Homework 2

EEE361 -Linear Algebra in Data Analysis and Machine Learning

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Introduction

In this assignment, we are asked to investigate the performance of the three alternative least square solver for $\boldsymbol{Ax = b}$. These are Generalized Minimum Residual (GMRES), Conjugate Gradients (CG) and Pseudoinverse of A. There are total of 6 different A matrices which are constructed according to specifications in the report. With the following lines of code, I generate my 6 different matrix and corresponding 10 x0s.

```
#Function that generates mxm A matrix with cor
def generate(size,tau):
    c = np.random.uniform(-1,1,(size,size))
    c = np.triu(c,1)
    c = c + c.T
    c[abs(c) > tau] = 0
    c = c + np.diag(np.ones(size))

    x = np.random.normal(0,1,(size,10))
    return c,x
```

After that we are asked to find corresponding b values for A and x. Then we add 3 different levels for each of b that comes out from 10 x0. Therefore, in total we have 30 b vectors.

```
#calculating b0s given Ai and xi
def find_b30(A,x):
    w1 = 0.0001 * np.random.normal(0,1,(x.shape[0],1))
    w2 = 0.01 * np.random.normal(0,1,(x.shape[0],1))
    w3 = np.random.normal(0,1,(x.shape[0],1))
    w = np.array([w1,w2,w3])
    b_30 = []
    for i in w:
        for col in x.T:
            b0 = (A @ col).reshape(x.shape[0],1)
            b = b0 + i
            b_30.append(b)
    return np.array(b_30)
```

Return of this function will be in the format of (30 x m x 1) where m is the dimension of the column vector b. To follow same notation with assignment, we can simply cast it to (3 x 10 x m x 1) where 3 denotes different noise levels. In my function first 10 elements corresponds to first noise level and so on. In my code I investigate the matrixes in the order of Pseudo Inverse, CG and GMRES.

Pseudo Inverse Method

For the full column rank or full row rank matrices there is closed form solution that can be find that minimizes the following equation.

$$\hat{x} = argmin \parallel b - A\hat{x} \parallel^2, where\ \hat{x}\ \in R^{nx1}$$

However, a general way to evaluate this problem when our matrix is not column rank or row rank is finding the SVD of matrix A since SVD is valid for every matrix.

$$\text{the pseudoinverse of } A = U\Sigma V^T \text{ is } A^+ = V\Sigma^+ U^T$$

$$\Sigma^+ \ has \frac{1}{\sigma_i} values\ and\ is\ transposed\ version\ of\ \Sigma$$

$$\hat{x} = A^+ b$$

We can simply use the "pinv" method of the numpy library to evaluate this. But it also simple to write our own algorithm. After constructing my algorithm, I evaluate the following error function:

$$E_{S,i} = \sqrt{\frac{1}{10}\sum_{j=1}^{10} \|e_{i,j}\|_2^2}, where\ e_{i,j} = x_{0,j} - \hat{x}_{i,j}\ and\ \hat{x}_{i,j} = A^+ b_{i,j}$$

Where each i corresponds to different noise level in w.

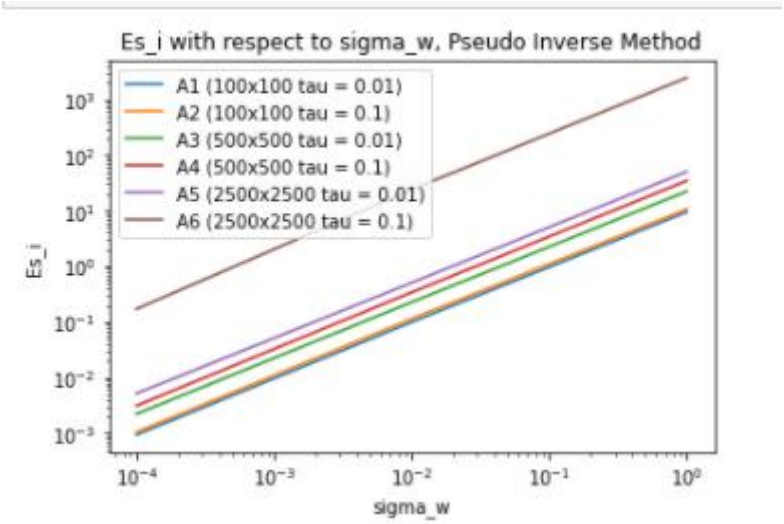


Figure 1: $E_{S,i}$ values with respect to different noise levels in logarithmic scale by Pseudo Inverse

From this figure above it is obvious that error increases as noise level increases. For the A6 matrix where size is 2500x2500 and tau = 0.1 error is considerably large. This might occur due to less sparsity and failure in being positive definite matrix. One can also observe that, small tau values have less error and as size of the matrix getting smaller error also decreases. For the small size matrix like 100x100 tau value is not even important. However, when we observe 2500x2500 error changes a lot as tau changes.

Conjugate Gradient Method:

For symmetric positive definite matrices, we can propose a powerful algorithm that solves

Sx = b where S is used for symmetric positive definite matrix. For the convention with previous notations, A will be used instead of S. For Hessenberg matrix generated in GMRES is tridiagonal when A is symmetric. Also, A norm of x which is $\| x \|^2_A = x^T Ax$ gives an appropriate norm to measure the error after k steps, where k is the iteration number. Using the derivations in the class we can construct the procedure as:

$$
\begin{aligned}
&x_0 = 0, r_0 = b, d_0 = r_0 \\
&\text{for } k = 1 \text{ to } N \\
&\alpha_k = \left(r_{k-1}^T r_{k-1}\right)/\left(d_{k-1}^T S d_{k-1}\right) \\
&x_k = x_{k-1} + \alpha_k d_{k-1} \\
&r_k = r_{k-1} - \alpha_k S d_{k-1} \\
&\beta_k = \left(r_k^{\mathrm{T}} r_k\right)/\left(r_{k-1}^{\mathrm{T}} r_{k-1}\right) \\
&d_k = r_k + \beta_k d_{k-1}
\end{aligned}
$$

To stop iteration, I also propose a condition such that residual square is smaller than a specific tolerance that will be find by trial and error. After that we are asked to find another type of error that measures the RMSE for different values of tau values. Its formula was given in the assignment as follows:

$$E_{O,i} = \sqrt{\frac{1}{10}\sum_{j=1}^{10} \left\| b_{i,j} - A\hat{x}_{i,j} \right\|_2^2} \quad where\ \hat{x}_{i,j}\ is\ the\ estimated\ solution\ of\ CG$$

After following the algorithm I obtained the following graphs of $E_{S,I}$ and $E_{S,O.}$

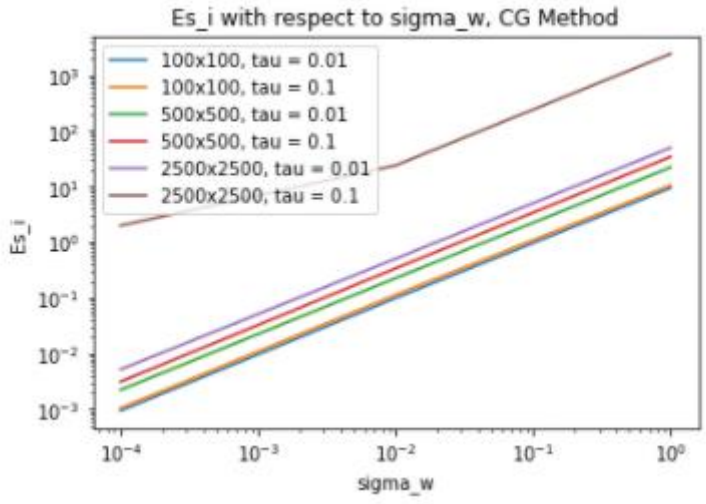


Figure 2: $E_{S,i}$ values with respect to noise levels logarithmic plot CG

We can also observe that error increases as noise level increases. It reaches its peak for 2500x2500 matrix with tau = 0.1. Again, it can occur because of the less sparsity and failure in being positive definite matrix. One can also observe that, small tau values have less error and as size of the matrix

getting smaller error also decreases. For the small size matrix like 100x100 tau value is not even important. However, when we observe 2500x2500 error changes a lot as tau changes.

For the other type of error, there are 6 different matrices with 3 different noise levels. Therefore, in total we have 18 graphs of error with respect to number of iterations.

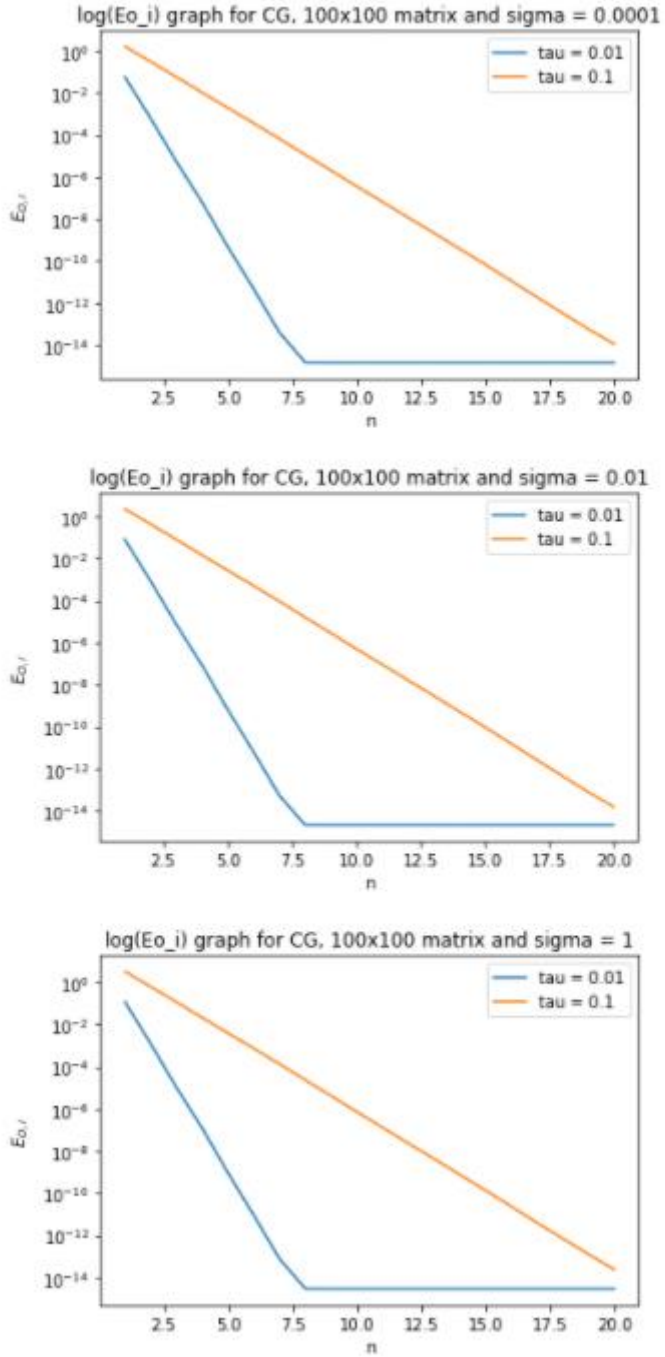


Figure 3: Log(E_oi) graph for CG with 100x100 matrix for all different sigma values

For 100x100 matrices we can conclude that regardless of the noise levels graphs are approximately equal. For tau = 0.01 we can obtain solution in 8 steps which is same as in the example given. Also, convergence is not a problem when tau = 0.1. We can still observe pretty good results.

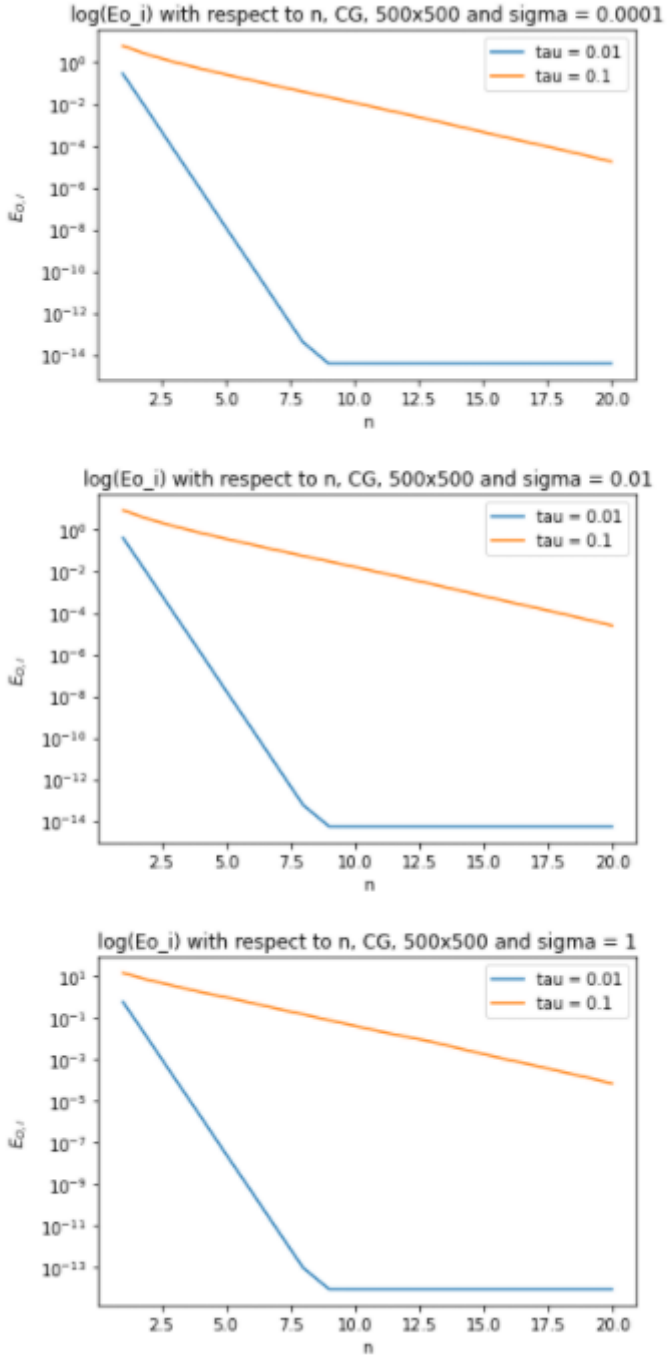


Figure 4: Log(E_oi) graph for CG with 500x500 matrix for all different sigma values

For 500x500 matrices we can conclude that regardless of the noise levels graphs are approximately equal. For tau = 0.01 we can obtain solution in 9 steps which is same as in the example given. Also, convergence is not a problem when tau = 0.1. We can still observe pretty good results.

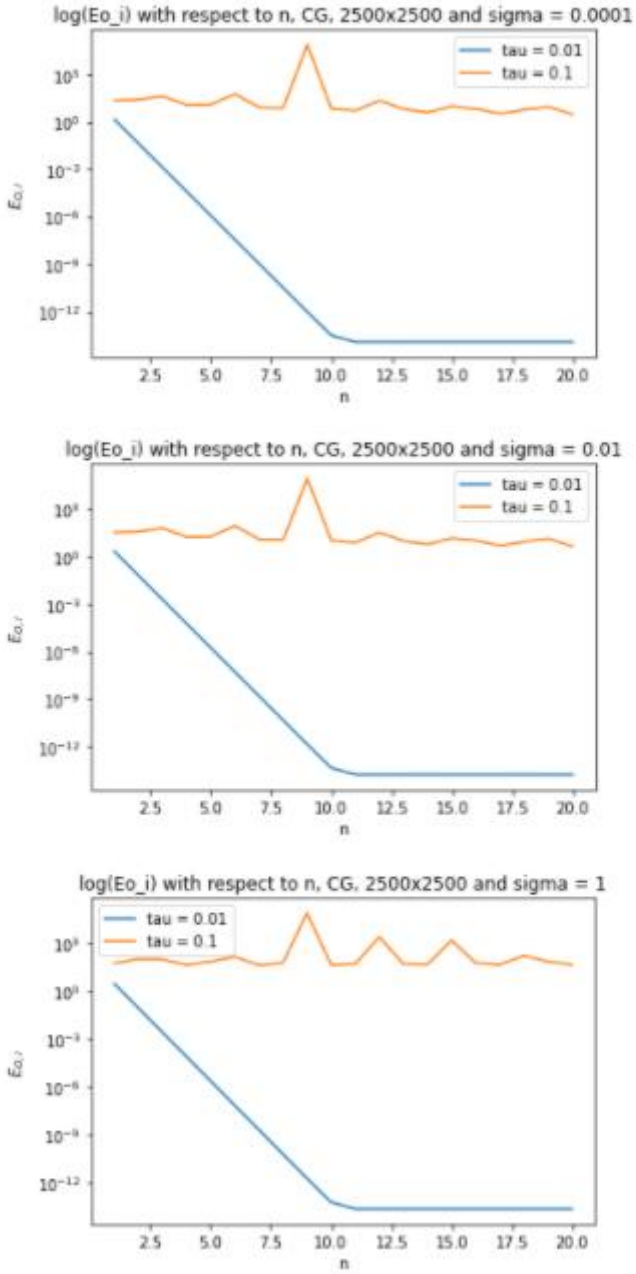


Figure 5: Log(E_oi) graph for CG with 2500x2500 matrix for all different sigma values

For 2500x2500 matrices we can observe that, regardless of the noise levels graphs are approximately equal. For tau = 0.01 we can obtain solution in 11 iterations. Unlike previous 100x100 and 500x500 matrices, we cannot converge with tau = 0.1. However, we can still solve large matrices with small tau values.

One can observe that for tau = 0.01 or close to zero, we find solutions in 8,9 and 11 iterations. When this compared with Pseudo Inverse method, we can say that it is much faster.

Stopping Criteria for CG

As it was already mentioned, there need to be a stopping criterion for CG. One can use a tolerance level for a residual. It can be formulated as follows

$$r_k = \|\mathrm{b} - A\hat{x}_k\|_2^1 < tolerance$$

However, it is also possible to check orthogonality condition for bases of Krylov subspace. Since we do not need to calculate each element of orthonormal vectors in a matrix for CG, choosing tolerance level for stopping criteria will be beneficial to decrease computational process.

