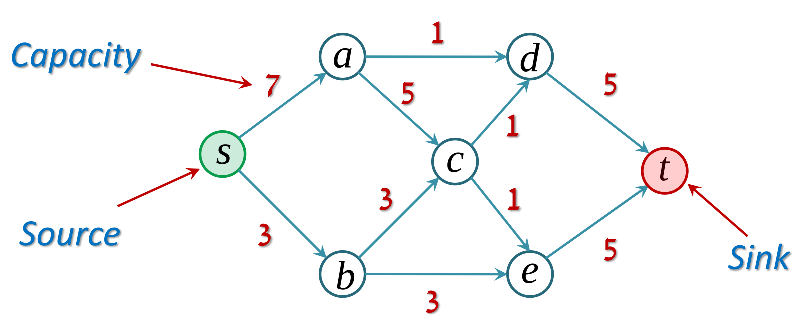
Ma/CS 6a

Class 14: Various (Flow) Execises

Reminder: Flow Networks

- A *flow network* is a digraph $G = (V, E)$, together with a *source* vertex $s \in V$, a *sink* vertex $t \in V$, and a *capacity function* $c: E \to \mathbb{N}$.



Reminder: Flow in a Network

- Given a flow network $G = (V, E, s, t, c)$, a *flow* in G is a function $f: E \rightarrow \mathbb{N}$ that satisfies
  - Every $e \in E$ satisfies $f(e) \leq c(e)$.
  - Every $v \in V \setminus \{s, t\}$ satisfies

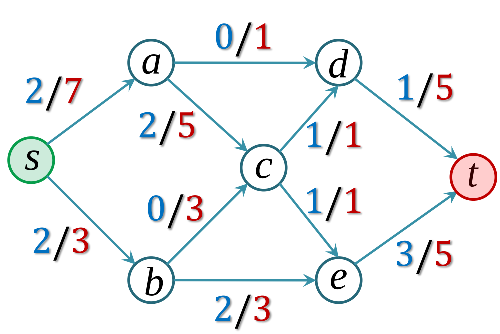
$$\sum_{(u,v)\in E} f(u,v) = \sum_{(v,w)\in E} f(v,w)$$

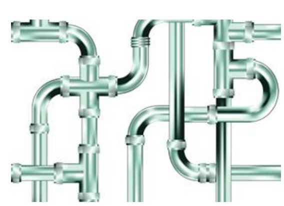
Total flow entering v.

Total flow exiting v.

Example: Flow

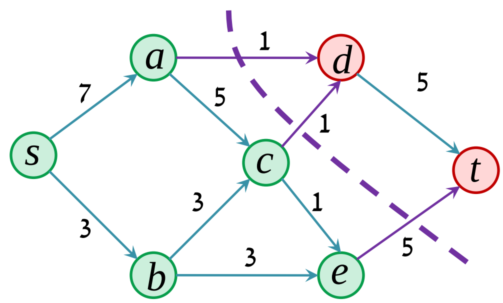
- The capacities are in red.
- The flow is in blue.





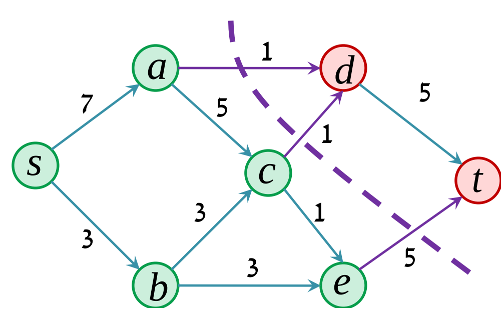
Reminder: Cuts

- A *cut* is a partitioning of the vertices of the flow network into two sets S, T such that $s \in S$ and $t \in T$.
- The *size of a cut* is the sum of the capacities of the edges from S to T.



Reminder: Max Flow – Min Cut

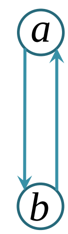
- **Max flow – min cut theorem.** In every flow network, the size of the minimum cut *is equal* to the size of the maximum flow.



Warm-up: Anti-Parallel Edges

- Two directed edges are said to be ***anti-parallel*** if they are between the same pair of vertices, but in opposite directions.

