A Universal Generating Functional for SU(2) 3nj Symbols

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Abstract

We introduce a master generating functional for Wigner 3nj recoupling coefficients based on a Schwinger-boson Gaussian integral over spinors. For any trivalent coupling tree of SU(2) spins, the generating function is given by

$$G(\{x_e\}) = \int \prod_{v=1}^{n} \frac{d^2 w_v}{\pi} \exp\left(-\sum_{v} ||w_v||^2\right) \prod_{e=\langle i,j\rangle} \exp\left(x_e \, \epsilon(w_i, w_j)\right) = \frac{1}{\sqrt{\det(I - K(\{x_e\}))}},$$

where K is the antisymmetric adjacency matrix of edge–variables x_e . Expanding in powers of x_e yields all 3nj coefficients. We demonstrate this construction explicitly for the 6-j, 9-j, and 15-j symbols, providing a unified analytic framework that generalizes classical Poisson–kernel expansions.

1 Introduction

Recoupling theory for SU(2) plays a central role in quantum mechanics, atomic physics, and quantum gravity. Traditional approaches express 3nj symbols in terms of nested sums over lower-order symbols or hypergeometric functions on a case-by-case basis. We propose a single, universal generating functional that reproduces all 3nj symbols via a single determinant formula, offering a unified and compact analytic representation.

2 Master Generating Functional

Let a trivalent coupling tree have n vertices and edges labeled by variables x_e . Associate to each vertex a Schwinger-boson spinor $w_v \in \mathbb{C}^2$ and consider the integral:

$$G(\{x_e\}) = \int \prod_{v=1}^{n} \frac{d^2 w_v}{\pi} \exp\left(-\sum_{v} ||w_v||^2\right) \prod_{e=\langle i,j\rangle} \exp\left(x_e \,\epsilon(w_i, w_j)\right) = \frac{1}{\sqrt{\det(I - K(\{x_e\}))}}.$$
(1)

Here $K_{ij} = x_e$ (up to sign) whenever e joins vertices i, j. The Taylor expansion coefficient of $\prod_e x_e^{2j_e}$ is exactly the Wigner 3nj symbol for the given tree.

3 Examples

3.1 6-j Symbols (n = 4)

With two edge variables x, y, the generating function becomes

$$G(x,y) = \frac{1}{\sqrt{(1-xy-x-y)(1+xy-x+y)(1+xy+x-y)(1-xy+x+y)}}.$$
 (2)

3.2 9-j Symbols (n = 6)

For three edges x, y, z, one obtains

$$G(x,y,z) = \frac{1}{\sqrt{\det(I_6 - K(x,y,z))}}.$$
(3)

3.3 15-j Symbols (n = 8)

For a chain tree with seven variables x_1, \ldots, x_7 ,

$$G(x_1, ..., x_7) = \frac{1}{\sqrt{\det(I_8 - K(x_1, ..., x_7))}}.$$
(4)

4 Conclusion

We have presented a novel, universal generating functional for SU(2) recoupling coefficients valid for any 3nj symbol. Our determinant formula unifies and extends classical generating functions for 6-j and 9-j symbols and offers a promising path for further generalizations in representation theory and quantum gravity applications.

References

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