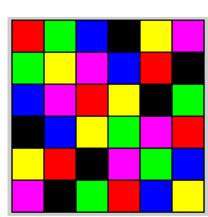
Ma/CS 6a

Class 28: Latin Squares

By Adam Sheffer

Latin Squares

- A ***Latin square*** is an $n \times n$ array filled with n different symbols, each occurring exactly once in each row and exactly once in each column.



A	B	F	C	E	D
B	C	A	D	F	E
C	D	B	E	A	F
D	E	C	F	B	A
E	F	D	A	C	B
F	A	E	B	D	C

Partial Latin Squares

- In this class, our goal is to investigate when we can solve ***partial filled Latin squares*** (i.e., Latin squares that are incomplete).

8			4		6			7
						4		
	1					6	5	
5		9		3		7	8	
				7				
	4	8		2		1		3
	5	2					9	
		1						
3			9		2			5

An EASY Warm-up

- **Problem.** What happens when only one row is missing?
  - There is a unique way of completing each column, and we can check whether the additional row is valid (it always is).

A	B	F	C	E	D
C	D	B	E	A	F
D	E	C	F	B	A
E	F	D	A	C	B
F	A	E	B	D	C

An Empty Square

- **Problem.** Find an easy way of building an $n \times n$ Latin Square.
  - **Hint.** Where have we seen Latin squares in this course before?
  - Consider a finite group $G = \{g_1, g_2, \dots, g_n\}$.
  - The multiplication table of G is a Latin square.
  - For example, consider the set $G = \{0,1,2,3,4\}$ under addition $mod\ 5$.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Reminder: Why a Group is a Latin Square

- Why is the multiplication table of $G = \{g_1, g_2, \dots, g_n\}$ a Latin square?
  - Consider the row/column that corresponds to g_i.
  - If two of the elements in the row/column are identical, we have $g_i g_j = g_i g_k$.
  - Multiplying by g_i^{-1} implies from the left yields $g_j = g_k$.

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Latin Rectangles

- **Problem.** Consider an $m \times n$ *Latin rectangle* with $m \leq n$ (i.e., no two elements in the same row/column are identical).
  - Can we always complete the rectangle to a Latin square?
  - We already know that this is true when $m = 0, n-1, n$.

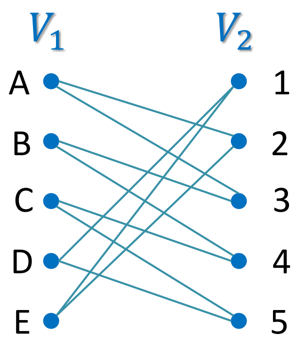
A	B	F	C	E	D
B	C	A	D	F	E
C	D	B	E	A	F

Rephrasing the Problem

- **Claim.** Every $m \times n$ Latin rectangle (where $m < n$) can be extended to an $(m+1) \times n$ Latin rectangle.
- To prove the claim, we build a bipartite graph $G = (V_1 \cup V_2, E)$.
  - V_1 consists of n vertices that correspond to the n symbols of the rectangle.
  - V_2 consists of n vertices that correspond to the n entries of the new row (row $m+1$).
  - An edge $(v_1, v_2) \in V_1 \times V_2$ means that symbol v_1 can be placed in cell v_2.

The Graph

- Completing the additional row is equivalent to finding a perfect matching in the graph.



A	B	C	D	E
B	C	D	E	A
C	D	E	A	B

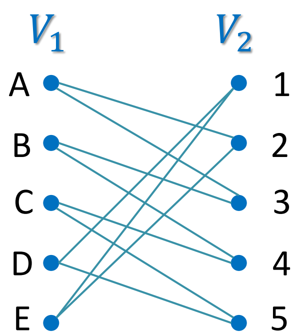
Reminder: Hall's Marriage Theorem

- **Theorem.** Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with $|V_1| = |V_2|$.
- There exists a perfect matching in G if and only if for every $A \subset V_1$, we have $|A| \leq |N(A)|$.

Philip Hall

Degrees

- Every vertex of V_1 has degree $n - m$.
  - Every symbol is in m columns of the rectangle.
- Every vertex of V_2 has degree $n - m$.
  - Every column already contains m symbols.



A	B	C	D	E
B	C	D	E	A
C	D	E	A	B

Concluding the Proof

- Consider a subset $A \subset V_1$.
  - The sum of the degrees of the vertices of A is $(n - m)|A|$.
  - Since every vertex of V_2 has degree $n - m$, we have $|A| \geq N(A)$.
- By Hall's theorem there exists a perfect matching, which implies that we can always add another row to the Latin rectangle, as long as it is not a square.

