Ma/CS 6a

Class 26: More Partitions

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Shameless Advertising

- Where can you see more discrete mathematics and more of Adam?
  - Winter quarter. Ma/CS 6b. More graphs, more algorithms, etc.
- For the more ambitious.
  - Spring quarter. A graduate level course on very recent algebraic methods in discrete math (completely different from the Ma/CS 6 material, and more advanced).
  - Summer research projects.

Recall: Partitions of a Positive Integer

- For a positive integer n, we denote by $p(n)$ the number of ways to write n as a sum of positive integers.
- Example. We can write $n = 5$ as

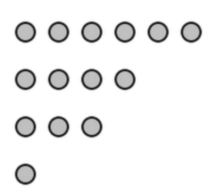
$$5, \quad 4+1, \quad 3+2, \quad 3+1+1,$$
$$2+2+1, \quad 2+1+1+1,$$
$$1+1+1+1+1.$$

so $p(5) = 7$.
  - $p(20) = 627$.
  - $p(100) = 190569292$.

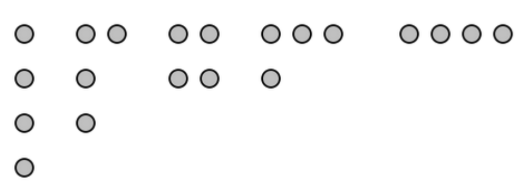
Recall: Ferrers Diagrams

- *Ferrers diagrams* are a graphic way of representing partitions.

$14 = 6 + 4 + 3 + 1$



$p(4) = 5$



The Generating Function

$$P(x) = p(0) + p(1)x + p(2)x^2 + \cdots$$
$$= \prod_{i=1}^{\infty}(1 - x^i)^{-1}$$
$$= (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^4 + x^8 + \cdots)\cdots.$$

Intuitive Explantion

- When opening the parentheses, $p(4)$ is the coefficient of x^4.

$$(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^4 + x^8 + \cdots)\cdots$$
$$1 + 1 + 2$$
$$(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^4 + x^8 + \cdots)\cdots$$
$$4$$
$$(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^4 + x^8 + \cdots)\cdots$$
$$1 + 3$$

Restricted Partitions #1

- Consider partitions of n with no more than k identical parts.
- For example, when $n = 12$ and $k = 2$:
  - $3 + 3 + 3 + 3$ and $4 + 4 + 4$ are not valid.
  - $5 + 5 + 2$ and $2 + 2 + 4 + 4$ are valid.
- **Problem.** What is the generating function of partitions that have no more than k identical parts?

$$\prod_{n=1}^{\infty}(1 + x^n + x^{2n} + x^{3n} + \cdots).$$

Restricted Partitions #1 (cont.)

- **Special case.** Taking $k = 1$, we get the generating function for $p(n \mid \text{each part is distinct})$:

$$(1 + x)(1 + x^2)(1 + x^3)\cdots$$

- What about the case of a general k?

$$\prod_{n=1}^{\infty}\left(1 + x^n + x^{2n} + \cdots + x^{kn}\right).$$

Restricted Partitions #2

- Consider partitions of n with only odd parts.
- For example, when $n = 12$:
  - $1 + 1 + 1 + \cdots + 1, 3 + 3 + 3 + 3, 11 + 1,$ etc…
- **Problem.** What is the generating function of partitions with only odd parts?

$$(1 - x)^{-1}(1 - x^3)^{-1}(1 - x^5)^{-1} \cdots$$
$$= \prod_{n=1}^{\infty}(1 - x^{2n-1})^{-1}.$$

Restricted Partitions #3

- Consider partitions of n with only even parts.
- For example, when $n = 12$:
  - $10 + 2, 2 + 2 + \cdots + 2, 4 + 4 + 4,$ etc…
- **Problem.** What is the generating function of partitions with only even parts?

$$(1 - x^2)^{-1}(1 - x^4)^{-1}(1 - x^6)^{-1} \cdots$$
$$= \prod_{n=1}^{\infty}(1 - x^{2n})^{-1}.$$

Restricted Partitions #4

- Consider partitions of n with each part equals to at most k.
- For example, when $n = 12$ and $k = 4$:
  - $5 + 5 + 2$ and $10 + 1 + 1$ are not valid.
- **Problem.** What is the generating function of partitions whose parts equal to at most k?

