Ma/CS 6a

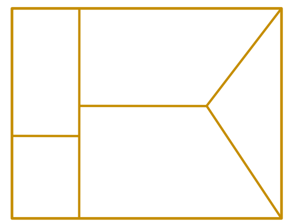
Class 9: Coloring

By Adam Sheffer

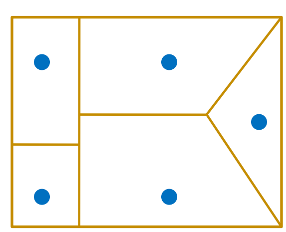
Map Coloring

- Can we color each state with one of three colors, so that no two adjacent states have the same color?

Map Coloring and Graphs

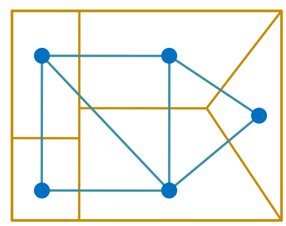


Map Coloring and Graphs



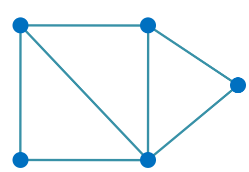
- Place a vertex in each state/face.

Map Coloring and Graphs



- Place a vertex in each state/face.
- Place an edge between any pair of vertices that represent adjacent faces.

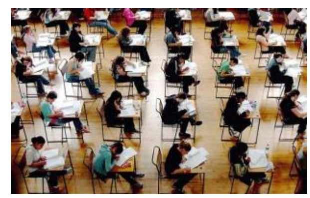
Map Coloring and Graphs



- **The problem.** Can we color the vertices using k colors, such that every edge is adjacent to two different colors?
- Such a coloring is called a *k-coloring*

Scheduling Exams

- **Problem.** We wish to set dates for the exams of every course in Caltech.
  - Two exams cannot be on the same day if the classes have at least one student in common.
  - How many exam days are necessary?

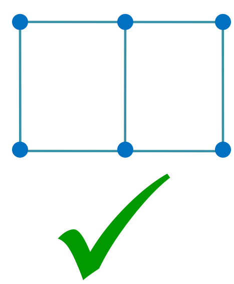


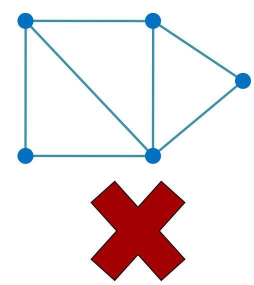
Solution

- **Solution.** Build a graph:
  - A vertex for every class.
  - An edge between every pair of classes with at least one common student.
  - Find the minimum k such that the graph is k-colorable. Every color corresponds to a different date.

The Case of Two Colors

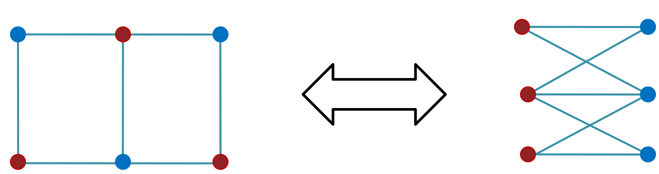
- **Problem.** Given a graph $G = (V, E)$, check whether it has a 2-coloring.





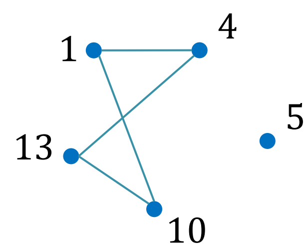
Bipartite Graphs

- An undirected graph is ***bipartite*** if it admits a 2-coloring.
- We can partition the vertices of a bipartite graph into two sets, with every edge having one vertex in each set.

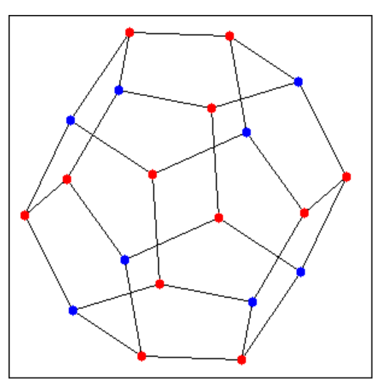


Problem: Prime Sums

- **Question.**
  - Let $G = (V, E)$ be an undirected graph with a vertex set $V = \{1,2, \dots, n\}$.
  - There is an edge between vertices i and j if and only if $i + j$ is prime.
  - Is G bipartite?
  - Yes! We can put every odd number on one side and every even number on the other.

