Ma/CS 6a

Class 25: Partitions

Explain the significance of the following sequence: un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu...

By Adam Sheffer

Answer

These are the Catalan numbers!

(The numbers one to ten in Catalan.)

Partitions of a Positive Integer

- For a positive integer n, we denote by $p(n)$ the number of ways to write n as a sum of (unordered) positive integers.
- Example. We can write $n = 5$ as

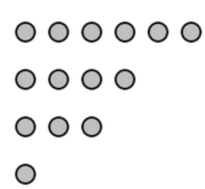
$$5, \quad 4+1, \quad 3+2, \quad 3+1+1,$$
$$2+2+1, \quad 2+1+1+1,$$
$$1+1+1+1+1.$$

so $p(5) = 7$.
  - $p(20) = 627.$
  - $p(100) = 190569292.$

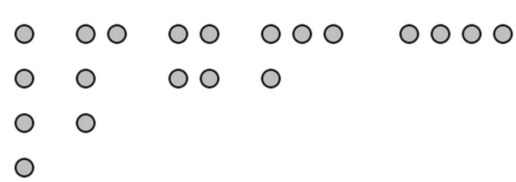
Ferrers Diagrams

- *Ferrers diagrams* are a graphic way of representing partitions.

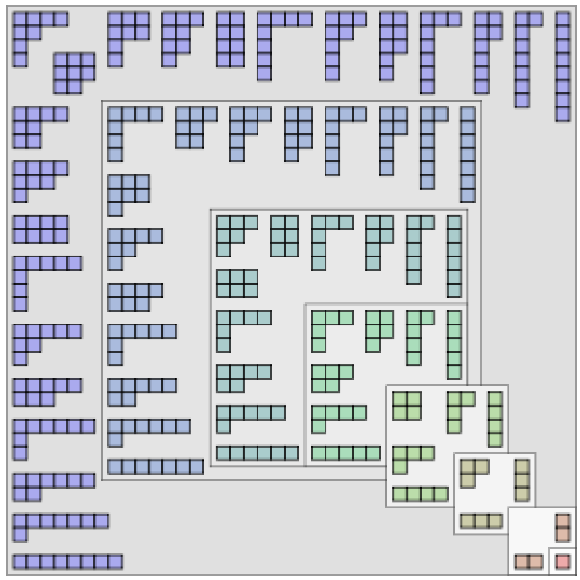
$14 = 6 + 4 + 3 + 1$



$p(4) = 5$



Ferrers Diagrams of 1 to 8

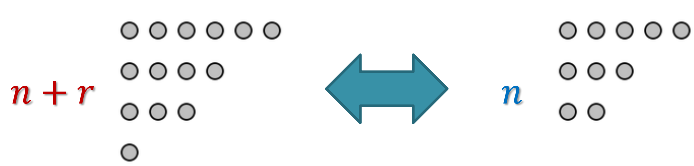


A Simple Observation

- **Claim.** Let n and r be positive integers. Then

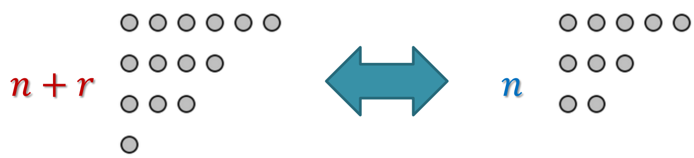
$$p(n \mid \text{number of parts} \leq r)$$
$$= p(n + r \mid \text{number of parts} = r).$$

- **Proof.** We find a bijection between the two sets of partitions:



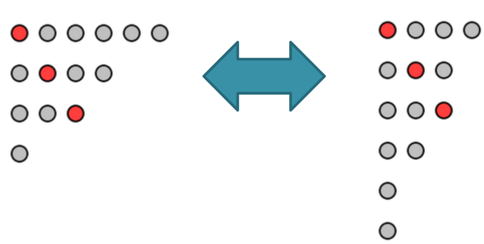
Detailed Proof

- We describe a bijection between the sets:
  - P_n - Partitions of n with at most r parts.
  - P_{n+r} - Partitions of $n+r$ with exactly r parts.
- Given a partition of P_n, we add a new first column with r elements, obtaining a partition of P_{n+r}.
- Given a partition of P_{n+r}, we remove the first column to obtain a partition of P_n



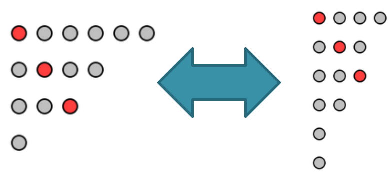
Conjugate Partitions

- Two partitions of a number n are said to be *conjugate* if one is obtained from the other by switching the rows and columns in the Ferrers Diagrams.



