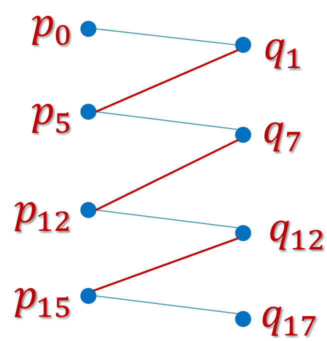
Ma/CS 6a

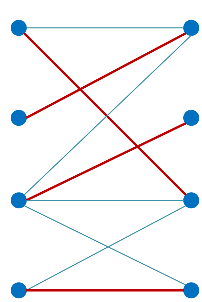
Class 12: More Matchings

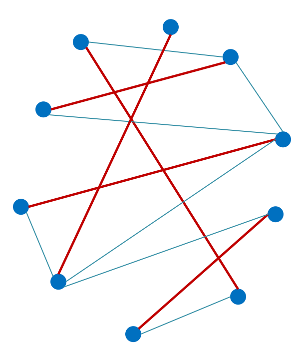


By Adam Sheffer

Reminder: Perfect Matchings

- A *perfect matching* of a graph $G = (V, E)$ is a matching of size $|V|/2$.





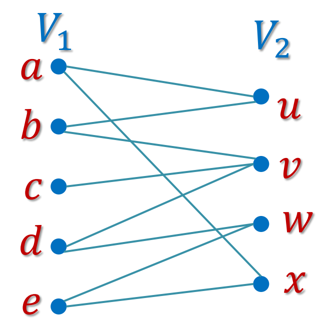
Reminder: Neighbor Sets

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.
- For any subset $A \subset V_1$, we define

$N(A) = \{y \in Y \mid (x, y) \in E \text{ for some } x \in A\}.$

$N(\{b, c, d\}) = \{u, v, w\}$

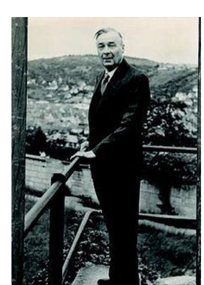
$N(\{a, e\}) = \{u, w, x\}$



Reminder: Variant of Hall's Theorem

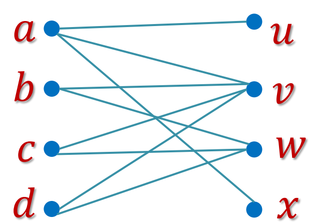
- **Theorem.** Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.
- There exists a matching of size $|V_1|$ in G if and only if for every $A \subset V_1$, we have $|A| \leq |N(A)|$.

Philip Hall



Deficiency

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.
- The ***deficiency*** of G is
$$\operatorname{def}(G) = \max_{A \subset V_1}\{|A| - |N(A)|\}.$$
- What is the deficiency of



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$A = \{b, c, d\}$

Deficiency

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.
- The ***deficiency*** of G is
$$\operatorname{def}(G) = \max_{A \subset V_1}\{|A| - |N(A)|\}.$$
- The deficiency cannot be smaller than 0 since when $A = \emptyset$ we have
$$|A| - |N(A)| = 0.$$

Deficiency and Maximum Matchings

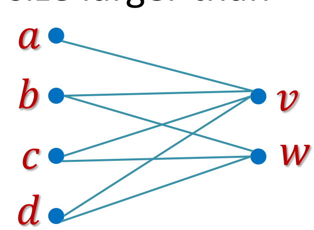
- **Theorem.** Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. The size of the maximum matching in G is

$$|V_1| - \mathrm{def}(G).$$

- This implies Hall's theorem.
  - $\mathrm{def}(G) = 0$ if and only if there exists a matching of size $|V_1|$.
  - When $|V_1| = |V_2|$, we have $\mathrm{def}(G) = 0$ if and only if there exists a perfect matching.

Proof: One Direction

- Set $d = \mathrm{def}(G)$.
- There exists a subset $A \subset V_1$ such that $|A| - |N(A)| = d$.
- In any matching of G, at least d vertices of A are unmatched.
- No matching can have size larger than $|V_1| - d$.



