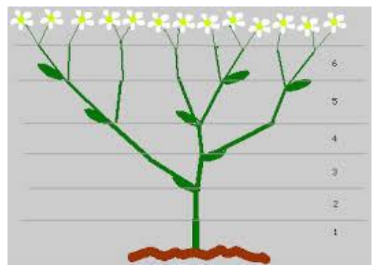
Ma/CS 6a

Class 23: Generating Functions



By Adam Sheffer

Recall: Power Series

- We define sums and products of power series as in the case of polynomials.
  - $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$
  - $B(x) = b_0 + b_1 x + b_2 x^2 + \cdots$

$$A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$$

$$A(x)B(x) = (a_0 b_0) + (a_1 b_0 + a_0 b_1)x + (a_2 b_0 + a_1 b_1 + a_0 b_2)x^2 + \cdots$$

Reminder: Inverse Elements

- **Theorem.** A power series
  $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots \in \mathbb{R}[[x]]$
  has an inverse if and only if $a_0 \neq 0$.

- **Example.**
  - What is the inverse of $1 - x$?

$$(1 - x)A(x) = 1.$$
$$A(x) = 1 + x + x^2 + x^3 + \cdots$$

Negative Exponents Formula

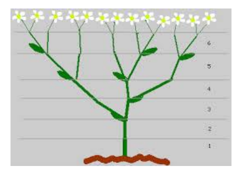
- **Theorem.** For any positive integer m,

$$(x+1)^{-m} = \sum_{n \geq 0} (-1)^n \binom{m+n-1}{n} x^n$$

- **Examples.**
  - $(x+1)^{-1} = \sum_{n\geq 0} (-1)^n x^n = 1 - x + x^2$
    $- x^3 + \cdots$
  - $(x+1)^{-2} = \sum_{n\geq 0} (-1)^n (n+1) x^n = 1 - 2x$
    $+ 3x^2 - 4x^3 + \cdots$

Generating Functions

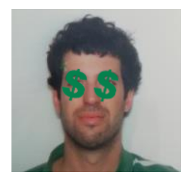
- Given an infinite sequence of numbers a_0, a_1, a_2 ..., the *generating function* of the sequence is the power series
  $$a_0 + a_1 x + a_2 x^2 + \cdots$$
- **Example.**
  - Recall the *Fibonacci numbers*:
    $F_0 = F_1 = 1 \qquad F_i = F_{i-1} + F_{i-2}.$
  - The corresponding generating function is
    $1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$



Why Mathematicians Should ***Not*** Watch Disney's 1939 Snow White and the Seven Dwarfs

Helping Adam Make Money

- **Problem.** The story of the millionaire Adam:
  - He started with nothing, and after working hard for a year managed to get 1$.
  - After the second year he had 5$.
  - Afterwards, at the beginning of each year Adam bought assets of value six times his worth at the beginning of the previous year.
  - At the end of each year Adam sold these assets for four times his worth at the beginning of the year.
  - *How many years did it take Adam to become a millionaire?*



Rephrasing with Generating Functions

- Consider the generating function of the problem $A(x) = a_0 + a_1x + a_2x^2 + \cdots$ (a_i is the money at the end of i\`th year).
- We already know

$$a_0 = 0, \qquad a_1 = 1, \qquad a_2 = 5.$$

- What other information do we have?

$$a_i = a_{i-1} - 6a_{i-2} + 4a_{i-1}, \qquad \text{for } i \geq 2.$$

- Rearranging and replacing i with $i + 2$:

$$a_{i+2} - 5a_{i+1} + 6a_i = 0.$$

Using the Recurrence Relation

- By applying the *recurrence relation* $a_{i+2} - 5a_{i+1} + 6a_i = 0$, we have

$$\begin{aligned} A(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ &= 0 + x + x^2(5a_1 - 6a_0) \\ &+ x^3(5a_2 - 6a_1) + \cdots \\ &= x + 5(a_1 x^2 + a_2 x^3 + \cdots) \\ &- 6(a_0 x^2 + a_1 x^3 + \cdots) \\ &= x + 5xA(x) - 6x^2 A(x). \end{aligned}$$

Looking for $A(x)$

- We have $A(x) = x + 5xA(x) - 6x^2A(x)$.
- That is,

$$A(x) = \frac{x}{1 - 5x + 6x^2} = \frac{1}{1-3x} - \frac{1}{1-2x}.$$

- Last time we proved

$$(1 - ax)^{-m} = \sum_{n \geq 0} \binom{m+n-1}{n} a^n x^n.$$

  - $(1-2x)^{-1} = \sum_{n\geq 0} 2^n x^n$.
  - $(1-3x)^{-1} = \sum_{n\geq 0} 3^n x^n$.

