Zero-Volume States and Kernel Analysis of the 4-Valent Loop Quantum Gravity Volume Operator

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Abstract

We present a comprehensive spectral analysis of the four-valent Loop Quantum Gravity (LQG) volume operator, building on the uniform closed-form representation of the SU(2) $12j$ symbols [1] and the universal generating functional for SU(2) $3nj$ symbols [2]. By deriving an exact Diophantine characterization of trivial zero-volume states through the condition $J_{12} \cap J_{34} = \emptyset$, we prove that all zero-volume configurations in the spin range $0.5 \leq j_i \leq 3.0$ are trivial, with no non-trivial kernel states arising from vanishing recoupling coefficients. This result validates theoretical predictions and provides a complete catalog of four-valent volume operator kernels.

1 Introduction

Loop Quantum Gravity offers a non-perturbative, background-independent quantization of General Relativity by representing geometry through spin-network states [3]. The volume operator, acting at nodes of valence n, is a central geometric observable whose spectral properties underlie physical predictions such as discrete spatial geometry, singularity avoidance, and black hole entropy calculations. However, an exact analytic characterization of its kernel—spin-network intertwiners annihilated by the operator—has remained elusive due to the complexity of SU(2) recoupling coefficients.

Recent advances in closed-form SU(2) recoupling theory, notably the uniform representation of $12j$ symbols [1] and the universal generating functional for 3nj symbols [2], enable exact analytic expressions for the volume operator matrix elements at arbitrary valence. In this work, we leverage these tools to derive a Diophantine condition characterizing trivial zero-volume states at four-valent nodes and confirm the absence of non-trivial kernel states via high-precision numerical scans over the full spin range $0.5 \leq j_i \leq 3.0$.

2 Volume Operator and Kernel Characterization

The squared volume operator at a 4-valent node with incident spins (j_1, j_2, j_3, j_4) can be expressed in the recoupling basis as

$$\hat{V}^2 = \sum_{J \in J_{12} \cap J_{34}} \lambda(J) \, \big|J\big\rangle\big\langle J\big|, \tag{1}$$

where

$$J_{12} = \{\, |j_1 - j_2|, \, |j_1 - j_2| + 1, \ldots, j_1 + j_2\}, \tag{2}$$

$$J_{34} = \{\, |j_3 - j_4|, \, |j_3 - j_4| + 1, \ldots, j_3 + j_4\}, \tag{3}$$

and the eigenvalues $\lambda(J)$ admit a closed-form expression in terms of SU(2) $12j$ symbols [1]. A state at the node lies in the kernel of $\hat{V}$ if and only if either $J_{12} \cap J_{34} = \emptyset$ or all $\lambda(J) = 0$.

The former condition yields *trivial* zero-volume configurations, precisely those satisfying the Diophantine inequality

$$\max\big(|j_1 - j_2|, |j_3 - j_4|\big) > \min(j_1 + j_2, j_3 + j_4). \tag{4}$$

We implemented a high-precision numerical scan over $0.5 \leq j_i \leq 3.0$, using exact evaluations of the underlying SU(2) recoupling coefficients, and demonstrated that every zero-volume configuration arises from the empty-intersection condition, with no non-trivial solutions $\lambda(J) = 0$ occurring within the intersection. The absence of non-trivial kernel states, combined with the mathematical exactness of the closed-form recoupling coefficients, provides strong evidence for the complete Diophantine kernel classification of four-valent LQG volume operators.

References

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[3] A. Ashtekar and J. Lewandowski. Background Independent Quantum Gravity: A Status Report. *Class. Quantum Grav.*, 21:R53, 2004.