Ma/CS 6a

Class 5: Basic Counting

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

By Adam Sheffer

No Collaboration Problems

- Every assignment will have (at most) one problem marked *NO COLLABORATION* .
  - These are problems that you are supposed to do on your own.
  - No asking for hints in office hours either (asking for clarifications is OK).
  - Usually medium difficulty problems.

Permutations

- **Problem.** Given a set $\{1,2,\ldots,n\}$, in how many ways can we order it?
- **The case $n = 3$.** Six distinct orders / permutations: 123, 132, 213, 231, 312, 321.
- **The general case.**

$$\underbrace{n}_{\text{Options for placing 1}} \cdot \underbrace{(n-1)}_{\text{Options for placing 2}} \cdot \cdots \cdot 2 \cdot \underbrace{1}_{\text{Options for placing } n}$$

$$n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1$$

Total Number of Subsets

- **Problem.** How many subsets does the set $S = \{1,2,\ldots,n\}$ have?
  - Two options for every element $i \in S$. Either i is in the subset or not.
  - Since there are n element in S, the number of subsets is $2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$.

Subsets of Size k

- Given a set $\{1,2,\dots,n\}$, how many (unordered) subsets of size k does it have?
- **Example.** Consider the case $n = 5$ and $k = 3$.
  - The possible subsets are $(1,2,3)$, $(1,2,4)$, $(1,2,5)$, $(1,3,4)$, $(1,3,5)$, $(1,4,5)$, $(2,3,4)$, $(2,3,5)$, $(2,4,5)$, $(3,4,5)$.
  - 10 distinct subsets!

Subsets of Size k (cont.)

- Given a set $S = \{1,2,\dots,n\}$, how many (unordered) subsets of size k does it have?
- Look at the $n!$ orderings of S and consider the first k numbers as the subset.
  - For example, when $n = 5$ and $k = 3$
  - **123**45 **342**51
  - **135**24 **341**52
  - **543**21 **135**42

Binomial Coefficients

- Given a set $S = \{1,2,\dots,n\}$, how many (unordered) subsets of size k does it have?
- Look at the $n!$ orderings of S and consider the first k numbers as the subset.
  - Every subset is obtained $k!\,(n-k)!$ times, so

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Pronounced "n choose k"

Warm-up Problem

- **Prove or disprove.** For every $n \geq k \geq 0$

$$\binom{n}{k} = \binom{n}{n-k}.$$

- **True.** Deciding which k elements to choose is like deciding which $n-k$ elements not to take.

Pascal's Inequality

- **Prove.** For every $n \geq k \geq 0$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

# of subsets containing 1

# of subsets not containing 1

Pascal's Triangle

- Pascal's inequality: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
- $\binom{n}{k}$ is element $k+1$ of row $n+1$.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Every number is the sum of the two numbers above it.

A Sum of Binomial Coefficients

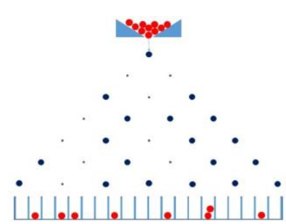
- **Prove.** For every $n \geq k \geq 0$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

- The left-hand side is the number of subsets of $\{1,2,3, \dots, n\}$, which is 2^n.

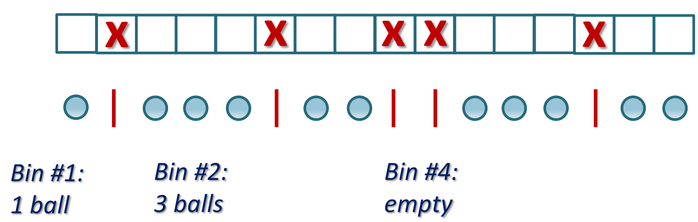
Partitioning into k Subsets

- **Problem.** For $n \geq k \geq 0$, we have n identical balls and k bins. In how many can place the balls in the bins?
- **Exmple.** If we have three balls and two bins, there are four options: (3,0), (2,1), (1,2), (0,3).



Partitioning into k Subsets

- **Problem.** For $n \geq k \geq 0$, we have n identical balls and k bins. In how many can place the balls in the bins?
- **Answer.** $\binom{n+k-1}{k-1}$. The $k-1$ choices correspond to the end of each bin.



The Binomial Theorem

- **Recall.**
  - $(x+y)^2 = x^2 + 2xy + y^2$.
  - $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.
- **The binomial theorem.** What is $(x+y)^n$?

$$\sum_{\substack{0 \leq i,j \leq n \\ i+j=n}} \binom{n}{i} x^i y^j$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots$$

The Binomial Theorem and Pascal's Triangle

$$(x+y)^1 = x + y$$
$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$\cdots$$

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The Binomial Theorem – Proof

- **The binomial theorem.**
$$(x+y)^n = \sum_{0 \le i \le n} \binom{n}{i} x^i y^{n-i}.$$
- **Proof.** We have
$$(x+y)^n = (x+y)(x+y)\cdots(x+y).$$
- The coefficient of $x^i y^{n-i}$ is the number of ways to choose x from i of the parentheses and y from the remaining ones.
- That is, the coefficient of $x^i y^{n-i}$ is $\binom{n}{i}$.