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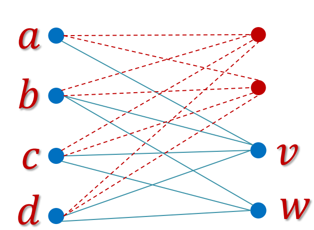
- Set $d = \mathrm{def}(G)$.
- There exists a subset $A \subset V_1$ such that $|A| - |N(A)| = d$.
- In any matching of G, at least d vertices of A are unmatched.
- No matching can have size larger than $|V_1| - d$.
- It remains to prove that a matching of this size does exist.

Proof: The Other Direction

- We add d new vertices to V_2.
  - We connect every new vertex to each vertex of V_1.
- Originally, every set $A \subset V_1$ satisfied $|A| \geq |N(A)| - d$.
  - Now $|A| \geq |N(A)|$.



Proof: The Other Direction

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  - We connect every new vertex to each vertex of V_1.
- Originally, every set $A \subset V_1$ satisfied $|A| \geq |N(A)| - d$.
  - Now $|A| \geq |N(A)|$.
- By the variant of Hall's theorem, there exists a matching M of size $|V_1|$.
- Removing the new vertices, we obtain a matching of G of size $|V_1| - d$.

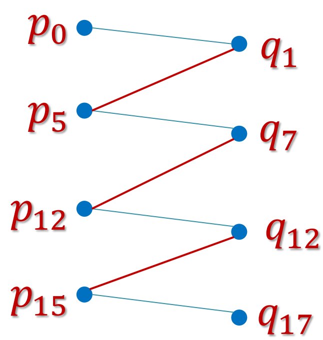
The Size of a Maximum Matching

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.
- Can we use the deficiency theorem to find the size of the maximum matching of M?

- We can check the deficiency of every subset $A \subset V_1$.
  - But there are $2^{|V_1|}$ such subsets!

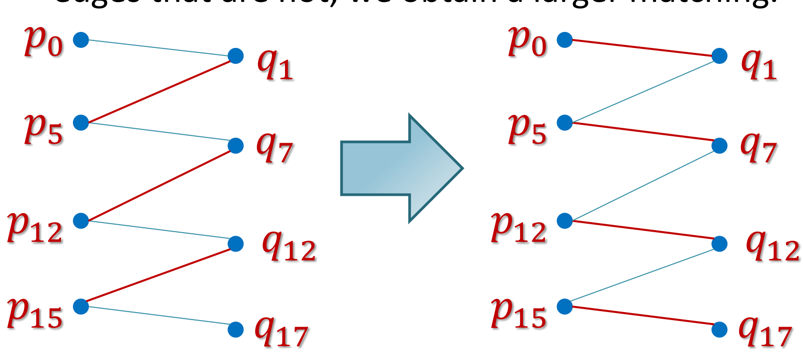
Alternating Paths

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, and let M be a matching of G.
- A path is *alternating* for M if every other edge of it is in M, and its two extreme vertices are not matched.



Alternating Paths

- A *maximal* path is *alternating* for M if every other edge of it is in M, and its two extreme vertices are not matched.
- By switching the edges that are in M with the edges that are not, we obtain a larger matching.

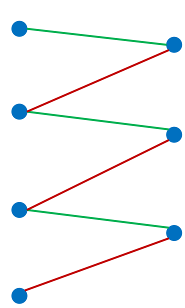


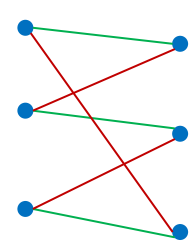
Existence of Alternating Paths

- **Theorem.** If a matching M in a bipartite graph $G = (V_1 \cup V_2, E)$ is not a maximum matching, then there exists an alternating path for M.
- **Proof.**
  - Let M^* be a maximum matching of G.
  - Let F be the set of edges that are either in M or in M^*, but not in both.
  - In the graph $G' = (V, F)$, every vertex is of degree at most two.

