Ma/CS 6a

Class 20: Subgroups, Orbits, and Stabilizers

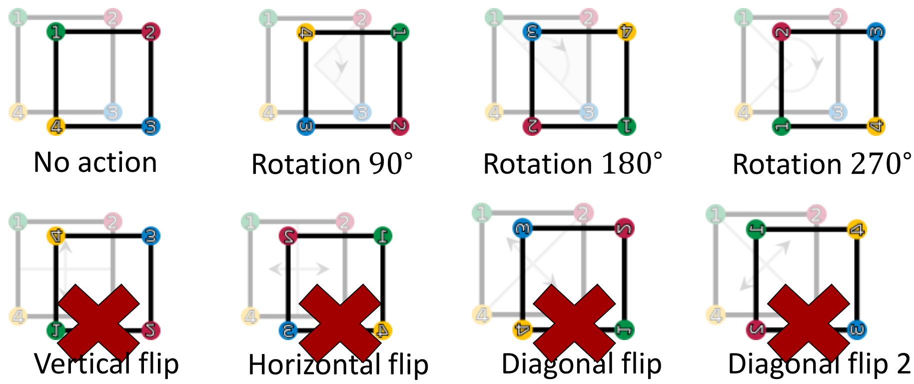
By Adam Sheffer

A Group

- A group consists of a set G and a binary operation $*$, satisfying the following.
  - **Closure.** For every $x, y \in G$, we have $x * y \in G$.
  - **Associativity.** For every $x, y, z \in G$, we have
    $$(x * y) * z = x * (y * z).$$
  - **Identity.** The exists $e \in G$, such that for every $x \in G$, we have
    $$e * x = x * e = x.$$
  - **Inverse.** For every $x \in G$ there exists $x^{-1} \in G$ such that $x * x^{-1} = x^{-1} * x = e$.

Reminder: Subgroups

- A *subgroup* of a group G is a group with the same operation as G, and whose set of members is a subset of G.



Lagrange's Theorem

- **Theorem.** If G is a group of a finite order n and H is a subgroup of G of order m, then $m|n$.
  - We will not prove the theorem.

- **Example.** The symmetry group of the square is of order 8.
  - The subgroup of rotations is of order 4.
  - The subgroup of the identity and rotation by $180°$ is of order 2.

Reminder: Parity of a Permutation

- **Theorem.** Consider a permutation $\alpha \in S_n$. Then
  - Either every decomposition of α consists of an even number of transpositions,
  - or every decomposition of α consists of an odd number of transpositions.

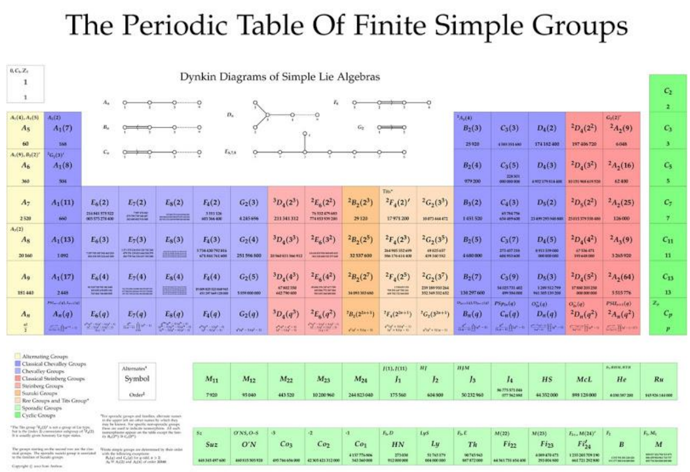
- $(1\ 2\ 3)(4\ 5\ 6)$:
  - $(1\ 3)(1\ 2)(4\ 6)(4\ 5)$.
  - $(1\ 4)(1\ 6)(1\ 5)(3\ 4)(2\ 4)(1\ 4)$.

