Weak Measurements

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Basic Ideas

- A eigenstate of the system the object to be measured in is mentioned by $|\, s_i \,\rangle$. A system is described by the pure state

$$|\, s \,\rangle \;=\; \sum_{i=1}^{d_s} c_i \,|\, s_i \,\rangle \tag{1}$$

The corresponding density matrix is

$$|\, s \,\rangle\langle\, s \,| \;=\; \Pi_s \tag{2}$$

- Trace is the sum over the diagonal elements of a matrix. It frees up an observable from a mix of two independent observables.

$$trace(A) \;=\; \sum_i \langle\, s_i \,|\, A \,|\, s_i \,\rangle \tag{3}$$

Measurement Rules

Assuming $\sigma_0 = \sum_a p_a \, \Pi_{s(a)}$ is the initial incoherent sum of pure states

- The possible result of measuring the observable S on the system is one possible eigenvalue $| \, s_i \, \rangle$
- The probability of obtaining the system in the state $| \, s_i \, \rangle$ is given by

$$prob \; (s_i \mid \sigma_0) = \langle \, s_i \, | \sigma_0 | \, s_i \, \rangle = Tr \, (\Pi_{s_i} \, \sigma_0) \tag{4}$$

- Measurement of observable S with result s_i transforms the original density matrix σ_0 into (By Luders rule)

$$\sigma_0' = \frac{\Pi_{s_i} \, \sigma_0 \, \Pi_{s_i}}{prob \; (s_i \mid \sigma_0)} \tag{5}$$

- Measurement of S without registering the result transforms density matrix into

$$\sigma_0 = \sum_i (\Pi_{s_i} \, \sigma_0 \, \Pi_{s_i}) \tag{6}$$

Post Selection, Time Evolution

- Using post selected state f, the probability of obtaining certain intermediate eigenvalue s_i is given by the ABL rule as

$$prob\left(s_i \mid |f\rangle, |s\rangle\right) = \frac{\left|\langle f| \Pi_{s_i} |s\rangle\right|^2}{\sum_j \left|\langle f| \Pi_{s_j} |s\rangle\right|^2} \tag{7}$$

- A state S under time evolution with Hamiltonian H gets the expression

$$U\ |s\rangle = exp\left(-i \int_{t_0}^{t} dt'\ H_s\right) |s\rangle \tag{8}$$

- Hence time evolution of the density matrix is

$$\sigma_t = U\ \sigma_0\ U* \tag{9}$$

Ancilla Measurement

- Assuming meter state to be μ_0 and system state to be σ_0, the density matrix $\tau_0 \;=\; \mu_0 \otimes \sigma_0$ evolves to

$$\tau_1 \;=\; U\left(\mu_0 \otimes \sigma_0\right) U^* \;=\; \sum_{i.j} \left(\left| \; m^{(i)} \right\rangle \Pi_{s_i} \sigma_0 \Pi_{s_j} \left\langle m^{(j)} \; \right| \right) \tag{10}$$

- The system state evolves to

$$\sigma_1 \;=\; Trace_{Meas} \tau_1 \;=\; \sum_{i.j} \left(\Pi_{s_i} \sigma_0 \Pi_{s_j} \left\langle m^{(j)} \mid m^{(i)} \right\rangle \right) \tag{11}$$

- The measurement apparatus evolves to

$$\sum_i \left| m^{(i)} \right\rangle \langle s_i \mid \sigma_0 \mid s_i \rangle \left\langle m^{(i)} \right| \tag{12}$$

- The measurement operator states $|m^{(i)} >$ and $|m^{(j)} >$ for i!=j overlap for ancilla measurement, while they don't overlap for projective (sharp) measurement.

Von Neumann Protocol

- The meter is supposed to have a continuous pointer variable Q which replaces the measurement states $|m_k>$ with continuous pointer states $|q>$. Hence the initial waveform of measurement apparatus becomes

$$|m^{(i)}> = \int dq|q>< q|m^{(i)}> = \int dq|q> \phi_i(q) \qquad (13)$$

- A particular choice for $\phi_0(q)$ would be a Gaussian centered around q=0, which would allow us express $\phi_i(q)$ in terms of $\phi_0(q)$

$$\phi_0(q) \;=\; \sqrt{\frac{1}{2\pi\Delta^2} exp\left(\frac{-q^2}{2\Delta^2}\right)} \qquad (14)$$

- Now the particular choice for Interaction Hamiltonian is made as

$$H_{interaction} \;=\; g\; S \otimes P \qquad (15)$$

Where g is the coupling constant, which exists only in the time of measurement, P is the meter conjugate variable to the pointer variable Q, S is the system observable

Von Neumann Protocol

- the final state becomes after time evolution

$$\phi_i(q) \;=\; \phi_0(q - gs_i) \tag{16}$$

- The measurement apparatus state becomes

$$\mu_1 \;=\; \sum_i < s_i|\sigma_0|s_i > \int dqdq'|q > \phi_0(q - gs_i)\phi_0(q' - gs_i)^* < q'| \tag{17}$$

- Amplification can be brought about buy post selecting a state f

$$\tau_f \;=\; \frac{\Pi_f \; \tau_1 \Pi_f}{prob\,(f|\tau_1)} \tag{18}$$

- By choosing denominator as small as possible, we can get a large amplification in the output state

Weak Measurements

- After post selection by a state f, nay measurement can be shown as

$$|m^{(f)}> = < f|exp\left(-igS \otimes P\right)\left(|s> \otimes |m^{(0)}>\right) \tag{19}$$

- As g is very small here, we can apply euler approximation to the exponential

$$|m^{(f)}> =< f|s> \left(1 - ig\frac{< f|S|s>}{< f|s>}P\right)|m^{(0)}> = < f|s> exp(-igS_W \tag{20}$$

- Here S_W is the weak value given by $S_W = \dfrac{< f|S|s>}{< f|s>}$
- hence the final equation wave function is given by

$$\phi_f(q) = < q|m^{(f)}> = \phi_0(q - gS_W) \tag{21}$$

- The noticeable factor here is the shift of the wave function can be controlled by controlling the weak value.