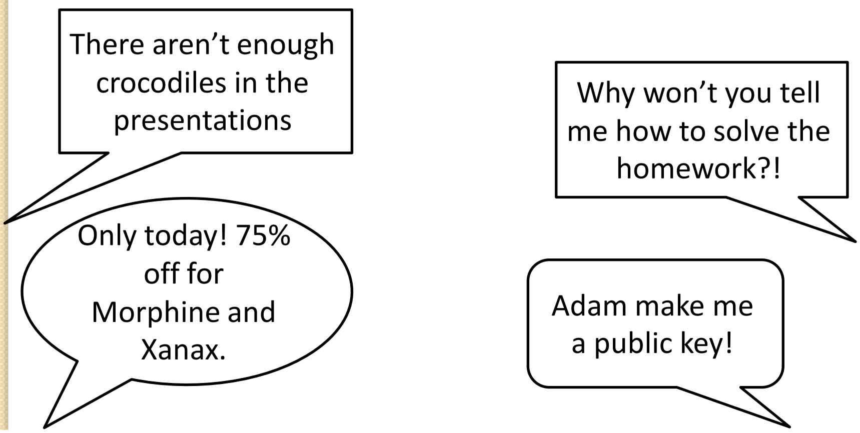
Ma/CS 6a

Class 4: Primality Testing

Is $2^{6972593} - 1$ prime?

By Adam Sheffer

- Send anonymous suggestions and complaints *from* here.
- Email: *adamcandobetter@gmail.com*
- Password: anonymous



There aren't enough crocodiles in the presentations

Why won't you tell me how to solve the homework?!

Only today! 75% off for Morphine and Xanax.

Adam make me a public key!

Reminder: Euler's Totient Function

- *Euler's totient $\varphi(n)$* is defined as follows: Given $n \in \mathbb{N}$, then

$$\varphi(n) = |\{x \mid 1 \leq x < n \ \text{ and GCD}(x, n) = 1\}|.$$

- In more words: $\varphi(n)$ is the number of natural numbers $1 \leq x \leq n$ such that x and n are coprime.

$$\varphi(12) = |\{1,5,7,11\}| = 4.$$

Reminder #2: The RSA Algorithm

- Bob wants to generate keys:
  - Arbitrarily chooses primes p and q. ❓
    $n = pq$ ✓ find $\varphi(n)$. ❓
  - Chooses e such that $\text{GCD}(\varphi(n), e) = 1$. ❓
  - Find d such that $de \equiv 1 \ mod \ \varphi(n)$. ❓
- Alice wants to pass bob m.
  - Receives n, e from Bob.
  - Returns $X \equiv m^e \ mod \ n$. ✓
- Bob receives X.

  Calculates $X^d \ mod \ n$. ✓

Finding $\varphi(n)$

- **Problem.** Given $n = pq$, where p, q are large primes, find $\varphi(n)$.
  - We need the number of elements in $\{1,2,\dots,n\}$ that are not multiplies of p or q.
  - There are $\frac{n}{p} = q$ numbers are divisible by p.
  - There are $\frac{n}{q} = p$ numbers are divisible by q.
  - Only $n = pq$ is divided by both.
  - Thus: $\varphi(n) = n - p - q + 1$.

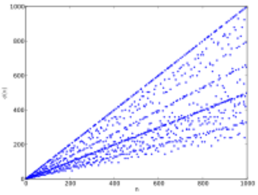
The RSA Algorithm

- Bob wants to generate keys:
  - Arbitrarily chooses primes p and q. ?
    $n = pq$ ✓ find $\varphi(n)$.✓
  - Chooses e such that $\mathrm{GCD}(\varphi(n), e) = 1$.?
  - Find d such that $de \equiv 1 \; mod \; \varphi(n)$. ?
- Alice wants to pass bob m.
  - Receives n, e from Bob.
  - Returns $X \equiv m^e \; mod \; n$. ✓
- Bob receives X.