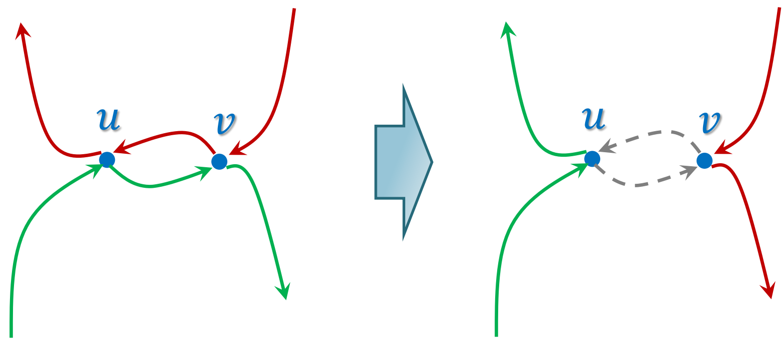
- **Problem.** Consider a flow network (V, E, s, t, c), and let $e, e' \in E$ be anti-parallel edges. Prove that there exists a maximum flow in which at least one of e, e' has no flow through it.

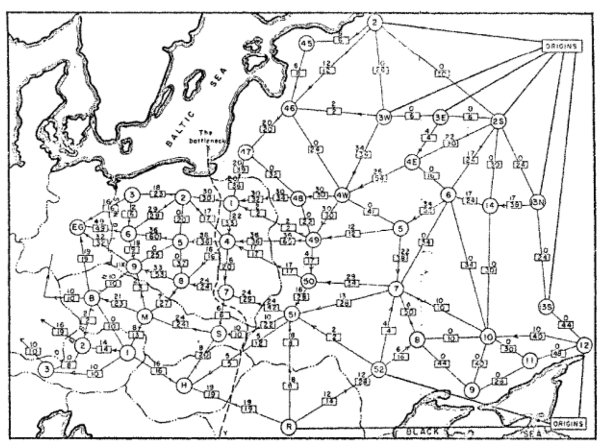


Solution

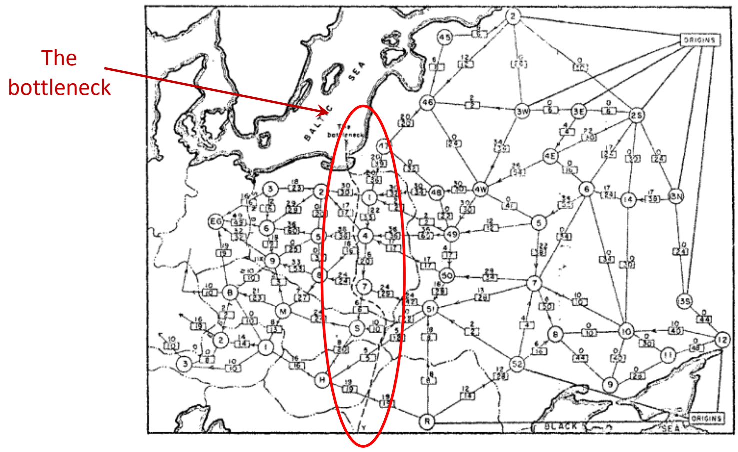
- Consider a maximum flow f.
  - If either e or e' has no flow through it in f, we are done.
  - Assume, WLOG, that $f(e) \leq f(e')$.
  - By decreasing $f(e')$ by $f(e)$ and then setting $f(e) = 0$, we obtain a valid flow of the same size.
    - Let $e = (v, u)$. Both the incoming and the outgoing flows of v and u were decreased by the same amount, so they remain equivalent.

An Illustration





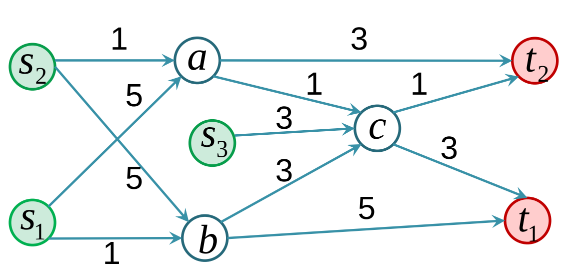
- **Recall:** The Rand corporation studied the Soviet train system.
- They studied the Soviet ability to transport things from the Asian side to European side.



- They also studied the minimum cut.
- **Problem.** In this scenario there are several sources and several sinks!

