Ma/CS 6a

Class 4: Primality Testing

Is $2^{6972593} - 1$ prime?

By Adam Sheffer

- Send anonymous suggestions and complaints *from* here.
- Email: *adamcandobetter@gmail.com*
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There aren't enough crocodiles in the presentations

Why won't you tell me how to solve the homework?!

Only today! 75% off for Morphine and Xanax.

Adam make me a public key!

Reminder: Euler's Totient Function

- *Euler's totient $\varphi(n)$* is defined as follows: Given $n \in \mathbb{N}$, then

$$\varphi(n) = |\{x \mid 1 \leq x < n \text{ and } \mathrm{GCD}(x,n) = 1\}|.$$

- In more words: $\varphi(n)$ is the number of natural numbers $1 \leq x \leq n$ such that x and n are coprime.

$$\varphi(12) = |\{1,5,7,11\}| = 4.$$

Reminder #2: The RSA Algorithm

- Bob wants to generate keys:
  - Arbitrarily chooses primes p and q.
    $n = pq$ find $\varphi(n)$.
  - Chooses e such that $\mathrm{GCD}(\varphi(n), e) = 1$.
  - Find d such that $de \equiv 1 \; mod \; \varphi(n)$.
- Alice wants to pass bob m.
  - Receives n, e from Bob.
  - Returns $X \equiv m^e \; mod \; n$.
- Bob receives X.
  Calculates $X^d \; mod \; n$.

Finding $\varphi(n)$

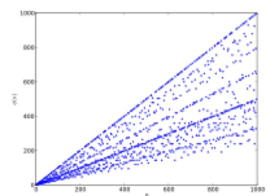
- **Problem.** Given $n = pq$, where p, q are large primes, find $\varphi(n)$.
  - We need the number of elements in $\{1,2,\ldots,n\}$ that are not multiples of p or q.
  - There are $\frac{n}{p} = q$ numbers are divisible by p.
  - There are $\frac{n}{q} = p$ numbers are divisible by q.
  - Only $n = pq$ is divided by both.
  - Thus: $\varphi(n) = n - p - q + 1$.

The RSA Algorithm

- Bob wants to generate keys:
  - Arbitrarily chooses primes p and q. ?
    $n = pq$ ✓ find $\varphi(n)$. ✓
  - Chooses e such that $\mathrm{GCD}(\varphi(n), e) = 1$. ?
  - Find d such that $de \equiv 1 \; mod \; \varphi(n)$. ?
- Alice wants to pass bob m.
  - Receives n, e from Bob.
  - Returns $X \equiv m^e \; mod \; n$. ✓
- Bob receives X.
  Calculates $X^d \; mod \; n$. ✓

Choose e s.t. $\text{GCD}(\varphi(n), e) = 1$

- **Problem.** Given $n = pq$, where p, q are large primes, find $e \in \mathbb{N}$ such that $\text{GCD}(\varphi(n), e) = 1$.
  - We can choose arbitrary numbers until we find one that is relatively prime to $\varphi(n)$.
  - For the "worst" values of $\varphi(n)$, a random number is good with probability $1/\log \log n$.



The RSA Algorithm

- Bob wants to generate keys:
  - Arbitrarily chooses primes p and q. ?
    $n = pq$ ✓ find $\varphi(n)$. ✓
  - Chooses e such that $\text{GCD}(\varphi(n), e) = 1$. ✓
  - Find d such that $de \equiv 1 \; mod \; \varphi(n)$. ?
- Alice wants to pass bob m.
  - Receives n, e from Bob.
  - Returns $X \equiv m^e \; mod \; n$. ✓
- Bob receives X.

  Calculates $X^d \; mod \; n$. ✓

Find d such that $de \equiv 1 \; mod \; \varphi(n)$

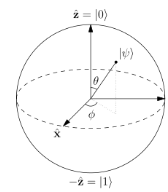
- **Recall.** Since $\text{GCD}\big(e, \varphi(n)\big) = 1$ then there exist $s, t \in \mathbb{Z}$ such that
  $$se + t\varphi(n) = 1.$$
- That is, $se \equiv 1 \; mod \; \varphi(n)$.
- We can find s, t by the ***extended Euclidean algorithm*** from lecture 2.