  Calculates $X^d \; mod \; n$. ✓

Choose e s.t. $\mathrm{GCD}(\varphi(n), e) = 1$

- **Problem.** Given $n = pq$, where p, q are large primes, find $e \in \mathbb{N}$ such that $\mathrm{GCD}(\varphi(n), e) = 1$.
  - We can choose arbitrary numbers until we find one that is relatively prime to $\varphi(n)$.
  - For the “worst” values of $\varphi(n)$, a random number is good with probability $1/\log\log n$.



The RSA Algorithm

- Bob wants to generate keys:
  - Arbitrarily chooses primes p and q. ?
    $n = pq$ ✓ find $\varphi(n)$. ✓
  - Chooses e such that $\mathrm{GCD}(\varphi(n), e) = 1$. ✓
  - Find d such that $de \equiv 1 \; mod \; \varphi(n)$. ?
- Alice wants to pass bob m.
  - Receives n, e from Bob.
  - Returns $X \equiv m^e \; mod \; n$. ✓
- Bob receives X.

  Calculates $X^d \; mod \; n$. ✓

Find d such that $de \equiv 1 \ mod \ \varphi(n)$

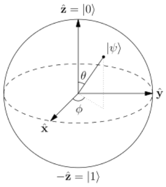
- **Recall.** Since $\mathrm{GCD}\big(e, \varphi(n)\big) = 1$ then there exist $s, t \in \mathbb{Z}$ such that
$$se + t\varphi(n) = 1.$$
- That is, $se \equiv 1 \ mod \ \varphi(n)$.
- We can find s, t by the ***extended Euclidean algorithm*** from lecture 2.

The RSA Algorithm

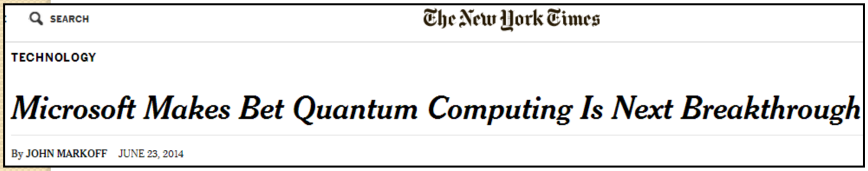
- Bob wants to generate keys:
  - Arbitrarily chooses primes p and q. ?
    $n = pq$ ✓ find $\varphi(n)$. ✓
  - Chooses e such that $\mathrm{GCD}(\varphi(n), e) = 1$. ✓
  - Find d such that $de \equiv 1 \ mod \ \varphi(n)$. ✓
- Alice wants to pass bob m.
  - Receives n, e from Bob.
  - Returns $X \equiv m^e \ mod \ n$. ✓
- Bob receives X.
    Calculates $X^d \ mod \ n$. ✓

Quantum Computing

- A *bit* of a computer contains a value of either 0 or 1.
- A quantum computer contains *qubits*, which can be in superpositions of states.
- Theoretically, a quantum computer can easily factor numbers and decipher almost any known encryption.



Should We Stop Ordering Things Online?

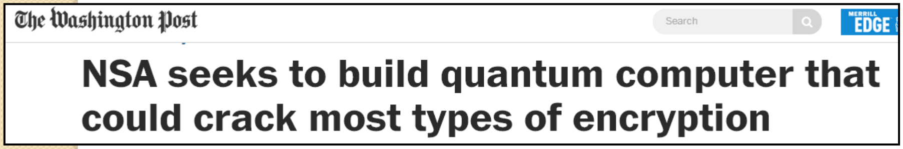


The New York Times

TECHNOLOGY

Microsoft Makes Bet Quantum Computing Is Next Breakthrough

By JOHN MARKOFF JUNE 23, 2014



The Washington Post

NSA seeks to build quantum computer that could crack most types of encryption

Finding Large Primes

- Let n be a LARGE integer (e.g., 2^{4000}).
- *The prime number theorem.* The probability of a random $p \in \{1, \dots, n\}$ being prime is about $1/\log n$.
- If we randomly choose numbers from $\{1, \dots, n\}$, we expect to have about $\log n$ iterations before finding a prime.
  - *But how can we check whether our choice is a prime or not?!*

Primality Testing

- Given a LARGE $q \in \mathbb{Z}$, how can we check whether q is prime?
- **The naïve approach.** Go over every number in $\{2, \dots, \sqrt{q}\}$ and check whether it divides q.
  - *But we chose our numbers to be too large for a computer to go over all of them!*

Recall: Fermat’s Little Theorem

- For any prime p and integer a relatively prime to p, we have

$$a^p \equiv a \; mod \; p.$$

- Pick a random integer a and check whether $a^q \equiv a \; mod \; q$.
  - If not, q is not a prime!
  - If yes, ???