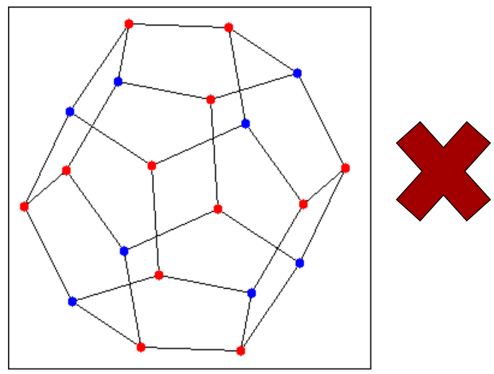


Is this Graph Bipartite?



Bipartite Graphs Characterization

- **Claim.** A graph $G = (V, E)$ is bipartite if and only if it does not contain cycles of odd length.

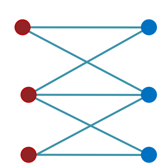


Proof: One Direction

- Assume that G is bipartite and prove that G contains no odd-length cycles:
  - Every edge connects the two sides of G.

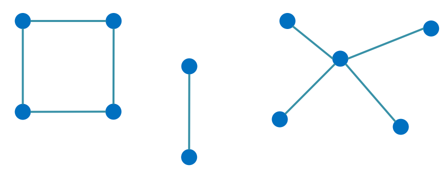
    $\Downarrow$
  - A path that starts and ends in the same side must have an even number of edges.

    $\Downarrow$
  - Any cycle must have an even number of edges.



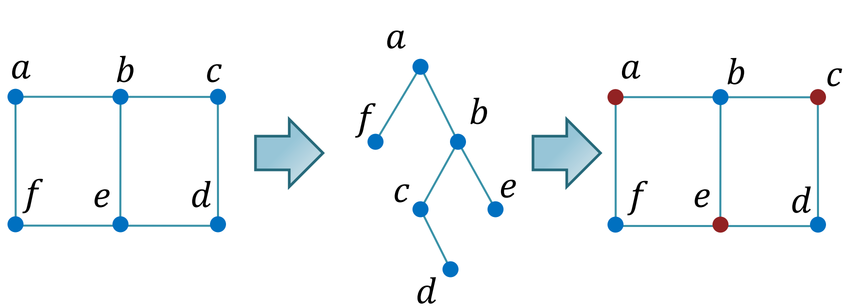
Proof: The Other Direction

- Assume that G contains no odd-length cycles and prove that G is bipartite:
  - If G is not connected, we prove the claim for each connected component separately. Thus, assume that G is connected.
  - We prove the claim by describing an algorithm that finds a 2-coloring of G.



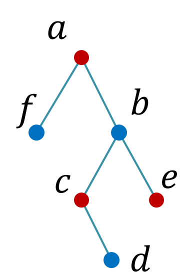
2-Coloring Algorithm

- Run the BFS algorithm from an arbitrary vertex v.
- Colored vertices of odd levels of the BFS tree red, and vertices of even levels blue.



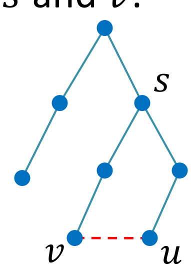
Correctness of the Algorithm

- **Prove.** No edge is *monochromatic*:
  - An edge of G either connects vertices in consecutive levels of the BFS tree, or vertices in the same level.
  - An edge between consecutive levels connects a blue vertex and a red vertex.
  - It remains to prove that no edge connects two vertices from the same level.