From figures we can observe that for tau = 0.01, there is no need to iterate over all size of A. Even very small number like 8 is enough for convergence. However, for cases like 2500x2500 with tau = 0.1 there is no convergence

Orthogonality Check of the Basis of Krylov Subspace with CG

$$\left[\!\left[Q_k^T Q_k - I_k\right]\!\right]_F$$

Formula above is used to check orthogonality of the basis of Krylov subspace. Simply I stored the error in each iteration and tried to whether this error is negligible or not. The output of the code is as follows:

```
The number of iterations are 8
Frobenius_norm of(Q_k.T Q_k - I_k) for iterations up to 8:[2.2204460492
50313e-16, 2.0615012487730452e-14, 2.0834224059487134e-14, 3.3182166466
83312e-14, 5.540714045129809e-14, 1.1134973711241468e-13, 1.64285356189
37574e-13, 2.27978310343056e-13]
```

Since the errors are in the order of 1e-13 at most, they are negligible for finite precision arithmetic. Even this output is for 100x100 matrix, I also observe orthogonality for higher dimensions. However, when we run algorithm for size of A even it is 100x100 matrix, we can observe that orthogonality fails.

```
: x_hat = CGSolver(A1 ,b = b_30_1[1].reshape(-1), tol=1e-15, max_iter = A1.shape[0])

  Frobenius_norm(Q_k.T Q_k - I_k) :[0.0, 1.5286021254169548e-14, 1.5290664174389183e-14, 3.466991831202
  984e-14, 5.52135051100503e-14, 8.258532061170629e-14, 1.0822092329298796e-13, 1.4434049048624823e-13,
  1.8617171780440088e-13, 2.8072217573323183e-13, 4.699230325310197e-13, 1.1808631069633995e-12, 3.0431
  72936361466e-12, 9.839112037659297e-12, 2.478106940780125e-11, 7.513916864381528e-11, 2.4737031515533
  06e-10, 8.40453770802019e-10, 2.8676537334203823e-09, 9.353110822745055e-09, 2.987305731170652e-08,
  7.679787252069066e-08, 2.198156835680745e-07, 8.433371809943986e-07, 1.954410566659652e-06, 6.4960725
  97777226e-06, 1.8516554914675114e-05, 4.749353564604476e-05, 0.00011991261251890846, 0.00045750879070
  21563, 0.0017455039671490239, 0.00527127317496325, 0.0200601872281514, 0.09308952209801656, 0.2241347
  149189267, 0.6110282156891614, 1.291913162017699, 1.7882935652119567, 1.9620562088927833, 1.998635699
  3122738, 1.9998959978866724, 2.0003100387664103, 2.009553906744469, 2.254563570483105, 2.624091828011
  454, 2.7896607980079082, 2.82486727237655, 2.828423299178655, 2.8284270970307306, 2.8284271236777347,
  2.8284271426965386, 2.8284273547946683, 2.8284283992290975, 3.1629259141607533, 3.7412733995886738,
  4.241715457068231, 4.472136109664967, 4.690900219846764, 4.900971463476021, 5.241392728662411, 5.5990
  68139380408, 5.821458450177403, 5.999606641156979, 6.1643217601082245, 6.3245425455344195, 6.48073792
  3591773, 6.633248204805956, 6.782324365290654, 6.928190161993513, 7.0696598674225, nan, nan, nan, na
  n, nan, nan, nan, nan, nan, nan, nan, nan, nan, nan, nan, nan, nan, nan, nan, nan, nan, nan, nan, na
  n, nan, nan, nan, nan, nan, nan]
```

One should note that orthogonality is not satisfied after higher number of iterations especially when it exceeds the number of iterations where convergence occurs. Therefore, we need to use stopping criteria to rely on our method.

GMRES Method

The key idea of GMRES is to compute Ab is fast when A is a sparse matrix. Therefore, we can quickly obtain Krylov subspace. There are two type of iterations which are Lanczos and Arnoldi. Since our matrix is symmetric, I used Lanczos iteration. It procedure can be formulated as follows:

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{S}\boldsymbol{q}_k \\ a_k &= \boldsymbol{q}_k^{\mathrm{T}}\boldsymbol{v} \\ \boldsymbol{v} &= \boldsymbol{v} - b_{k-1}\boldsymbol{q}_{k-1} - a_k\boldsymbol{q}_k \\ b_k &= \| \boldsymbol{v} \| \\ \boldsymbol{q}_{k+1} &= \boldsymbol{v}/b_k \end{aligned}$$

Then I evaluate the performance of the GMRES solver with Lanczos iteration using the following formula:

$$E_{S,i} = \sqrt{\frac{1}{10}\sum_{j=1}^{10} \|e_{i,j}\|_2^2}, where\ e_{i,j} = x_{0,j} - \hat{x}_{i,j}$$

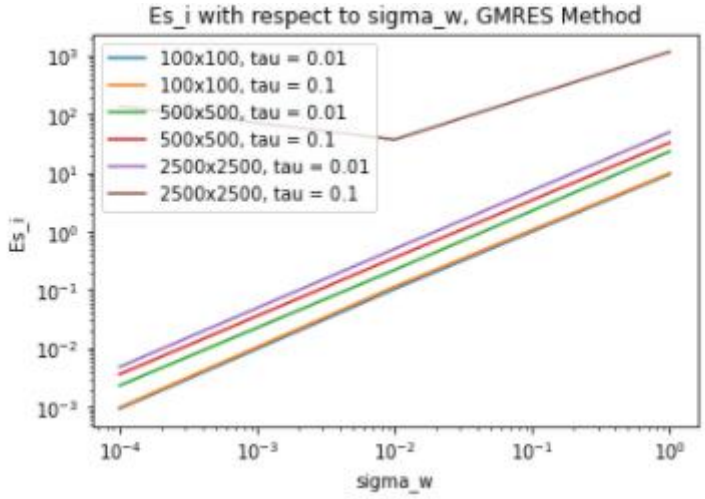


Figure 6: $E_{S,i}$ values with respect to noise levels in logarithmic scale plot, GMRES Method

From figure above, one can observe that error increases as noise level increase. For the matrix 2500x2500 with tau = 0.1, error reaches its peak. Again, we can say that since the matrix has less sparsity and fails on being positive definite this error value occurs.

Stopping Condition for GMRES

The stopping criteria used here is like CG. The same formulation above can be used to check for tolerance value.

$$r_k = \|\mathrm{b} - A\hat{x}_k\|_2^1 < tolerance$$

From that tolerance, what we expect is matrix converges to some certain value so that we do not need the iterate all over Krylov subspace. Possibly some k value smaller than n will be sufficiently enough for convergence. However, for all matrices choosing tolerance as 1e-15 was problematic. Error does not converge to that value so iterates for max iteration. It is also possible to use orthogonality concept for stopping condition. Using columns of Q matrix, we can observe whether it is orthogonal or not by taking following Frobenius norm:

$$\left[\!\left[Q_k^T Q_k - I_k\right]\!\right]_F$$

Therefore, I implement my stopping criteria such that when the frobenius norm of the error reaches some certain threshold level the algorithm stops. With that approach, computational time is decreased significantly.

Orthogonality of Basis in GMRES Method

By checking the formula above in each iteration, we can store the frobenius norm to see when orthogonality is not valid for the matrices that converge to some tolerance level. We can observe some of the results as follows:

```
Stopped at Iteration:  15
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 14 : [2.738403775149492e-14, 2.7512940514375
922e-14, 5.0582398423737836e-14, 7.801038455485599e-14, 1.1283951153407029e-13, 1.2815480875456268e
-13, 1.574133051273942e-13, 1.7198934905568496e-13, 2.0646956357245393e-13, 2.5152488121521703e-13,
3.380279834396841e-13, 4.974858453868019e-13, 6.781268420883936e-13, 9.940135287420976e-13, 1.32773
94827394852e-12]
```

Since my stopping criteria relies on the orthogonality concept, I iterate through some certain threshold (1e-12) value. When the frobenius norm of the error reaches this value, I assume that orthogonality does not exists.

In [50]:

```
import numpy as np
from numpy import inf
import matplotlib.pyplot as plt
```

Generating Random Matricies with Desired Conditions

In [51]:

```
#Function that generates mxm A matrix with corresponding tau value and its 10 x0s
def generate(size,tau):
    c = np.random.uniform(-1,1,(size,size))
    c = np.triu(c,1)
    c = c + c.T
    c[abs(c) > tau] = 0
    c = c + np.diag(np.ones(size))

    x = np.random.normal(0,1,(size,10))
    return c,x
```

In [52]:

```
A1,x_1 = generate(100,0.01)
A2,x_2 = generate(100,0.1)
A3,x_3 = generate(500,0.01)
A4,x_4 = generate(500,0.1)
A5,x_5 = generate(2500,0.01)
A6,x_6 = generate(2500,0.1)
```

In [53]:

```
#calculating b0s given Ai and xi
def find_b30(A,x):
    w1 = 0.0001 * np.random.normal(0,1,(x.shape[0],1))
    w2 = 0.01 * np.random.normal(0,1,(x.shape[0],1))
    w3 = np.random.normal(0,1,(x.shape[0],1))
    w = np.array([w1,w2,w3])
    b_30 = []
    for i in w:
        for col in x.T:
            b0 = (A @ col).reshape(x.shape[0],1)
            b = b0 + i
            b_30.append(b)
    return np.array(b_30)
```

In [54]:

```
#here first 10 colmns corresponds to first noise level and so on
b_30_1 = find_b30(A1,x_1)
b_30_2 = find_b30(A2,x_2)
b_30_3 = find_b30(A3,x_3)
b_30_4 = find_b30(A4,x_4)
b_30_5 = find_b30(A5,x_5)
b_30_6 = find_b30(A6,x_6)
```

In [41]:

```
def pseudo(A):
    U,S,V_T = np.linalg.svd(A, full_matrices = False)
    S_pinv = np.diag(1/S)
    S_pinv[S_pinv == inf] = 0
    A_pinv = np.matmul(V_T.transpose(),np.matmul(S_pinv,U.transpose()))
    return A_pinv
```

In [45]:

```
def find_err_pseudo(A,x,b):
    err_n1 = []
    err_n2 = []
    err_n3 = []
    A_pinv = pseudo(A) #we can use np.linalg.pinv also
    for i in range(30):
        x_hat = A_pinv @ b[i]
        x_0 = x[:,np.mod(i,10)].reshape(-1,1)
        if i <10:
            err1 = np.linalg.norm(-x_hat + x_0) ** 2
            err_n1.append(err1)
        elif i >= 10 and i < 20:
            err2 = np.linalg.norm(-x_hat + x_0) ** 2
            err_n2.append(err2)
        else:
            err3 = np.linalg.norm(-x_hat + x_0) ** 2
            err_n3.append(err3)
```

```
    err_n1 = (np.sum(err_n1) / 10) ** 0.5
    err_n2 = (np.sum(err_n2) / 10) ** 0.5
    err_n3 = (np.sum(err_n3) / 10) ** 0.5
    err_full = [err_n1,err_n2,err_n3]
    return err_full
```

Finding $E_{S,i}$ errors for Pseudo Inverse Algortihm

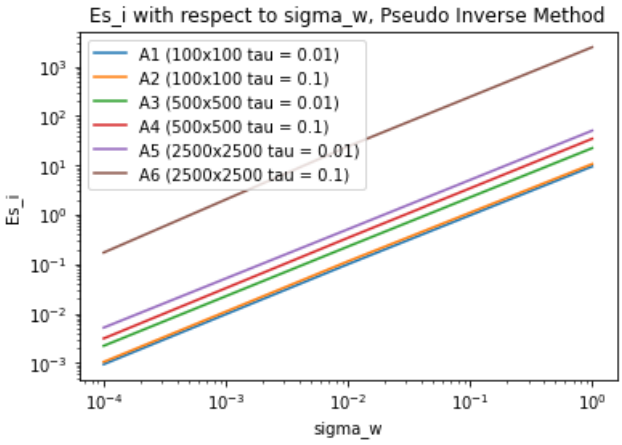
In [46]:

```
##Pseudo inverse method

err1 = find_err_pseudo(A1,x_1,b_30_1)
err2 = find_err_pseudo(A2,x_2,b_30_2)
err3 = find_err_pseudo(A3,x_3,b_30_3)
err4 = find_err_pseudo(A4,x_4,b_30_4)
err5 = find_err_pseudo(A5,x_5,b_30_5)
err6 = find_err_pseudo(A6,x_6,b_30_6)
```

In [47]:

```
w = [0.0001,0.01,1]
plt.figure()
plt.plot(w,err1, label = "A1 (100x100 tau = 0.01)")
plt.plot(w,err2, label = "A2 (100x100 tau = 0.1)")
plt.plot(w,err3, label = "A3 (500x500 tau = 0.01)")
plt.plot(w,err4, label = "A4 (500x500 tau = 0.1)")
plt.plot(w,err5, label = "A5 (2500x2500 tau = 0.01)")
plt.plot(w,err6, label = "A6 (2500x2500 tau = 0.1)")
plt.title("Es_i with respect to sigma_w, Pseudo Inverse Method")
plt.yscale('log')
plt.xscale('log')
plt.ylabel('Es_i')
plt.xlabel('sigma_w')
plt.legend()
plt.show(block =False)
```



CG Solver and Errors

In [60]:

```
def CGSolver(A,b,max_iter,tol):

    r = b.copy()
    d = r.copy()
    x = np.zeros((b.shape))

    condition = 13
    Q = np.zeros((b.shape[0],b.shape[0] + 1))
    orth_check = []

    for k in range(max_iter):
        A_d = A @ d

        Q[:,k] = r / np.linalg.norm(r)
        prev = r.T @ r # will be used in improvement
        alpha = (prev) / ( d.T @ A_d)
        x = x + alpha*d #approximate solution
        r = r - alpha*(A_d) #new residual

        if( k < condition ):
            norm = np.linalg.norm( Q[:,:k+1].T @ Q[:,:k+1] - np.eye(k+1,k+1) )
```

```
            orth_check.append(norm)

        if np.linalg.norm(r) < tol:

            print("The number of iterations are", k + 1)
            break

        else:
            beta = (r.T @ r ) /prev #improvement
            d = r + beta*d #next search direction
    print(f'Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to {condition}:{orth_check[:condition]}')
    x = x.reshape(-1,1)
    return x
```

In [61]:

```
x_hat = CGSolver(A1 ,b = b_30_1[1].reshape(-1), tol=1e-15, max_iter = A1.shape[0])
```

```
0.04903121809782604
0.0003684536813354624
2.0053941429551965e-06
1.5678475340496474e-08
9.81230023729969e-11
6.801515796304619e-13
4.091091779840583e-15
2.929436746789689e-17
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 3.3616326471535004e-15,
4.4021209043996844e-14, 8.289050617415349e-14, 9.71497660473368e-14, 1.109133053330141e-13, 1.33359135538
91805e-13, 1.52398162273344e-13]
```

In [57]:

```
def getCGError(A,b,x):

    x_init = np.zeros((A.shape[0])).reshape(-1,1)
    err_n1 = []
    err_n2 = []
    err_n3 = []
    for i in range(30):
        x_hat = CGSolver(A = A ,b = b[i].reshape(-1), tol=1e-15, max_iter = A.shape[0])
        x_0 = x[:,np.mod(i,10)].reshape(-1,1)
        if i <10:
            err1 = np.linalg.norm(-x_hat + x_0) ** 2
            err_n1.append(err1)
        elif i >= 10 and i < 20:
            err2 = np.linalg.norm(-x_hat + x_0) ** 2
            err_n2.append(err2)
        else:
            err3 = np.linalg.norm(-x_hat + x_0) ** 2
            err_n3.append(err3)

    err_n1 = (np.sum(err_n1) / 10) ** 0.5
    err_n2 = (np.sum(err_n2) / 10) ** 0.5
    err_n3 = (np.sum(err_n3) / 10) ** 0.5
    err_full = [err_n1,err_n2,err_n3]

    return err_full
```

Finding $E_{S,i}$ errors for CG algortihm

In [113]:

```
err_es_a1 = getCGError(A1,b_30_1,x_1)
err_es_a2 = getCGError(A2,b_30_2,x_2)
err_es_a3 = getCGError(A3,b_30_3,x_3)
err_es_a4 = getCGError(A4,b_30_4,x_4)
err_es_a5 = getCGError(A5,b_30_5,x_5)
err_es_a6 = getCGError(A6,b_30_6,x_6)
```

```
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 2.0615012487730452e-14,
2.0834224059487134e-14, 3.318216646683312e-14, 5.540714045129809e-14, 1.1134973711241468e-13,
1.6428535618937574e-13, 2.27978310343056e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 1.5286021254169548e-14,
1.5290664174389183e-14, 3.466991831202984e-14, 5.52135051100503e-14, 8.258532061170629e-14, 1.08220923292
98796e-13, 1.4434049048624823e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 3.603539097141439e-14,
4.1505547046445554e-14, 5.689143064672139e-14, 6.789421383081953e-14, 8.394958892337966e-14,
```