Pierre de Fermat

Example: Fermat Primality Testing

- Is $n = 355207$ prime?

$$2^{355207} \equiv 84927 \; mod \; 355207.$$

- n is not prime since $2^n \not\equiv 2 \; mod \; n.$

- We can try 1000 different values of a and see if $a^n \equiv a \; mod \; n$ for each of them.

Carmichael Numbers

- A number $q \in \mathbb{N}$ is said to be a *Carmichael number* if it is not prime, but still satisfies $a^q \equiv a \; mod \; q$ for every a that is relatively prime to q.
  - The smallest such number is 561.
  - Very rare – about one in 50 trillion in the range 1 – 10^{21}.

R. D. Carmichael

Example: Carmichael Numbers

- **Claim.** Let $k \in \mathbb{N} \setminus \{0\}$ such that $6k + 1, 12k + 1$, and $18k + 1$ are primes. Then

$$n = (6k + 1)(12k + 1)(18k + 1)$$

is a Carmichael number.

- **Example.**
  - For $k = 1$, we have that $7,13,19$ are primes.
  - $7 \cdot 13 \cdot 19 = 1729$ is a Carmichael number.

Proof

- We need to prove that for any a that is relatively prime to n, we have
$$a^n \equiv a \mod n.$$
- **Recall.** Since $GCD(a, n) = 1$, this is equivalent to $a^{n-1} \equiv 1 \mod n$.
- We rewrite $n = 1296k^3 + 396k^2 + 36k + 1$.
- For any such a, we have
$$\begin{aligned} a^{n-1} &= a^{1296k^3+396k^2+36k} \\ &= \left(a^{6k}\right)^{216k^2+66k+6}. \end{aligned}$$

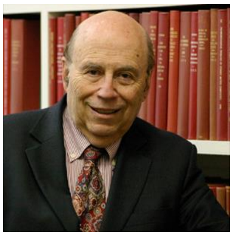
Proof (cont.)

- For any a relatively prime to n, we have
$$a^{n-1} = \left(a^{6k}\right)^{216k^2+66k+6}.$$
- **Recall.** If $a \in \mathbb{N}$ is not divisible by a prime p then $a^{p-1} \equiv 1 \mod p$.
- Since a and $6k - 1$ are relatively prime
$$a^{n-1} \equiv 1^{216k^2+66k+6} \equiv 1 \mod 6k + 1.$$
- Similarly, we have $a^{n-1} \equiv 1 \mod 12k + 1$ and $a^{n-1} \equiv 1 \mod 18k + 1$.
- Since $a^{n-1} - 1$ is divisible by the three pairwise coprime integers $6k + 1, 12k + 1$, and $18k + 1$, it is also divisible by their product n. That is, $a^{n-1} \equiv 1 \mod n$.

Miller–Rabin Primality Test

- The *Miller–Rabin primality test* works on *every number*.