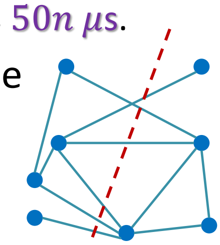
Solving the Problem

$$A(x) = \frac{1}{1-3x} - \frac{1}{1-2x}$$
$$= \sum_{n\geq 0} 3^n x^n - \sum_{n\geq 0} 2^n x^n$$
$$= \sum_{n\geq 0} (3^n - 2^n) x^n.$$

- That is, after the i`th year, Adam had $3^n - 2^n$. Millionaire after the 13th year.

Generating Functions and Algorithms

- We wish to compute some property of a graph.
  - If the graph has one vertex, we can compute this property in $1\ \mu s$.
  - If the graph has $n > 1$ vertices, we cut the graph into two parts of $\sim n/2$ points that can be handled separately. This takes $50n\ \mu s$.
- How long does it take to handle a graph with n vertices?
  - Solve $a_n = 50n + 2a_{n/2}$ and $a_1 = 1$.



Alan Turing

- English mathematician. Known for:
  - Invented the *Turing machine* (and thus helped formalizing the idea of algorithms).
  - Discovered the *halting problem*.
  - One of the main breakers of the *Enigma code* in World War II.
  - Invented the *Turing test* (and thus sometimes considered as the father of artificial intelligence).

Alan Turing's Death

- Turing was obsessed with Disney's *Snow white and the seven dwarfs*.
- He especially liked citing the wicked witch's lines about giving snow white the poisoned apple.
- On June 8th 1954, Turing committed suicide by biting a poisoned apple (he injected cyanide into it).

Recap: Using Generating Functions

- Solving problems via generating functions:
  - In the problem, we identify the first few values of $a_0, a_1, a_2, \ldots$, and also a recursion relation.
  - We use these to obtain an equation of the form $A(x) = x + 5xA(x) - 6x^2A(x)$.
  - Isolate $A(x)$ to obtain an expression of the form $A(x) = \frac{x}{1-5x+6x^2}$.
  - *Simplify the expression* to obtain a sum of "simple" terms, each which can be written as a power series.

A Problematic Step

- In the millionaire question, we had the step $\frac{x}{1-5x+6x^2} = \frac{1}{1-3x} - \frac{1}{1-2x}$.
- Can we rewrite every fraction as a sum of *"nice"* parts?
  - Not when working over $\mathbb{R}$:

$$\frac{1}{x^2+2x+8} = ???.$$

  - If we work over $\mathbb{C}$, every polynomial is a product of linear terms:

$$x^2 + 2x + 8 = \left(x + 1 + i\sqrt{7}\right)\left(x + 1 - i\sqrt{7}\right).$$

Polynomial Division

- **Recall.** Given two integers $a, b \in \mathbb{Z}$, there are *unique* $q, r \in \mathbb{Z}$ such that $r < b$ and
$$a = qb + r.$$

- **The polynomial variant.** Given two polynomials $a(x)$ and $b(x) \neq 0$, there are *unique* $q(x), r(x)$ such that
$$a(x) = q(x)b(x) + r(x).$$
and either $\deg r(x) < \deg b(x)$ or $r(x) = 0$.

Example: Polynomial Division

- Divide $a(x) = x^4 + 3x^3 - 2x^2 + 5$ by $b(x) = x^2 - 2x + 1.$

$$a(x) = q(x)b(x) + r(x)$$

$$a(x) = x^2 \cdot b(x) + 5x^3 - 3x^2 + 5$$

$$a(x) = (x^2 + 5x) \cdot b(x) + 7x^2 - 5x + 5$$

$$a(x) = (x^2 + 5x + 7) \cdot b(x) + 9x - 2$$

$q(x) = x^2 + 5x + 7$ and $r(x) = 9x - 2$

Larger Numerator

- We wish to write expressions of the form $\frac{a(x)}{b(x)}$ as sums of "simple" terms.
- If $\deg a(x) \geq \deg b(x)$, by the division property we can rewrite the expression as $q(x) + \frac{r(x)}{b(x)}$.
- Thus, it suffices to consider cases where $\deg a(x) < \deg b(x)$.