9.917122513579535e-14, 1.1337681544397809e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 1.3447265584762209e-14,
3.962578582809 24e-14, 7.323345875634886e-14, 9.029651772847642e-14, 1.2542823385466396e-13, 1.55985908504
84097e-13, 2.2235659112837813e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 1.828327410221544e-14,
4.4733126560204777e-14, 5.5557055585779146e-14, 7.465796223236529e-14, 1.1586256423258878e-13, 1.66206806
89727822e-13, 2.489498407041728e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 5.324028471810433e-14,
7.595492750739752e-14, 1.0430480255364933e-13, 1.1132505506397428e-13, 1.8970020766809313e-13, 2.41142622
48346285e-13, 3.373562008138344e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 5.144553231239119e-16, 1
.3207737277914558e-14, 3.3651447822894194e-14, 4.347331053105801e-14, 7.498245544239362e-14, 1.0659763789
124376e-13, 1.3745046553026684e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 8.55521767799067e-15,
3.213235607063152e-14, 6.847673034370563e-14, 1.0718417572688567e-13, 1.8140732438101478e-13, 2.753565990
5503673e-13, 3.6542921994903767e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 2.2402236251591758e-14,
2.328222606906975e-14, 4.3098931861139987e-14, 5.652543206685802e-14, 1.0135368052891418e-13, 1.650663585
6326315e-13, 2.6978297376262453e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 2.6418966112614355e-14,
5.6198397719126966e-14, 6.837395630512939e-14, 7.47904905411771e-14, 1.061160756125442e-13, 1.35794471424
61657e-13, 1.6740338354119776e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 3.16954892348942e-14, 3.40867070230485e-
14, 5.307798230932146e-14, 5.95898189043368e-14, 9.146753188907507e-14, 1.0775001377773192e-13, 1.4677972
102501265e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 1.3929509358388264e-14,
1.3949090859556503e-14, 3.950399537479148e-14, 6.618563088038386e-14, 1.0614905976119807e-13,
1.5558050768519753e-13, 2.278880162387879e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 3.219605112904118e-16,
5.549522870889872e-15, 1.42696872727583e-14, 1.6389195155785428e-14, 2.7366770382678166e-14, 3.4069487160
50087e-14, 4.843698976480022e-14]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 2.2880495203590606e-14,
3.2145180711434464e-14, 3.818981948202319e-14, 4.767127522541371e-14, 7.577282475356605e-14,
8.956790672671012e-14, 1.3887719501532214e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 3.1757341254212946e-14,
3.1990914233469536e-14, 5.105088901591445e-14, 6.096397627176048e-14, 1.0639279656127556e-13,
1.6808973215545805e-13, 2.888200674725166e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 8.909963489861811e-15,
1.7368117893547052e-14, 4.2965715169630156e-14, 4.975048832792723e-14, 9.740247392211434e-14,
1.4612932538537026e-13, 2.2319177978212973e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 4.052588989051215e-14, 4.343234356992059e-
14, 6.060120958728415e-14, 6.815952926015824e-14, 1.0854470446002846e-13, 1.402637603719685e-13,
1.8413182715794472e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 1.6938392692521363e-15,
7.134494650015552e-15, 2.670469897285843e-14, 4.899296738005985e-14, 1.0470789089825187e-13, 1.6068802309
240116e-13, 2.2373776383952841e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 8.002639340822694e-15, 9.379745183003216e-
15, 1.4081954793063615e-14, 1.6174781970853905e-14, 4.595763328883259e-14, 7.352954868684291e-14, 1.23072
20968850353e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[4.440892098500626e-16, 2.5922810826014554e-14,
6.082578780978322e-14, 7.875808110358104e-14, 9.53220528772131e-14, 1.4315876299235895e-13, 2.09702263710
2752e-13, 3.286742570842264e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 1.0545130578596763e-14,
1.070514495964878e-14, 3.589989977573114e-14, 6.104667038140926e-14, 9.383534773534274e-14, 1.46113507269
21654e-13, 2.1684740688878937e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 1.3524445924675837e-14,
1.3906641721908525e-14, 2.8587489722085 03e-14, 3.4379777224993414e-14, 4.651732888936322e-14,
6.761215323086182e-14, 1.0516010590369167e-13]

0.7012153250001026-14, 1.0516010590509107e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 1.7676897949373728e-14, 1.9742993905499642e-14, 4.025900744419494e-14, 4.911767554473225e-14, 7.919323566561639e-14, 9.382626185221025e-14, 1.28963224831624e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 2.575693119273424e-15, 3.198629840326568e-14, 7.246742370987529e-14, 9.595750866722514e-14, 1.3189181110646072e-13, 1.687926729441726e-13, 2.180283795696589e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 7.176163773829736e-14, 7.367898933480936e-14, 1.249495864894369e-13, 1.522618785924517e-13, 2.328580240046167e-13, 2.4488620979555153e-13, 3.4857715916855335e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 1.1983261789646981e-14, 1.9567694923875087e-14, 4.34402790168088e-14, 5.721603477302299e-14, 9.38194256102284e-14, 1.2963967381293002e-13, 1.7544357554425834e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 4.236558742675644e-14, 4.3685216752641535e-14, 5.828265538787121e-14, 7.806560447098048e-14, 1.1716490385881516e-13, 1.658594653631003e-13, 3.1702241197133786e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 2.843243664760714e-14, 2.98581329153797e-14, 4.140224889556919e-14, 4.7749594672205375e-14, 6.58123525627246e-14, 7.235083593620645e-14, 1.0767383341297296e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 1.1653642265948437e-15, 1.812218981616372e-15, 7.946226620920758e-15, 2.2320254156655638e-14, 9.695963834765313e-14, 2.1758120743748089e-13, 3.9285297670787264e-13]
The number of iterations are 8
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 1.1029455425629619e-14, 4.012973840373954e-14, 7.745052470548829e-14, 9.428765928678539e-14, 1.2137638698968815e-13, 1.4826569972201893e-13, 1.7496947097283879e-13]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 9.477263213641288e-16, 1.393294239594247e-15, 1.7033111919063826e-15, 2.0598875287503064e-15, 2.4260446373960323e-15, 2.9511112708369914e-15, 3.779511253597908e-15, 4.908405379810684e-15, 6.0614392301573985e-15, 7.471004826558747e-15, 8.86346867936667e-15, 9.978209484541109e-15]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 1.0219687088977656e-15, 1.431730341032669e-15, 2.1726246583164522e-15, 2.994429714013743e-15, 3.3323043703655125e-15, 3.6592885853247865e-15, 3.956660343779941e-15, 4.2846981914550925e-15, 4.529750625490465e-15, 5.476390389357287e-15, 7.281614240414526e-15, 1.0317506122515461e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 9.801871946272908e-16, 1.345454566815501e-15, 2.5720518060453994e-15, 3.251891964192074e-15, 3.8276511945969816e-15, 4.625334017497649e-15, 5.3987609172605496e-15, 7.29186394488088e-15, 9.184336089042866e-15, 1.2068407734743438e-14, 1.6436512389367828e-14, 2.1865958100875665e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 1.0978765707609207e-15, 1.9791093470177067e-15, 2.1096633989177213e-15, 3.844987876790338e-15, 7.51758301549358e-15, 1.012032885821985e-14, 1.215483347098312e-14, 1.5035742150040762e-14, 1.789789335219348e-14, 2.0934552794106442e-14, 2.5250049011470694e-14, 3.095473436692663e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 3.443946507802457e-16, 4.32481072944615e-16, 1.3673302379511695e-15, 2.7691927733458386e-15, 5.47598047330798e-15, 7.412363700501016e-15, 9.273050132656093e-15, 1.1373302607504771e-14, 1.3720189120260179e-14, 1.6553413797187563e-14, 1.9516358057476416e-14, 2.2205747693455275e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 6.812687184325428e-16, 1.0874719456006108e-15, 1.285221871048214e-15, 1.3362311067621076e-15, 2.0968444248333318e-15, 2.9653175923154538e-15, 3.993105567140282e-15, 5.352627302908821e-15, 6.708512891592309e-15, 9.735834487180887e-15, 1.3943658488774731e-14, 1.918323950515336e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 1.1293129273656663e-15, 2.1578056243073808e-15, 3.306941722871483e-15, 4.27749481630819e-15, 4.716947814147365e-15, 4.829570071222052e-15, 5.24526355535141e-15, 6.2432089610264115e-15, 7.730155962435453e-15, 9.169245233732502e-15, 1.0655086527043422e-14, 1.1789846283888092e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 1.592730811712604e-15, 1.7515635741539165e-15, 2.5892280608515746e-15, 2.8579515303341827e-15, 3.166642601664658e-15, 4.288740291116195e-15, 5.721557493184151e-15, 7.080690550724839e-15, 8.869874752144346e-15, 1.0725333196702217e-14, 1.2661504463087416e-14, 1.5553645876740667e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 6.013346660658531e-16, 8.497867180662578e-16, 9.513475649530878e-16, 1.875991356583955e-15, 3.0445641656148945e-15, 3.915765879124571e-15, 4.526219869231741e-15, 5.2078954537369344e-15, 6.493445565313497e-15, 8.386076889577397e-15, 1.0655243173805022e-14, 1.3828800972865691e-14]

0655245175805022e-14, 1.5028600972865691e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 9.79714804521952e-16, 2.1017876380848995e-15, 3.0971436385930214e-15, 3.666778697517715e-15, 4.504340345285568e-15, 5.205274298391585e-15, 6.034679055658731e-15, 7.0155644632670185e-15, 8.143919229644071e-15, 9.650400952315779e-15, 1.1184209467 45687e-14, 1.3295625283825153e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 5.347232739971291e-16, 9.32513833356246e-16, 2.122418399363236e-15, 3.376287982976537e-15, 4.3722866590383136e-15, 5.86350406192196e-15, 7.920720126966679e-15, 9.93613582015952e-15, 1.286265735220148e-14, 1.8086437862208712e-14, 2.370593840718563e-14, 2.8925915797506985e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 9.117437148199453e-16, 1.8502914112806377e-15, 2.8589132994961225e-15, 3.739923861000921e-15, 4.284540414326062e-15, 5.498931665270413e-15, 6.924635038729913e-15, 8.959684136967343e-15, 1.096693403849635e-14, 1.329423069369187e-14, 1.6669568697509972e-14, 2.0830725858980142e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 8.58248099067739e-16, 2.474074788156823e-15, 2.8959131520454966e-15, 3.2195488645988317e-15, 3.7515662045313025e-15, 4.292833036512877e-15, 5.283788855736991e-15, 7.296781761065049e-15, 1.0134593066401278e-14, 1.3758493420284054e-14, 1.9925756950392814e-14, 2.857628976128393e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 1.5060217890223186e-15, 2.9171354899806153e-15, 4.037437619384779e-15, 4.7044427753144564e-15, 5.091617816935573e-15, 5.373284816388129e-15, 5.557825831094148e-15, 6.134410384212135e-15, 6.9477413818100115e-15, 8.418031932334607e-15, 1.0784257004242725e-14, 1.386405808237097e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 4.968617647951461e-16, 6.374575706708107e-16, 1.557210740733909e-15, 3.4436279685178825e-15, 6.103809271997506e-15, 8.590448864770945e-15, 1.0343204325260335e-14, 1.1479186544982509e-14, 1.1838672184542737e-14, 1.2033400843599648e-14, 1.2293714958560156e-14, 1.2745670209162908e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 4.634478434877626e-16, 5.945173333296e-16, 1.327274428986882e-15, 2.1168744874103014e-15, 3.3489819242695076e-15, 5.365222063672617e-15, 9.14783422216228e-15, 1.3942286484092715e-14, 1.9526268697830733e-14, 2.7530007088214297e-14, 3.919151642349466e-14, 5.1837188069413005e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 4.3671898002592676e-16, 1.6752101136834123e-15, 2.7236771707403722e-15, 5.65487605548814e-15, 8.00402701077409e-15, 9.701508091366237e-15, 1.0785414941362036e-14, 1.1883880387418195e-14, 1.3161864537999022e-14, 1.519886990713186e-14, 1.8016232785643096e-14, 2.1462729407295138e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 6.146601603632139e-16, 1.0940845373227976e-15, 1.6366805728367676e-15, 3.5186619733889807e-15, 5.027714987152726e-15, 6.434554739349296e-15, 7.148700857860916e-15, 8.325404722836192e-15, 9.645218716489974e-15, 1.1410712825024566e-14, 1.3593894148525707e-14, 1.6795033335942964e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 4.695245831620258e-16, 8.727196849884202e-16, 1.691543557187333e-15, 2.8495781079197722e-15, 4.386211948866064e-15, 5.3564916251876236e-15, 5.886169084948722e-15, 6.156292418286821e-15, 6.8173354253247824e-15, 8.037724148091607e-15, 1.0345130401793522e-14, 1.39669700078692e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 6.698204406721846e-16, 7.885738221570011e-16, 1.4326127468015947e-15, 1.8678485499215226e-15, 2.46070425597596e-15, 3.0483571723356013e-15, 4.068518987914546e-15, 5.052221068647524e-15, 6.011805614194097e-15, 6.7704227954487715e-15, 7.536443254949294e-15, 8.642207420568086e-15]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 1.2555816072239378e-15, 1.8938025180545784e-15, 2.1184925205529128e-15, 3.0482400400817473e-15, 3.990771762741388e-15, 5.3920160139573234e-15, 6.504384838905428e-15, 7.545204348606852e-15, 9.352630491996625e-15, 1.2132384387871804e-14, 1.4470148927210655e-14, 1.5919191539246644e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 8.368767034041726e-16, 1.622128225185444e-15, 3.0841880849872253e-15, 4.230560589220568e-15, 5.0131012817675635e-15, 5.648614586 5032195e-15, 7.222503525356929e-15, 8.984215360124791e-15, 1.0537379890790726e-14, 1.2145861333438679e-14, 1.3879018077093582e-14, 1.5998848862735338e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 1.9339847104307365e-15, 2.6379200317197165e-15, 2.641441056370077e-15, 3.3812390566542944e-15, 4.241399321684 53e-15, 5.04018743457079e-15, 6.1766766923227716e-15, 7.0449861918505935e-15, 7.474952228892502e-15, 8.227404822086176e-15, 8.875008321326984e-15, 9.550350545497257e-15]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[4.440892098500626e-16, 1.3876621347565342e-15, 1.7343803289589938e-15, 2.076596962969225e-15, 3.0224762713666972e-15, 5.272162673801757e-15, 7.707820619439453e-15, 1.09484476466729e-14, 1.5009154871915834e-14, 1.9940407418884867e-14, 2.5071997568585136e-14, 3.024105138583144e-14, 3.5509689905833e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 2.899747890763511e-16,

Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 2.899747890763511e-16, 4.894707875017775e-16, 4.702124699337447e-16, 9.402153230719437e-16, 3.452845576824465e-15, 6.994311058850094e-15, 1.0704767247519158e-14, 1.4524447779119765e-14, 1.893556446428075e-14, 2.315126270498629e-14, 2.68322459742632e-14, 2.8719056075524536e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 9.850417075053345e-16, 2.545661551820451e-15, 4.473913390145688e-15, 5.776077996323676e-15, 7.301189488230145e-15, 9.417690166457071e-15, 1.1915613831115283e-14, 1.4420676589973803e-14, 1.649113828010872 5e-14, 1.907117011686547e-14, 2.225983117180667e-14, 2.5169149950003984e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 5.22895428082315e-16, 8.790740205303898e-16, 1.728784023204336e-15, 2.764557131236618e-15, 4.046920274361415e-15, 5.76672052285232e-15, 8.356062241827678e-15, 1.0806245647145542e-14, 1.2815675191324206e-14, 1.4756490290359138e-14, 1.6889655242335344e-14, 1.898284779181175e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 6.773945804624358e-16, 1.0148459354107037e-15, 1.9407734209815496e-15, 2.489692653494284e-15, 2.937012123716531e-15, 3.4811805285717257e-15, 4.571310123454324e-15, 5.80156665920203e-15, 7.574978274493226e-15, 1.0945603387591093e-14, 1.5212827710763084e-14, 2.0491019753374063e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 1.454279218131786e-15, 4.15364146348402e-15, 6.050133567902814e-15, 7.075685925292763e-15, 7.369665509256809e-15, 7.566633867117032e-15, 8.261897764424754e-15, 9.307972078388524e-15, 1.1764017874881835e-14, 1.4616907287988018e-14, 1.7108420610246675e-14, 2.0188007918441456e-14]
The number of iterations are 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 1.095816803644294e-15, 1.5191652554305227e-15, 1.596857952509554e-15, 2.928168009189203e-15, 3.4305861879213905e-15, 4.181775739908319e-15, 5.521100256578075e-15, 6.357370341469465e-15, 7.474855835330739e-15, 8.432700484074935e-15, 9.262931550394292e-15, 1.0170338881605035e-14]
The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 1.4441651182179666e-14, 1.4697987605049444e-14, 2.010779977219168e-14, 2.3669997536018788e-14, 2.6365001499348972e-14, 3.671415725005869e-14, 4.314050915209624e-14, 5.322397344024775e-14]
The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 5.440977811676313e-15, 1.1556778312661305e-14, 2.540271486166159e-14, 4.2623069419519235e-14, 6.488643354083515e-14, 8.540193714321712e-14, 1.164506137379893e-13, 1.6382038536480427e-13]
The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 1.7224202785686503e-14, 3.0135221385758 29e-14, 4.989756085389536e-14, 6.336751086215918e-14, 7.746110131656574e-14, 8.912084934238 76e-14, 1.0009881787956329e-13, 1.0737445826395955e-13]
The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 2.660105980180033 2e-14, 2.8800227746900275e-14, 5.0619638945422543e-14, 5.723901481272972e-14, 6.935659338283 05e-14, 7.217868950950 92e-14, 8.193505309439641e-14, 8.816484790541149e-14]
The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 5.760101173064295e-15, 1.0390326933028902e-14, 3.026912539211659e-14, 5.44077001385385e-14, 7.59471091473628e-14, 9.313569764424866e-14, 1.11316839543573e-13, 1.4012384824748917e-13]
The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 7.982283741511642e-16, 2.072846738210891e-14, 2.705365558192291e-14, 3.36402032077685e-14, 5.018286158334111e-14, 5.7665582468164e-14, 6.651929532868712e-14, 8.013672518013106e-14]
The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 8.36961921417935e-15, 2.1201353586429576e-14, 2.5274316235990316e-14, 2.5850646046701486e-14, 3.2158829226115507e-14, 4.5313071148629315e-14, 6.008052664017023e-14, 7.976476390162326e-14]
The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 1.1087551354720814e-14, 1.3344635540944 77e-14, 2.7711013258544485e-14, 4.066780939734661e-14, 5.44063081417712 93e-14, 6.798388086774556e-14, 8.074588531447533e-14, 1.0148566261411206e-13]
The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 9.401267899436285e-15, 1.087641240602105e-14, 2.3671763022153246e-14, 3.56418276520223 3e-14, 4.2329495078103254e-14, 5.5532051580456597e-14, 7.778219697924239e-14, 1.0033977423739372e-13]
The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 4.949902808377287e-15, 6.109890262930452e-15, 1.7881771809254507e-14, 5.378641521522138e-14, 9.542217597132757e-14, 1.3049848040943602e-13, 1.521267392141353e-13, 1.664348665513828e-13]
The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 1.508024561581292e-15, 2.7688588664425 73e-15, 3.672117461884334e-15, 5.562379397036932e-15, 2.163198466945719e-14, 4.798388924808468 3e-14, 6.513425087754729e-14, 8.411699114330667e-14]
The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 7.30800273852802e-15, 3.6325238821138134e-14, 6.023946453020871e-14, 7.312031911177312e-14, 7.857378821752451e-14, 8.230859895179628e-14, 9.439172814400485e-14, 1.067990806849981e-13]
The number of iterations are 9

