Ma/CS 6a: Problem Set 8*

Due noon, Monday, December 1

1. We have a regular three-dimensional cube C (that is, a cube that is composed out of six identical squares), and four colors. We wish to color each of the six faces of C. The colors of the different faces are not necessarily distinct. Two colorings of C are considered identical if one can be obtained from the other by a series of reflections and rotations. use the method that we saw in Lecture 21 to count how many distinct colorings of C exist.

2. (NO COLLABORATION) Let G be a permutation group of a set X, such that $|X| - 2 \geq |G| \geq 2$, that there are exactly two orbits, and that these orbits have the same size. Prove that there exists a permutation $g \in G$ that does not contain any cycles of length one in its cycle structure.

3. Consider a finite group G and two subgroup H, N of G. For $g \in G$, we write

$$gH = \{a \mid a = g \cdot h \text{ for some } h \in H\},$$
$$Hg = \{a \mid a = h \cdot g \text{ for some } h \in H\},$$
$$HN = \{a \mid a = h \cdot n \text{ for some } h \in H, n \in N\}.$$

A subgroup H of G is a *normal subgroup* if it satisfies $gH = Hg$ for every $g \in G$.

Let H, N be two normal subgroups of G such that their orders are relatively prime. Prove that for any $x \in H$ and $y \in N$, we have $xy = yx$ (hint: what is $N \cap H$?).

4. Consider the sequence with the initial conditions $a_0 = 2$, $a_1 = 8$, and the recurrence relation $a_{i+2} = \sqrt{a_{i+1}a_i}$. Use generating functions to find the value of $\lim_{n\to\infty} a_n$. Specifically, use the techniques that we saw in Lecture 23 and write down the steps of you calculation. Do not use a computer, although you may skip writing uninteresting technical steps like the detailed steps of solving a set of linear equations. Do not any techniques that were learned after Lecture 23.

*The awesome students who helped correcting this assignment: Xiangyi Huang