GCD of Polynomials

- A polynomial $a(x)$ is a ***divisor*** of a polynomial $b(x)$ if there exists a polynomial $f(x)$ such that
$$b(x) = a(x)f(x).$$
- The ***greatest common divisor*** of two polynomials $a(x), b(x)$ (denoted $\mathrm{GCD}(a,b)$) satisifies:
  - $\mathrm{GCD}(a,b)$ is a divisor of both $a(x)$ and $b(x)$.
  - Any other divisor of both $a(x)$ and $b(x)$ is also a divisor of $\mathrm{GCD}(a,b)$.

GCD Property

- **Recall.** For any $a, b \in \mathbb{Z}$, there exist $s, t \in \mathbb{Z}$ such that
$$\mathrm{GCD}(a, b) = as + bt.$$

- **Theorem.** For any two polynomials $a(x), b(x)$, there exist polynomials $s(x), t(x)$ such that
$$GCD(a, b) = a(x)s(x) + b(x)t(x).$$

$$\mathrm{GCD}(x^4(x-1)^2(x+1), x^5(x+1)^3(x+4)^2)$$
$$= x^4(x+1).$$

Partial Fractions

- **Theorem.** Consider polynomials $a(x), b(x)$ so that
  - $\deg a(x) < \deg b(x)$.
  - $b(x)$ has a nonzero constant term.
  - $b(x) = s(x)t(x)$, such that $\mathrm{GCD}(s, t) = 1$.

  Then there exist $f(x), g(x)$ such that
$$\frac{a(x)}{b(x)} = \frac{f(x)}{s(x)} + \frac{g(x)}{t(x)},$$

$\deg f(x) < \deg s(x)$ and $\deg g(x) < \deg t(x)$.

Example: Partial Fractions

- Consider the expression
$$\frac{a(x)}{b(x)} = \frac{5-3x}{x^2-3x+2}.$$
  - $b(x)$ has a nonzero constant term.
  - $\deg b(x) > \deg a(x)$.
  - $b(x) = (x-1)(x-2)$.

$$\frac{a(x)}{b(x)} = \frac{5-3x}{x^2-3x+2} = -\frac{2}{x-1} - \frac{1}{x-2}.$$

Proof

- For simplicity, we write $b = st$ (etc.).
  - Since $GCD(s,t) = 1$, there exist polynomials u, v such that
$$1 = su + tv \quad \Rightarrow \quad a = asu + atv.$$
  - Dividing av by s, we have
$$av = qs + r,$$
  where the ***remainder*** satisfies $\deg r < \deg s$.

$$\frac{a}{b} = \frac{asu + atv}{st} = \frac{asu + t(qs + r)}{st}.$$

Proof (cont.)

- We have $\frac{a}{b} = \frac{asu+t(qs+r)}{st}$, where $\deg r < \deg s$.
- Setting $f = r$ and $g = au + tq$, we have
$$\frac{a}{b} = \frac{asu + t(qs + r)}{st} = \frac{sg + tf}{st} = \frac{f}{s} + \frac{g}{t}.$$
- It remains to bound $\deg f$ and $\deg g$:
  - $\deg f = \deg r < \deg s$.
  - $\deg ft < \deg st = \deg b$.
  - Since $a = tf + sg$, we have
  $\deg sg = \deg(a - ft) < \deg b = \deg st$.

Using Partial Fractions

- We wish to decompose $\frac{a(x)}{b(x)}$ where
$\deg a(x) < \deg b(x)$ and
$b(x) = p_1(x)^{m_1} p_2(x)^{m_2} \cdots p_d(x)^{m_d}$.
- By repeatedly applying the partial fractions technique, we obtain
$$\frac{a(x)}{b(x)} = \frac{h_1(x)}{p_1(x)^{m_1}} + \frac{h_2(x)}{p_2(x)^{m_2}} + \cdots + \frac{h_d(x)}{p_d(x)^{m_d}}$$

(where $\deg h_i(x) < m_i \deg p_i(x)$).

