Ma/CS 6a: Problem Set 2*

Due noon, Thursday, October 16

1. Take two arbitrary composite four-digit natural numbers (use a computer program to generate random four digit numbers, and then to verify that they are not prime) and find a composite witness for each of them (witnesses of the same type as in the Miller-Rabin algorithm).

2. (a) Let p be a prime number and let $s \in \mathbb{N} \setminus \{0\}$. Find $\varphi(p^s)$ and explain your answer.
(b) Let $a, b \in \mathbb{N} \setminus \{0\}$ such that a and b are relatively prime. Explain why $\varphi(ab) = \varphi(a)\varphi(b)$. **This section is optional. You are allowed not to solve it and still rely on it in (c).**
(c) Let $p_1, p_2, \ldots, p_k$ be prime numbers and let $s_1, s_2, \ldots, s_k \in \mathbb{N} \setminus \{0\}$. Find $\varphi(p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k})$ and *prove* your claim.

3. (NO COLLABORATION) Prove the following identity for every $n \in \mathbb{N} \setminus \{0\}$.

$$2\left[\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots\right] = 2^n$$

(hint: express $(x - y)^n$ using binomial coefficients, and assign specific values to x and y).

4. Consider two integers n and k, such that $k > 1$ and $n > 2k$. What is the number of subsets of $\{1, 2, 3, \ldots, n\}$ of size k that do not contain two consecutive elements? (hint: consider k-tuples of the form $(a_1, a_2 - 1, a_3 - 2, a_4 - 3, \cdots, a_k - k + 1)$).

5. A function is *monotonically increasing* if for every $i > j$ we have $f(i) \geq f(j)$. How many monotonically increasing functions are there from $\{0, 1, 2, 3, \ldots, n\}$ to $\{0, 1, 2, 3, \ldots, n\}$? (hint: define $k_i = f(i) - f(i-1)$, $k_0 = f(0)$, and $k_{n+1} = n - f(n)$. What is the meaning of these k_i's?).

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