Ma/CS 6a: Problem Set 4

Due noon, Thursday, October 30*

1. Let $G = (V, E)$ be a connected undirected graph such that $V = \{v_1, v_2, \ldots, v_n\}$, the degree of v_i is d_i, and $d_1 \geq d_2 \geq \ldots \geq d_n$. Prove that G is k-colorable, where

$$k = 1 + \max_i \min\{d_i, i - 1\}.$$

2. (NO COLLABORATION) Prove that adding any edge to an MST creates *exactly* one circle.

3. Let $G = (V, E)$ be a connected undirected graph, let $V' \subset V$ be a subset of vertices, and let $w : E \to \mathbb{R}$ be a weight function on the edges. Describe an efficient algorithm for finding an MST of G in which every vertex of V' is a leaf (that is, has a degree of 1). If such an MST does not exist, the algorithm should announce this fact. Explain why your algorithm is correct.

4. A *connected component* of an undirected graph is a maximal set of vertices such that there is a path between every two vertices. For example, a connected graph has one connected component, which is the entire graph. We think of a connected components as a set of vertices, so connected components containing the same vertices but different edges are still considered to be equivalent.

Let G be an undirected graph with an edge weight function $w : E \to \mathbb{R}$. Let T_1, T_2 be two MSTs of G, and let $k \in \mathbb{R}$. We throw from T_1 and T_2 every edge of weight at least k, and denote the resulting forests as T_1' and T_2', respectively. Prove that T_1' and T_2' have the same connected components.

5. (a) Prove or disprove: No spanning tree contains two edge-disjoint perfect matchings.

(b) A matching M is *maximal* if we cannot increase M by adding additional edges to it (that is, edges that have no common vertices with the existing edges of M. Notice that a maximal matching is not necessarily a *maximum* matching). Consider an undirected graph G with maximum matchings of size k. Prove that the size of any maximal matching of G is at least $k/2$.

*The awesome students who helped correcting this assignment: Muammad ibn Mūsā al-Khwārizmī.