The RSA Algorithm

- Bob wants to generate keys:
  - Arbitrarily chooses primes p and q. **?**
    $n = pq$ ✓ find $\varphi(n)$. ✓
  - Chooses e such that $\text{GCD}(\varphi(n), e) = 1$. ✓
  - Find d such that $de \equiv 1 \; mod \; \varphi(n)$. ✓
- Alice wants to pass bob m.
  - Receives n, e from Bob.
  - Returns $X \equiv m^e \; mod \; n$. ✓
- Bob receives X.
  Calculates $X^d \; mod \; n$. ✓

Quantum Computing

- A *bit* of a computer contains a value of either 0 or 1.
- A quantum computer contains *qubits*, which can be in superpositions of states.
- Theoretically, a quantum computer can easily factor numbers and decipher almost any known encryption.



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Finding Large Primes

- Let n be a LARGE integer (e.g., 2^{4000}).
- *The prime number theorem.* The probability of a random $p \in \{1, \ldots, n\}$ being prime is about $1/\log n$.
- If we randomly choose numbers from $\{1, \ldots, n\}$, we expect to have about $\log n$ iterations before finding a prime.
  - *But how can we check whether our choice is a prime or not?!*

Primality Testing

- Given a LARGE $q \in \mathbb{Z}$, how can we check whether q is prime?
- **The naïve approach.** Go over every number in $\{2, \ldots, \sqrt{q}\}$ and check whether it divides q.
  - *But we chose our numbers to be too large for a computer to go over all of them!*

Recall: Fermat's Little Theorem

- For any prime p and integer a relatively prime to p, we have

$$a^p \equiv a \ mod \ p.$$

- Pick a random integer a and check whether $a^q \equiv a \ mod \ q$.
  - If not, q is not a prime!
  - If yes, ???

Pierre de Fermat

Example: Fermat Primality Testing

- Is $n = 355207$ prime?

$$2^{355207} \equiv 84927 \ mod \ 355207.$$

- n is not prime since $2^n \not\equiv 2 \ mod \ n.$

- We can try 1000 different values of a and see if $a^n \equiv a \ mod \ n$ for each of them.

Carmichael Numbers

- A number $q \in \mathbb{N}$ is said to be a *Carmichael number* if it is not prime, but still satisfies $a^q \equiv a \bmod q$ for every a that is relatively prime to q.
  - The smallest such number is 561.
  - Very rare – about one in 50 trillion in the range $1 - 10^{21}$.

R. D. Carmichael

Example: Carmichael Numbers

- **Claim.** Let $k \in \mathbb{N} \setminus \{0\}$ such that $6k+1, 12k+1$, and $18k+1$ are primes. Then
$$n = (6k+1)(12k+1)(18k+1)$$
is a Carmichael number.

- **Example.**
  - For $k = 1$, we have that $7, 13, 19$ are primes.
  - $7 \cdot 13 \cdot 19 = 1729$ is a Carmichael number.

Proof

- We need to prove that for any a that is relatively prime to n, we have
$$a^n \equiv a \bmod n.$$
- **Recall.** Since $GCD(a, n) = 1$, this is equivalent to $a^{n-1} \equiv 1 \bmod n$.
- We rewrite $n = 1296k^3 + 396k^2 + 36k + 1$.
- For any such a, we have
$$a^{n-1} = a^{1296k^3+396k^2+36k}$$
$$= \left(a^{6k}\right)^{216k^2+66k+6}.$$

Proof (cont.)

