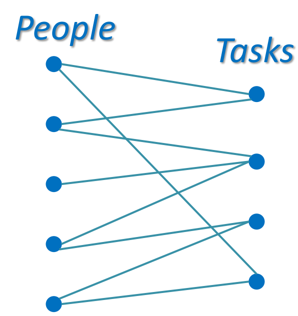
By Adam Sheffer

Task Assignment Problem

- **Problem.** We have
  - A set of tasks that need to be done.
  - A set of people, each qualified to do a different subset of tasks.
  - Each person can perform at most one task. Each task performed by at most one person.
  - Assign tasks to as many people as possible.

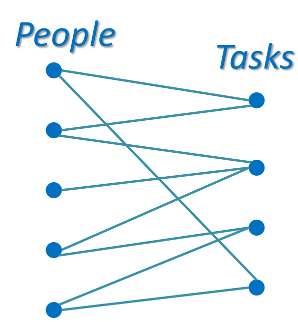
How to Assign Tasks?

- What kind of graph do we have?
  - *Bipartite:* One set of vertices that correspond to people. Another set of vertices that correspond to tasks.
- What are the edges?
  - An edge between every person and every task that s/he is qualified to do.



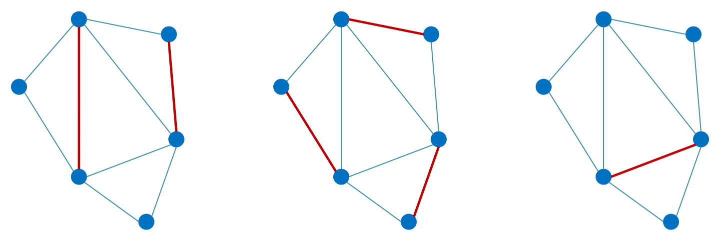
How to Assign Tasks? (cont.)

- Bipartite graph with a vertex for every person and a vertex for every task.
- Every person is connected by an edge to the tasks that she can do.
- *What should we do with this graph?*



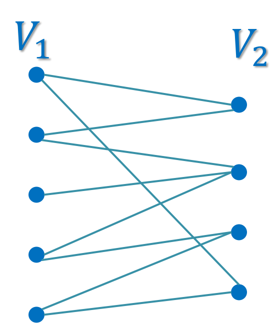
Matchings

- A *matching* in an undirected graph is a set of vertex-disjoint edges.
- The *size* of a matching is the number of edges in it.
- A *maximum matching* of G is a matching of maximum size.



Exactly the Same Problem

- **Problem.** Given a bipartite graph $G = (V_1 \cup V_2, E)$. Find a maximum matching of G.

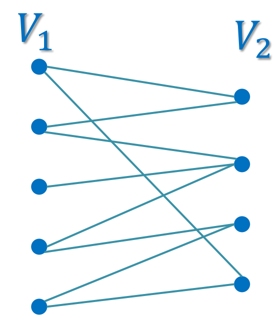


Degrees in Bipartite Graphs

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.
- We have

$$\sum_{v \in V_1} d(v) = \sum_{u \in V_2} d(u) = |E|.$$

(Every edge of E contributes 1 to the sum of the degrees in each side.)



Task Assignment: A Special Case

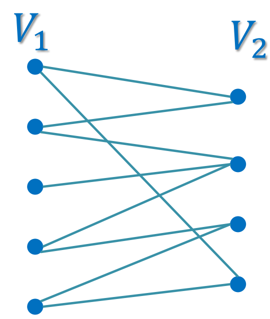
- **Problem.** In the task assignment problem we have the additional information:
  - Each person is qualified to do exactly k of the jobs.
  - Every job has exactly k people that are qualified for it.

  Prove that the number of people is equal to the number of jobs.

Solution

- We build a bipartite graph $G = (V, E)$, as before.
  - Denote by V_1 the set of vertices that correspond to people.
  - Denote by V_2 the set of vertices that correspond to tasks.
  - Recall that

$$\sum_{v \in V_1} d(v) = \sum_{u \in V_2} d(u) = |E|.$$



Solution (cont.)

- Since each person is qualified to do exactly k of the tasks:

$$\sum_{v \in V_1} d(v) = k|V_1|.$$

- Since every task has exactly k people that are qualified to do it:

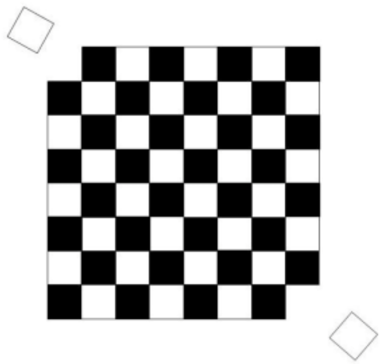
$$\sum_{u \in V_2} d(u) = k|V_2|.$$

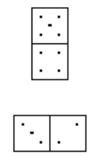
- Thus:

$$k|V_1| = \sum_{v \in V_1} d(v) = \sum_{u \in V_2} d(u) = k|V_2|.$$

A Classic Riddle

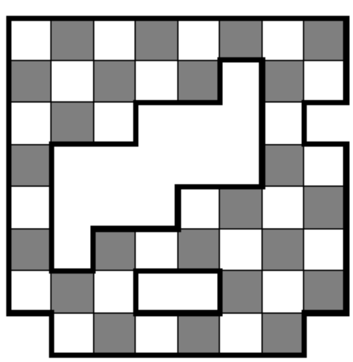
- Given a chessboard with two opposite corners removed, can we cover it by domino tiles such that each square is covered by exactly one tile?