The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 9.859932624763205e-15,
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The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 8.499799585178745e-15, 1
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200074e-14, 7.761700560342027e-14, 9.522550437835196e-14]
The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 2.1393085922526072e-14,
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The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 2.1788080950248492e-14,
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The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 5.336927450758683e-15, 5.860243170053156e-
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The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 1.3548101112109768e-14,
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The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 4.831047600592665e-15,
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The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 4.771467369658979e-15, 4.90824871762971e-1
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The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 1.7442745908190065e-14,
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The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 7.44262643575997e-15, 1.3065535777952048e-
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The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[4.440892098500626e-16, 2.005341838854472e-15, 7
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The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 5.396760699463418e-16, 7.224450077577017e-
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The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 7.83345277106719e-15, 2.327379004365881e-1
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The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 2.3418214891621075e-14,
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4579046e-14, 5.010868156419255e-14, 5.814327495999295e-14]
The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 2.2187185142180648e-14, 4.10869390415539e-
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The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 1.11229692817725e-14,
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The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 2.385068575519322e-14, 3
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The number of iterations are 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 8.808629967024962e-15, 8.830392052905175e-
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The number of iterations are 54
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 5.495323605393213e-16, 7.404739851862878e-
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-15, 4.166377535535871e-15]
The number of iterations are 54

The number of iterations are 54
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 6.315584771133016e-16,
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The number of iterations are 53
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 9.678699938266253e-16, 1
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The number of iterations are 53
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 2.58246670560308e-16, 3.668929899173809e-1
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The number of iterations are 53
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 4.740479068156532e-16, 5.557413475166007e-
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The number of iterations are 54
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 2.5215027719414395e-16,
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The number of iterations are 54
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[4.440892098500626e-16, 3.6329102379526464e-16,
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The number of iterations are 54
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[3.3306690738754696e-16, 9.892902786314526e-16,
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The number of iterations are 54
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 1.263471883812634e-15,
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The number of iterations are 54
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 5.03057844280837e-16, 5.639332289373551e-16, 8.980962553119527e-16, 1.0694920455985792e-15, 1.5760461611759971e-15, 2.0518818396534e-15, 2.40058944
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The number of iterations are 54
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[4.440892098500626e-16, 4.2769289799955053e-16,
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The number of iterations are 53
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 6.096411645789702e-16,
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The number of iterations are 53
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 2.5391338980276673e-16,
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The number of iterations are 54
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 1.7915994452782828e-15,

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Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 2.775108412471059e-16, 9.389097111459737e-16, 1.199100189222226e-15, 1.611065267162962 4e-15, 2.036526362636886e-15, 2.6293491139071726e-15, 3.0806257759863235e-15, 3.503584340020865e-15, 4.007562723326504 4e-15, 4.407973751293908e-15, 4.70626218442581e-15, 5.067790178748694e-15]
The number of iterations are 54
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 9.710485573708405e-16, 1.2385488285507493e-15, 1.4442631786532256e-15, 1.728863120493416e-15, 2.0618609796396138e-15, 2.620816994022207e-15, 3.2879926473560742e-15, 3.9343437756213684e-15, 4.65659607425922e-15, 5.100057491833037e-15, 5.462686895271643e-15, 5.774154427579673e-15]
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Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 5.130149563429823e-16, 6.999388888571482e-16, 8.286435136683909e-16, 1.0612377334781564e-15, 1.2567645078466106e-15, 1.4159054078473856e-15, 1.8202421392092744e-15, 2.173944666507363e-15, 2.623273678977589e-15, 3.0599540024830766e-15, 3.613907785322949e-15, 4.13654361900651e-15]
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The number of iterations are 55
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The number of iterations are 55
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The number of iterations are 55
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The number of iterations are 55
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The number of iterations are 55
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The number of iterations are 55
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The number of iterations are 11
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The number of iterations are 11
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4]
The number of iterations are 11
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The number of iterations are 11
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The number of iterations are 11
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The number of iterations are 11
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The number of iterations are 11
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The number of iterations are 11
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The number of iterations are 11
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The number of iterations are 11
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The number of iterations are 11
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The number of iterations are 11
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The number of iterations are 11
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 3.8431173006685104e-15,
5.723362919231098e-15, 5.762194257954047e-15, 7.047840411669586e-15, 1.0269497439893795e-14, 1.5087448324
515273e-14, 1.946840105363716e-14, 2.3223587707112602e-14, 2.6182080705116826e-14, 3.012112267120165e-14]
The number of iterations are 12

Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 1.0147842945947883e-14, 2.1216654371219555e-14, 2.7781817757166885e-14, 3.1857779708633655e-14, 3.304852668505044e-14, 3.594012036505871e-14, 3.903858623990509e-14, 4.080684485360176e-14, 4.5883620498003334e-14, 5.489830139357913e-14, 6.71720799058084e-14]
The number of iterations are 12
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 3.6120855396626546e-15, 3.7627143615684085e-15, 1.0939973504273863e-14, 2.2224375240835192e-14, 3.14640205619076e-14, 3.7840537796589695e-14, 4.08590772872559e-14, 4.3004823203175735e-14, 4.614118226500454e-14, 5.070625153372919e-14, 5.967756515460846e-14]
The number of iterations are 12
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[4.440892098500626e-16, 4.541444809311768e-15, 1.1151158841185028e-14, 2.8145256696752888e-14, 4.196104913312881e-14, 5.426793543216624e-14, 6.405141972920234e-14, 7.402200286604022e-14, 8.700356540405611e-14, 1.0288839298508216e-13, 1.1531511674836314e-13, 1.248557444637114e-13]
The number of iterations are 12
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 2.8829895997150926e-16, 3.6510263870060205e-16, 1.0886642473819297e-14, 2.1445019726694308e-14, 3.214806355623746e-14, 3.9321909051826704e-14, 4.294499856519994e-14, 4.447904000623426 6e-14, 4.773415874168175e-14, 5.414719641918824e-14, 6.441431290691607e-14]
The number of iterations are 12
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 2.6594566609666763e-15, 2.709349218013859e-15, 5.743034897319222e-15, 6.003266714272016e-15, 7.463172526452638e-15, 1.146069476735592e-14, 1.6212584244404333e-14, 1.9152130321157215e-14, 2.3968055502148864e-14, 2.6113742136398295e-14, 2.9454174910009906e-14]
The number of iterations are 12
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[4.440892098500626e-16, 3.5196058203566705e-16, 3.638607423427983e-15, 6.365517930410148e-15, 8.678238627904736e-15, 1.6965422830241667e-14, 2.490158180844051e-14, 3.645353874476898e-14, 5.593748533627243e-14, 7.325404361714096e-14, 9.056475213891387e-14, 1.0714346591300965e-13]
The number of iterations are 12
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 5.438915594073881e-15, 1.3255303067087754e-14, 1.6309241862525468e-14, 1.6549973768604157e-14, 1.7903915605967543e-14, 2.1010245676115223e-14, 3.1716057791023627e-14, 4.927026874287144e-14, 6.812044532992884e-14, 9.10305377245909e-14, 1.1524728450013178e-13]
The number of iterations are 12
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 5.4474122855843474e-15, 1.0892761728244783e-14, 2.3065355244323655e-14, 3.273249197821158e-14, 4.372587098174651e-14, 5.2957768870954877e-14, 5.862641074781813e-14, 6.28937202303571e-14, 6.837696254116331e-14, 7.599873023894743e-14, 8.467192700448031e-14]
The number of iterations are 12
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[6.661338147750939e-16, 1.4008591815425155e-14, 2.2594474705509626e-14, 3.1021762965057515e-14, 3.3271447472327606e-14, 3.912355956913675e-14, 4.102153417556433e-14, 4.318236471673408e-14, 4.848185715109419e-14, 6.096669755772925e-14, 7.789748542640873e-14, 9.812248091292656e-14]
The number of iterations are 12
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 1.0379783113579769e-14, 1.4469296644291791e-14, 1.4911788944665632e-14, 1.7754184042520042e-14, 2.0714215552426553e-14, 2.764493170541098e-14, 3.7707523425303164e-14, 5.370621366478246e-14, 7.271753607435603e-14, 1.007962623897727e-13, 1.279481332289966e-13]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 4.858778217026724e-16, 6.877308465222102e-16, 9.55787330993635e-16, 1.3011241257030551e-15, 1.6509644423944028e-15, 2.212446772910573 4e-15, 5.773287807474961e-15, 9.23879258335085e-15, 1.2583824496710615e-14, 1.9438178629884304e-14, 2.8150936849038977e-14, 3.639896223728452e-14]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 9.298103671868174e-16, 1.3947477798498698e-15, 1.8454248794969965e-15, 2.3031772474317807e-15, 2.6898546033275728e-15, 3.4239354870886246e-15, 3.889849286254892e-15, 4.086008426946057e-15, 4.351158476499324e-15, 4.4166312080732737e-13, 7.643476274964211e-13, 1.0386127467279107e-12]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[8.881784197001252e-16, 7.566664011648003e-16, 1.0924086898578244e-15, 1.402747028761804e-15, 1.8676450327702898e-15, 2.000571735773355e-15, 2.438222023874 2273e-15, 2.8673604241718475e-15, 3.561496234415671e-15, 3.893820614773686e-15, 4.681628376769063e-15, 5.618006627016811e-15, 6.551667568798823e-15]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[6.661338147750939e-16, 1.9502871836071272e-16, 5.129256333635216e-16, 6.886902434685875e-16, 1.1928134712196952e-15, 1.9056998405483232e-15, 2.4046974577667544e-15, 2.7513332765577063e-15, 3.1886472779079333e-15, 3.501872038226064e-15, 6.352508735572225e-15, 9.465678938747257e-15, 1.2637219201750188e-14]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[6.661338147750939e-16, 1.2772188048396653e-15, 1.8745450027575873e-15, 2.3788284007763014e-15, 2.7472874419288858e-15, 3.2685079521628912e-15, 3.7751846542728816e-15, 4.627424911393493e-15, 5.518298333011869e-15, 6.153827913710679e-15, 7.073618455837092e-14, 1.2386208264855822e-13, 1.70647070121453e-13]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[5.551115123125783e-16, 3.6714351262581434e-16, 5.580794154964348e-16, 1.0314101536621478e-15, 1.453596562796156e-15, 2.069284613868829e-15, 2.679912032934723 3e-15, 3.5507920266871655e-15, 4.741857980154324e-15, 5.7701423671564064e-15, 1.0444147297506593e-13, 1.8518948040960666e-13, 2.564868829781574e-13]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 6.658447020420676e-16, 9.681511986273587e-16, 1.1779582257296323e-15, 1.3169032922707832e-15, 1.5406049216923914e-15, 1.7066956819490876e-15, 2.885561818906168e-15, 4.16602692665928e-15, 5.35298058842467e-15, 7.732372695462776e-15, 1.1078048327998283e-14, 1.41370603415145e-14]

Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 2.6620111052967165e-16, 4.7733985228029 32e-16, 5.754709106439828e-16, 6.414322608621078e-16, 8.778883725835726e-16, 1.05409907949 2911e-15, 2.567867377405552e-15, 3.989084158950344e-15, 5.315661308136437e-15, 2.6070377734903704e-14, 4. 5743054976310414e-14, 6.354579976666838e-14]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 7.644975489842486e-16, 8 .784041837389243e-16, 1.0001473830225746e-15, 1.7522388797995943e-15, 2.7145404135316625e-15, 3.430008590 7471457e-15, 4.099466893647894e-15, 4.88088983331572e-15, 5.507797971696 54e-15, 1.2177958125974735e-14, 1 .9871954216837144e-14, 2.6825050185506065e-14]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 2.286189845300709e-16, 3 .4214861977591 42e-16, 6.154783203567206e-16, 9.871883271340956e-16, 1.4897866808070117e-15, 1.841547702201 2293e-15, 7.82250606485234e-15, 1.3096932426399827e-14, 1.764070693177038e-14, 7.913067193391514e-09, 1. 3746925759386393e-08, 1.8948771182602757e-08]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 5.304106959243414e-16, 8 .420328978742023e-16, 1.1067358648511083e-15, 2.0681178937699084e-15, 2.9826672519422222e-15, 3.911504381 7968734e-15, 4.676898372476439e-15, 5.615935386517669e-15, 6.423177173301704e-15, 8.312461915022685e-15, 1.1331990962069036e-14, 1.4718550276991737e-14]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[4.440892098500626e-16, 3.187274104835396e-16, 4 .1660549951554124e-16, 8.062852553509039e-16, 1.3755428708147895e-15, 2.0871631894891944e-15, 2.690711725 1682506e-15, 3.807242230812537e-15, 5.065274862992919e-15, 6.39813930513863e-15, 1.6466716042715686e-12, 2.862457641088223e-12, 3.907729328141631e-12]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[4.440892098500626e-16, 7.097064896574335e-16, 9 .105025757124682e-16, 1.5484700723992123e-15, 1.7446174460508294e-15, 1.9111561271806044e-15, 2.003242395 0951364e-15, 4.463333592806931e-15, 7.58054077357782e-15, 1.0411844987777142e-14, 1.2547873604176588e-14, 1.4842426582071244e-14, 1.678351859485855e-14]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[5.551115123125783e-16, 8.005982021365895e-16, 1 .0653160909836467e-15, 1.6850645790959378e-15, 2.2213326147200566e-15, 2.6738210902408033e-15, 3.024102947 0788373e-15, 3.418714941469371e-15, 3.695657356303361e-15, 4.043292287855853e-15, 1.189440084082218e-14, 1.986818907963849e-14, 2.7253168053767442e-14]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 1.9084367603610151e-16, 2.6025357470795436e-16, 3.265454017792844e-16, 7.045247195789006e-16, 1.3994329851414499e-15, 1.7388111383765108e-15, 3.306160091487886e-15, 5.089836945705952e-15, 6.746013198812614e-15, 4.1955895657602585e-14, 7.415223031729553e-14, 1.042584359740489e-13]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[4.440892098500626e-16, 1.0429361332776627e-15, 1.6333740914719642e-15, 1.8906487334869903e-15, 2.12994766695943e-15, 2.3839342705567047e-15, 2.885701263 950554e-15, 3.6539644473545304e-15, 4.660599754577228e-15, 5.314697545817 6415e-15, 2.8846290212323957e-13, 5.110728744364151e-13, 7.064038613275086e-13]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 1.2616606095639662e-15, 1.6130483003055317e-15, 2.343097505061249e-15, 2.6227111475091285e-15, 2.841336337205183e-15, 3.41991789831551e-15, 4.669063260466185e-15, 6.006027401414106e-15, 7.649232189228821e-15, 3.123858045921203e-14, 5.301164643283234e-14, 7.185019700511097e-14]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[5.551115123125783e-16, 3.741235936707433e-17, 7 .169782322684533e-17, 3.7071831794821163e-16, 1.8947901566908403e-15, 3.155101911588129e-15, 4.626323467026873e-15, 6.67640183024133e-15, 8.882598007381188e-15, 1.1171982663285326e-14, 1.4817881920708902e-14, 1 .9239199732256053e-14, 2.3697996437139894e-14]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 7.603602053332852e-16, 1 .1608890209639206e-15, 1.5640747531568692e-15, 1.9348366819475626e-15, 2.2491342724898576e-15, 2.777217447 0568505e-15, 2.9455802235358793e-15, 3.0101989845774325e-15, 3.129973374640553e-15, 1.3167913453374645e-14, 2.3078462968902494e-14, 3.207303935624639e-14]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 9.989795397325817e-16, 1.3798011380632406e-15, 1.7511632652962189e-15, 3.089054004934657e-15, 4.364869819136202e-15, 5.600179690021208e-15, 9.714569748212168e-15, 1.4171838151480124e-14, 1.8424631494784176e-14, 2.7471669278466566e-12, 4.770898368929996e-12, 6.568717595159667e-12]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 4.013200977766824e-16, 5.801685278971575e-16, 7.449281529237109e-16, 7.930847265743473e-16, 9.661252427888318e-16, 1.2764674667523205e-15, 1.7382794946779497e-15, 2.1842703586004135e-15, 2.6987113245298584e-15, 3.246800168584083e-15, 3.8383077860657695e-15, 4.423380684117886e-15]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[1.1102230246251565e-16, 6.288320446336525e-16, 1.0703591467846456e-15, 1.4869861946142464e-15, 1.7355659184276869e-15, 1.9472738668735315e-15, 2.5529894 27689548e-15, 3.1371666406758426e-15, 3.5604679574105535e-15, 3.9060477434362434e-15, 4.783870709445193e-15, 5.8706978410855634e-15, 6.8657611129810765e-15]
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Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[0.0, 6.553095261261512e-16, 8.288934938927836e-16, 9.41487866014991e-16, 1.1210780673170464e-15, 1.4447462381460644e-15, 1.8810417448951923e-15, 2.3812790895518805e-15, 2.8399761201903876e-15, 3.3525904834820772e-15, 3.881998150016136e-15, 4.431780302237 02e-15, 4.969605931609761e-15]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[7.771561172376096e-16, 3.2785646482232245e-16, 5.4826154027420 49e-16, 1.5128217452606472e-15, 2.4843192939757543e-15, 3.5358403620183778e-15, 4.597161950521664e-15, 6.001604217345457e-15, 7.708313191655242e-15, 9.557829712957479e-15, 1.2061693515473309e-14, 1.490041607782316e-14, 1.7943237621297785e-14]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 4.556533479436251e-16, 5 .188840778859921e-16, 9.962172118849815e-16, 1.2551314044210858e-15, 1.5229346641741553e-15, 1.633996663480594e-15, 1.741898834628403e-15, 1.8297213129842855e-15, 1.9575354434038274e-15, 3.128045303589151e-15, 4.329038214813395e-15, 5.731038719518068e-15]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[4.440892098500626e-16, 3.1168170664846484e-16,

```
4.071109210883448e-16, 6.357390648385112e-16, 8.18769170461314e-16, 9.583631439037215e-16, 1.770986782535
6561e-15, 2.7128594021629515e-15, 3.5523400254037217e-15, 4.382615567440656e-15, 6.4531262204286125e-15,
8.897516190411613e-15, 1.1148772469763612e-14]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[2.220446049250313e-16, 3.161030172945078e-16, 4
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77402e-15, 1.8486049699126703e-15, 2.3799696458456344e-15, 3.0134757751431958e-15, 6.111833735834265e-15,
9.337074881329066e-15, 1.250518914080772e-14]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[4.440892098500626e-16, 3.7008993014893356e-16,
4.969507628825812e-16, 1.0226672438560963e-15, 1.6776057140524375e-15, 2.256016610205942e-15, 2.648743979
141804e-15, 3.5332843779869564e-15, 4.760261679539662e-15, 6.022257455546663e-15, 1.1572763833586284e-14,
1.7439966407789095e-14, 2.3544478112665592e-14]
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to 13:[4.440892098500626e-16, 6.092625126534275e-16, 8
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345412e-15, 2.1010329286292682e-15, 2.427080870311774e-15, 2.9230356911884653e-15, 4.454292287008403e-15,
6.401037389025929e-15, 8.210396272407026e-15]
```
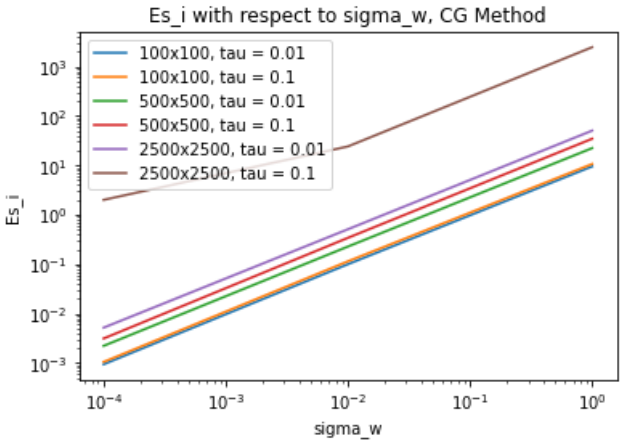

