Ma/CS 6a

Class 13: Network Flow

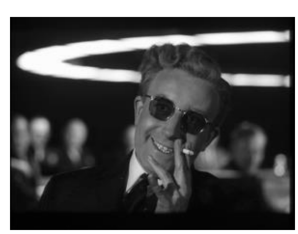
By Adam Sheffer

The RAND Corporation

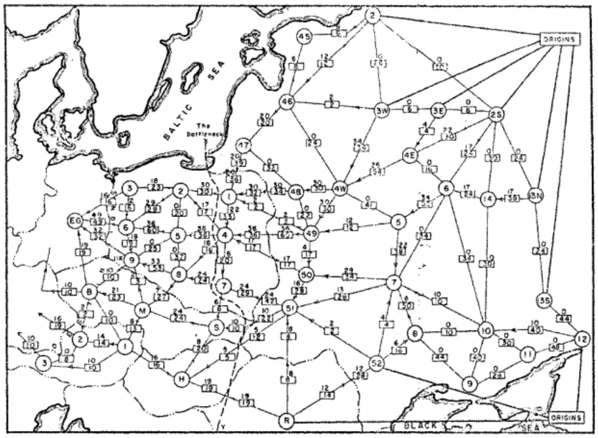
- American think tank composed of scientists.
- In the 50's helped decision concerning the nuclear race, space program, etc.
- Contributed to the development of many scientific techniques, such as game theory.

John Nash

John von Neumann



Dr. Strangelove



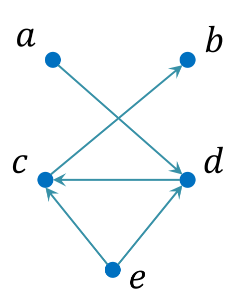
- A document that was declassified in 1999 describes how RAND studied the Soviet train system.
- They studied the Soviet ability to transport things from place to place (e.g., Asian side to European side).

Recall: Directed Graphs

- In a ***directed graph*** (or ***digraph***) every edge has a direction.
- A ***directed path*** is a path that follows the direction of the edges.

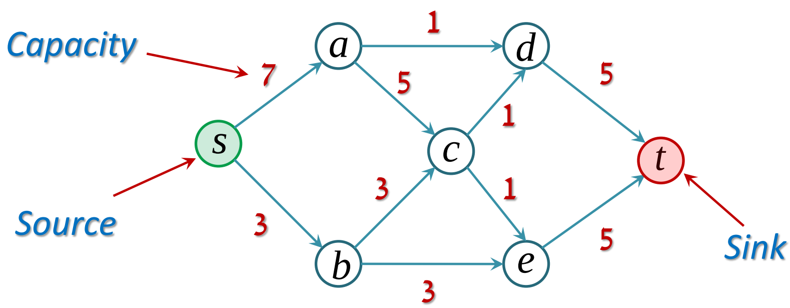
✓ $a \rightarrow d \rightarrow c \rightarrow b$

✗ $a \rightarrow d \rightarrow e \rightarrow c \rightarrow b$



Flow Networks

- A *flow network* is a digraph $G = (V, E)$, together with a *source* vertex $s \in V$, a *sink* vertex $t \in V$, and a *capacity function* $c: E \to \mathbb{N}$.



Flow in a Network

- Given a flow network $G = (V, E, s, t, c)$, a *flow* in G is a function $f: E \to \mathbb{N}$ that satisfies
  - Every $e \in E$ satisfies $f(e) \leq c(e)$.
  - Every $v \in V \setminus \{s, t\}$ satisfies

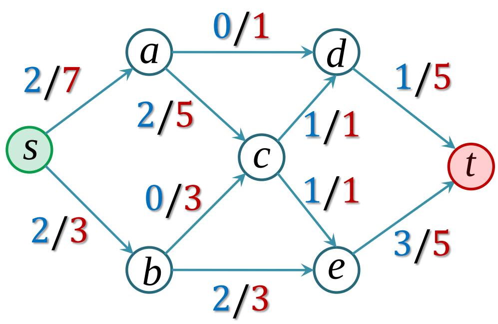
$$\sum_{(u,v) \in E} f(u, v) = \sum_{(v,w) \in E} f(v, w)$$

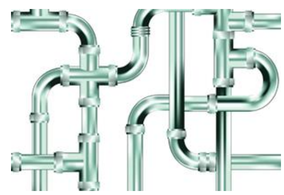
Total flow entering v.