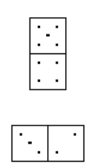




An Advanced Variant

- Given an $n \times n$ chessboard, with various squares removed, can we cover it by domino tiles?



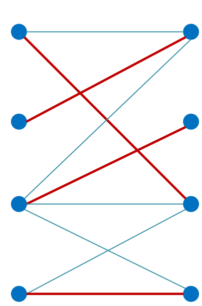


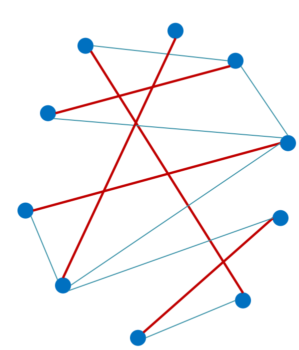
Solving the Advanced Variant

- We build the following graph:
  - A vertex for every square.
  - There's an edge between two squares if they are adjacent on the board.
  - This graph is bipartite! We can partition the vertices to black squares and white squares.
- What should we do with the graph?
  - An edge in a matching corresponds to a tile on two adjacent squares.
  - We need to know whether there is a matching that touches all of the vertices.

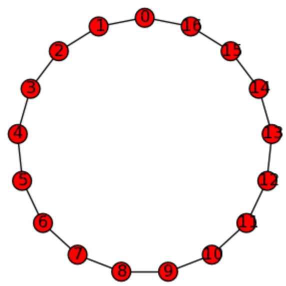
Perfect Matchings

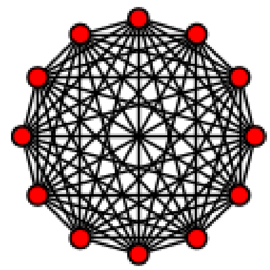
- A *perfect matching* of a graph $G = (V, E)$ is a matching of size $|V|/2$.
  - For a bipartite graph to have a perfect matching, both sides must have the same size.





How Many Perfect Matchings Can a Graph Have?





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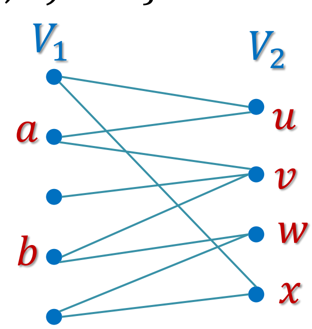
$$\frac{|V|!}{(|V|/2)!\,2^{|V|/2}}$$

Neighbor Sets

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.
- For any vertex $v \in V_1$, we define the *neighbor set* of v as
$$N(v) = \{u \in V_2 \mid (v,u) \in E\}.$$

$N(a) = \{u, v\}$

$N(b) = \{v, w\}$



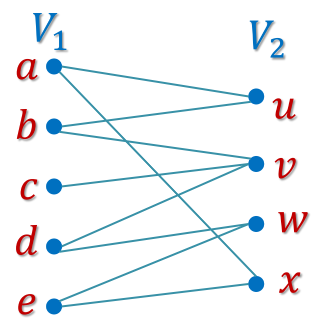
More Neighbor Sets

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.
- For any subset $A \subset V_1$, we define

$N(A) = \{y \in V_2 \mid (x, y) \in E \text{ for some } x \in A\}$.

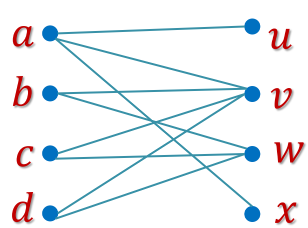
$N(\{b, c, d\}) = \{u, v, w\}$

$N(\{a, e\}) = \{u, w, x\}$



Neighbor Sets and Perfect Matchings

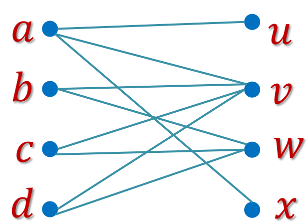
- Explain why there's no perfect matching in this graph:



- $N(\{b, c, d\}) = \{v, w\}$, so we cannot find a match for all three vertices b, c, d.

A Necessary Condition for Perfect Matchings

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.
- If there exists a subset $A \subset V_1$ such that
  $$|A| > |N(A)|,$$
  then there is no perfect matching in G.
  - We cannot simultaneously find a match for each of the vertices of A.



Hall's Marriage Theorem

- **Theorem.** Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with $|V_1| = |V_2|$.
- There exists a perfect matching in G if and only if for every $A \subset V_1$, we have $|A| \leq |N(A)|$.

