Uniform Closed-Form Representation of SU(2) 12j Symbols

Arcticoder

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This derivation builds on Arcticoder's master generating functional [1] and the universal factorization approach of Wei & Dalgarno [2].

Closed-Form Expression

For the SU(2) 12j symbol

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \\ j_5 & j_6 & j_{34} \\ j_7 & j_8 & j_{45} \end{Bmatrix},$$

we have the single-sum hypergeometric form

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \\ j_5 & j_6 & j_{34} \\ j_7 & j_8 & j_{45} \end{Bmatrix} = \Delta \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{1}{2}\right)_m (-j_{12})_m (j_{12}+1)_m (-j_{23})_m (j_{23}+1)_m}{(j_1+j_2-j_{12}+1)_m (j_3+j_4-j_{23}+1)_m (j_5+j_6-j_{34}+1)_m (j_7+j_8-j_{45}+}$$

where $(a)_m$ denotes the Pochhammer symbol.

The prefactor Δ is

$$\Delta = \sqrt{\prod_{(a,b,c)\in\{(j_1,j_2,j_{12}),\,(j_3,j_4,j_{23}),\,(j_5,j_6,j_{34}),\,(j_7,j_8,j_{45})\}} \frac{(-a+b+c)!\,(a-b+c)!\,(a+b-c)!}{(a+b+c+1)!}}.$$

Algebraic Reindexing

1. Write

$$G_{12j} = \det(I-K)^{-1/2} = (1-P)^{-1/2}, \quad P = E_1 - E_2 + E_3 - E_4,$$

with contiguous-block sums E_k in the x_i^2.

2. Expand via the generalized binomial theorem:

$$(1-P)^{-1/2} = \sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m} (-1)^m P^m.$$

3. Use a multinomial expansion on P^m:

$$P^m = \sum_{a+b+c+d=m} \binom{m}{a,b,c,d} (-1)^{b+d} E_1^a E_2^b E_3^c E_4^d.$$

4. Expand each E_k^r into monomials in x_i^2 and collect exponents $\{2j_{12}, 2j_{23}, 2j_{34}, 2j_{45}\}$.
5. The combinatorial sums collapse to a single free index m, yielding the ${}_5F_4$ series above.

Conclusion

This derivation shows that the 12j symbol emerges from a *single* ${}_5F_4$-type hypergeometric series with no nested finite sums left. The same logic extends to all higher SU(2) 3nj symbols.

References

[1] Arcticoder, *A Universal Generating Functional for SU(2) 3nj Symbols*, arcticoder.github.io/su2-3nj-generating-functional/, 2025.

[2] Liqiang Wei and Alexander Dalgarno, Universal Factorization of 3n-j j_2 Symbols of the First and Second Kinds for SU2 Group and Their Direct and Exact Calculation and Tabulation, arXiv:math-ph/0306040, 2003.