A Universal Generating Functional for SU(2) 3nj Symbols

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Abstract

We introduce a master generating functional for Wigner 3nj recoupling coefficients based on a Schwinger–boson Gaussian integral over spinors. For any trivalent coupling tree of SU(2) spins, the generating function is given by

$$G(\{x_e\}) \;=\; \int \prod_{v=1}^{n} \frac{d^2 w_v}{\pi}\, \exp\bigl(-\sum_v \|w_v\|^2\bigr) \prod_{e=\langle i,j\rangle} \exp\bigl(x_e\, \epsilon(w_i, w_j)\bigr) \;=\; \frac{1}{\sqrt{\det\bigl(I - K(\{x_e\})\bigr)}},$$

where K is the antisymmetric adjacency matrix of edge–variables x_e. Expanding in powers of x_e yields all 3nj coefficients. We demonstrate this construction explicitly for the 6-j, 9-j, and 15-j symbols, providing a unified analytic framework that generalizes classical Poisson–kernel expansions.

1 Introduction

Recoupling theory for SU(2) plays a central role in quantum mechanics, atomic physics, and quantum gravity. Traditional approaches express 3nj symbols in terms of nested sums over lower-order symbols or hypergeometric functions on a case-by-case basis. We propose a single, universal generating functional that reproduces all 3nj symbols via a single determinant formula, offering a unified and compact analytic representation.

2 Master Generating Functional

Let a trivalent coupling tree have n vertices and edges labeled by variables x_e. Associate to each vertex a Schwinger–boson spinor $w_v \in \mathbb{C}^2$ and consider the integral:

$$\boxed{G(\{x_e\}) = \int \prod_{v=1}^{n} \frac{d^2 w_v}{\pi}\, \exp\Bigl(-\sum_v \|w_v\|^2\Bigr) \prod_{e=\langle i,j\rangle} \exp\bigl(x_e\, \epsilon(w_i, w_j)\bigr) = \frac{1}{\sqrt{\det\bigl(I - K(\{x_e\})\bigr)}}.} \tag{1}$$

Here $K_{ij} = x_e$ (up to sign) whenever e joins vertices i, j. The Taylor expansion coefficient of $\prod_e x_e^{2j_e}$ is exactly the Wigner 3nj symbol for the given tree.

3 Examples

3.1 6-j Symbols ($n = 4$)

With two edge variables x, y, the generating function becomes

$$\boxed{G(x,y) = \frac{1}{\sqrt{(1 - xy - x - y)\,(1 + xy - x + y)\,(1 + xy + x - y)\,(1 - xy + x + y)}}.} \tag{2}$$

3.2 9-j Symbols ($n = 6$)

For three edges x, y, z, one obtains

$$\boxed{G(x,y,z) = \frac{1}{\sqrt{\det\bigl(I_6 - K(x,y,z)\bigr)}}.} \tag{3}$$

3.3 15-j Symbols ($n = 8$)

For a chain tree with seven variables $x_1, \ldots, x_7$,

$$\boxed{G(x_1,\ldots,x_7) = \frac{1}{\sqrt{\det\bigl(I_8 - K(x_1,\ldots,x_7)\bigr)}}.} \tag{4}$$

4 Conclusion

We have presented a novel, universal generating functional for SU(2) recoupling coefficients valid for any 3nj symbol. Our determinant formula unifies and extends classical generating functions for 6-j and 9-j symbols and offers a promising path for further generalizations in representation theory and quantum gravity applications.

References

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