Closed-Form Matrix Elements for Arbitrary-Valence SU(2) Nodes via Generating Functionals

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Abstract

We derive closed-form expressions for SU(2) operator matrix elements on arbitrary-valence nodes by extending the universal generating functional approach with source terms. Our central result is a determinant-based formula incorporating group-element dependence, which yields all matrix elements via a single Gaussian integral and hypergeometric expansion.

1 Introduction

The computation of SU(2) recoupling coefficients has seen recent advances: uniform closed-form representation of 12j symbols [1], a universal generating functional [2], closed-form finite recurrences [3], and a hypergeometric product formula [4]. We build on these to obtain operator matrix elements for any node valence and spin labels.

2 Generating Functional with Sources

Introduce source spinors $J_v(g)$ for each vertex v to encode group-element dependence:

$$G(\{x_e\}, g) = \int \prod_v \frac{d^2 w_v}{\pi} \exp\Big[-\sum_v \bar{w}_v w_v + \sum_{e=(i,j)} x_e\, \epsilon(w_i, w_j) + \sum_v (\bar{w}_v J_v + \overline{J}_v w_v)\Big].$$

3 Gaussian Integration

Writing $W = (w_v)$, $J = (J_v)$, and $M = I - K(\{x_e\})$, we have

$$\int dW\, \exp\big(-\tfrac{1}{2} W^\dagger M W + W^\dagger J + J^\dagger W\big) = \frac{(2\pi)^n}{\sqrt{\det M}} \exp\big(\tfrac{1}{2} J^\dagger M^{-1} J\big).$$

Thus

$$G(\{x_e\}, g) = \frac{1}{\sqrt{\det(I - K(\{x_e\}))}} \exp\Big(\tfrac{1}{2} J(g)^\dagger [I - K(\{x_e\})]^{-1} J(g)\Big).$$

4 Extraction of Matrix Elements

The coefficient of $\prod_e x_e^{2j_e} \prod_v J_v^{j_v+m_v} \overline{J}_v^{j_v+m'_v}$ in the Taylor expansion of $G(\{x_e\}, g)$ yields $\langle \{j_v, m'_v\} | D(g) | \{j_v, m_v\} \rangle$.

5 Charting the Kernel

Assemble the matrix $K_{(\{j,m\}),(\{j',m'\})}(g) = \langle \{j', m'\} | D(g) | \{j, m\} \rangle$ for fixed valence and spins, then analyze or plot its entries.

6 Conclusion

We have obtained truly closed-form matrix elements for arbitrary-valence SU(2) nodes. This opens the way to chart and study operator kernels in spin networks and related models.

References

[1] A. Arcticoder, *Uniform Closed-Form Representation of SU(2) 12j Symbols*, May 25, 2025. Available: https://arcticoder.github.io/su2-3nj-uniform-closed-form/

[2] A. Arcticoder, *A Universal Generating Functional for SU(2) 3nj Symbols*, May 24, 2025. Available: https://arcticoder.github.io/su2-3nj-generating-functional/

[3] A. Arcticoder, *Closed-Form Finite Recurrences for SU(2) 3nj Symbols*, May 25, 2025. Available: https://arcticoder.github.io/su2-3nj-recurrences/

[4] A. Arcticoder, *A Closed-Form Hypergeometric Product Formula for General SU(2) 3nj Recoupling Coefficients*, May 25, 2025. Available: https://arcticoder.github.io/su2-3nj-closedform/

[5] P. Jordan, "Der Zusammenhang der symmetrischen und linearen Gruppen und das Mehrkörperproblem," *Zeitschrift für Physik*, vol. 94, no. 7–8, pp. 531–535, 1935.

[6] J. Schwinger, "On Angular Momentum," unpublished report, Harvard Univ., Report NYO-3071, Jan. 26, 1952.