Recap: Simplifying $a(x)/b(x)$

- We decompose $a(x)/b(x)$ to
$$\frac{a(x)}{b(x)} = \frac{h_1(x)}{p_1(x)^{m_1}} + \frac{h_2(x)}{p_2(x)^{m_2}} + \cdots + \frac{h_d(x)}{p_d(x)^{m_d}}.$$
- Assume that $p_i(x)$ is ***linear***. That is, $p_i(x) = (ax + b)$. Then we have
$$\frac{h_i(x)}{p_i(x)^{m_i}} = \frac{h_i(x)}{a^{m_i}}(x + b/a)^{-m_i},$$
and we already know how to compute $(x + b/a)^{-m_i}$.

Kurt Gödel

- An Austrian mathematician.
  - Considered as one of the top logicians ever.
  - Famous for his two ***incompleteness theorems*** (which also pushed Turing to come up with the Turing machine and the halting problem).

The Death of Kurt Gödel

- Gödel was also obsessed with Disney's snow white and the seven dwarfs. He used to try to convince his good friend Albert Einstein to see the movie with him.
  - Due to the movie, Gödel became paranoid about people trying to poison his food. He only agreed to eat his wife's cooking.
  - When his wife was hospitalized for several months, he *died of starvation*.

Who will be Next?

Stop Disney before it is too late!

Finding the Power Series

- **Problem.** Find the power series of
$$G(x) = \frac{12x^2 - 24x + 4}{9x^3 - 9x^2 - x + 1}.$$
- **Solution.**
  - First we factorize
  $9x^3 - 9x^2 - x + 1 = (1 - x)(1 - 3x)(1 + 3x)$.
  - Thus, we would like to rewrite the expression:
  $$\frac{12x^2 - 24x + 4}{9x^3 - 9x^2 - x + 1} = \frac{A}{1 - x} + \frac{B}{1 - 3x} + \frac{C}{1 + 3x}.$$

Finding the Partial Fractions

- We would like to solve
$$\frac{12x^2 - 24x + 4}{9x^3 - 9x^2 - x + 1} = \frac{A}{1 - x} + \frac{B}{1 - 3x} + \frac{C}{1 + 3x}.$$
- Multiply both sides by $(1 - x)(1 - 3x)(1 + 3x)$:
$$\begin{aligned} 12x^2 - 24x + 4 \\ &= A(1 - 3x)(1 + 3x) + B(1 - x)(1 + 3x) \\ &+ C(1 - x)(1 - 3x). \end{aligned}$$
- Equating coefficients of $x^2, x, 1$, we obtain
$$\begin{aligned} 12 &= -9A - 3B + 3C, \\ -24 &= 2B - 4C, \\ 4 &= A + B + C. \end{aligned}$$

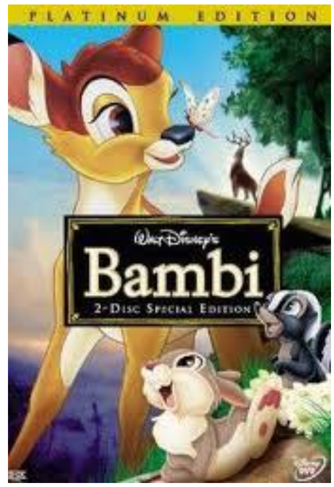
Finding the Partial Fractions (cont.)

- We would like to solve
$$\frac{12x^2-24x+4}{9x^3-9x^2-x+1}=\frac{A}{1-x}+\frac{B}{1-3x}+\frac{C}{1+3x}.$$
- We have
$$12=-9A-3B+3C,$$
$$-24=2B-4C,$$
$$4=A+B+C.$$
- The solution to the system is $A=1, B=-2, C=5$, so
$$G(x)=\frac{12x^2-24x+4}{9x^3-9x^2-x+1}=\frac{1}{1-x}-\frac{2}{1-3x}+\frac{5}{1+3x}.$$

The Power Series

- We have $G(x)=(1-x)^{-1}-2(1-3x)^{-1}+5(1+3c)$. Recall that
$$(1-ax)^{-1}=\sum_{n\geq 0}a^nx^n.$$
$$(1-x)^{-1}=\sum_{n\geq 0}x^n, \qquad (1-3x)^{-1}=\sum_{n\geq 0}3^nx^n,$$
$$(1+3x)^{-1}=\sum_{n\geq 0}(-3)^nx^n.$$
- We thus have
$$G(x)=\sum_{n\geq 0}(1-2\cdot 3^n+5(-3)^n)x^n$$

Next Week…


PLATINUM EDITION
Walt Disney's
Bambi
2-DISC SPECIAL EDITION