Example: The Graph G'

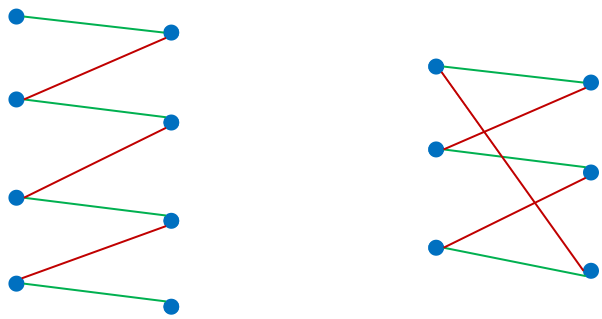
- The graph $G' = (V, F)$.
  - Every vertex has a degree of at most two.
  - The graph is a union of paths and cycles.





Finding an Alternating Path

- By definition, M^* has more edges than M.
- In at least one of the paths of G', M^* has more edges than M.
- This must be an alternating path for M!



Find a Maximum Matching

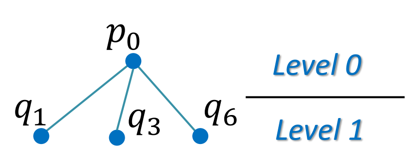
- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.
- Start with any matching M. A single edge is fine.
- Repeatedly find an alternating path for M and use it to obtain a larger matching.
- The process terminates after at most $|V_1|$ steps.

Finding an Alternating Path

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, and let M be a non-maximum matching.
- We wish to find whether there is an alternating path for M starting at a specific unmatched vertex $p_0 \in V_1$.
  - We run a variant of BFS from p_0.

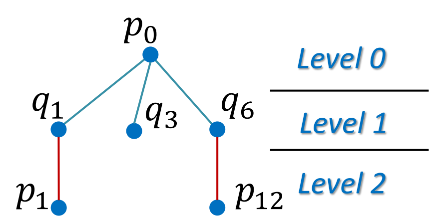
BFS Variant

- The root of the BFS tree is p_0.
- At the first level we have vertices that are adjacent to p_0 in G.



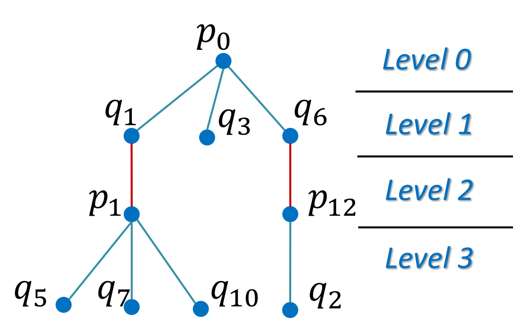
BFS Variant (2)

- For each vertex of level 1, if it is matched in M, we connect it to its match.



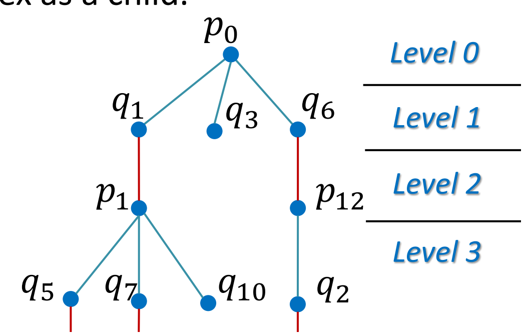
BFS Variant (3)

- For each vertex of level 2, we connect it (by edges not in M) to any of its neighbors in G that are not in the tree yet.



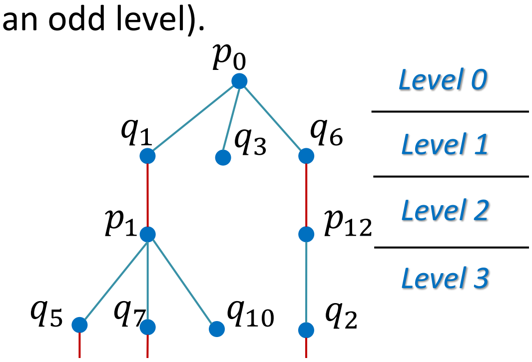
BFS Variant (4)

- We repeat this process:
  - Vertices of even levels (p_i's) have as their children every new vertex adjacent to them.
  - Vertices of odd levels (q_i's) have only their matching vertex as a child.