In [125]:

```
plt.figure()

plt.plot(w,err_es_a1, label = '100x100, tau = 0.01')
plt.plot(w,err_es_a2, label = '100x100, tau = 0.1')
plt.plot(w,err_es_a3,label = '500x500, tau = 0.01')
plt.plot(w,err_es_a4, label = '500x500, tau = 0.1')
plt.plot(w,err_es_a5, label = '2500x2500, tau = 0.01')
plt.plot(w,err_es_a6, label = '2500x2500, tau = 0.1')

plt.title("Es_i with respect to sigma_w, CG Method")
plt.yscale('log')
plt.xscale('log')
plt.ylabel('Es_i')
plt.xlabel('sigma_w')
plt.legend()
plt.show(block=False)
```



Finding $E_{O,i}$ errors for CG algortihm

In [118]:

```
def CGSolverS(A,x_init,b,max_iter):

    r = b.copy()
    d = r.copy()
    x = x_init.copy()
    residual = np.zeros((max_iter))

    for k in range(max_iter):
        A_d = A @ d
        prev = r.T @ r
        alpha = prev/(d.T @ A_d)
        x = x + alpha*d
        r = r - alpha*A_d
        beta = (r.T @ r)/prev
        d = r + beta*d
        residual[k] = np.linalg.norm(b - np.matmul(A,x))
    return residual
```

In [119]:

```
def getCGErrorO(A,b,x,maxiter):
    x_init = np.zeros((A.shape[0])).reshape(-1,1)
    err_n1 = []
```

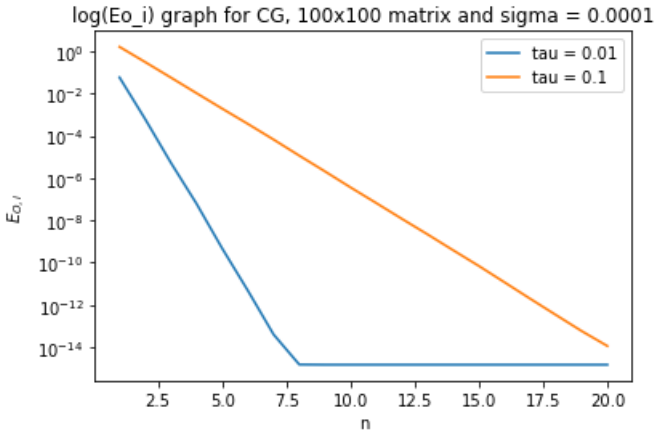
```
    err_n2 = []
    err_n3 = []
    err_sum = 0
    for i in range(30):
        res = CGSolverS(A = A, x_init = x_init ,b = b[i], max_iter = maxiter)
        err_sum = err_sum + res**2
        if i == 9:
            err_n1.append(err_sum)
        elif i == 19:
            err_n2.append(err_sum)
        elif i == 29:
            err_n3.append(err_sum)

    err_n1 = (np.array(err_n1) / 10) ** 0.5
    err_n2 = (np.array(err_n2) / 10) ** 0.5
    err_n3 = (np.array(err_n3) / 10) ** 0.5
    err_full = [err_n1,err_n2,err_n3]

    return np.array(err_full).reshape(3,maxiter)
```

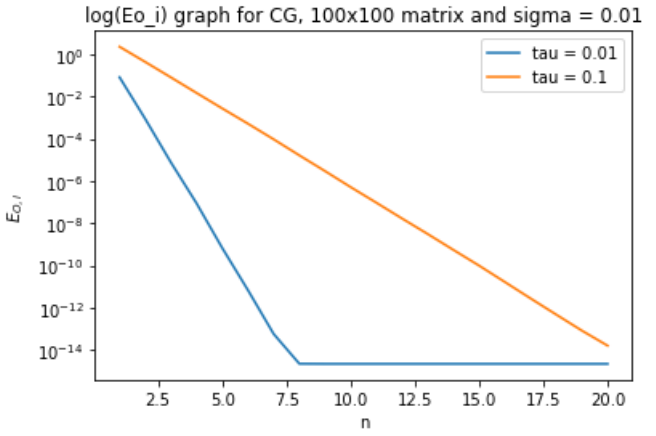
In [120]:

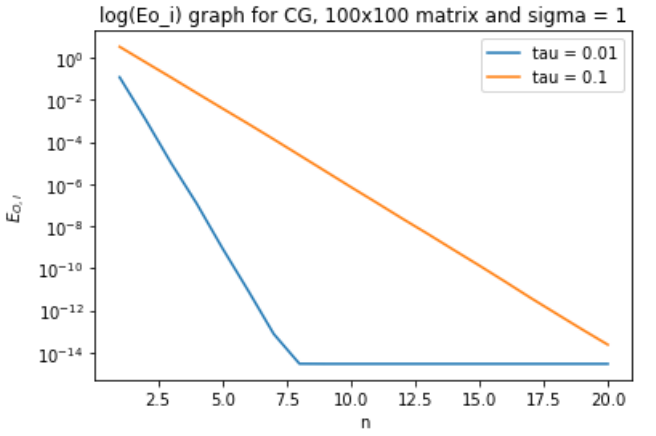
```
err_eo_a1 = getCGErrorO(A1,b_30_1,x_1,20)
err_eo_a2 = getCGErrorO(A2,b_30_2,x_2,20)
err_eo_a3 = getCGErrorO(A3,b_30_3,x_3,20)
err_eo_a4 = getCGErrorO(A4,b_30_4,x_4,20)
err_eo_a5 = getCGErrorO(A5,b_30_5,x_5,20)
err_eo_a6 = getCGErrorO(A6,b_30_6,x_6,20)
```

In [121]:

```
plt.figure()
plt.plot(np.arange(1,21),err_eo_a1[0,:], label = 'tau = 0.01')
plt.plot(np.arange(1,21),err_eo_a2[0,:],label = 'tau = 0.1')
plt.title('log(Eo_i) graph for CG, 100x100 matrix and sigma = 0.0001')
plt.yscale('log')
plt.ylabel('$E_{O,i}$')
plt.xlabel('n')
plt.legend()
plt.show(block=False)

plt.figure()
plt.plot(np.arange(1,21),err_eo_a1[1,:], label = 'tau = 0.01')
plt.plot(np.arange(1,21),err_eo_a2[1,:],label = 'tau = 0.1')
plt.title('log(Eo_i) graph for CG, 100x100 matrix and sigma = 0.01')
plt.yscale('log')
plt.ylabel('$E_{O,i}$')
plt.xlabel('n')
plt.legend()
plt.show(block=False)

plt.figure()
plt.plot(np.arange(1,21),err_eo_a1[2,:], label = 'tau = 0.01')
plt.plot(np.arange(1,21),err_eo_a2[2,:],label = 'tau = 0.1')
plt.title('log(Eo_i) graph for CG, 100x100 matrix and sigma = 1')
plt.yscale('log')
plt.ylabel('$E_{O,i}$')
plt.xlabel('n')
plt.legend()
plt.show(block=False)
```



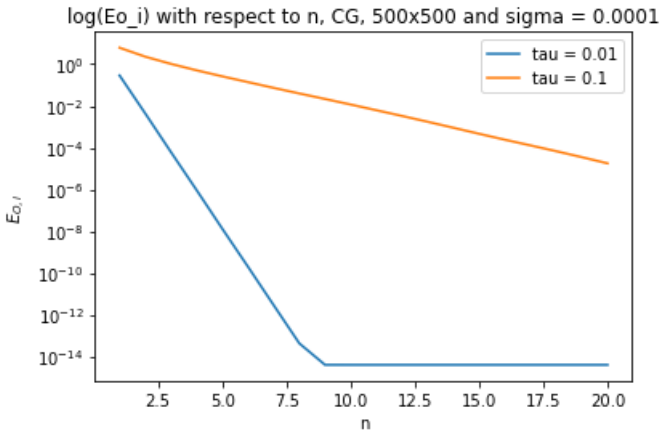


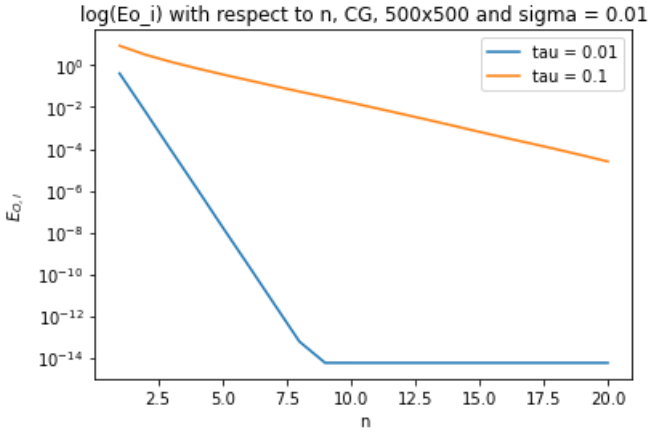


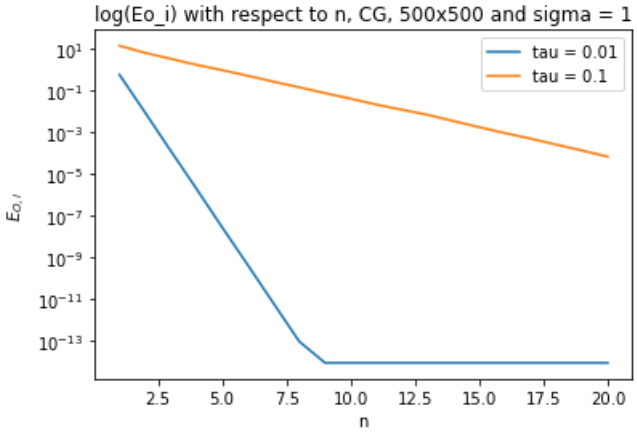
In [124]:

```
plt.figure()
plt.plot(np.arange(1,21),err_eo_a3[0,:], label = 'tau = 0.01')
plt.plot(np.arange(1,21),err_eo_a4[0,:],label = 'tau = 0.1')
plt.title('log(Eo_i) with respect to n, CG, 500x500 and sigma = 0.0001')
plt.yscale('log')
plt.ylabel('$E_{O,i}$')
plt.xlabel('n')
plt.legend()
plt.show(block=False)

plt.figure()
plt.plot(np.arange(1,21),err_eo_a3[1,:], label = 'tau = 0.01')
plt.plot(np.arange(1,21),err_eo_a4[1,:],label = 'tau = 0.1')
plt.title('log(Eo_i) with respect to n, CG, 500x500 and sigma = 0.01')
plt.yscale('log')
plt.ylabel('$E_{O,i}$')
plt.xlabel('n')
plt.legend()
plt.show(block=False)

plt.figure()
plt.plot(np.arange(1,21),err_eo_a3[2,:], label = 'tau = 0.01')
plt.plot(np.arange(1,21),err_eo_a4[2,:],label = 'tau = 0.1')
plt.title('log(Eo_i) with respect to n, CG, 500x500 and sigma = 1')
plt.yscale('log')
```

```
plt.ylabel('$E_{O,i}$')
plt.xlabel('n')
plt.legend()
plt.show(block=False)
```



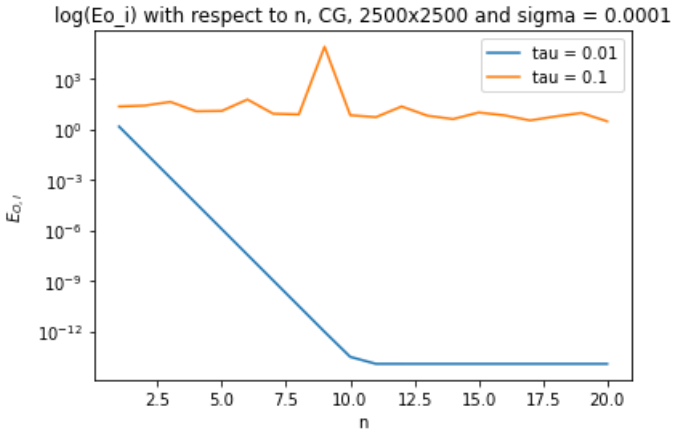


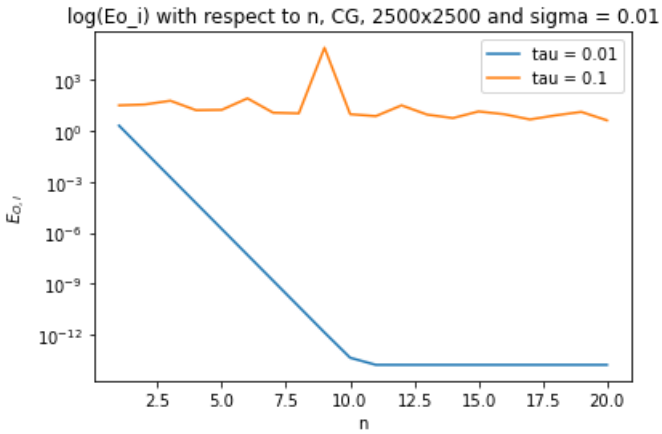


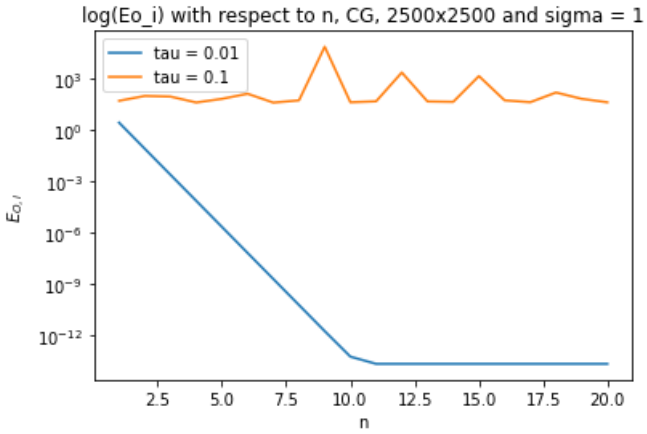
In [123]:

```
plt.figure()
plt.plot(np.arange(1,21),err_eo_a5[0,:], label = 'tau = 0.01')
plt.plot(np.arange(1,21),err_eo_a6[0,:],label = 'tau = 0.1')
plt.title('log(Eo_i) with respect to n, CG, 2500x2500 and sigma = 0.0001')
plt.yscale('log')
plt.ylabel('$E_{O,i}$')
plt.xlabel('n')
plt.legend()
plt.show(block=False)

plt.figure()
plt.plot(np.arange(1,21),err_eo_a5[1,:], label = 'tau = 0.01')
plt.plot(np.arange(1,21),err_eo_a6[1,:],label = 'tau = 0.1')
plt.title('log(Eo_i) with respect to n, CG, 2500x2500 and sigma = 0.01')
plt.yscale('log')
plt.ylabel('$E_{O,i}$')
plt.xlabel('n')
plt.legend()
plt.show(block=False)

plt.figure()
```

```
plt.plot(np.arange(1,21),err_eo_a5[2,:], label = 'tau = 0.01')
plt.plot(np.arange(1,21),err_eo_a6[2,:],label = 'tau = 0.1')
plt.title('log(Eo_i) with respect to n, CG, 2500x2500 and sigma = 1')
plt.yscale('log')
plt.ylabel('$E_{O,i}$')
plt.xlabel('n')
plt.legend()
plt.show(block=False)
```