Philip Hall

Proving Hall's Theorem

- **Already proved:** If there exists a subset $A \subset V_1$ such that $|A| > |N(A)|$, then there is no perfect matching in G.
- **It remains to prove:** If every subset $A \subset V_1$ satisfies $|A| \leq |N(A)|$, then there is a perfect matching in G.

Hall's Theorem: Proof

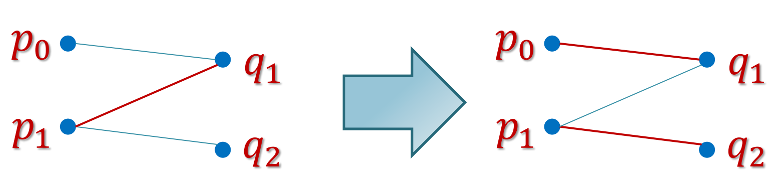
- Let M be a maximum matching of G.
- Assume, for contradiction, that there is a vertex p_0 that is not matched in M.
- By the assumption, $|N(p_0)| \geq 1$, so there exists an edge (p_0, q_1).
- q_1 must be matched in M, since otherwise the matching $M' = M \cup \{(p_0, q_1)\}$ contradicts the maximality of M.

Hall's Theorem: Proof (2)

- Let M be a maximum matching of G.
- Assume, for contradiction, that there is a vertex p_0 that is not matched in M.
- There exists an edge (p_0, q_1).
- In M, q_1 is matched with p_1.
- By assumption $|N(\{p_0, p_1\})| \geq 2$, so a vertex q_2 is adjacent to either p_0 or p_1.

Hall's Theorem: Proof (3)

- If q_2 is unmatched in M:
  - If q_2 is connected to p_0: a contradiction by creating a larger matching M', as before.
  - If q_2 is connected to p_1: remove (p_1, q_1) from M and insert (p_0, q_1) and (p_1, q_2). This again yields a larger matching!

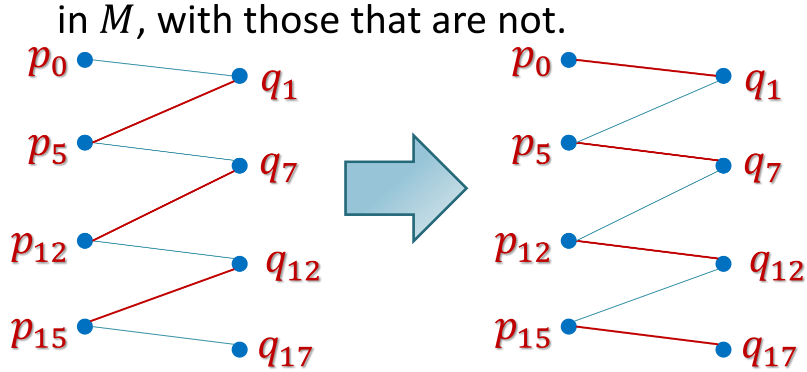


Hall's Theorem: Proof (4)

- If q_2 is matched to p_2 in M, then $|N(\{p_0, p_1, p_2\})| \geq 3$.
- So there is a vertex q_3 connected to either p_0, p_1, or p_2.
- We repeat this process. After $|V_1|$ steps no vertices of V_1 remain and we obtain a contradiction.

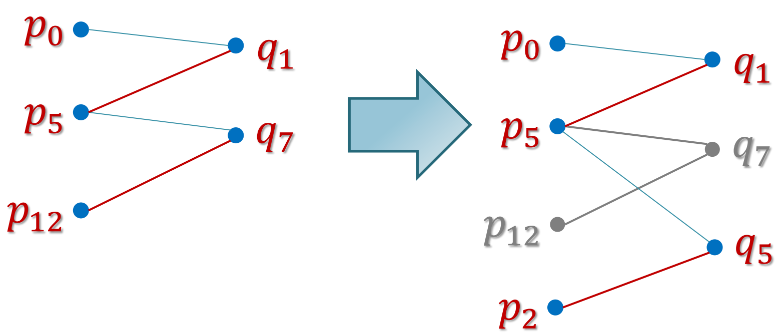
Chains

- During the proof, we encounter "chains" of vertices, starting with p_0.
- We obtain a larger matching by switching between the edges of the chain that are in M, with those that are not.



A Detail Worth Noticing

- Sometimes, we discover a vertex in the middle of the chain.
  - We then cut the chain at this point, and ignore it's original end.

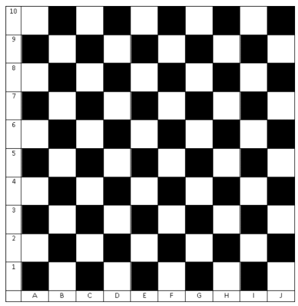


A Stronger Result

- Our proof works also when $|V_1| \neq |V_2|$:
- **Theorem.** Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. There exists a matching of size $|V_1|$ in G if and only if for every $A \subset V_1$, we have $|A| \leq |N(A)|$.

The End: A Riddle

- Can we tile a 10×10 board with 1×4 and 4×1 tiles?



(hint: use the technique from the classic riddle)