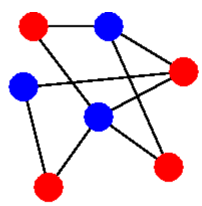
Correctness of the Algorithm (2)

- For contradiction, assume that the edge $(u, v) \in E$ where $u, v \in V$ are in the same level of the BFS tree.
- Let s be their *lowest common ancestor*.
- Let P denote the path between s and u. Let Q denote the path between s and v.
- If P is of length n, so is Q.
- Connecting P, Q, and the edge (u, v) yields a cycle of length $2n + 1$. *Contradiction!*

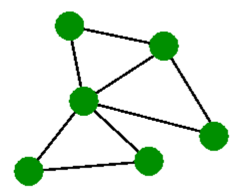


A More General Algorithm

- **Problem.** Change the previous algorithm so that it receives any graph G.
  - If G is bipartite, output a 2-coloring.
  - Otherwise, output an error message.



A) A Bipartite Graph

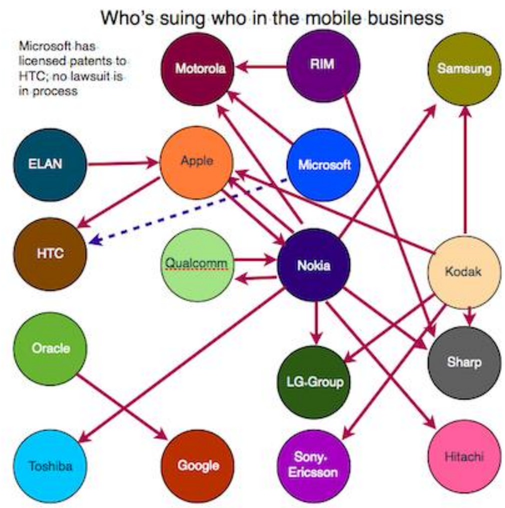


B) A non–Bipartite Graph

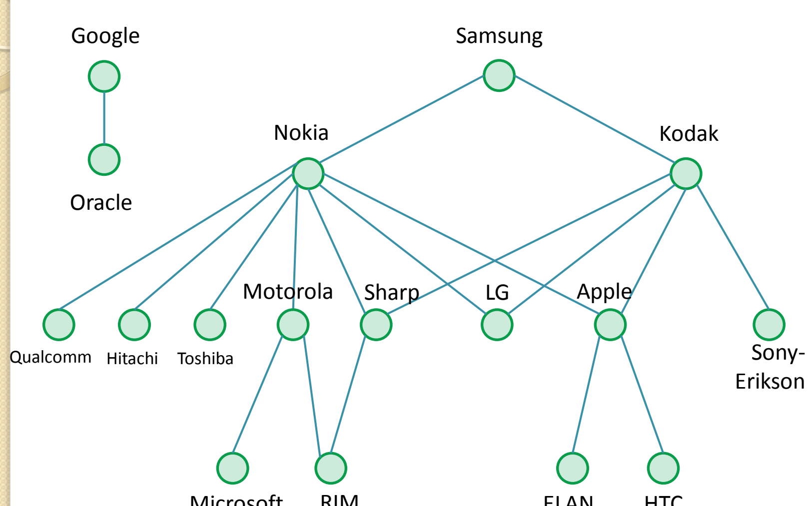
Solution

- Change the BFS so that when it examines an edge, it checks whether both of its endpoints are in the same level:
  - When examining an edge, if it leads to a vertex that we already visited (non-white), check its level in the tree.
  - If the vertex is in the same level, stop the algorithm and output an error message.

Example

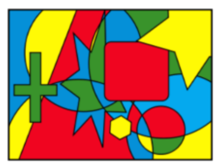


Example (cont.)



The Four Color Theorem

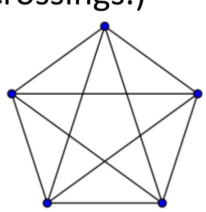
- **Theorem.** Every map has a 4-coloring.
  - Asked over 150 years ago.
  - Over the decades several false proofs were published.
  - Proved in 1976 by Appel and Haken. Extremely complicated proof that relies on a computer program.



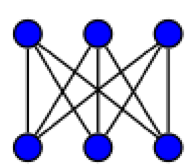
The Four Color Theorem

- **Question.** Does the four color theorem imply that every graph has a 4-coloring?
  - **No!** While every map corresponds to a graph, most graphs do not correspond to a map.
  - (Graphs that correspond to a map are called *planar* and can be drawn without edge crossings.)