Partially Filled Latin Square

- We define a ***partially filled Latin square of order n*** to be an $n \times n$ table such that:
  - Some of the cells of the table are filled with one of n symbols.
  - Any row and column contains each symbol at most once.

	2			7		
		5		4		
			5			
	4					

Completing Partially Filled Latin Squares

- What is the minimum number of filled entries in a partially filled Latin square of order n, such that the table cannot be completed into a Latin square?

2			
4			
	3		1

A Tight Bound

- **Theorem.** Any partially filled Latin square of order n with at most $n-1$ entries can be completed to a valid Latin square.

	2			7		
		5		4		
			5			
	4					

Matrix Representation

- An $n \times n$ Latin square can be represented as by $3 \times n^2$ matrix.

$$\begin{array}{c} R \\ C \\ E \end{array} = \begin{pmatrix} 1 & 2 & 2 & 4 & \cdots \\ 1 & 3 & 1 & 1 & \cdots \\ 1 & 4 & 2 & 4 & \cdots \end{pmatrix}$$

In Row 2, Column 3,
the Entry is 4

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Valid Matrix Representation

- **Claim.** A $3 \times n^2$ matrix M with entries in $S_n = \{1,2,\dots,n\}$ represents an $n \times n$ Latin square iff when taking any two rows of M every element of S_n^2 appears exactly once.

$$\begin{pmatrix} 1 & 2 & 2 & 2 & \cdots \\ 1 & 3 & 1 & 1 & \cdots \\ 1 & 4 & 2 & 4 & \cdots \end{pmatrix}$$

Not valid!

Proof: Valid Matrix Representation

- **Proof.**
  - Asking the first two rows to contain every pair exactly once corresponds to having a unique entry in every cell of the square.
  - Asking the first and third rows to contain every pair once corresponds to having every symbol exactly once in every row.
  - Asking last two rows to contain every pair once corresponds to having every symbol exactly once in every column.

Conjugate Latin Squares

- Two Latin squares of order n are *conjugate* if their matrix representations are identical up to a permutation of the rows.
  - By the previous claim, permuting the rows of such a matrix results in a different Latin square.

1	3	2
2	1	3
3	2	1

R: 1 1 1 2 2 2 3 3 3
C: 1 2 3 1 2 3 1 2 3
E: 1 3 2 2 1 3 3 2 1

1	2	3
3	1	2
2	3	1

R: 1 3 2 2 1 3 3 2 1
C: 1 1 1 2 2 2 3 3 3
E: 1 2 3 1 2 3 1 2 3

A Weaker Theorem

- We prove a weaker variant of the theorem:
  - **Theorem.** Any partially filled $n \times n$ Latin square with at most $n - 1$ entries consisting of *$m \leq n/2$ distinct symbols*, can be completed to a valid Latin square.

	2			7		
		5		4		
			5			
	4					

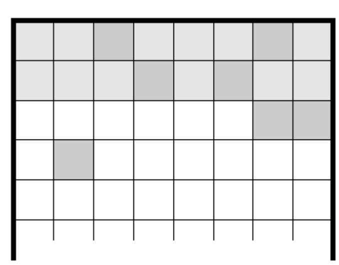
First Steps of the Proof

- By conjugation, we may replace the condition at most $n/2$ distinct symbols with all the entries being in the first $n/2$ rows.
  - m – the number of non-empty rows ($\leq n/2$).
  - r_i – the number of entries in row i.
  - We reorder the rows of the matrix so that $r_1 \geq r_2 \geq \cdots \geq r_m$.
  - Notice that $r_1 + r_2 + \cdots + r_m \leq n - 1$.

The Plan

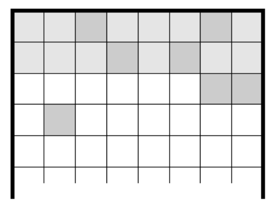
- Suffices to prove: The partial square can be completed to an $m \times n$ Latin rectangle.
- Suffices to prove #2: For every $\ell \leq m$, if we completed rows $1, \ldots, \ell - 1$, we can complete row ℓ.

$$r_1 = r_2 = r_3 = 2$$
$$r_4 = 1$$
$$\ell = 3$$



Perfect Matchings, Yet Again

- To prove that we can complete row ℓ, we again use ***Hall's theorem***.
- Define a bipartite graph $G = (V_1, V_2, E)$:
  - Every **vertex** of V_1 corresponds to a symbol that is missing in the ℓ'th row.
  - Every **vertex** of V_2 corresponds to an empty cell of the ℓ'th row.
  - An **edge** $(v_1, v_2) \in E$ implies that "symbol" v_1 may be placed in "cell" v_2.
  - We have $|V_2| = n - r_\ell$.



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