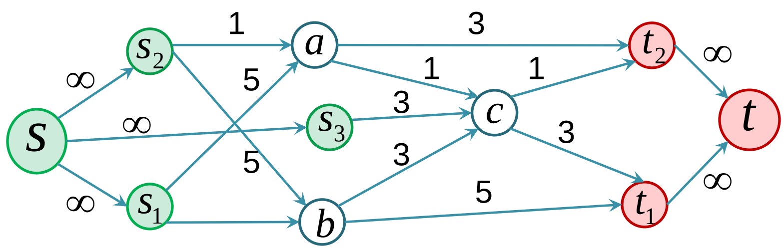
Problem 2: Several Sources and Sinks

- **Problem.** We are given a flow network with several sources and several sinks. Explain how to use an algorithm for finding a maximum flow (in a standard flow network) for this case.



Solution

- We add a "super source" S, and add an edge from it to each of the sources. Each of these edges has an infinite capacity.
- We symmetrically add a "super sink" T.
- Run the original algorithm from S to T.



Problem 3: Even Flow

- **Problem.** Given a flow network (V, E, s, t, c) such that all of the capacities are even, prove that the size of the maximum flow is even.

The capacities are even.

Every cut has an even size.

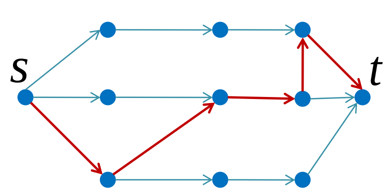
The minimum cut has an even size.

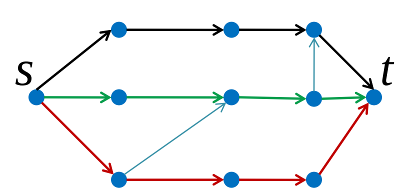
Max flow – min cut

The maximum cut has an even size.

Problem 4: Edge-disjoint Paths

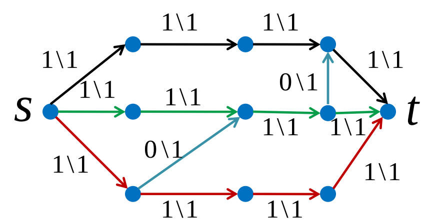
- **Problem.** Given a digraph $G = (V, E)$ and vertices $s, t \in V$, describe an algorithm that finds the maximum number of edge-disjoint paths from s to t.





Solution: Edge-disjoint Paths

- We give every edge a capacity of 1. A network with only 1-capacities is called a ***0-1 network*** (max flow can be computed more efficiently in such networks).
- Find a max flow in the resulting 0-1 network.

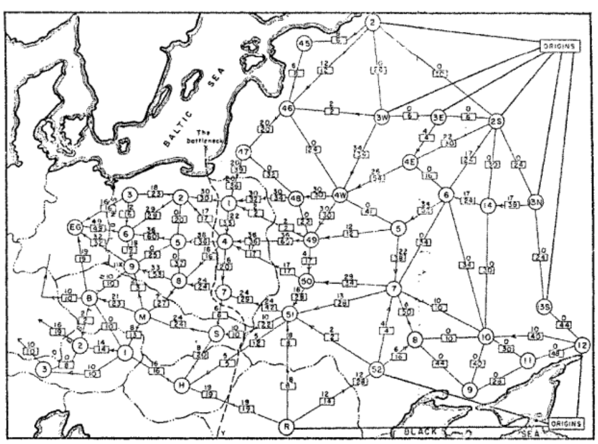


Correctness of Solution

- Given a 0-1 flow network with a max flow f and maximum number of edge-disjoint paths k, we need to prove $|f| = k$.
- $|f| \geq k$: By having a flow of 1 through every disjoint path, we obtain a flow of size k.
- $|f| \leq k$: Proof by induction on $|f|$.
  - Induction basis: obvious when $|f| = 0$.

Correctness of Solution (2)

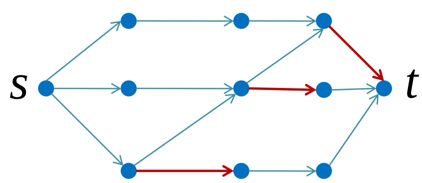
- Induction step (show that $m = |f| \leq k$).
  - Consider a maximum flow (of size m).
  - Remove every edge with 0 flow through it.
  - Find a path from s to t (it exists since $|f| > 0$).
  - Remove the edges of the path, to obtain a network with a flow of size at least $m - 1$.
  - By induction hypothesis, there are at least $m - 1$ edge-disjoint paths in this network.
  - Bring back the path that was removed, obtaining at least m edge-disjoint paths.