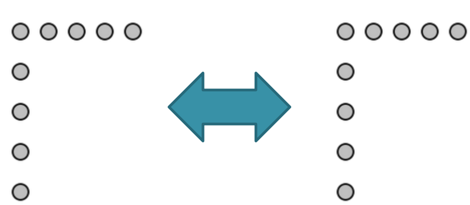
Using Conjugate Partitions

- Consider a pair of conjugate partitions α, β. The size of the largest part of α is the number of elements of β.
- Using a bijection argument as before, we have
$$p(n \mid \text{largest part} = m)$$
$$= p(n \mid \text{number of parts} = m).$$



Self-Conjugation

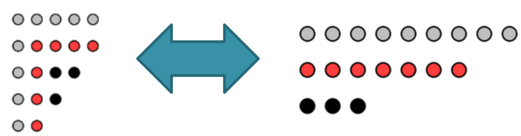
- A partition is ***self-conjugate*** if it is its own conjugate.
- **Claim.**
$$p(n \mid \text{self–conjugate})$$
$$= p(n \mid \text{the parts are distinct and odd}).$$



Self Conjugation Proof

$p(n \mid \text{self–conjugate})$
$= p(n \mid \text{the parts are distinct and odd})$.

- **Proof.** As before, we find a bijection between the two sets of partitions.
- Given a self conjugate partition, let k_i be the number of elements in the 1st row and column after removing the first $i-1$ rows and columns. For $i < j$, we have $k_i > k_j$.
- We use the $2k_i - 1$ elements in the i'th "row and column" to create the i'th row.



Partitions and Generating Functions

- To calculate $p(i)$, we define a generating function for the number of partitions:
  $P(x) = p(0) + p(1)x + p(2)x^2 + \cdots$
  - By convention, we write $p(0) = 1$.
- We have as many initial values as we like:
  - $p(1) = 1,\ p(2) = 2,\ p(3) = 3,\ \ p(4) = 5,$ $p(5) = 7, \ldots$
- Not clear how to find a recursion relation.

Warm-Up Question

- For any positive integer n, we have $(1-x^n)^{-1} = 1 + x^n + x^{2n} + x^{3n} + \cdots$
- Let $p_n(i)$ denote the number of partitions of i where each part is of size n.

$$p_n(i) = \begin{cases} 1, & \text{if } n|i, \\ 0, & \text{otherwise.} \end{cases}$$

- The corresponding generating function:

$$\begin{aligned} P_n(x) &= p_n(0) + p_n(1)x + p_n(2)x^2 + \cdots \\ &= (1-x^n)^{-1}. \end{aligned}$$

A Bit of Progress

- Let $p_{n,m}(i)$ denote the number of partitions of i where each part is equal to either i or j.
- Let

$$\begin{aligned} P_{n,m}(x) &= p_{n,m}(0) + p_{n,m}(1)x + p_{n,m}(2)x^2 + \cdots \\ &= (1 + x^n + x^{2n} + \cdots)(1 + x^m + x^{2m} + \cdots) \\ &= (1-x^n)^{-1}(1-x^m)^{-1}. \end{aligned}$$

Changing a Dollar

- **Problem.** In how many ways can a dollar be exchanged for quarters (25c), dimes (10c), and nickels (5c)?
- To make the numbers simpler, we can divide everything by 5:
  - In how many ways can we write 20 as a sum of 1's, 2's, and 5's.
  - The coefficient of y^{20} in
    $$(1-y)^{-1}(1-y^2)^{-1}(1-y^5)^{-1}.$$

Number Crunching

- First, let us calculate

$$(1-y^2)^{-1}(1-y^5)^{-1} = (1+y^2+y^4+\cdots+y^{20})(1+y^5+y^{10}+y^{15}+y^{20}).$$

Table 26.3.1

1	y	y^2	y^3	y^4	y^5					y^{10}					y^{15}					y^{20}	
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	
					1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	$(\times y^5)$
										1	0	1	0	1	0	1	0	1	0	1	$(\times y^{10})$
															1	0	1	0	1	0	$(\times y^{15})$
																				1	$(\times y^{20})$
1	0	1	0	1	1	1	1	1	1	2	1	2	1	2	2	2	2	2	2	3	

$$1+y^2+y^4+y^5+y^6+y^7+y^8+y^9+2y^{10}+y^{11}+2y^{12}+y^{13}+2y^{14}+2y^{15}+2y^{16}+2y^{17}+2y^{18}+2y^{19}+3y^{20}.$$

Number Crunching (cont.)