- For any a relatively prime to n, we have
$$a^{n-1} = \left(a^{6k}\right)^{216k^2+66k+6}.$$
- **Recall.** If $a \in \mathbb{N}$ is not divisible by a prime p then $a^{p-1} \equiv 1 \bmod p$.
- Since a and $6k - 1$ are relatively prime
$$a^{n-1} \equiv 1^{216k^2+66k+6} \equiv 1 \bmod 6k + 1.$$
- Similarly, we have $a^{n-1} \equiv 1 \bmod 12k + 1$ and $a^{n-1} \equiv 1 \bmod 18k + 1$.
- Since $a^{n-1} - 1$ is divisible by the three pairwise coprime integers $6k + 1, 12k + 1$, and $18k + 1$, it is also divisible by their product n. That is, $a^{n-1} \equiv 1 \bmod n$.

Miller–Rabin Primality Test

- The *Miller–Rabin primality test* works on *every number*.

Gary Miller

Michael Rabin

Root of Unity

- **Claim.** For any prime p, the only numbers $a \in \{1, \ldots, p\}$ such that $a^2 \equiv 1 \ mod \ p$ are 1 and $p - 1$.

- **Example.** The solutions to
$$a^2 \equiv 1 \ mod \ 1009$$
are exactly the numbers satisfying
$$a \equiv 1 \text{ or } 1008 \ mod \ 1009.$$

Root of Unity

- **Claim.** For any prime p, the only numbers $a \in \{1, \dots, p\}$ such that $a^2 \equiv 1 \ mod\ p$ are 1 and $p - 1$.
- **Proof.**

$$a^2 \equiv 1 \ mod\ p$$

$$a^2 - 1 \equiv 0 \ mod\ p$$

$$(a+1)(a-1) \equiv 0 \ mod\ p$$

- That is, either $p | (a+1)$ or $p | (a-1)$.

Roots of Unity Properties

- Given a prime $p > 2$, we write

$$p - 1 = 2^s d$$

  where d is odd and $s \geq 1$.
- **Claim.** For any ***odd*** prime p and any $1 < a < p$, one of the following holds.
  - $a^d \equiv 1 \ mod\ p$.
  - There exists $0 \leq r < s$ such that

$$a^{2^r d} \equiv -1 \ mod\ p.$$

Roots of Unity Properties (2)

- **Claim.** For any ***odd*** prime p and any $1 < a < p$, one of the following holds.
  - $a^d \equiv 1 \, mod \, p$.
  - There exists $0 \le r < s$ such that $a^{2^r d} \equiv -1 \, mod \, p$.
- **Proof.**
  - By Fermat's little theorem $a^{p-1} \equiv 1 \, mod \, p$.
  - Consider $a^{(n-1)/2}, a^{(n-1)/4}, \ldots, a^{(n-1)/2^s}$. By the previous claim, each such root is $\pm 1 \, mod \, n$.
  - If all of these roots equal 1, we are in the first case. Otherwise, we are in the second.

Composite Witnesses

- Given a composite (non-prime) ***odd*** number n, we again write $n - 1 = 2^s d$ where d is odd and $s \ge 1$.
- We say that $a \in \{2,3,4, \ldots, n-2\}$ is a ***witness*** for n if
  - $a^d \not\equiv 1 \, mod \, p$.
  - For every $0 \le r < s$, we have $a^{2^r d} \not\equiv -1 \, mod \, p$.

Example: Composite Witness

- **Problem.** Prove that 91 is not a prime.

$$90 = 2 \cdot 45.$$

$$2^{45} \equiv 57 \; mod \; 91.$$

- 2 is a witness that 91 is not a prime.

There are Many Witnsses

- Given an odd composite n, the probability of a number $\{2, \dots, n-2\}$ being a witness is at least ¾.
- Given an odd $n \in \mathbb{N}$, take i numbers and check if they are witnesses.
  - If we found a witness, n is composite.
  - If we did not find a witness, n is prime with probability at least

The End

I'VE DISCOVERED A WAY TO GET COMPUTER SCIENTISTS TO LISTEN TO ANY BORING STORY.