Total flow exiting v.

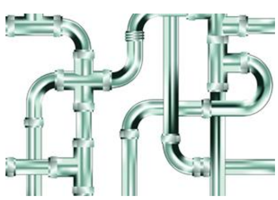
Example: Flow

- The capacities are in red.
- The flow is in blue.

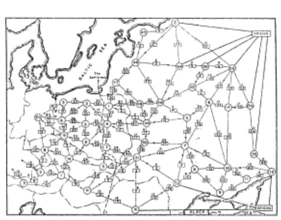




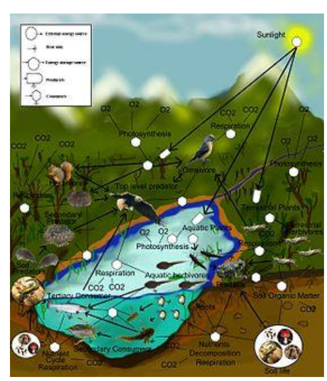
What is it Good For?



Pipe Network analysis



Transportation networks

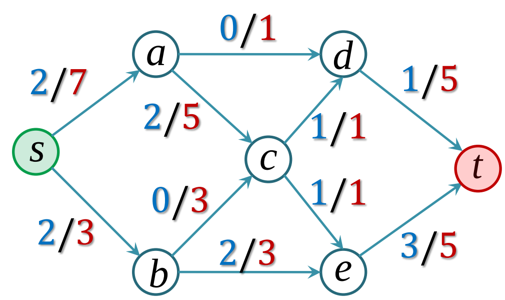


Ecological food webs

Size of a Flow

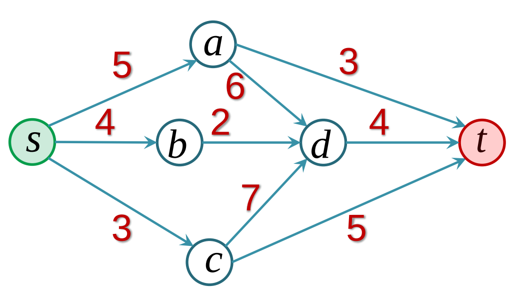
- In any flow f, the flow coming out of s is equal to the flow getting into t, since nothing is allowed to accumulate at intermediate vertices.
  - We refer to this quantity as the size of f (or $|f|$).

$\boxed{|f| = 4}$



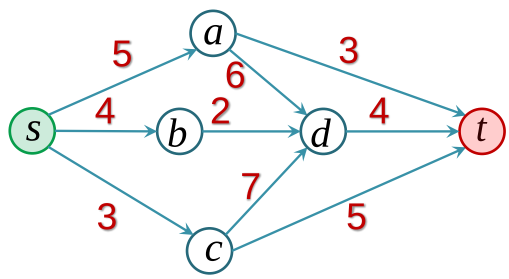
Maximum Flow

- We are usually interested in finding the *maximum flow* of a network.
- We wish to have an algorithm that receives a flow network, and finds a maximum flow of it.



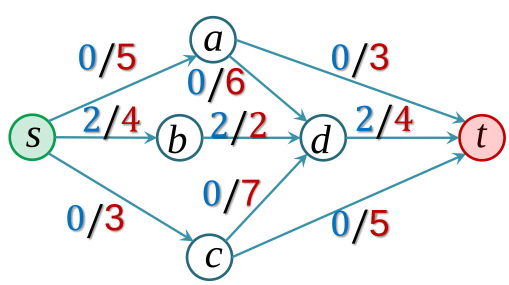
An Initial Flow

- Find a path from s to t.
  - For example, the path $s \to b \to d \to t$.
  - The flow that can pass in it is the minimum edge capacity – 2.