- We have

$(1-x^2)^{-1}(1-x^5)^{-1}$
$$= 1 + y^2 + y^4 + y^5 + y^6 + y^7 + y^8 + y^9 + 2y^{10} + y^{11} + 2y^{12} + y^{13} + 2y^{14} + 2y^{15} + 2y^{16} + 2y^{17} + 2y^{18} + 2y^{19} + 3y^{20}.$$

- What is the coefficient of y^{20} in $(1-y)^{-1}(1-y^2)^{-1}(1-y^5)^{-1}$?
  - Every element of $(1-y^2)^{-1}(1-y^5)^{-1}$ corresponds to one way of writing 20:

$$1+1+1+1+1+1+1+1+2+1+2+1+2+2+2+2+2+2+3 = 29$$

Back to General Partitions

- **Theorem.** The generating function of the number $p(n)$ of partitions can be written as

$$P(x) = p(0) + p(1)x + p(2)x^2 + \cdots = \prod_{i=1}^{\infty}\left(1-x^i\right)^{-1} = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^4 + x^8 + \cdots)\cdots.$$

Proof Sketch

- We need to verify that the coefficient of x^n in $P(x)$ is $p(n)$.
  - Consider a partition $n = m_1 s_1 + m_2 s_2 + \cdots + m_k s_k$, where $s_1, \dots, s_k$ are distinct numbers and m_i is the number of parts of size s_i in the partition.
  - In $\prod_{i=1}^{\infty} (1 - x^i)^{-1}$, this partition corresponds to taking $x^{m_i s_i}$ from $(1 + x^{s_i} + x^{2 s_i} + \cdots)$.
  - Similarly, any choice of elements from the parentheses in $\prod_{i=1}^{\infty} (1 - x^i)^{-1}$ that yields x^n corresponds to a partition of n.

A Small Issue

- Our proof is fine if we have a product of finitely many terms, but in $\prod_{i=1}^{\infty} (1 - x^i)^{-1}$ we have products of infinitely many terms!
  - When proving that the coefficient of x^n is $p(n)$, it suffices to consider $\prod_{i=1}^{n} (1 - x^i)^{-1}$.

Restricted Partitions #1

- Consider partitions of n with no more than k identical parts.
- For example, when $n = 12$ and $k = 2$:
  - $3 + 3 + 3 + 3$ and $4 + 4 + 4$ are not valid.
  - $5 + 5 + 2$ and $2 + 2 + 4 + 4$ are valid.
- **Problem.** What is the generating function of partitions that have no more than k identical parts?

$$\prod_{n=1}^{\infty}(1 + x^n + x^{2n} + x^{3n} + \cdots).$$

Restricted Partitions #1 (cont.)

- **Special case.** Taking $k = 1$, we get the generating function for $p(n \mid \text{each part is distinct})$:

$$(1 + x)(1 + x^2)(1 + x^3)\cdots$$

- What about the case of a general k?

$$\prod_{n=1}^{\infty}\left(1 + x^n + x^{2n} + \cdots + x^{kn}\right).$$

Restricted Partitions #2

- Consider partitions of n with only odd parts.
- For example, when $n = 12$:
  - $1 + 1 + 1 + \cdots + 1$, $3 + 3 + 3 + 3$, $11 + 1$, etc…
- **Problem.** What is the generating function of partitions with only odd parts?

$$(1-x)^{-1}(1-x^3)^{-1}(1-x^5)^{-1}\cdots$$
$$= \prod_{n=1}^{\infty}(1-x^{2n-1})^{-1}.$$

Restricted Partitions #3

- Consider partitions of n with only even parts.
- For example, when $n = 12$:
  - $10 + 2$, $2 + 2 + \cdots + 2$, $4 + 4 + 4$, etc…
- **Problem.** What is the generating function of partitions with only even parts?

$$(1-x^2)^{-1}(1-x^4)^{-1}(1-x^6)^{-1}\cdots$$
$$= \prod_{n=1}^{\infty}(1-x^{2n})^{-1}.$$

Restricted Partitions #4

- Consider partitions of n with each part equals to at most k.
- For example, when $n = 12$ and $k = 4$:
  - $5 + 5 + 2$ and $10 + 1 + 1$ are not valid.
- **Problem.** What is the generating function of partitions whose parts equal to at most k?

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\cdots\left(1-x^k\right)^{-1}$$
$$=\prod_{n=1}^{k}(1-x^n)^{-1}.$$

Happy Thanksgiving!