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1} \cdots \left(1-x^k\right)^{-1}$$
$$= \prod_{n=1}^{k} (1-x^n)^{-1}.$$

Are These the Same? #1

- The generating function of $p(n \mid \text{each part is odd})$ is
$$\prod_{n=1}^{\infty} (1-x^{2n-1})^{-1}.$$
- The generating function of $p(n \mid \text{each part is even})$ is
$$\prod_{n=1}^{\infty} (1-x^{2n})^{-1}.$$
- Does $p(n \mid \text{each part is odd})$ $= p(n \mid \text{each part is even})$ for every n?

Answer

- No!
  - For example, ***when n is odd the number of even partitions is zero*** and the number of odd partitions is not.

Are These the Same? #2

- The generating function of $p(n \mid \text{each part is odd})$ is
$$\prod_{n=1}^{\infty}(1-x^{2n-1})^{-1}.$$
- The generating function of $p(n \mid \text{each part is distinct})$ is
$$\prod_{n=1}^{\infty}(1+x^{n}).$$
- Does $p(n \mid \text{each part is odd}) = p(n \mid \text{each part is distinct})$ for every n?

Answer

- Yes! We now prove that
$$\prod_{n=1}^{\infty}(1-x^{2n-1})^{-1}=\prod_{n=1}^{\infty}(1+x^{n}).$$
- **Proof.** Since $1+y=(1-y^2)/(1-y)$, we have
$$\prod_{n=1}^{\infty}(1+x^{n})=\frac{\prod_{n=1}^{\infty}(1-x^{2n})}{\prod_{n=1}^{\infty}(1-x^{n})}$$
$$=\prod_{n=1}^{\infty}(1-x^{2n-1})^{-1}$$

Odd VS Even Number of Parts

- We define

$e_n = p(n \mid \text{distinct parts and their \# is even})$
$o_n = p(n \mid \text{distinct parts and their \# is odd})$.

- **Problem.** What is $e_n - o_n$?
  - When $n=2$, $e_n-o_n=0-1=-1$.
  - When $n=3$, $e_n-o_n=1-1=0$.
  - When $n=4$, $e_n-o_n=1-1=0$.
  - When $n=5$, $e_n-o_n=2-1=1$.
  - When $n=6$, $e_n-o_n=2-2=0$.

The Generating Function

- Set $q(n) = e_n - o_n$, and
  $Q(x) = 1 + q(1)x + q(2)x^2 + q(3)x^3 + \cdots$
- **Claim.** $Q(x) = (1-x)(1-x^2)(1-x^3)\cdots$.
- **Proof.**
  - A partition $n = s_1 + \cdots + s_k$ (where the parts are distinct), corresponds to $(-x^{s_1})(-x^{s_2})\cdots(-x^{s_k})$.
  - That is, every even partition of n contributes x^n and every odd partition contributes $-x^n$.

The Correct Bound

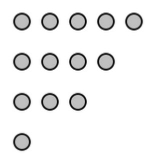
- **Theorem.** We have

$$e_n - o_n = \begin{cases} (-1)^m, & \text{if } n = \frac{1}{2}m(3m \pm 1), \\ 0, & \text{otherwise.} \end{cases}$$

- **Proof.**
  - Refer to partitions that are counted in e_n as *even*, and partitions that are counted in o_n as *odd*.
  - We define a map from even to odd partitions, or from odd to even partitions, which is almost a bijection.

Preparation for the Map

- λ – a partition with distinct parts.
- $s(\lambda)$ – the size of the smallest part of λ.
- $t(\lambda)$ – the length of the sequence that starts with the first part of λ and continues as long as parts decrease by 1 at each step.



$s(\lambda) = 1$
$t(\lambda) = 3$

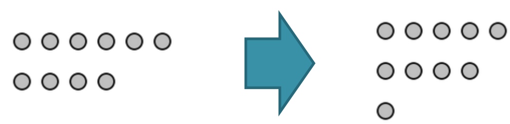
The Mapping – Case 1

- If $s(\lambda) \leq t(\lambda)$, we remove the last part of λ and add 1 to each of the first $s(\lambda)$ parts.

- This map takes an even partition to an odd partition (or vice versa).

The Mapping – Case 2

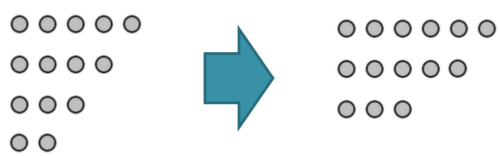
- If $s(\lambda) > t(\lambda)$, we remove 1 from each of the $t(\lambda)$ largest parts, and add a new smallest part of size $t(\lambda)$.