Generalized Minimum Residual (GMRES)

In [68]:

```
def GMRES_by_Lancsoz(A,b_original,max_iter,tol):

    b = b_original.copy() #nx1
    n = A.shape[0]
    T_tilda = np.zeros((n+1, n))
    Q = np.zeros((n,n + 1))
    q1 = (b / np.linalg.norm(b)).reshape(-1)
    Q[:,0] = q1

    e1 = np.zeros(n+1)
    e1[0] = np.linalg.norm(b)
    e1 = e1.reshape(-1,1)

    frob_err = []
```

```
    bk = 0
    for k in range(0,max_iter):# 0,1,.., m-1

        qk = Q[:,k]
        v = A @ qk
        ak = qk.T @ v

        if (k == 0):
            v = v - ak * qk
        else:
            v = v - bk * qk_prev - ak * qk

        bk = np.linalg.norm(v)
        qk_prev = qk
        qk =  v / bk
        Q[:,k + 1] = qk

        T_tilda[k,k] = ak
        T_tilda[k+1,k] = bk

        if k != max_iter - 1:
            T_tilda[k,k+1] = bk

        frob_err.append(np.linalg.norm(np.matmul(Q[:,:k+2].T,Q[:,:k+2])-np.eye(k+2,k+2)))

        Q_lst,R_lst = np.linalg.qr(T_tilda[:k+1,:k+1])
        R_lst_inv = np.linalg.pinv(R_lst)
        y_k = R_lst_inv @ (Q_lst.T @ e1[:k+1,0])
        x_hat = Q[:,:k+1] @ y_k


        if( frob_err[k] > tol):
            print('Stopped at Iteration: ', k + 1)
            print('Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to :', k, ':' ,frob_err)
            break

    x_hat = x_hat.reshape(-1,1)
    return x_hat
```

In [103]:

```
def err_es_lancsoz(A,b,x,max_iter,tolerance):

    err_n1 = []
    err_n2 = []
    err_n3 = []
    for i in range(30):
        x_hat = GMRES_by_Lancsoz(A,b[i],max_iter,tolerance)
        x_0 = x[:,np.mod(i,10)].reshape(-1,1)
        x_hat = x_hat.reshape(-1,1)
        if i <10:
            err1 = np.linalg.norm(-x_hat + x_0) ** 2
            err_n1.append(err1)
        elif i >= 10 and i < 20:
            err2 = np.linalg.norm(-x_hat + x_0) ** 2
            err_n2.append(err2)
        else:
            err3 = np.linalg.norm(-x_hat + x_0) ** 2
            err_n3.append(err3)

    err_n1 = (np.sum(err_n1) / 10) ** 0.5
    err_n2 = (np.sum(err_n2) / 10) ** 0.5
    err_n3 = (np.sum(err_n3) / 10) ** 0.5
    err_full = [err_n1,err_n2,err_n3]
    return err_full
```

In [104]:

```
gmres_err1_es = err_es_lancsoz(A1,b_30_1,x_1,A1.shape[0],1e-12)
gmres_err2_es = err_es_lancsoz(A2,b_30_2,x_2,A2.shape[0],1e-12)
gmres_err3_es = err_es_lancsoz(A3,b_30_3,x_3,A3.shape[0],1e-12)
gmres_err4_es = err_es_lancsoz(A4,b_30_4,x_4,A4.shape[0],1e-12)
gmres_err5_es = err_es_lancsoz(A5,b_30_5,x_5,A5.shape[0],1e-12)
gmres_err6_es = err_es_lancsoz(A6,b_30_6,x_6,A6.shape[0],1e-12)
```

```
Stopped at Iteration:  15
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 14 : [2.738403775149492e-14, 2.7512940514375922e-
14, 5.0582398423737836e-14, 7.801038455485599e-14, 1.1283951153407029e-13, 1.2815480875456268e-13, 1.5741
33051273942e-13, 1.7198934905568496e-13, 2.0646956357245393e-13, 2.5152488121521703e-13, 3.38027983439684
1e-13, 4.974858453868019e-13, 6.781268420883936e-13, 9.940135287420976e-13, 1.3277394827394852e-12]
```

```
Stopped at Iteration:  10
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 9 : [2.566768711203053e-15, 4.140040424358951e-14
, 1.2763340390923038e-13, 2.149934675250701e-13, 3.131117812857598e-13, 3.881815440604242e-13, 5.32609687
0630121e-13, 6.236908140433446e-13, 8.123190145323653e-13, 1.0016070385928602e-12]
Stopped at Iteration:  9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 8 : [3.642529915077027e-14, 1.0236014375683071e-1
3, 2.1754932886639087e-13, 2.505488811991961e-13, 4.571938467596748e-13, 4.851630106139851e-13,
7.610848874534248e-13, 8.427926133931015e-13, 1.042479125496636e-12]
Stopped at Iteration:  14
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 13 : [5.326245944324137e-14, 5.472522910609583e-1
4, 1.1519085458748233e-13, 2.2192746009075396e-13, 2.4858397720676516e-13, 3.4108430982054406e-13, 4.2707
433684184766e-13, 4.781019362400366e-13, 5.763609898763839e-13, 6.411345057608512e-13, 7.422658796421051e
-13, 8.763097952297585e-13, 9.968880517661192e-13, 1.1725582279730741e-12]
Stopped at Iteration:  13
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 12 : [2.200825158361058e-14, 2.7603395528922782e-
14, 4.386127940118167e-14, 6.178995625543739e-14, 1.0586368843109252e-13, 1.5512491203051003e-13, 2.41777
492494708e-13, 2.925153022254592e-13, 4.3502173635725765e-13, 5.513678467258232e-13, 7.372217280711825e-1
3, 9.054942698177782e-13, 1.1403112491499458e-12]
Stopped at Iteration:  10
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 9 : [4.762170168962732e-14, 7.299229363908571e-14
, 1.4343739732218564e-13, 1.8304640403161684e-13, 3.257900304563767e-13, 3.6568400356263746e-13, 5.388098
611632559e-13, 6.705745828513363e-13, 9.31805055455497e-13, 1.2563187543661995e-12]
Stopped at Iteration:  14
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 13 : [1.0642069132529804e-13,
1.1291046925623084e-13, 2.4975482092061657e-13, 2.6913500827201897e-13, 3.9630553807288543e-13, 4.4278094
99116132e-13, 5.18327116750321e-13, 5.775783455408372e-13, 6.670173905163452e-13, 7.013912143417656e-13,
7.94978507982305e-13, 8.341864134653867e-13, 9.770377682873257e-13, 1.0182018611212215e-12]
Stopped at Iteration:  10
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 9 : [5.49035032925213e-15, 5.480104077758929e-14,
1.2510570558434324e-13, 2.1952835952927684e-13, 3.3800588072936986e-13, 4.096575037959446e-13, 5.49212578
243098e-13, 6.654996461997364e-13, 8.485582798410842e-13, 1.0846734426307096e-12]
Stopped at Iteration:  15
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 14 : [3.423923844179998e-14, 4.0300304578771947e-
14, 1.2369091824443 71e-13, 1.4580341930318602e-13, 2.1627570233160864e-13, 2.2475205277467328e-13, 2.9599
196921647606e-13, 3.212637206572319e-13, 3.726504060268231e-13, 3.973104089540377e-13, 5.32541642847142e-
13, 5.779446294636524e-13, 7.229616955417989e-13, 8.069862748329969e-13, 1.0397411182008547e-12]
Stopped at Iteration:  15
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 14 : [7.357850360642954e-14, 7.400359918843697e-1
4, 1.539545358525896e-13, 2.3257837290210707e-13, 2.793636444407101e-13, 3.8128106039572394e-13,
4.031464823752311e-13, 4.712586822467201e-13, 5.145258396929095e-13, 5.811928510475285e-13, 6.39127973675
2236e-13, 7.446042218063486e-13, 8.172755621195342e-13, 9.44234560906027e-13, 1.1356580894789904e-12]
Stopped at Iteration:  11
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 10 : [5.849402975226109e-14, 6.225520748265415e-1
4, 9.081206779293684e-14, 1.2864011657046394e-13, 1.8284582301498502e-13, 2.26393704730628e-13, 2.6309596
25464886e-13, 3.4805649193438205e-13, 5.303951528182184e-13, 7.664396027852466e-13, 1.1552570444843536e-12]
Stopped at Iteration:  13
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 12 : [2.5887352203102452e-14, 3.841597704666898e-
14, 8.77568099510355e-14, 9.74874775843357e-14, 1.5039351679563872e-13, 1.6313805910937202e-13, 2.3108572
140987775e-13, 2.571067889310571e-13, 3.556745413075496e-13, 4.1636650637096534e-13, 6.037111033408286e-13, 8.009173876725079e-13, 1.0514353278145454e-12]
Stopped at Iteration:  14
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 13 : [7.760503356469795e-16, 7.613140911687472e-1
5, 2.319210458040553e-14, 2.909745308932977e-14, 3.133130462565917e-14, 8.677790522007102e-14,
1.4175600143576432e-13, 2.2583868706269676e-13, 3.078400088661212e-13, 3.8996562602347415e-13, 5.23992928
5238269e-13, 6.975282092469123e-13, 9.2583910140012e-13, 1.2581568235208174e-12]
Stopped at Iteration:  14
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 13 : [6.141110680960987e-14, 7.641399875582453e-1
4, 1.5907290119950138e-13, 1.7105748103998583e-13, 2.179488550010468e-13, 2.3545208084662577e-13, 2.96826
87179661284e-13, 3.2668428986396903e-13, 4.151838896690858e-13, 4.3613054095925174e-13, 5.462419583793253
e-13, 6.607159557317544e-13, 8.816891528360692e-13, 1.259689203886134e-12]
Stopped at Iteration:  14
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 13 : [6.080235001786493e-14, 6.384900519029099e-1
4, 1.0964320793443272e-13, 1.1062607978005793e-13, 1.5217379360891107e-13, 1.7886368209927032e-13, 2.3268
166689480916e-13, 3.1432151356640295e-13, 3.6953454091633816e-13, 4.931281896194244e-13, 5.67865951708423
4e-13, 7.614605636008483e-13, 8.646713355418352e-13, 1.219928329019125e-12]
Stopped at Iteration:  14
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 13 : [3.734170441654497e-14, 4.958672192404996e-1
4, 1.0399935108744317e-13, 1.215285378207912e-13, 1.947534759032541e-13, 2.0856275883923498e-13,
2.880336228525439e-13, 3.6280591067027867e-13, 4.829039654878767e-13, 5.580399311771012e-13, 6.9528575975
2328e-13, 8.007451243757143e-13, 9.966709140841245e-13, 1.1692265258970558e-12]
Stopped at Iteration:  12
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 11 : [1.6517025016101614e-14,
2.1706344643123174e-14, 7.947484813633774e-14, 1.7613787773674647e-13, 2.716799543799529e-13,
3.4548846627167217e-13, 3.9633113965167064e-13, 5.089510081067111e-13, 6.254398187706157e-13,
7.493661037321048e-13, 8.709486284800876e-13, 1.0622718630091314e-12]
Stopped at Iteration:  11
```

Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 10 : [4.3015389294077185e-14, 4.549554145924827e-14, 9.955765863043365e-14, 1.1982286686477738e-13, 1.9294121479852916e-13, 3.032864531732833e-13, 3.6933493225338606e-13, 5.356579445024599e-13, 6.189488936194602e-13, 8.314959069532114e-13, 1.1520539045942762e-12]
Stopped at Iteration: 17
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 16 : [3.81256183918413e-15, 2.159057960053209e-14, 2.953019492917418e-14, 3.156934315958675e-14, 6.648077216591585e-14, 9.283390216296845e-14, 1.0585687728010685e-13, 1.289789177099751e-13, 1.5232670758772204e-13, 1.778491817817047e-13, 2.1450668606312666e-13, 2.84780459351118e-13, 3.471843924477512e-13, 4.2453148806395634e-13, 5.700105537660503e-13, 7.784313515466901e-13, 1.3011783653147575e-12]
Stopped at Iteration: 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 8 : [7.087380098689185e-14, 7.729229637449712e-14, 2.312676480851732e-13, 2.9484592384492903e-13, 4.97987623161306e-13, 5.732537691673194e-13, 8.07447308104293e-13, 9.147738736285642e-13, 1.0872514801879814e-12]
Stopped at Iteration: 11
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 10 : [3.9658802785711747e-14, 4.5822214456316136e-14, 6.35645570999413e-14, 1.0139330925477716e-13, 1.4290206283482287e-13, 2.2168748572155098e-13, 2.926919928669126e-13, 3.959282417508901e-13, 5.786247934346357e-13, 8.257060755914523e-13, 1.233573946766814e-12]
Stopped at Iteration: 13
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 12 : [2.46538952885875e-14, 3.14040975716797e-14, 6.0623053475313 56e-14, 7.996645166837611e-14, 1.156715212900651e-13, 1.4019585138553982e-13, 2.326468266423424e-13, 2.945834412411518e-13, 3.881239299438802e-13, 4.938798437673689e-13, 7.007977550256138e-13, 8.936162415305632e-13, 1.1572607659439442e-12]
Stopped at Iteration: 10
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 9 : [3.198053710642971e-14, 5.69869899158226e-14, 1.4225295735416628e-13, 1.6863442346737487e-13, 2.776315545796153e-13, 3.162155356353813e-13, 4.957171310523833e-13, 5.800741308109313e-13, 8.094425730333788e-13, 1.0016879610356358e-12]
Stopped at Iteration: 17
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 16 : [6.728620911323059e-14, 6.782262109717635e-14, 1.1266192394712386e-13, 1.2413644436612377e-13, 1.7773344143568914e-13, 1.9067754584580009e-13, 2.544492885722059e-13, 2.975409578719845e-13, 3.421423522408613e-13, 4.126157186096078e-13, 4.459839206462644 7e-13, 5.126720998039367e-13, 5.6171062983118e-13, 6.594100012183951e-13, 7.704460083205347e-13, 9.78378132795253e-13, 1.3084550234853574e-12]
Stopped at Iteration: 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 8 : [1.0198305151880798e-14, 4.0462504678878696e-14, 7.873084972394539e-14, 1.4663056841939158e-13, 2.5591051706009875e-13, 4.231364541305606e-13, 5.860209991317745e-13, 7.810543158862351e-13, 1.0666137909665795e-12]
Stopped at Iteration: 10
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 9 : [4.0305749350262486e-14, 6.39512619855456e-14, 1.2582168925590762e-13, 1.3981860188176294e-13, 2.805974803400441e-13, 3.5849003141633195e-13, 5.392051861779612e-13, 6.550729850082159e-13, 8.850993572806924e-13, 1.0513832863755998e-12]
Stopped at Iteration: 9
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 8 : [4.106528965938715e-14, 4.806812147735745e-14, 1.2101068978729828e-13, 1.673968367838303e-13, 3.1415356138640635e-13, 4.0315363189830854e-13, 6.309122970496453e-13, 8.138296442573024e-13, 1.2372398510323782e-12]
Stopped at Iteration: 12
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 11 : [2.8325638702161495e-14, 4.6635377976970174e-14, 8.32861342531366e-14, 9.276733252720323e-14, 1.930243265732 59e-13, 2.2831381376085 3e-13, 3.2308347306553965e-13, 3.9416890258126526e-13, 5.417443856820148e-13, 6.646276240749133e-13, 9.243817786260183e-13, 1.0938760054923121e-12]
Stopped at Iteration: 13
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 12 : [2.930890961191198e-14, 3.0482497860403904e-14, 5.4919335705918695e-14, 6.962773216363346e-14, 8.926933626695857e-14, 1.1209114766340692e-13, 1.765979473781724e-13, 2.90008730667655e-13, 3.6487143600228723e-13, 5.008786566495788e-13, 6.329840096171754e-13, 8.365437056714333e-13, 1.0569647418440736e-12]
Stopped at Iteration: 13
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 12 : [2.3843922513830224e-14, 4.526834881445768e-14, 8.72998282947304e-14, 9.726668444235591e-14, 1.500559720014091e-13, 1.813280499083456e-13, 2.879253208399734e-13, 3.980904808345362e-13, 5.293692040408464e-13, 6.360248277721402e-13, 7.662059584385 47e-13, 8.960346210178045e-13, 1.136795002288489e-12]
Stopped at Iteration: 26
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 25 : [2.692946917937489e-15, 3.653651381463119e-15, 6.8303366145934206e-15, 6.928197160356842 6e-15, 8.65668193650263e-15, 9.727548337213377e-15, 1.0777090763212392e-14, 1.1559452173053833e-14, 1.2487716104464808e-14, 1.3848006224769568e-14, 1.435188837232638e-14, 1.5970738515998743e-14, 1.75714585990282e-14, 1.9203399577121 59e-14, 2.2094263128073325e-14, 2.6944510412871098e-14, 3.157472043184437e-14, 3.712544470228979e-14, 4.563283275490370 4e-14, 6.001552811777478e-14, 8.856767238110265e-14, 1.3845594493694567e-13, 2.1944174540423283e-13, 3.6985091663302647e-13, 6.53714067484505e-13, 1.2438426253905605e-12]
Stopped at Iteration: 27
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 26 : [2.6152050514427237e-16, 6.167605901819989e-16, 1.4163844557785023e-15, 2.6977989829664376e-15, 4.709905043749468e-15, 6.480180476837073e-15, 8.464638409034015e-15, 1.023750712851522e-14, 1.2578406467163785e-14, 1.4106455045886971e-14, 1.5967766964890845e-14, 1.834329112974177e-14, 2.1446235623769697e-14, 2.3934216698224665e-14, 2.963224259420192e-14, 3.3014502421666664e-14, 4.06429696750857e-14, 4.7577848935397334e-14, 6.442254360306019e-14, 8.180377619382092e-14, 1.1501984971886518e-13, 1.6171532172112665e-13, 2.4089110145364005e-13, 3.4031021001256264e-13, 5.141146942638264e-13, 8.814413853345411e-13, 1.3617684190010272e-12]