BFS Variant (5)

- How can we tell whether an alternating path for M starts at p_0?
  - Every such path corresponds to an unmatched vertex at an odd level of the tree (i.e., a leaf at an odd level).



Concluding Remarks

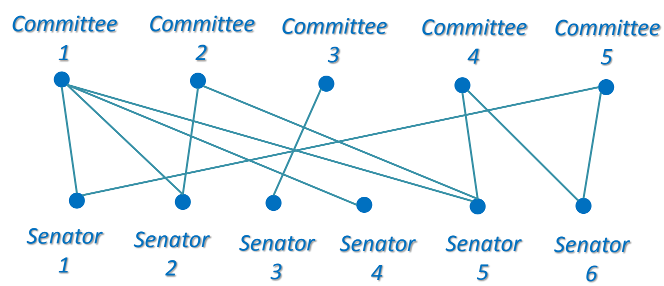
- Given a matching M in a bipartite graph $G = (V_1 \cup V_2, E)$, for every vertex of V_1 that is unmatched in M:
  - Run the BFS variant to check whether there is an alternating path starting from it.
- If no alternating paths were found – M is a maximum matching.
- Otherwise, we use the alternating path to obtain a larger matching.

A Committee of Committees

- The US senate has 20 committees and each senator may serve on several committees.
- The *committee of committees* should have a representative from each committee, and no senator is allowed to represent more than one committee.
- Is this always possible?
  - No! What if senator Bob is the only person on two committees?

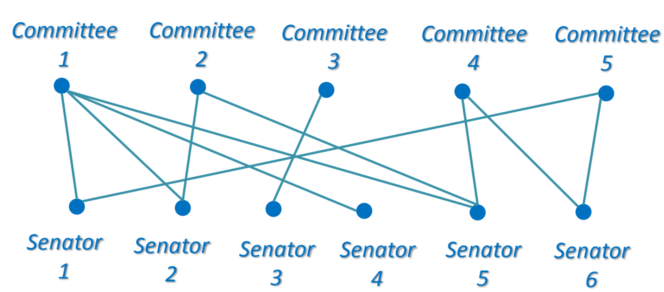
A Committee of Committees?

- How can we find out whether a committee of committees is possible?
  - Build a graph!



A Committee of Committees?

- A committee of committees is possible if the graph has a matching of size 20.

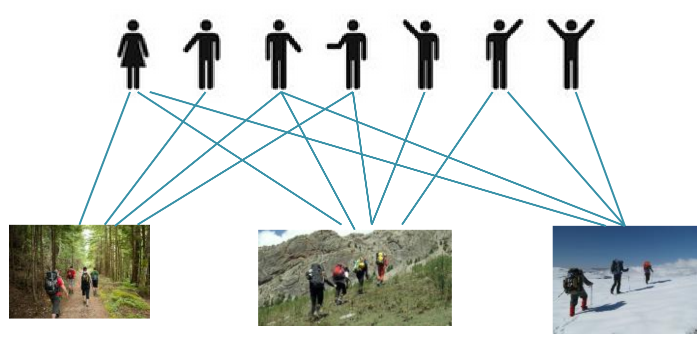


Problem: Retreat Resort

- **Problem.** A retreat resort currently has n guests staying in it. On Saturdays, the resort offers hikes with travelling guides.
  - Every guest has a list of hikes that he is interested in.
  - Every guide is allowed to take up to 5 people with him.
  - Describe an efficient algorithm that finds whether every guest can go on a hike that he is interested in.

Building a Graph

- Create a bipartite graph with a vertex for every guest and for every hike.
  - An edge between every guest and every hike that he is interested in.



Fixing the Graph

- A matching in the graph does not take into account that up to 5 people can go on a hike.
- Split every hike vertex v into five vertices, and connect each of them to each of the vertices that v was connected to.
- There is a valid hiking assignment if and only if the graph has a matching of size n.

The End