Subgroup of Even Permutations

- Consider the group S_n:
  - **Recall.** A product of two even permutations is even.
  - The subset of even permutations is a subgroup. It is called the ***alternating group*** A_n.
  - **Recall.** Exactly half of the permutations of S_n are even. That is, the order of A_n is half the order of S_n.

Atlas of Finite Groups

- (only in class)



Application of Lagrange's Theorem

- **Problem.** Let G be a finite group of order n and let $g \in G$ be of order m. Prove that
$$m|n \qquad \text{and} \qquad g^n = 1.$$
- **Proof.**
  - Notice that $\{1, g, g^2, \dots, g^{m-1}\}$ is a cyclic subgroup of order m.
  - By Lagrange's theorem $m|n$.
  - Write $n = mk$ for some integer k. Then
$$g^n = g^{mk} = (g^m)^k = 1.$$

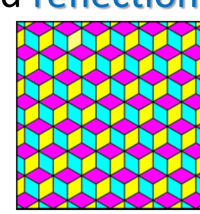
Groups of a Prime Order

- **Claim.** Every group G of a prime order p is isomorphic to the cyclic group C_p.

- Proof.
  - By Lagrange's theorem, G has no subgroups.
  - Thus, by the previous slide, every element of $G \setminus \{1\}$ is of order p.
  - G is cyclic since any element of $G \setminus \{1\}$ generates it.

Symmetries of a Tiling

- Given a repetitive tiling of the plane, its symmetries are the transformations of the plane that
  - Map the tiling to itself (ignoring colors).
  - Preserve distances.
- These are combinations of translations, rotations, and reflections.


Example: Square Tiling

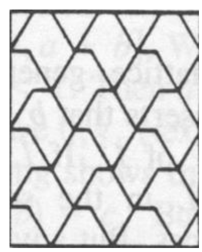
- What symmetries does the square tiling has?
  - Translations in every direction.
  - Rotations around a vertex by $0°, 90°, 180°, 270°$.
  - Rotations around the center of a square by $0°, 90°, 180°, 270°$.
  - Reflections across vertical, horizontal and diagonal lines.
  - Rotations around the center of an edge by $180°$.

Wallpaper Groups

- Given a tiling, its set of symmetries is a group called a *wallpaper group* (not accurate! More technical conditions).
  - **Closure.** Composing two symmetries results in a transformation that preserves distances and takes the lattice to itself.
  - **Associativity.** Holds.
  - **Identity.** The “no operation” element.
  - **Inverse.** Since symmetries are bijections from the plane to itself, inverses are well defined.

Wallpaper Groups

- There are exactly 17 different wallpaper groups.
- That is, the set of all repetitive tilings of the plane can be divided into 17 classes. Two tilings of the same class have the same "behavior".


Equivalence Relations

- **Recall.** A relation R on a set X is an *equivalence relation* if it satisfies the following properties.
  - **Reflexive.** For any $x \in X$, we have xRx.
  - **Symmetric.** For any $x, y \in X$, we have xRy if and only if yRx.
  - **Transitive.** If xRy and yRz then xRz.

Example: Equivalence Relations

- **Problem.** Consider the relation of congruence $mod\ 31$, defined over the set of integers $\mathbb{Z}$. Is it an equivalence relation?
- **Solution.**
  - **Reflexive.** For any $x \in \mathbb{Z}$, we have
    $$x \equiv x\ mod\ 31.$$
  - **Symmetric.** For any $x, y \in \mathbb{Z}$, we have $x \equiv y\ mod\ 31$ iff $y \equiv x\ mod\ 31$.
  - **Transitive.** If $x \equiv y\ mod\ 31$ and $y \equiv z\ mod\ 31$ then $x \equiv z\ mod\ 31$.