Stopped at Iteration: 26
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 25 : [9.366183383432055e-16, 1.7192414707391093e-15, 4.0086394934718325e-15, 5.519369168283374e-15, 8.533191592719408e-15, 1.0849780981867612e-14, 1.4396335795943605e-14, 1.5961172964698082e-14, 1.875404846765216e-14, 2.165789651642033e-14, 2.3613690072209312e-14, 2.6188757031235127e-14, 2.8907812246171186e-14, 3.1254900016572946e-14, 3.510074817916572e-14, 4.018023736093725e-14, 4.964092794752567e-14, 5.900484085133661e-14, 7.31231965884149e-14, 9.151957455728 73e-14, 1.225813080896565e-13, 1.8579806807384323e-13, 3.161967063204943e-13, 4.860852531844764e-13, 7.195033842446917e-13, 1.1172677386654622e-12]
Stopped at Iteration: 26
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 25 : [2.500275240052666 5e-15, 4.423935359823848e-15, 7.161911808301053e-15, 8.494306744864206e-15, 1.1371077628276828e-14, 1.3230192952546723e-14, 1.7744886224150998e-14, 2.1252344027640353e-14, 2.637689131750705e-14, 2.768201710389018e-14, 3.037702654859726e-14, 3.1872742031145433e-14, 3.5002136495611446e-14, 3.6233130988476255e-14, 4.0907432506187324e-14, 4.272923037743711e-14, 4.817284040424827e-14, 5.212788330620476e-14, 5.978038233073084e-14, 6.832164403203314e-14, 8.795871926518205e-14, 1.231470967202811e-13, 1.989965333647575e-13, 3.5037196027166754e-13, 6.177436568598417e-13, 1.097739044996614e-12]
Stopped at Iteration: 26
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 25 : [3.2729245380346086e-15, 4.4593759432108345e-15, 6.8091854138397826e-15, 7.998476050696107e-15, 9.041420317715451e-15, 1.0324274351940483e-14, 1.186818320411436e-14, 1.3325145388176682e-14, 1.4619843668225325e-14, 1.573279634550525e-14, 1.7459585710537936e-14, 1.980024803029833e-14, 2.32122115970563 1e-14, 2.8540869383294626e-14, 3.687737996057118e-14, 5.0479103264831884e-14, 7.483275770719769e-14, 1.1279028600888354e-13, 1.4446225113233802e-13, 1.793055567614919e-13, 2.309782964660053e-13, 3.083278600093221e-13, 4.375571848941716e-13, 6.501987114011022e-13, 9.95665545079592e-13, 1.549668806376811e-12]
Stopped at Iteration: 26
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 25 : [5.725355553557655e-16, 1.4833774771546268e-15, 1.9546188731196593e-15, 2.1783616939167183e-15, 2.968168165912237e-15, 4.1344200193438356e-15, 5.476421216307918e-15, 6.133485530410998e-15, 7.605850163089207e-15, 9.011572594626056e-15, 1.1201185019143204e-14, 1.275189509728088e-14, 1.579162301144645e-14, 1.8354444782741957e-14, 2.201429010410124e-14, 2.6198991511792464e-14, 3.399396345542975e-14, 4.4726485952256047e-14, 6.005661192128332e-14, 8.698802347614797e-14, 1.363711343067595e-13, 2.1113894678024783e-13, 3.796758098362624e-13, 5.989262711617606e-13, 8.480760991580751e-13, 1.2322662557518212e-12]
Stopped at Iteration: 21
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 20 : [1.948948012459365e-15, 2.111364687199207e-15, 3.384171189031608 7e-15, 4.195877087646668e-15, 6.655304226599617e-15, 7.763450711263295e-15, 1.0706763701627 68e-14, 1.270839625464329e-14, 1.6876385721966637e-14, 2.2217467812276598e-14, 3.369973465978421e-14, 4.717891608836776e-14, 6.653348871531062e-14, 1.0089430997030983e-13, 1.6770936389552486e-13, 2.6382589174921196e-13, 3.6542082915789726e-13, 4.80041558702747e-13, 6.894612433054481e-13, 9.348435152272937e-13, 1.1032183482449686e-12]
Stopped at Iteration: 25
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 24 : [1.542622181382843e-15, 2.6038894992731076e-15, 4.510464013794792e-15, 5.015343245614596e-15, 6.661121235223134e-15, 7.391384506945657e-15, 8.802432138428116e-15, 1.0582468478750545e-14, 1.2000509122262093e-14, 1.4607012427123293e-14, 1.6041501802528957e-14, 1.8155826405791768e-14, 2.1698506605715748e-14, 2.48639452091293e-14, 3.4708352821907776e-14, 5.088176590713918e-14, 8.263333777881437e-14, 1.1782395305950496e-13, 1.4880422034299907e-13, 2.0151122415243107e-13, 3.0398775730833783e-13, 4.5110792950989593e-13, 6.563643084290211e-13, 8.298061159331774e-13, 1.1868303490256231e-12]
Stopped at Iteration: 27
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 26 : [2.3266046511434065e-15, 2.9549228098090205e-15, 4.655785326900289e-15, 5.4195644652011375e-15, 7.90226117194602e-15, 9.484326063267094e-15, 1.4157845531681138e-14, 1.687892005541707e-14, 2.208703312409519e-14, 2.5219018519144653e-14, 2.9869832803029063e-14, 3.247783453049761e-14, 3.967423954542488e-14, 4.158825064090057e-14, 5.010245197589456e-14, 5.372060874401647e-14, 6.380781043069954e-14, 7.247314384679948e-14, 8.282235680653908e-14, 9.332669372599036e-14, 1.1275449659722147e-13, 1.4140421048770887e-13, 1.9173509744877026e-13, 2.845488247199059e-13, 4.04578868929033e-13, 6.152462926301313e-13, 1.0630553886031018e-12]
Stopped at Iteration: 27
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 26 : [1.8321464268187376e-15, 2.262523039118783e-15, 4.2837838043663235e-15, 5.6243995489845015e-15, 8.247017624387548e-15, 9.145611223133246e-15, 1.3026725513714771e-14, 1.340202637368278e-14, 1.6619119233909483e-14, 1.681504838432063e-14, 2.121090622428361e-14, 2.2184635396785696e-14, 2.5975188496120772e-14, 2.805094618233863e-14, 3.202691749303472e-14, 3.601251125693931e-14, 4.0280317772852355e-14, 4.7492044864099094e-14, 5.70056405310145e-14, 7.564837790657971e-14, 1.0129889902876309e-13, 1.4105259768903195e-13, 2.0935215786071863e-13, 3.3045062260922597e-13, 4.9698793642927e-13, 7.963046487196908e-13, 1.4887703871506363e-12]
Stopped at Iteration: 24
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 23 : [2.0509876126094263e-15, 4.363458918020903e-15, 8.524411237553492e-15, 1.0151251262362422e-14, 1.5266127652941346e-14, 1.6134013564829728e-14, 2.0025287397355783e-14, 2.1047214117634227e-14, 2.5512499760527768e-14, 2.7135403621734827e-14, 3.245262618140521e-14, 3.472556491695483e-14, 4.244792338477513e-14, 4.6840313937524216e-14, 6.197633534977347e-14, 7.495623349783156e-14, 9.50442396480727e-14, 1.1514898069338092e-13, 1.4800959335994823e-13, 2.025500546045932e-13, 3.148392449604167e-13, 5.073208684086722e-13, 8.171598248937362e-13, 1.384338771850223e-12]
Stopped at Iteration: 28
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 27 : [3.545894581919458e-15, 7.247038211925443e-15, 1.3486300382705453e-14, 1.5985632042473247e-14, 2.3949846639658086e-14, 2.6219447850339498e-14, 3.439017550670748e-14, 3.9162080298959986e-14, 4.861337231517704e-14, 5.366450332138375e-14, 6.044904534830515e-14, 6.690413231919497e-14, 7.393892359035954e-14, 7.693600951144365e-14, 8.761388386703184e-14, 8.924408858339851e-14, 9.822489148722446e-14, 9.96899862594987e-14, 1.138035379331435e-13, 1.1870537847470836e-13, 1.3757892454826976e-13, 1.5429989678520423e-13, 1.9359759693226997e-13, 2.490260176110735e-13,

3.56787119818615e-13, 5.876115211046573e-13, 9.729451393764308e-13, 1.6071734335959354e-12]
Stopped at Iteration: 28
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 27 : [7.961416993959742e-16, 1.8254286697171027e-15, 3.1481785347739023e-15, 3.576871089002802e-15, 5.056919378784612e-15, 5.99608848648732e-15, 7.889427703742024e-15, 8.944626552934657e-15, 1.1222024351888244e-14, 1.3601889429977872e-14, 1.5378950857328667e-14, 1.7695102444875887e-14, 2.0926372208963197e-14, 2.2565433650898868e-14, 2.6435121779839303e-14, 2.837475277020837e-14, 3.5063283704065666e-14, 4.0103450572755235e-14, 4.8829044251475364e-14, 5.827471974246295e-14, 7.614102432175435e-14, 1.1149967688031345e-13, 1.863754688261877e-13, 2.807391435050535e-13, 4.0923232624361606e-13, 6.249770847707746e-13, 9.555514182154423e-13, 1.6133905987296854e-12]
Stopped at Iteration: 25
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 24 : [2.377044786818747e-15, 2.772967368143602e-15, 4.447964274543615e-15, 4.84380397182224e-15, 6.167003845883744e-15, 7.0636556794858545e-15, 9.131174385498622e-15, 1.0583666041984654e-14, 1.3154051153207329e-14, 1.4413300394763116e-14, 1.6909170626849198e-14, 1.926776740580353e-14, 2.3964825134913354e-14, 2.882835747590282e-14, 3.630765882715417e-14, 4.254419278022956e-14, 5.274193582426736e-14, 6.421618732748219e-14, 8.26802596590135e-14, 1.1104742752626645e-13, 1.6368666399805229e-13, 2.6576277134541665e-13, 4.5003575650097646e-13, 8.059472223469556e-13, 1.4385247430548718e-12]
Stopped at Iteration: 23
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 22 : [7.60948540406141e-16, 1.9414554688497168e-15, 4.552678567131633e-15, 7.323304093777526e-15, 1.004677813781379e-14, 1.2343377609022791e-14, 1.5441474780850633e-14, 1.8605903276303958e-14, 2.234532934893914e-14, 2.4961330755003625e-14, 2.919756345890572e-14, 3.4393017379280346e-14, 4.450890525214996e-14, 6.214040331378791e-14, 8.666067375472565e-14, 1.2557239586683363e-13, 1.9628793881224031e-13, 3.0134346814199604e-13, 3.778900888056762e-13, 4.565136536610414e-13, 5.69639819122185e-13, 7.475111729375704e-13, 1.0611366868613873e-12]
Stopped at Iteration: 31
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 30 : [2.0890560349322558e-15, 2.9512817867572872e-15, 5.137914538936743e-15, 5.7226704794923935e-15, 7.492233224124133e-15, 7.970908915765865e-15, 9.382795194879577e-15, 1.0242024310965246e-14, 1.187972724686607e-14, 1.3281799176485613e-14, 1.7846962408135562e-14, 1.865982067168668e-14, 2.2611189672639352e-14, 2.4819328802691252e-14, 2.787985254174695e-14, 3.0121820825851624e-14, 3.3772558886755746e-14, 3.584080767036411e-14, 3.953780375992816e-14, 4.185728979143538e-14, 4.636871086139666e-14, 5.200625438072984e-14, 6.312346137465154e-14, 8.193427333162385e-14, 1.1158210407213628e-13, 1.7331264032872107e-13, 2.878383709485083e-13, 4.370401922496137e-13, 6.112899969364786e-13, 9.08782602236806e-13, 1.5136939103288381e-12]
Stopped at Iteration: 23
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 22 : [6.890028263629845e-16, 1.3757160025943086e-15, 2.7440540041506253e-15, 5.882160694679767e-15, 1.0790357436524149e-14, 1.4000467585677417e-14, 2.0197008408136738e-14, 2.6042402018099175e-14, 3.4057060234104364e-14, 4.127091211845121e-14, 5.184362066369442e-14, 6.236151546528696e-14, 7.73156572148833e-14, 9.663712807757819e-14, 1.35523171987394e-13, 1.9169926330615913e-13, 2.5475863172620323e-13, 3.2980454924130945e-13, 4.650540743059401e-13, 6.263732503805029e-13, 7.822170869532823e-13, 9.657793707088958e-13, 1.2444586503927048e-12]
Stopped at Iteration: 26
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 25 : [1.724903304153833e-15, 2.2254157282883242e-15, 4.227488188808098e-15, 4.856514823170166e-15, 6.760826468452102e-15, 8.370900843159921e-15, 9.496549552587525e-15, 1.0728339668938657e-14, 1.1470545564069185e-14, 1.2714253905404233e-14, 1.3815650609180746e-14, 1.713364218215034e-14, 1.951644634642382e-14, 2.4878243718844762e-14, 3.247636888463977e-14, 4.538077347388003e-14, 6.57663975938266e-14, 8.711335924540598e-14, 1.047254429378896e-13, 1.3296994199573138e-13, 1.880935819658829e-13, 2.697066316104376e-13, 3.8631791954075884e-13, 4.864288520652003e-13, 6.954652562646773e-13, 1.0581626757576701e-12]
Stopped at Iteration: 28
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 27 : [4.609022724757759e-16, 2.0996434513646395e-15, 5.023674276507986e-15, 7.964859719533285e-15, 1.1340675136716971e-14, 1.3340607140666071e-14, 1.7142333241570054e-14, 1.8981937048252028e-14, 2.2009230095041554e-14, 2.3310356613873024e-14, 2.496398739270268e-14, 2.6220494679837053e-14, 2.910221432622304e-14, 3.0788716195950554e-14, 3.5778342471797155e-14, 3.85360831547020 3e-14, 4.399661953982678e-14, 4.9473413143747446e-14, 5.707962099099763e-14, 6.637769776899832e-14, 8.526828433988365e-14, 1.159874427552217e-13, 1.6568493177417712e-13, 2.491503535971279e-13, 3.5419238902682414e-13, 5.382596745768405e-13, 9.061509258149275e-13, 1.478547814184929e-12]
Stopped at Iteration: 26
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 25 : [1.2833209020223676e-15, 2.232788296568361e-15, 4.563943727731992e-15, 5.218435745954165e-15, 7.21778367043419e-15, 7.598628472903778e-15, 1.0157495183619664e-14, 1.0895339773471757e-14, 1.3386505900004742e-14, 1.4193516675408317e-14, 1.8206064729203474e-14, 1.9940738288554888e-14, 2.4837381938343663e-14, 2.851197613640934e-14, 3.664363873824 85e-14, 4.495490767393892e-14, 5.643702924708032e-14, 7.274672923868436e-14, 1.0068449242504712e-13, 1.4675315374777055e-13, 2.0954774541720697e-13, 2.978823911756379e-13, 4.443310029763107e-13, 6.67451427135017e-13, 9.314989668127703e-13, 1.3246654207075165e-12]
Stopped at Iteration: 26
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 25 : [2.525142149744041e-16, 1.2812421846408986e-15, 2.308474120291873e-15, 2.6717780819382247e-15, 4.1654275044576116e-15, 6.034252934211549e-15, 6.431429889457042e-15, 7.94334423994942e-15, 9.3553482505737e-15, 1.0785708189275786e-14, 1.2760025625283042e-14, 1.6216341817082393e-14, 1.9381779407890004e-14, 2.271029316866637e-14, 2.6902209999208435e-14, 3.010588714602929e-14, 3.583617583694707e-14, 4.2588758001275684e-14, 5.5800282253199794e-14, 7.803345332957908e-14, 1.2121182486866846e-13, 1.9446805934941068e-13, 3.0949804958653685e-13, 5.297730322506248e-13, 8.796177205003169e-13, 1.505498838621494e-12]
Stopped at Iteration: 25
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 24 : [5.305432017121034e-16, 1.22608565997739e-15, 2.3584278610734645e-15, 2.900009411131792e-15, 3.822910953927266e-15, 4.4736463067905355e-15, 5.633783593713926e-15, 7.712863122009459e-15, 9.373505787064396e-15, 1.2038584854227524e-14, 1.3245070710784 76e-14, 1.7273446978098267e-14, 1.9467232170061512e-14, 2.357486632353918e-14, 2.737637095