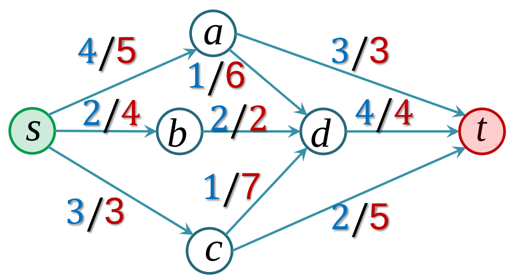
Increasing the Flow

- Find another path from s to t, among the edges that are not ***saturated*** $(f(e) \neq c(e))$.
  - For example, the path $s \to a \to t$.
  - The flow that can pass in it is the minimum edge capacity – 3.



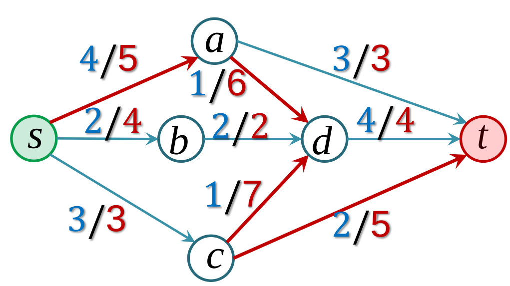
Maximum Flow?

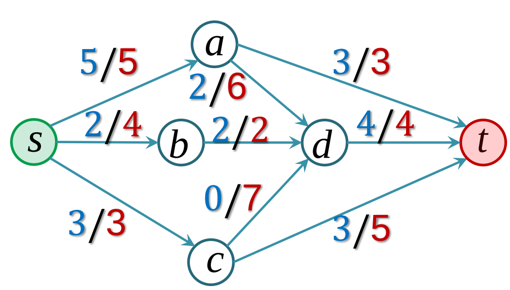
- We keep increasing the flow until there are no more paths of unsaturated edges.
- Does that mean that we have a maximum flow?
  - No! This network has a flow of size 10.



A Different Kind of Path

- We can go in the opposite direction of an edge e, if $f(e) > 0$.
- $s \to a \to d \to c \to t$.



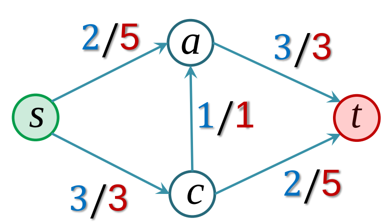


Residual Network

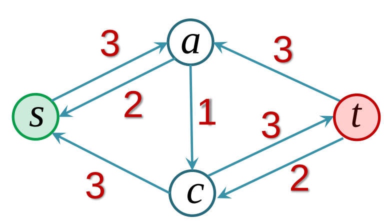
- Given a flow network (V, E, s, t, c), and a flow f, the *residual network* of f is a network with
  - The same set of vertices V, source s, and sink t.
  - Every edge $e \in E$ has the new capacity $c'(e) = c(e) - f(e)$.
  - For every edge $(u, v) \in E$, we add the edge (v, u) with capacity $f(e)$.