Equivalence Via Permutation Groups

- Let G be a group of permutations of the set X. We define a relation on X:
  $$x \sim y \quad \Leftrightarrow \quad g(x) = y \text{ for some } g \in G.$$
- **Claim.** $\sim$ is an equivalence relation.
  - **Reflexive.** The group G contains the identity permutation id. For every $x \in X$ we have $\mathrm{id}(x) = x$ and thus $x \sim x$.
  - **Symmetric.** If $x \sim y$ then $g(x) = y$ for some $g \in G$. This implies that $g^{-1} \in G$ and $x = g^{-1}(y)$. So $y \sim x$.

Equivalence Via Permutation Groups

- Let G be a group of permutations of the set X. We define a relation on X:
  $x{\sim}y \quad \Leftrightarrow \quad g(x) = y \text{ for some } g \in G.$
- **Claim.** $\sim$ is an equivalence relation.
  - **Transitive.** If $x{\sim}y$ and $y{\sim}z$ then $g(x) = y$ and $h(y) = z$ for $g, h \in G$. Then $hg \in G$ and $hg(x) = z$, which in turn implies $x{\sim}z$.

Orbits

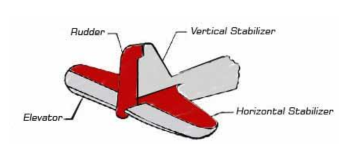
- Given a permutation group G of a set X, the equivalence relation $\sim$ partitions X into ***equivalence classes*** or ***orbits***.
  - For every $x \in X$ the orbit of x is
    $\boldsymbol{Gx} = \{y \in X \mid x{\sim}y\}$
    $\qquad = \{y \in X \mid g(x) = y \text{ for some } g \in G\}.$

Example: Orbits

- Let $X = \{1,2,3,4,5\}$ and let
  $G = \{\mathrm{id}, (1\ 2), (3\ 4), (1\ 2)(3\ 4)\}$.
- What are the equivalence classes that G induces on X?
  - $G1 = G2 = \{1,2\}$.
  - $G3 = G4 = \{3,4\}$.
  - $G5 = \{5\}$.

Stabilizers

- Let G be a permutation group of the set X.
- Let $G(x \to y)$ denote the set of permutations $g \in G$ such that $g(x) = y$.
- The ***stabilizer of x*** is
  $$G_x = G(x \to x).$$



Example: Stabilizer

- Consider the following permutation group of $\{1,2,3,4\}$:

$$G = \{\mathrm{id}, (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 3\ 2),$$
$$(2\ 4), (1\ 3), (1\ 2)(3\ 4), (1\ 4)(2\ 3)\}.$$

- The stabilizers are
  - $G_1 = \{\mathrm{id}, (2\ 4)\}$.
  - $G_2 = \{\mathrm{id}, (1\ 3)\}$.
  - $G_3 = \{\mathrm{id}, (2\ 4)\}$.
  - $G_4 = \{\mathrm{id}, (1\ 3)\}$.

Stabilizers are Subgroups

- **Claim.** G_x is a subgroup of G.
  - **Closure.** If $g, h \in G_x$ then $g(x) = x$ and $h(x) = x$. Since $gh(x) = x$ we have $gh \in G_x$.
  - **Associativity.** Implied by the associativity of G.
  - **Identity.** Since $\mathrm{id}(x) = x$, we have $id \in G_x$.
  - **Inverse.** If $g \in G_x$ then $g(x) = x$. This implies that $g^{-1}(x) = x$ so $g^{-1} \in G_x$.

Cosets

- Let H be a subgroup of the group G. The *left coset* of H with respect to $g \in G$ is
$gH = \{a \in G \mid a = gh \ \text{ for some } \ h \in H\}$.

- **Example.** The coset of the alternating group A_n with respect to a transposition $(x\ y) \in S_n$ is the subset of odd permutations of S_n.