K_5



$K_{3,3}$

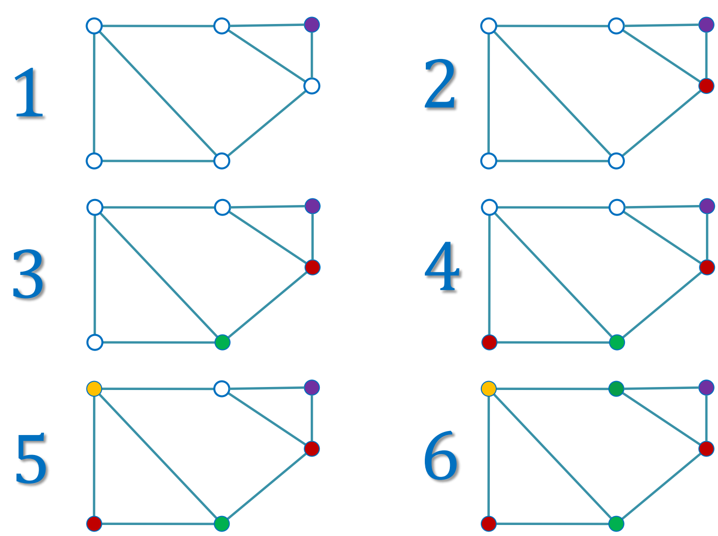


Coloring Graphs with Bounded Degrees

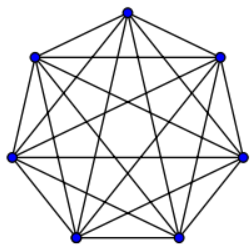
- **Problem.** Show that any graph $G = (V, E)$ such that every vertex of V has degree at most k, admits a $(k+1)$-coloring.

- **Proof.**
  - At each step choose an arbitrary uncolored vertex v.
  - Since v has at most k neighbors, one of the $k+1$ colors must be OK for v.

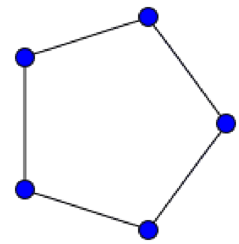
Example: $k + 1$-coloring



Sometimes We Cannot Do Better



- K_n - complete graph of n vertices.
- Max degree: $n-1$.
- Colors needed: n.



- C_n - cycle of *odd* length n.
- Max degree: 2.
- Colors needed: 3.

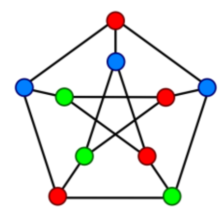
Better Graph Coloring

- **Problem.** Show that if a graph $G = (V, E)$ satisfies:
  - Every vertex of V has degree at most k.
  - G is connected.
  - At least one vertex has degree $< k$.

  Then G has a *k-coloring*.

3-Colorable Graphs

- **Problem.** Let $G = (V, E)$ be a graph that is 3-colorable, and let $n = |V|$.
  - Describe an efficient algorithm for coloring G with $c\sqrt{n}$ colors.
- **Observation.** For any $v \in V$, the set of neighbors of v can be colored using two colors.
  - Otherwise, we would need four colors to color v and its neighbors.



Solution

- Algorithm:
  - As long as there is a vertex v of degree at least $\sqrt{n}$, color v with one new color and then color v's neighbors with two other new colors. Then remove v and it's neighbors
    - At each step we remove at least $\sqrt{n} + 1$ vertices, so there are less than $\sqrt{n}$ steps.
    - This step requires less than $3\sqrt{n}$ colors.
  - When all the remaining vertices have degree at most $\sqrt{n} - 1$, we know how to color the graph using at most $\sqrt{n}$ colors.

3-Colorable Graphs are Frustrating!

- It is probably impossible to color any 3-colorable graph in a reasonable time, using a constant number of colors.
- In 2007, **Chlamtac** presented an efficient algorithm for coloring using $cn^{0.2072}$ colors.
  - This algorithm WAY TOO complicated for us to discuss.

The End: Three utilities problem