Example: Residual Network

A network and a flow



The residual network

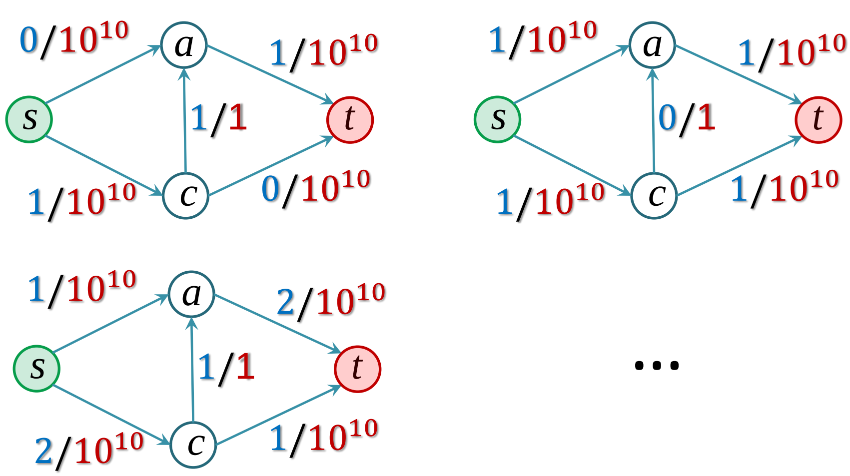


The Ford-Fulkerson Algorithm

- Start with a flow f of size zero.
- Repeat:
  - Build the residual network of f.
  - Find a path P from s to t in the residual network. If there is no such path, stop and return f.
  - Set $d = \min_{e \in P} c(e)$.
  - Increase the flow in f of every edge in P by d (for reverse edges, decrease the flow by d).

Not Efficient

- The algorithm can have a VERY large running time:



The Running Time

- What is an upper bound for the number of steps of the Ford-Fulkerson algorithm?
  - The size of the maximum flow $|f|$.
  - Running time of the algorithm: at most $c|f|(|V| + |E|)$.
- We can improve the running time by using BFS to find a path in every residual network.
  - This gives a running time of $c|V||E|^2$.
  - Not in the material of this course!

#	Year	Discoverer(s)	Exact	Ballpark	Reference
1	1951	Dantzig	$O(n^2mU)$	n^2mU	[Dantzig 1951]
2	1955	Ford & Fulkerson	$O(nmU)$	nmU	[Ford and Fulkerson 1956]
3	1970	Dinitz Edmonds & Karp	$O(nm^2)$	nm^2	[Dinitz 1970] [Edmonds and Karp 1972]
4	1970	Dinitz	$O(n^2m)$	n^2m	[Dinitz 1970]
5	1972	Edmonds & Karp Dinitz	$O(m^2 \log U)$	m^2	[Edmonds and Karp 1972] [Dinitz 1973]
6	1973	Dinitz Gabow	$O(nm \log U)$	nm	[Dinitz 1973] [Gabow 1985]
7	1974	Karzanov	$O(n^3)$		[Karzanov 1974]
8	1977	Cherkassky	$O(n^2\sqrt{m})$		[Cherkassky 1977]
9	1980	Galil & Naamad	$O(nm \log^2 n)$		[Galil and Naamad 1980]
10	1983	Sleator & Tarjan	$O(nm \log n)$		[Sleator and Tarjan 1983]
11	1986	Goldberg & Tarjan	$O(nm \log(n^2/m))$		[Goldberg and Tarjan 1988]
12	1987	Ahuja & Orlin	$O(nm + n^2 \log U)$		[Ahuja and Orlin 1989]
13	1987	Ahuja et al.	$O(nm \log(n\sqrt{\log U}/(m + 2))$		[Ahuja et al. 1989]
14	1989	Cheriyan & Hagerup	$E(nm + n^2 \log^2 n)$		[Cheriyan and Hagerup 1995]
15	1990	Cheriyan et al.	$O(n^3/\log n)$		[Cheriyan et al. 1996]
16	1990	Alon	$O(nm + n^{8/3} \log n)$		[Alon 1990]
17	1992	King et al.	$O(nm + n^{2+\epsilon})$		[King et al. 1992]
18	1993	Phillips & Westbrook	$O(nm(\log_{m/n} n + \log^{2+\epsilon} n))$		[Phillips and Westbrook 1993]
19	1994	King et al.	$O(nm \log_{m/(n \log n)} n)$		[King et al. 1994]
20	1998	Goldberg & Rao	$O(m^{3/2} \log(n^2/m) \log U)$ $O(n^{2/3}m \log(n^2/m) \log U)$	$m^{3/2}$ $n^{2/3}m$	this paper this paper

A Recent Development

- In 2012, an algorithm with a running time of $c|V||E|$ was discovered.
- Combining the results of two groups:

