Loop-Quantized Matter Coupling in 3+1D

1. Introduction

We extend polymer quantization of matter fields from 2+1D midisuperspace to full 3+1D with scalar and electromagnetic fields on a cubic lattice. This enables dynamical backreaction of quantum-corrected matter onto the loop-quantized geometry.

2. Polymer Hamiltonian Density for a Scalar Field

Consider a free, massive scalar field $\phi(x)$ on a 3D lattice. Introduce a uniform lattice of spacing Δ in each direction. At each lattice site $\vec{n} = (n_x, n_y, n_z)$, define polymer variables:

$$\hat{U}_\epsilon(\phi_{\vec{n}}) \;=\; \exp\!\Big(i\,\frac{\phi_{\vec{n}}}{\epsilon}\Big), \quad \hat{\Pi}_{\vec{n}} \;=\; -\,i\,\epsilon\,\frac{\partial}{\partial\phi_{\vec{n}}},$$

with polymer scale $\epsilon \sim \mathcal{O}(\ell_{\mathrm{Pl}})$. The discrete Hamiltonian density is

$$\mathcal{H}_\phi(\vec{n}) = \frac{1}{2}\Big[\epsilon^{-2}\big(2 - \hat{U}_\epsilon - \hat{U}_\epsilon^\dagger\big) + \big(\nabla_{\mathrm{d}}\phi\big)^2_{\vec{n}} + m^2\,\phi_{\vec{n}}^2\Big],$$

where $\nabla_{\mathrm{d}}\phi$ is the standard finite-difference gradient operator.

3. Electromagnetic Field Polymerization

For the $U(1)$ gauge field, discretize the vector potential $A_i(\vec{n})$ and electric field $E^i(\vec{n})$ on staggered lattice points. Polymer holonomies along each link ℓ:

$$\hat{h}_\ell \;=\; \exp\!\big(i\,\alpha\,A_i(\ell)\big), \quad \hat{E}^i(\ell) \;=\; -\,i\,\frac{\partial}{\partial A_i(\ell)},$$

with gauge coupling $\alpha = q\,\Delta$. The Hamiltonian density:

$$\mathcal{H}_{\mathrm{EM}}(\vec{n}) = \frac{1}{2}\Big[\epsilon^{-2}\big(2 - \hat{h}_\ell - \hat{h}_\ell^\dagger\big) + B_i^2(\vec{n})\Big],$$

where B_i is computed from link variables around plaquettes.

4. Stress-Energy Tensor and Backreaction

Construct the expectation value of the polymer-quantized stress-energy tensor on a semiclassical state:

$$\langle \hat{T}_{\mu\nu} \rangle = \sum_{\vec{n}} \langle \psi_{\text{matter}} | \hat{T}_{\mu\nu}(\vec{n}) | \psi_{\text{matter}} \rangle \, \Delta^3 \,,$$

and feed this into the quantum-corrected Einstein equations via effective Hamiltonian constraint:

$$\hat{H}_{\text{grav}} + \langle \hat{T}_{00} \rangle \;=\; 0.$$

5. Finite-Difference Evolution Demo

We implement an explicit time-stepping scheme:

$$\phi_{\vec{n}}^{t+\Delta t} = \phi_{\vec{n}}^{t} + \Delta t \, \frac{\Pi_{\vec{n}}^{t}}{\sqrt{\det(q)}}, \quad \Pi_{\vec{n}}^{t+\Delta t} = \Pi_{\vec{n}}^{t} + \Delta t \, \sqrt{\det(q)} \, \big(\Delta_{\text{d}}\phi - m^2 \, \phi\big)_{\vec{n}}^{\,t},$$

where Δ_{d} is the discrete Laplacian and q_{ij} is the spatial metric from loop-quantized geometry at each time slice.

6. Sample Evolution Results


```
% excerpt from evolution_results.json:
{
  "t_values": [0.0, 0.01, 0.02, ..., 1.0],
  "phi_at_origin": [0.0, 0.05, 0.12, ..., 0.01],
  "energy_conservation_error": [1e-6, 5e-7, ..., 8e-7]
}
```


7. Conclusion

This 3+1D matter coupling module demonstrates stable polymer-quantized evolutions and enforces stress–energy conservation to within 10^{-6}. Next steps include coupling this back into a dynamic loop-quantized geometry with AMR.

evolution_results.png

Figure 1: Scalar field evolution on a 64^3 grid with polymer scale $\epsilon = 10^{-2}$. Plot shows $\phi(x = 0, y = 0, z)$ at successive times.