- Perhaps RAND were also studying the minimum number of train tracks that are needed to be destroyed to prevent any transportation from the Asian side to the European side?

Problem 5: Disconnecting Edges

- **Problem.** Given a digraph $G = (V, E)$ and vertices $s, t \in V$, describe an algorithm that finds the minimum number of edges needed to be removed from G so that there would be no paths from s to t.



Solution: Disconnecting Edges

- As before, we give every edge a capacity of 1, to obtain a 0-1 network.
- Find a max flow in the network.

- We already proved that the max flow equals the maximum number of edge-disjoint paths.
- It remains to prove that the max number of edge-disjoint paths equals the min number of edges needed to disconnect s from t.

Proof

- k – maximum number of edge disjoint paths.
- ℓ – minimum number of disconnecting edges.

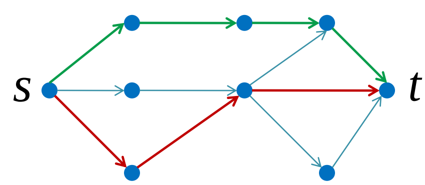
- $\ell \geq k$: There are k edge-disjoint paths, and we need to remove at least one edge from each.
- $\ell \leq k$: If we remove the min cut is a set of disconnecting edges. By the max flow – min cut theorem, there are k edges in the min cut.

Menger's Theorem

- The idea that the min number of disconnecting edges is equal to the max number of edge-disjoint paths is called *Menger's Theorem*, and is from 1927.

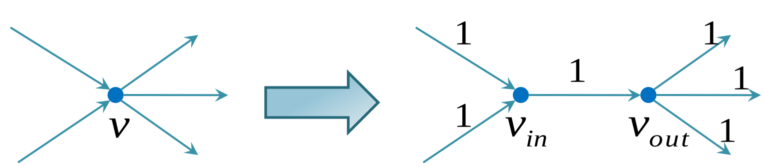
Problem 6: Vertex-disjoint Paths

- **Problem.** Given a digraph $G = (V, E)$ and vertices $s, t \in V$, describe an algorithm that finds the maximum number of *vertex-disjoint* paths from s to t.



Solution: Vertex-disjoint Paths

- We split every vertex of $V \setminus \{s, t\}$ as follows:



- In the resulting graph, two paths are edge disjoint if and only if they are vertex disjoint.
- As before, we add capacities of 1, to obtain a 0-1 network.

Solution: Vertex-disjoint Paths (2)

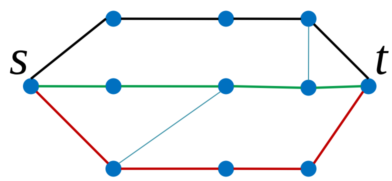
- Algorithm:
  - We build a flow network as described in the previous slide.
  - Find max flow in the resulting network.
- It remains to prove:
  - There is a one-to-one correspondence between the sets of vertex-disjoint paths in the original graph and the edge-disjoint ones in the new network.

Proof

- Every path in the new network is of the form $s \to v_{in} \to v_{out} \to u_{in} \to \cdots \to w_{in} \to w_{out} \to t$.
- It corresponds to the path in the original graph:
  $$s \to v \to u \to \cdots \to w \to t.$$
- A set of vertex-disjoint paths in the original graph corresponds to a vertex-disjoint set of paths in the new network, and these are edge-disjoint.
- In the new network, a set of edge-disjoint paths are also vertex-disjoint, and so also the corresponding paths in the original graph.

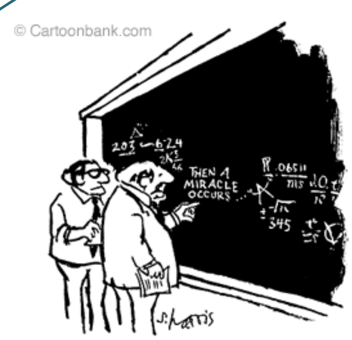
Problem 7, If Time Permits

- **Problem.** Given an undirected graph $G = (V, E)$ and vertices $s, t \in V$, describe an algorithm that finds the maximum number of edge-disjoint paths from s to t.



The End

How detailed should my homework solution be?



"I think you should be more explicit here in step two."