Gary Miller



Michael Rabin

Root of Unity

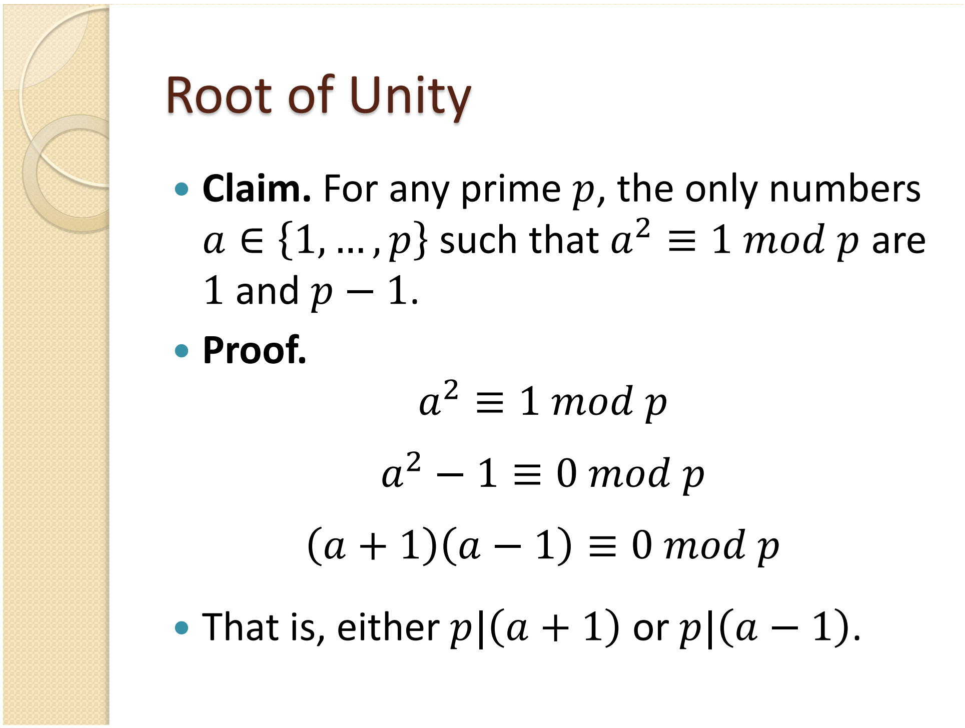
- **Claim.** For any prime p, the only numbers $a \in \{1, \dots, p\}$ such that $a^2 \equiv 1 \; mod \; p$ are 1 and $p - 1$.

- **Example.** The solutions to

$$a^2 \equiv 1 \; mod \; 1009$$

are exactly the numbers satisfying

$$a \equiv 1 \text{ or } 1008 \; mod \; 1009.$$



Root of Unity

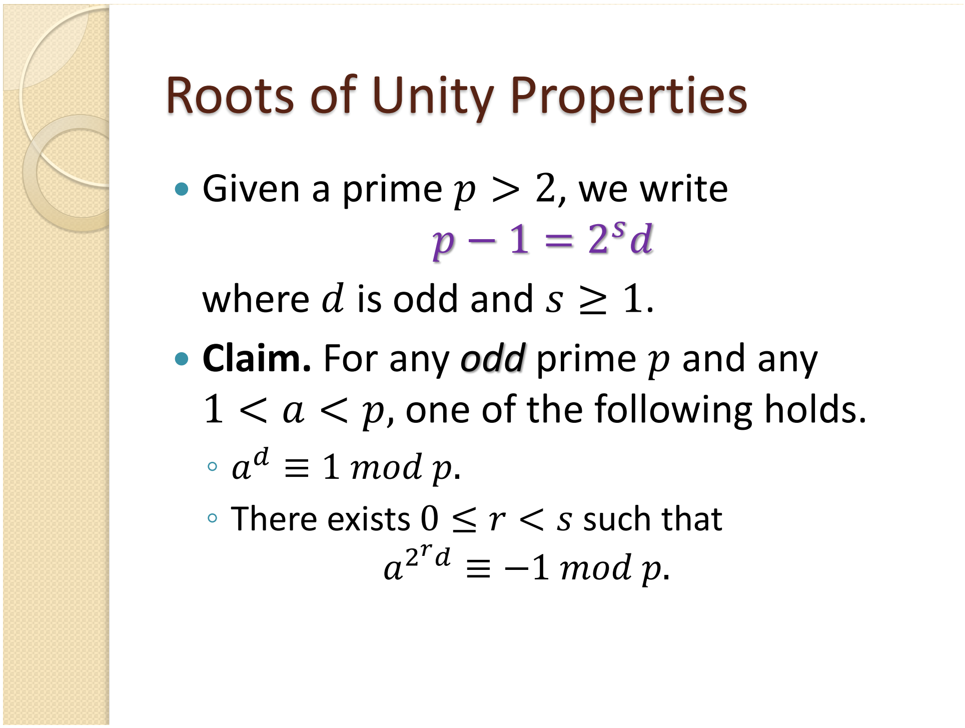
- **Claim.** For any prime p, the only numbers $a \in \{1, \dots, p\}$ such that $a^2 \equiv 1 \; mod \; p$ are 1 and $p - 1$.
- **Proof.**