J. Orlin

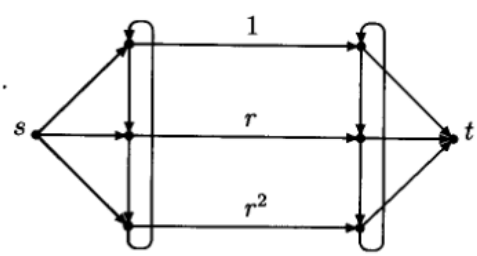
V. King

S. Rao

R. Tarjan

Irrational Capacities

$$r = \frac{-1+\sqrt{5}}{2} \approx .618\ldots$$

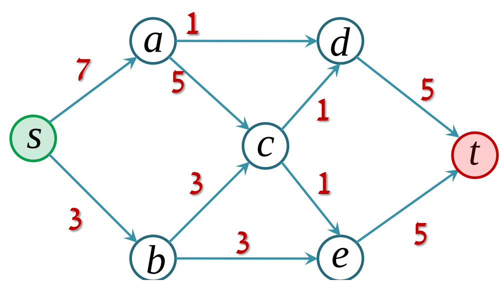


- The missing capacities are all $r + 2$.
- Even though the size of the maximum flow is 2, Ford-Fulkerson might run forever!

Bounding the Size of the Maximum Flow

- Given a flow network, what is an easy upper bound for the size of the maximum flow?

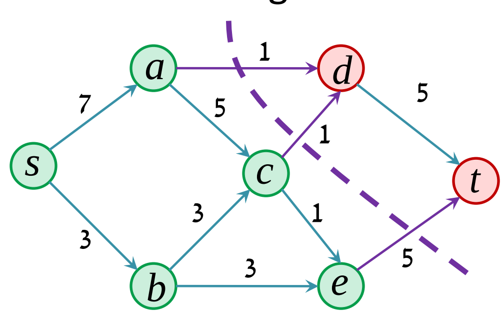
$$|F| \leq \min\left\{\sum_{(s,v)\in E} c(s,v)\,,\ \sum_{(u,t)\in E} c(u,t)\right\}.$$



$\boxed{|f| \leq 10}$

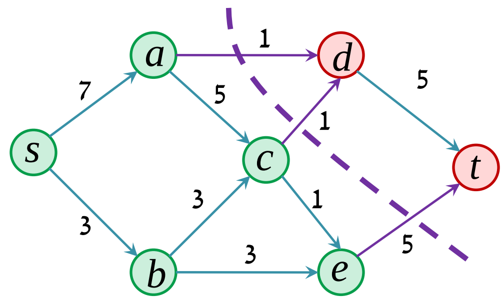
Cuts

- A *cut* is a partitioning of the vertices of the flow network into two sets S, T such that $s \in S$ and $t \in T$.
- The *size of a cut* is the sum of the capacities of the edges from S to T.



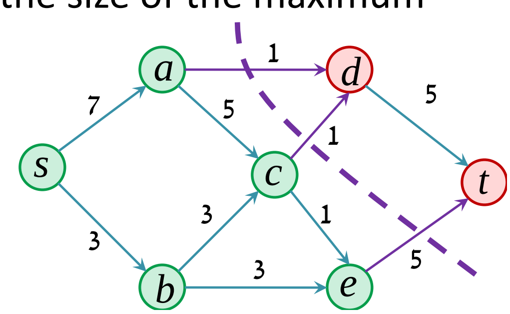
Cuts and Maximum Flow

- If there exists a cut of size d, no flow can be of size more than d.
  - Each part of the flow must pass through the cut.
  - At most d can pass through the cut.



