Ma/CS 6a: Problem Set 9*

Due noon, Thursday, December 4

1. Prove that $\frac{1}{2}\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]$ is an integer for every $n \in \mathbb{N}$. The question can be solved by using the binomial formula. Do NOT solve it this way, but rather find a recurrence relation.

2. (a) (NO COLLABORATION) Consider the sequence of numbers that satisfies $a_0 = 1$, $a_1 = 3$, and $a_{i+2} = 4a_{i+1} - 4a_i$ for every $i \geq 0$. Find a non-recursive formula for the value of a_n.
(b) (NO COLLABORATION) Consider the sequence of numbers that satisfies $c_0 = 1$ and $c_i = c_0 + c_1 + \cdots + c_{i-1}$ for every $i \geq 1$. Find a non-recursive formula for the value of c_n.

3. Let b_n denote the number of subsets $A \subset \{1, 2, \ldots, n\}$ such that there do not exist two elements $i, j \in A$ with $|i - j| \leq 2$ (notice that the empty set is also a valid subset). By convention, we set $b_0 = 1$. Find the generating function of b_n (it suffices to express it in the form $F(x)/G(x)$. There is no need to simplify the expression and find the formula for the b_n).

4. In class we considered partitions where the order of the summands does not matter. Now assume that the order does matter. For example, $1 + 4$ and $4 + 1$ are two distinct partitions of five. Let $p_k(n)$ denote the number of such partitions of n that have exactly k parts (for some positive integer k). Find a formula of the form $\sum_i d_i x^i$ for the generating function of $p_k(n)$ (we previously solved a very similar question by using balls and bins. Do NOT do that here. Instead, find the generating function directly).

*The awesome students who helped correcting this assignment: Tim Holland and Ramruthwick Pathireddy.