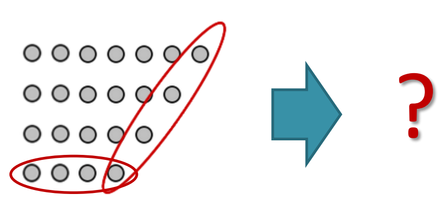
- Once again, the map takes an even partition to an odd partition (or vice versa).

Examining Case 1

- When does the mapping of case 1 fails? (assuming that $s(\lambda) \leq t(\lambda)$).



- When $s(\lambda) = t(\lambda) =$ the number of parts of λ.

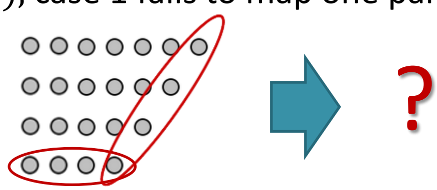


Case 1 is "Almost" Well Behaved

- For what values of n can the problem in case 1 occur?
  - Write the number of parts as $m = s(\lambda) = t(\lambda)$.

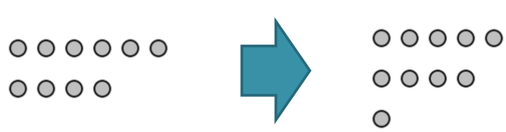
$$n = m + (m+1) + (m+2) + \cdots + (2m-1)$$
$$= \frac{1}{2}m(3m-1).$$

- That is, when $n = \frac{1}{2}m(3m-1)$ (for some $m \in \mathbb{N}$), case 1 fails to map one partition.

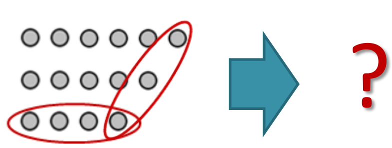


Examining Case 2

- When does the mapping of case 1 fails? (assuming that $s(\lambda) > t(\lambda)$).



- When $s(\lambda) - 1 = t(\lambda)$.

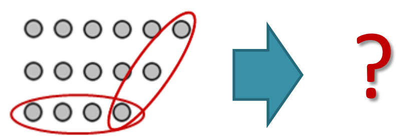


Case 2 is "Almost" Well Behaved

- For what values of n can the problem in case 2 occur?
  - The number of parts: $m = s(\lambda) - 1 = t(\lambda)$.

$$n = (m+1) + (m+2) + \cdots + m$$
$$= \frac{1}{2}m(3m+1).$$

- That is, when $n = \frac{1}{2}m(3m+1)$ (for some $m \in \mathbb{N}$), case 2 fails to map one partition.



Concluding the Proof

- If $n \neq \frac{1}{2}m(3m \pm 1)$, then the mapping is a bijection and thus $e_n - o_n = 0$.
- If $n = \frac{1}{2}m(3m \pm 1)$, then
  - If m is even, the mapping takes the even partitions to distinct odd partitions, with one exception, so $e_n - o_n = 1$.
  - If m is odd, the mapping takes the odd partitions to distinct even partitions, with one exception, so $e_n - o_n = -1$.

A Simple Observation

- **Recall.**
  - The partitions generating function is
$$P(x) = \prod_{n=1}^{\infty} (1 - x^n)^{-1}.$$
  - The generating function of $e_n - o_n$ is
$$Q(x) = \prod_{n=1}^{\infty} (1 - x^n).$$
- We thus have
$$P(x) \cdot Q(x) = 1.$$

Consequences of the Observation

- We have $P(x) \cdot Q(x) = 1$. Equivalently,
$$(1 + p(1)x + p(2)x^2 + \cdots)(1 - x - x^2 + x^5 + x^7 - x^{12} + \cdots) = 1.$$
- That is, for any $n \geq 1$, the coefficient of x^n in the product is zero.
- For example, by considering the coefficient of x^{13}, we have
$$p(13) - p(12) - p(11) + p(8) + p(6) - p(1) = 0.$$
  - or
$$p(13) = p(12) + p(11) - p(8) - p(6) + p(1).$$

Computing $p(n)$

- The above technique gives us an efficient recursive method for computing $p(n)$.

n	7	8	9	10	11	12	13	14
$p(n-1)$	11	15	22	30	42	56	77	101
$p(n-2)$	7	11	15	22	30	42	56	77
$p(n-5)$	2	3	5	7	11	15	22	30
$p(n-7)$	1	1	2	3	5	7	11	15
$p(n-12)$	-	-	-	-	-	1	1	2
$p(n)$	15	22	30	42	56	77	101	135

The End: Teaching Survey