Monomials and Degrees

- Polynomials are sums of ***monomials***:

$$x^7 + 3x^2y^4z + 5x^3z^3 + \cdots$$

- The ***degree of a monomial*** is the sum of the powers of its variables.

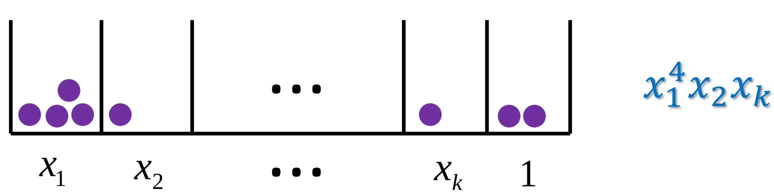
$$\deg(3x^2y^4z) = 2 + 4 + 1 = 7\,.$$

- The ***degree of a polynomial*** is the maximum of the degrees of its monomials

$$\deg(x^5 + 3x^2y^4z + 5x^3z^3) = 7$$

Number of Monomials

- **Problem.** How many distinct monomials can a polynomial of degree D in k variables have?
- **Answer.** Take $k + 1$ bins – one for every variable and one extra. Every placement of D balls in the bins corresponds to a monomial.



Number of Monomials

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$$\binom{D+k}{k}$$

Returning to Lecture 3

- To prove "Fermat's little theorem", we assumed, without proof, that for any prime p

$$(a+b)^p \equiv a^p + b^p \bmod p.$$

- **Proof.** By the binomial theorem:

$$(a+b)^p = \binom{p}{0} a^p + \binom{p}{1} a^{p-1} b + \binom{p}{2} a^{p-2} b^2 + \cdots$$

- To prove the claim, it suffices to prove that $p \mid \binom{p}{i}$ for every $1 \le i \le p-1$.
- This holds since in $\binom{p}{i} = \frac{p!}{i!(p-i)!}$ the numerator is divisible by p but the denominator is not.

Partitions of an Integer

- r, n – two positive integers.
- **Problem.** What is the number of solutions of

$$a_1 + a_2 + \cdots + a_r = n,$$

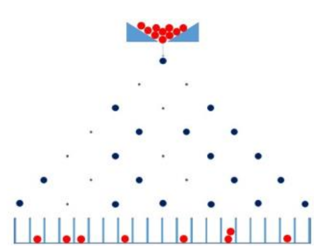
where each a_i is a natural number?

$$5 = 1 + 1 + 3 = 1 + 3 + 1 = 0 + 0 + 5 = 1 + 0 + 4 = \cdots$$

Solution

- Consider n as a sum of n unit elements.
- Dividing these elements across the r variables a_i is equivalent to placing n balls in r bins.
  - The value of a_i is the number of balls in the i'th bin.

$$\binom{n+r-1}{r-1}$$



Another Inequality

- **Problem.** Prove the identity

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

- **Proof.**
  - We begin with the identity
  $$(1+x)^n(1+x)^n = (1+x)^{2n}.$$
  - By the binomial theorem, we have

$$\left(\binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n\right)\left(\binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n\right) = \left(1 + \binom{2n}{1}x + \cdots + \binom{2n}{2n}x^{2n}\right).$$

Proof (cont.)

$$\left(\binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n\right)\left(\binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n\right) = \left(1 + \binom{2n}{1}x + \cdots + \binom{2n}{2n}x^{2n}\right).$$

- Consider the coefficient of x^n on each side.
  - On the right hand side, it is $\binom{2n}{n}$.
  - On the left hand side, it is

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

Summing Up

- In how many ways can we choose k elements from $\{1,2,3,\ldots,n\}$?

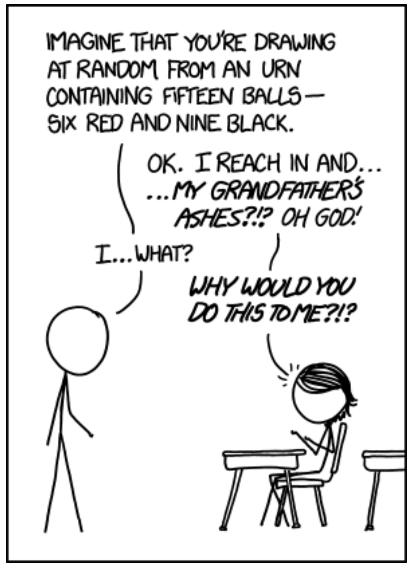
	Ordered	Unordered
No repetitions	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$
With repetitions	n^k	$\binom{n+k-1}{k-1}$

Summing Up #2

- In how many ways can we place k balls into n bins?

	At most 1 ball in each bin	Any number of balls in each bin
Each ball has a different color	$\frac{n!}{(n-k)!}$	n^k
Balls are indistinguishable	$\binom{n}{k}$	$\binom{k+n-1}{n-1}$

The End


IMAGINE THAT YOU'RE DRAWING AT RANDOM FROM AN URN CONTAINING FIFTEEN BALLS— SIX RED AND NINE BLACK.
OK. I REACH IN AND... ...MY GRANDFATHER'S ASHES?!? OH GOD!
I...WHAT?
WHY WOULD YOU DO THIS TO ME?!?