$$a^2 \equiv 1 \; mod \; p$$

$$a^2 - 1 \equiv 0 \; mod \; p$$

$$(a + 1)(a - 1) \equiv 0 \; mod \; p$$

- That is, either $p|(a + 1)$ or $p|(a - 1)$.



Roots of Unity Properties

- Given a prime $p > 2$, we write

$$p - 1 = 2^s d$$

where d is odd and $s \geq 1$.

- **Claim.** For any ***odd*** prime p and any $1 < a < p$, one of the following holds.
  - $a^d \equiv 1 \; mod \; p$.
  - There exists $0 \leq r < s$ such that

$$a^{2^r d} \equiv -1 \; mod \; p.$$

Roots of Unity Properties (2)

- **Claim.** For any ***odd*** prime p and any $1 < a < p$, one of the following holds.
  - $a^d \equiv 1 \, mod \, p$.
  - There exists $0 \leq r < s$ such that $a^{2^r d} \equiv -1 \, mod \, p$.
- **Proof.**
  - By Fermat's little theorem $a^{p-1} \equiv 1 \, mod \, p$.
  - Consider $a^{(n-1)/2}, a^{(n-1)/4}, \ldots, a^{(n-1)/2^s}$. By the previous claim, each such root is $\pm 1 \, mod \, n$.
  - If all of these roots equal 1, we are in the first case. Otherwise, we are in the second.

Composite Witnesses

- Given a composite (non-prime) ***odd*** number n, we again write $n - 1 = 2^s d$ where d is odd and $s \geq 1$.
- We say that $a \in \{2,3,4, \ldots, n-2\}$ is a ***witness*** for n if
  - $a^d \not\equiv 1 \, mod \, p$.
  - For every $0 \leq r < s$, we have $a^{2^r d} \not\equiv -1 \, mod \, p$.

Example: Composite Witness

- **Problem.** Prove that 91 is not a prime.

$$90 = 2 \cdot 45.$$

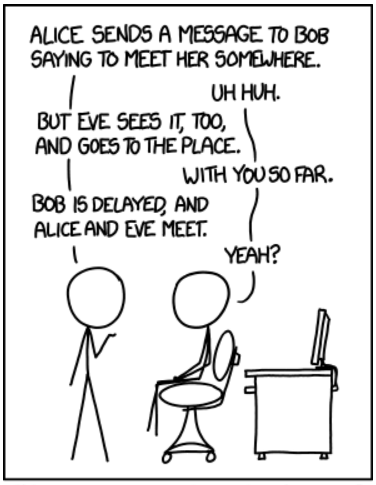
$$2^{45} \equiv 57 \; mod \; 91.$$

- 2 is a witness that 91 is not a prime.

There are Many Witnsses

- Given an odd composite n, the probability of a number $\{2, \dots, n-2\}$ being a witness is at least ¾.
- Given an odd $n \in \mathbb{N}$, take i numbers and check if they are witnesses.
  - If we found a witness, n is composite.
  - If we did not find a witness, n is prime with probability at least

The End


ALICE SENDS A MESSAGE TO BOB SAYING TO MEET HER SOMEWHERE.
UH HUH.
BUT EVE SEES IT, TOO, AND GOES TO THE PLACE.
WITH YOU SO FAR.
BOB IS DELAYED, AND ALICE AND EVE MEET.
YEAH?

I'VE DISCOVERED A WAY TO GET COMPUTER SCIENTISTS TO LISTEN TO ANY BORING STORY.