Problem 1

Let

$$x = 2793$$
$$y = 1467$$

We will first find a composite witness for $x = 2793$. We have that $x - 1 = 2792 = 2^3 \cdot 349$. This gives us $s = 2$ and $d = 349$. We will choose our witness to be $a = 2$. Now we must verify (for equivalency, we are using mod 2793).

$$a^d = 2^{349} \equiv 1724 \not\equiv 1$$

For r = 0, we get the following

$$a^{2^r d} = a^d \equiv 1724 \not\equiv -1$$

For r = 1, we get the following

$$a^{2^r d} = a^{2d} \equiv 424 \not\equiv -1$$

For r = 2, we get the following

$$a^{2^r d} = a^{4d} \equiv 1024 \not\equiv -1$$

So we have that $a = 2$ is a composite witness for 2793.

We will now find a composite witness for $y = 1467$. We have that $y - 1 = 1466 = 2 * 733$. This gives us $s = 1$ and $d = 733$. We will choose our witness to be $a = 2$. Now we must verify (for equivalency, we are using mod 1467).

$$a^d = 2^{733} \equiv 1451 \not\equiv 1$$

For r = 0, we get the following

$$a^d = 2^{733} \equiv 1451 \not\equiv -1$$

So we have that $a = 2$ is a composite witness for 1467.

Problem 2

(a) We are trying to find $\varphi(p^s)$, where p is a prime number and $s \in \mathbb{N}\backslash\{0\}$. We can see that the answer is

$$\varphi(p^s) = p^s - p^{s-1}$$

This is because there are p^{s-1} numbers that share a common factor with p^s. These factors are $1 \cdot p, 2 \cdot p, 3 \cdot p, ..., p^{s-1} \cdot p$.

(b) Given a number n, we have that

$$\varphi(n) = n \cdot \prod_{p|n}(1 - \frac{1}{p})$$

So given ab, we have that

$$\varphi(ab) = ab \cdot \prod_{p|ab}(a - \frac{1}{p})$$

But a and b are relatively prime, which means that for every p, p can only have a common factor with only one of a or b (never both). So because of this we have that

$$\varphi(ab) = a \cdot \prod_{p|a}(1 - \frac{1}{p}) \cdot b \cdot \prod_{q|b}(1 - \frac{1}{q})$$

But this is just equivalent to

$$\varphi(ab) = \varphi(a) \cdot \varphi(b)$$

So we have that $\varphi(ab) = \varphi(a) \cdot \varphi(b)$ when a and b are relatively prime.

(c) We have that all the numbers we are multiplying together $(p_1^{s_1}, p_2^{s_2}, \cdots, p_k^{s_k})$ are relatively prime, that $p_1, p_2, \cdots, p_k$ are prime numbers, and that $s_1, s_2, \cdots, s_k \in \mathbb{N}\backslash\{0\}$. Thus, using parts (a) and (b), we have that the answer is simply

$$\varphi(p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}) = \varphi(p_1^{s_1})\varphi(p_2^{s_2}) \cdots \varphi(p_k^{s_k})$$
$$\varphi(p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}) = (p_1^{s_1} - p_1^{s_1 - 1})(p_2^{s_2} - p_2^{s_2 - 1}) \cdots (p_k^{s_k} - p_k^{s_k - 1})$$

Problem 3

Writing $(x-y)^n$ using binomial coefficients, we get

$$(x-y)^n = \binom{n}{0}x^n - \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 - \binom{n}{3}x^{n-3}y^3 + \binom{n}{4}x^{n-4}y^4 + \cdots + (-1)^n \cdot \binom{n}{n}x^0y^n$$

Assigning $x = 1$ and $y = -1$, we get

$$(x-y)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \binom{n}{4} + \cdots + \binom{n}{n}$$

For any such sequence of binomial coefficients $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \binom{n}{4} + \cdots + \binom{n}{n}$ as we have on the right hand side, we have that $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$ divides it by 2. We can see this is true by induction. Take $n = 2$ to be the base case. Then we have that the binomial coefficient series for this is $\binom{2}{0} + \binom{2}{1} + \binom{2}{2}$. We can see that $\binom{2}{0} + \binom{2}{2} = 2 = \binom{2}{1}$. So the base case is satisfied. Now assume that our statement is true for some n. Now we want to show it is true for $n+1$. So consider

$$\binom{n+1}{0} + \binom{n+1}{1} + \binom{n+1}{2} + \binom{n+1}{3} + \binom{n+1}{4} + \cdots + \binom{n+1}{n+1}$$

This becomes

$$\left[\binom{n}{-1} + \binom{n}{0}\right] + \left[\binom{n}{0} + \binom{n}{1}\right] + \left[\binom{n}{1} + \binom{n}{2}\right] + \left[\binom{n}{2} + \binom{n}{3}\right] + \cdots + \left[\binom{n}{n} + \binom{n}{n+1}\right]$$

by Pascal's rule, which is equivalent to

$$2\left[\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}\right]$$

So half of this is just

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

which is just

$$\binom{n+1}{0} + \binom{n+1}{2} + \binom{n+1}{4} + \cdots$$

So we have proved our statement by induction. Finally, since we chose $x = 1$, $y = -1$, we can conclude that

$$(x-y)^n = 2^n = 2\left[\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots\right]$$

Problem 4

So we have a set of numbers $\{1, 2, 3, ..., n\}$. From this set we will take the set $\{1, 2, 3, ..., n-k+1\}$. Then from this set will will take sets of k numbers $\{a_1, a_2, a_3, ..., a_k\}$. Then we will map these sets to sets like so: $\{a_1, a_2+1, a_3+2, ..., a_k+k-1\}$. We clearly have that this mapping gets rid of consecutive elemnents. We also have that this mapping creates subsets of $\{1, 2, 3, \cdots, n\}$, since the largest number in the set we were mapping from was $n-k+1$. Finally, we have that this mapping is exhaustive in creating subsets of $\{1, 2, 3, \cdots, n\}$ of size k that do not contain two consecutive elements, since given any arbitrary such subset, our mapping can clearly create it. Another way to see this is that if we have a subset of $\{1, 2, 3, \cdots, n\}$ of size k that does not contain two consecutive elements, we can get back to a subset of $\{1, 2, 3, \cdots, n-k+1\}$ by applying the reverse of our transformation (and vice versa). So overall, the number of subsets of size k that do not contain two consecutive elements is $\binom{n-k+1}{k}$, since our mapping works on all k size subsets from the set $\{1, 2, 3, \cdots, n-k+1\}$.

Problem 5

Given $k_i = f(i) - f(i-1)$, $k_0 = f(0)$, and $k_{n+1} = n - f(n)$, we have that $\sum_{i=0}^{n+1} k_i = n$. We can see that this is analogous to dividing n balls up into $n+2$ bins, where we can place any number of balls in each bin. This is just $\binom{n+n+2-1}{n+2-1} = \binom{2n+1}{n+1}$. That is, there are $\binom{2n+1}{n+1}$ ways to put n balls into $n+2$ bins, where here our bins represent k_i's. In other words, there are $\binom{2n+1}{n+1}$ ways to choose our k_i values, where all the k_i values will be non-negative (since we can put at the very least 0 balls in each "bin"). And choosing a set of non-negative k_i values is the same as choosing a monotonically increasing function, given how we defined our k_is. So we have that there are $\binom{2n+1}{n+1}$ monotonically increasing functions.