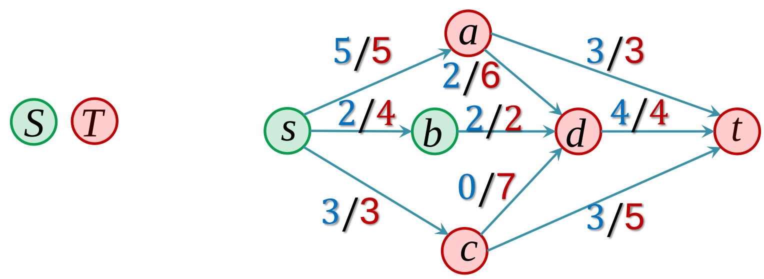
Max Flow – Min Cut

- We saw that the size of the *minimum cut* $\geq$ size of maximum flow.
- **Max flow – min cut theorem.** In every flow network, the size of the minimum cut *is equal* to the size of the maximum flow.



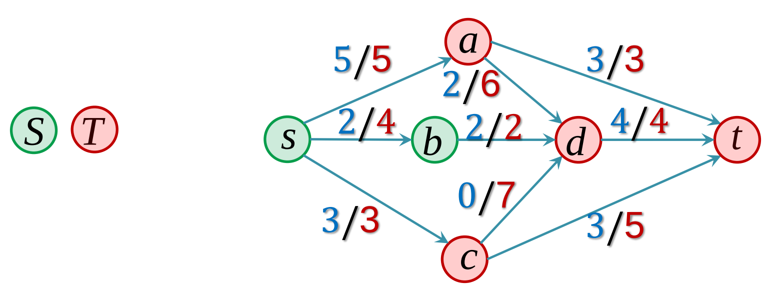
Proof: Max Flow – Min Cut

- Let f be a maximum flow of the network.
- In the residual network R of f, there is no path from s to t.
- Let S be the set of vertices that are accessible from s in R.
- Let $T = V \setminus S$. Then (S, T) is a cut.



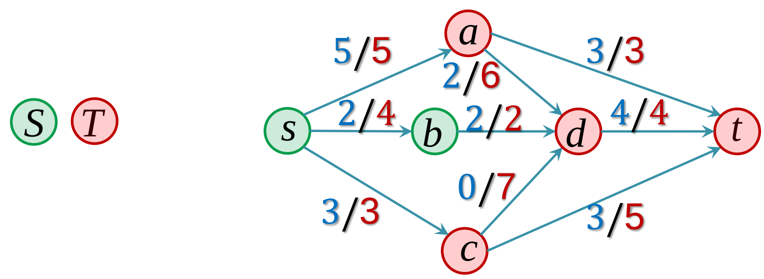
The Cut (S, T): Claim 1

- **Claim 1.** An edge (u, v) such that $u \in S$ and $v \in T$ must be saturated in f.
  - Since $u \in S$, there is a path from s to u in R.
  - If (u, v) is not saturated, there is a path from s to v in R. Contradicting that $v \in T$!



The Cut (S,T): Claim 2

- **Claim 2.** An edge (u,v) such that $u \in T$ and $v \in S$ cannot have flow through it in f.
  - Since $v \in S$, there is a path from s to v in R.
  - If $f(u,v) > 0$, there is a path from s to u in R. Contradicting that $u \in T$.



Completing the Proof

- f - maximum flow.
- (S,T) - a cut defined according to f.
  - Every edge from S to T is saturated in f.
  - Every edge e from T to S satisfies $f(e) = 0$.
- Thus, the size of the cut (S,T) is $|f|$.
- Thus, the minimum cut has size at most $|f|$.
- That is, the minimum cut is of size $\leq$ max flow.

Ford-Fulkerson Correctness

- **Claim.** A flow f is a maximum flow if and only if there are no s-t paths in the residual network R of f.

- ⇨: If there is an s-t path in R, we can increase f, so f is not a maximum flow.
- ⇦: If there are no s-t paths in R, we can define a cut of size $|f|$ as before. This implies that $|f|$ is the size of the min cut, so f is a maximum flow.

The End

- In the Vietnam war, the Vietcong used a series of underground tunnels called the ***Ho Chi Minh trail***.
- The US was looking for the ***min cut*** to efficiently disconnect the south and the north parts of the system.
- The capacity of an edge is the difficulty of destroying it.