$G(x \to y)$ are Cosets

- **Claim.** Let G be a permutation group and let $h \in G(x \to y)$. Then
$$G(x \to y) = hG_x.$$

- **Proof.**
  - $hG_x \subseteq G(x \to y)$. If $a \in hG_x$, then $a = hg$ for some $g \in G_x$. We have $a \in G(x \to y)$ since
$$a(x) = hg(x) = h(x) = y.$$
  - $G(x \to y) \subseteq hG_x$. If $b \in G(x \to y)$ then
$$h^{-1}b(x) = h^{-1}(y) = x.$$
That is, $h^{-1}b \in G_x$, which implies $b \in hG_x$.

Sizes of Cosets and Stabilizers

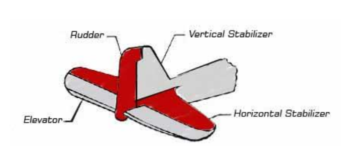
- **Claim.** Let G be a permutation group on X and let G_x be the stabilizer of $x \in X$. Then
$$|G_x| = |hG_x| \text{ for any } h \in G.$$
  - **Proof.** By the Latin square property of G.

- **Corollary.** The size of $G(x \to y)$:
  - If y is in the orbit Gx then
$$|G(x \to y)| = |G_x|.$$
  - If y is **not** in the orbit Gx then
$$|G(x \to y)| = 0.$$

Sizes of Orbits and Stabilizers

- **Theorem.** Let G be a group of permutations of the set X. For every $x \in X$ we have
$$|Gx| \cdot |G_x| = |G|.$$

The orbit of x *The stabilizer of x*



Example: Orbits and Stabilizers

- Consider the following permutation group of $\{1,2,3,4\}$:

$$G = \{\mathrm{id}, (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 3\ 2), (2\ 4), (1\ 3), (1\ 2)(3\ 4), (1\ 4)(2\ 3)\}.$$

  - We have $|G| = 8$.
  - We have the orbit $G1 = \{1,2,3,4\}$. So $|G1| = 4$.
  - We have the stabilizer $G_1 = \{\mathrm{id}, (2\ 4)\}$. So $|G_1| = 2$.
  - Combining the above yields

$$|G| = 8 = |G1| \cdot |G_1|.$$

A Useful Table

- Let $G = \{g_1, g_2, \dots, g_n\}$ be a group of permutations of $X = \{x_1, x_2, \dots, x_m\}$.
  - For an element $x \in X$, we build the following table, where ✓ implies that $g_i(x) = x_j$.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	…	x_m
g_1	✓								
g_2			✓						
g_3									✓
…									
g_n			✓						

Table Properties 1

- How many ✓'s are in the table?
  - Since $g_i(x)$ has a unique value, each row contains exactly one ✓.
  - The total number of ✓'s in the table is $|G|$.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	…	x_m
g_1	✓								
g_2			✓						
g_3									✓
…									
g_n			✓						

Table Properties 2

- How many ✓'s are in the column of x_i?
  - If x_i is not in the orbit Gx, then 0.
  - If x_i is in the orbit Gx, then
    $$|G(x \to y)| = |G_x|.$$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	…	x_m
g_1	✓								
g_2			✓						
g_3									✓
…									
g_n			✓						

Proving the Theorem

- **Theorem.** Let G be a group of permutations of the set X. For every $x \in X$ we have
$$|Gx| \cdot |G_x| = |G|.$$
- **Proof.**
  - Counting by rows, the number of ✓'s in the table is $|G|$.
  - Counting by columns, there are $|Gx|$ non-empty columns, each containing $|G_x|$ ✓'s.
  - That is, $|G| = |Gx| \cdot |G_x|$.

Double Counting

- Our proof technique was to count the same value (the number of ✓'s in the table) in two different ways.
- This technique is called *double counting* and is very useful in combinatorics.

The End: Alhambra

- *Alhmbra* is a palace and fortress complex located in Granada, Spain.
  - The Islamic art on the walls is claimed to contain all 17 wallpaper groups.
  - Mathematicians like to visit the palace and look for as many types as they can find.