8526874e-14, 3.280634721576419e-14, 3.9544828812127054e-14, 4.815308546841196e-14, 6.477156299228657e-14, 9.72326536238851e-14, 1.5790558311624543e-13, 2.8929243473996286e-13, 5.210232434208031e-13, 8.037586044510197e-13, 1.0849329250009923e-12]
Stopped at Iteration: 27
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 26 : [8.340307653846247e-16, 2.014816296023888e-15, 3.444674262760176e-15, 4.679547683692754e-15, 7.189981664847993e-15, 8.408819416486037e-15, 1.0731069228570786e-14, 1.235802684422927e-14, 1.5359240957660088e-14, 1.76095467982364e-14, 2.031546230881392e-14, 2.2124452641662904e-14, 2.59514757039911e-14, 2.8072129922828684e-14, 3.3062235253258166e-14, 3.740674441761306e-14, 4.549918788622843e-14, 5.597701865986504e-14, 7.097600246644068e-14, 9.726372198050528e-14, 1.338484751212052e-13, 1.8964949108517772e-13, 2.9651348412991226e-13, 3.9891138065807195e-13, 5.61515214280301e-13, 9.485015347993495e-13, 1.4645510153648876e-12]
Stopped at Iteration: 24
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 23 : [6.707195401143195e-16, 1.0466769871106295e-15, 1.561085357112974e-15, 3.600625744811872e-15, 6.087952901142975e-15, 1.0164039434935383e-14, 1.3262630321068856e-14, 1.632549851534745e-14, 2.1122139502794555e-14, 2.665260116939672e-14, 3.690384018498932e-14, 4.782878520582444e-14, 6.068335749597637e-14, 7.232413247785907e-14, 8.771646258355681e-14, 1.0301993594788661e-13, 1.2640425962882392e-13, 1.5078304471890383e-13, 1.95455280884145e-13, 2.4944828890066223e-13, 3.312269837683295e-13, 4.700330788968585e-13, 7.543260813413161e-13, 1.4915194303056232e-12]
Stopped at Iteration: 25
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 24 : [7.622640022690767e-16, 1.283523984724276e-15, 2.1325841237335336e-15, 2.3250381680596964e-15, 2.8738857920160202e-15, 3.3699728113493506e-15, 4.3920730965170275e-15, 4.8004773737292255e-15, 5.428015117496576e-15, 6.313799357177452e-15, 8.037535270430461e-15, 1.1338119198010355e-14, 1.7583177835486348e-14, 2.687057993051992e-14, 3.334031858926714e-14, 3.89975279916989e-14, 4.6091483884110155e-14, 5.838736439134498e-14, 7.887513948331118e-14, 1.1263383849601082e-13, 1.7573682988380268e-13, 2.815796424034905e-13, 4.487841445760089e-13, 8.000955505884942e-13, 1.3679096710851363e-12]
Stopped at Iteration: 25
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 24 : [2.604219899307372e-16, 1.249820931250682e-15, 2.4408896992420417e-15, 4.384143547050299e-15, 6.371261602453995e-15, 7.436289023543945e-15, 9.578515329747526e-15, 1.0365240943546955e-14, 1.2258945699105399e-14, 1.2996375922689452e-14, 1.4641542202979387e-14, 1.5230602869824114e-14, 1.7148606736015473e-14, 1.8196182031122933e-14, 2.0206142679136677e-14, 2.2088673466413947e-14, 2.5030933767427167e-14, 2.9573279253991846e-14, 4.0148319498723397e-14, 5.927523692765254e-14, 9.410295611785925e-14, 1.688749535555177e-13, 3.073407623563114e-13, 7.351038033994343e-13, 1.4810700787500804e-12]
Stopped at Iteration: 26
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 25 : [1.2811889954918567e-15, 1.654478582673638e-15, 2.3861321620128224e-15, 4.016870483483607e-15, 5.18547203234025e-15, 5.7534774460858105e-15, 6.6016353426695855e-15, 7.834893724804462e-15, 9.328201079313675e-15, 1.0506877548703751e-14, 1.2332766640984196e-14, 1.442796455481356e-14, 1.7609190337689213e-14, 2.1258950025302084e-14, 2.6597775681316067e-14, 3.2751221469723143e-14, 4.0944369575860114e-14, 5.5251847272028333e-14, 7.735988812802275e-14, 1.0553132026216141e-13, 1.3826321312178973e-13, 1.8991279769582623e-13, 2.703335613009026e-13, 3.8884696028626573e-13, 6.173094607244818e-13, 1.0965923598166366e-12]
Stopped at Iteration: 27
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 26 : [3.140838187698333e-16, 5.322114526976641e-16, 1.20116605238756e-15, 1.6781013500329528e-15, 2.6028376797166577e-15, 5.676016668472985e-15, 9.243568994229 91e-15, 1.3124181765315029e-14, 1.565965452763093e-14, 1.8823759125573095e-14, 2.1087578754007013e-14, 2.4498986200922752e-14, 2.827302015931149e-14, 3.133791629555883e-14, 3.624691801604015e-14, 4.233618969066512e-14, 5.1946653876451046e-14, 6.217251566586309e-14, 7.288998564240081e-14, 9.316222045900083e-14, 1.2843013714135598e-13, 1.8144162201286881e-13, 2.678718344104519e-13, 3.9881615272108046e-13, 6.233503221812638e-13, 9.508617581220196e-13, 1.5587915479679482e-12]
Stopped at Iteration: 26
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 25 : [6.081039482867528e-16, 6.521912992500816e-16, 1.8550416173925043e-15, 4.256515846908906e-15, 5.324569422457606e-15, 6.870228465198023e-15, 8.592708310288252e-15, 9.319077934576897e-15, 1.1614483764427357e-14, 1.350108444418077e-14, 1.5775395825834616e-14, 1.8738611973824562e-14, 2.2632342466639266e-14, 2.7196414366696276e-14, 3.1814448706009 92e-14, 3.5423721619570624e-14, 4.304587483720443e-14, 5.46103312785 53e-14, 7.317558105519978e-14, 1.0520736851363757e-13, 1.4879766044836536e-13, 2.0677035572756894e-13, 2.998944434640228e-13, 4.468801566868 1503e-13, 7.340251641146427e-13, 1.1831988377168996e-12]
Stopped at Iteration: 30
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 29 : [8.853186550431396e-16, 9.413853962598215e-16, 1.585084057448174e-15, 2.2270164190322202e-15, 2.5764727419407968e-15, 3.4461708255842864e-15, 4.589377000073343e-15, 5.848810173336346e-15, 7.191683221505278e-15, 7.842495247455588e-15, 9.449037008770579e-15, 1.0250632343130831e-14, 1.1570175792196794e-14, 1.295560612548167e-14, 1.5196573790742644e-14, 1.7547924964618604e-14, 2.0058793109408742e-14, 2.4816688210834476e-14, 2.712440575910028e-14, 3.3607389175104444e-14, 3.822024798786939e-14, 4.969519608434913e-14, 6.461667042313898e-14, 8.822550629456725e-14, 1.1391516905127385e-13, 1.5991515117886267e-13, 2.3538617535377646e-13, 3.9221902401500183e-13, 7.301088405118542e-13, 1.2409259779535455e-12]
Stopped at Iteration: 23
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 22 : [2.0541825295081723e-14, 2.165915364664782e-14, 5.0122652007992834e-14, 8.220858086891599e-14, 9.122045670669171e-14, 1.1735550464949096e-13, 1.3820705396900704e-13, 1.496303619324216e-13, 1.7467124680861322e-13, 2.0344193690425838e-13, 2.3017266896981395e-13, 2.5761291806718025e-13, 3.089764481247836e-13, 3.5071506056131555e-13, 4.0405143348609835e-13, 4.477977824326049e-13, 5.145237904040139e-13, 5.586369505629028e-13, 6.35756426860704e-13, 6.852980270122674e-13, 7.901855663354714e-13, 8.698542496522236e-13, 1.028493215565493e-12]
Stopped at Iteration: 26
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 25 : [1.5885570515605614e-14, 1.641997500670327e-

14, 4.0388561406291285e-14, 6.084943034100132e-14, 6.433316485387736e-14, 7.274164009829792e-14,
9.277247797216977e-14, 1.0255265218818754e-13, 1.1765295458346997e-13, 1.3704523515696256e-13, 1.59149916
8815628e-13, 1.778915998358884e-13, 1.8656654349941568e-13, 2.0791607904270463e-13, 2.333440807259224e-
13, 2.5035450412615856e-13, 2.7963682448202706e-13, 3.042105060847246e-13, 3.4811829158826804e-13, 3.958
856990149189e-13, 4.615431581784324e-13, 5.304238775308962e-13, 6.219460347064986e-13, 7.496826315151561e
-13, 9.523722715833893e-13, 1.2577460208002959e-12]
Stopped at Iteration: 17
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 16 : [2.580601219381409e-14, 4.192648170423936e-1
4, 9.384583108702824e-14, 1.2250070455532638e-13, 2.0052781010264745e-13, 2.282523783014581e-13,
3.265245644394398e-13, 3.4667447622448114e-13, 4.4831765874294416e-13, 4.658216897812811e-13, 5.642222019
674972e-13, 5.850037614594014e-13, 6.916552911503509e-13, 7.172062931774474e-13, 8.484163482702488e-13,
8.963182746329082e-13, 1.0329174952756837e-12]
Stopped at Iteration: 29
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 28 : [1.5682870409842946e-14,
2.1882972221042235e-14, 4.2872679114052255e-14, 4.884765622819741e-14, 7.608137369867492e-14,
8.116892868352407e-14, 1.0551377429706424e-13, 1.1118107991104695e-13, 1.4138760773251844e-13, 1.45426762
1953731e-13, 1.7482768339233528e-13, 1.8051571261469924e-13, 2.0365848412867986e-13, 2.0810350116588403e-
13, 2.3288664605448683e-13, 2.359911719257326e-13, 2.5748572215001064e-13, 2.6543953710650357e-13, 2.8553
83043209163e-13, 2.972865862802324e-13, 3.266527773093457e-13, 3.446547641399529e-13,
3.918988869510623e-13, 4.365354801449896e-13, 5.228387909872813e-13, 6.134769654280949e-13, 7.70496504973
3198e-13, 9.95461176670353e-13, 1.373273719530838e-12]
Stopped at Iteration: 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 21 : [1.7408754155158958e-14, 3.790498471682489e-
14, 6.088712021038181e-14, 6.77809944782094e-14, 8.512381308343076e-14, 9.437660613893451e-14, 1.14085931
00688647e-13, 1.5088350953762835e-13, 1.6976580317966603e-13, 2.1020864236847074e-13, 2.25548315995488e-1
3, 2.54468164093104e-13, 2.787558905593737e-13, 2.9992566674271073e-13, 3.4386443564203107e-13, 3.7345994
4533699e-13, 4.311355062046224e-13, 4.869862768083934e-13, 5.620487061347489e-13, 6.764258052628251e-13,
8.251009196809872e-13, 1.0518597939312489e-12]
Stopped at Iteration: 19
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 18 : [1.1445607492953953e-14,
1.2179057428126912e-14, 2.0711524734148988e-14, 2.2916246402382826e-14, 2.744848258751794e-14, 3.67062297
11317714e-14, 4.775365485343888e-14, 6.940146288962836e-14, 1.0201345939907764e-13, 1.1773042917439366e-1
3, 1.4397623860194552e-13, 1.6866565050517606e-13, 2.1034001683998645e-13, 2.5616444124686123e-13, 3.3496
9938405355e-13, 4.264348846825369e-13, 5.999807037437553e-13, 8.444953068377146e-13,
1.1662684225582442e-12]
Stopped at Iteration: 27
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 26 : [1.2508738670984016e-14,
3.1905140992668034e-14, 5.3614293490497746e-14, 5.5275705281977695e-14, 7.887933193654139e-14, 9.35947460
8874242e-14, 1.0142160041453424e-13, 1.2330532722580803e-13, 1.28192959163417e-13, 1.498585797815936e-13,
1.5516733455669002e-13, 1.7381271728981569e-13, 1.8316154750621e-13, 1.9323235957622653e-13, 2.0804625860
052824e-13, 2.1313035027047892e-13, 2.384080067115878e-13, 2.5196496162273503e-13, 2.911137773025775e-13,
3.111690770002879e-13, 3.5596163192593254e-13, 4.022901746827426e-13, 4.672322476530948e-13, 5.5174420958
087e-13, 6.909862619678929e-13, 8.8595923137633e-13, 1.1286055471515e-12]
Stopped at Iteration: 16
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 15 : [1.490813240747426e-14, 3.608278224508684e-1
4, 9.073947977907932e-14, 1.233251225736747e-13, 1.8370436206853394e-13, 2.1360114148379008e-13,
2.979081621721227e-13, 3.2862869806169105e-13, 4.45463400725989e-13, 4.749230321459149e-13, 6.08206644121
9772e-13, 6.425257900438055e-13, 7.748019702332563e-13, 8.16721083471982e-13, 9.687682186388938e-13, 1.02
77890335967436e-12]
Stopped at Iteration: 27
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 26 : [1.4210353474615669e-14,
1.5531038279894845e-14, 2.052449635984547e-14, 2.6316508500927463e-14, 2.862939281960635e-14,
4.529879749866462e-14, 6.542843765506408e-14, 8.873569745554827e-14, 1.2455516386514678e-13, 1.4314049484
944427e-13, 1.6561094248082843e-13, 1.8349002373270339e-13, 1.9810846559966138e-13, 2.1475161502338137e-1
3, 2.331081822002044e-13, 2.4505920158632184e-13, 2.6795802729169563e-13, 2.794941117131043e-13,
3.102954191724511e-13, 3.3163472866615285e-13, 3.67801344051959e-13, 4.004236675673882e-13, 4.63162276248
7012e-13, 5.553033935790219e-13, 7.202513982175454e-13, 9.637280260801018e-13, 1.372732964945655e-12]
Stopped at Iteration: 20
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 19 : [2.6332095211171253e-14, 3.205122028775412e-
14, 6.277762349123361e-14, 6.515387096784933e-14, 9.48085894763638e-14, 9.591232216443816e-14, 1.24141898
56876512e-13, 1.3594731597078286e-13, 1.8091647302011542e-13, 2.0389328626112808e-13, 2.7201703348728346e-
13, 2.904223975199894e-13, 3.479322259122525e-13, 3.730761532977983e-13, 4.386601516693035e-13,
4.729468504101083e-13, 5.58720347777531e-13, 6.398173290523931e-13, 8.187773908261456e-13, 1.043617597668
4214e-12]
Stopped at Iteration: 26
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 25 : [1.579499856931012e-14, 2.4858984257352148e-
14, 4.824905335890425e-14, 6.090849349795467e-14, 7.096813913650372e-14, 7.8663991075891e-14,
8.602687500583107e-14, 9.234994954323601e-14, 1.1653671274912053e-13, 1.2435134281171386e-13, 1.428908316
2253004e-13, 1.592603480432920 4e-13, 1.776690838479696 2e-13, 1.963454792609514 3e-13, 2.139136196878477 6e-
13, 2.302317102658153e-13, 2.576495726256272e-13, 2.8005201431616506e-13, 3.140349216132239e-13,
3.4729753384706806e-13, 3.9596459942061975e-13, 4.674252019607429e-13, 5.633505739637024e-13,
7.047861958375569e-13, 9.1395563949855e-13, 1.2186923983863983e-12]
Stopped at Iteration: 28
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 27 : [1.8013273017488066e-14,
2.7508676261418346e-14, 5.641988674225704e-14, 7.039411809568964e-14, 1.2181167564951686e-13,
1.471600802823131e-13, 2.0165276977405696e-13, 2.189030272020703e-13, 2.808378019247459e-13, 2.9034046899
728487e-13, 3.541151305170221e-13, 3.606195707427768e-13, 4.15362953911465e-13, 4.185374397865275e-13, 4.

742530676632699e-13, 4.761857181386428e-13, 5.325622477397593e-13, 5.359274443436292e-13,
5.878805462097839e-13, 5.945300523900815e-13, 6.436499709166796e-13, 6.53683960078378e-13, 7.097348241331
505e-13, 7.32706537090375e-13, 8.000318012982059e-13, 8.421226547251441e-13, 9.454051531760964e-13, 1.025
2499233427477e-12]
Stopped at Iteration: 29
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 28 : [3.3599669112145343e-14, 3.888057767686501e-
14, 6.699316784086085e-14, 8.500007256787193e-14, 1.1062963713034562e-13, 1.7037499923580888e-13, 1.82064
00296774945e-13, 2.1789846652847927e-13, 2.4548894805117924e-13, 2.575845897585735e-13, 2.859771656038811
e-13, 2.980069947489911e-13, 3.2235031622355137e-13, 3.389449067413925e-13, 3.6210278428149025e-13, 3.775
636772764898e-13, 3.9798833977464263e-13, 4.0771688076435044e-13, 4.298769705281665e-13, 4.426988332627604
5e-13, 4.623619933051985e-13, 4.8286553576557e-13, 5.042987996538492e-13, 5.348118329670867e-13,
5.720524732455722e-13, 6.331506768481472e-13, 7.119029712113402e-13, 8.297843487289301e-13, 1.01270839153
62715e-12]
Stopped at Iteration: 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 21 : [6.656970995457425e-15, 2.346002163301542e-1
4, 4.4710951256602826e-14, 4.9624697478392094e-14, 7.107961582151371e-14, 8.070307542490536e-14,
9.091491073330233e-14, 9.690942427643254e-14, 1.1381826433947464e-13, 1.2439575121846913e-13, 1.332369289
0964295e-13, 1.4218779542668696e-13, 1.565205325050456e-13, 1.6916488381600796e-13, 1.9718638965688915e-1
3, 2.1909539846543286e-13, 2.638448492193758e-13, 3.159528891623077e-13, 4.15859732500612e-13, 5.51323291
0950623e-13, 7.760303394103767e-13, 1.1370846481637066e-12]
Stopped at Iteration: 26
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 25 : [1.9003308429539482e-14,
2.0969906809216952e-14, 3.9224367540093075e-14, 4.2971398264290794e-14, 6.290842314263265e-14, 6.70817449
8542288e-14, 8.20898496698414e-14, 9.374820379231563e-14, 9.908216843080845e-14, 1.1653127974410948e-13,
1.2223909761785689e-13, 1.4365228993564784e-13, 1.498024562186787e-13, 1.7029908292660452e-13, 1.77640406
17618384e-13, 2.018495220915937e-13, 2.2370743764889205e-13, 2.569165796098571e-13, 2.784412189289749e-
13, 3.169622250433719e-13, 3.5537892445400973e-13, 4.2049578375629386e-13, 5.142661982299372e-13, 6.56186
1640924389e-13, 8.545879982299364e-13, 1.154825099383871e-12]
Stopped at Iteration: 17
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 16 : [2.921359044343335e-14, 6.918972643758l1e-
14, 1.2990485927064505e-13, 1.5386836224765069e-13, 2.451166102155197e-13, 2.7677802747387005e-13, 3.7278
98446343863e-13, 3.995191661418995e-13, 5.161342107797931e-13, 5.297191298857997e-13,
6.433212440557548e-13, 6.53175682116426e-13, 7.760482162615939e-13, 7.871874797941656e-13, 9.116635684774
178e-13, 9.25618625344993e-13, 1.051509114446629e-12]
Stopped at Iteration: 19
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 18 : [2.8390102609231032e-14, 2.921302599923737e-
14, 5.726144633962478e-14, 8.589507007354141e-14, 9.07315879724355e-14, 1.1356925069635299e-13, 1.3225039
208025818e-13, 1.5939649535239269e-13, 1.9364852787710118e-13, 2.3497412892856903e-13,
2.8186543320363973e-13, 3.3796381244674917e-13, 4.164249913658443e-13, 4.87740751787183e-13, 5.9651293925
52583e-13, 6.892358595259302e-13, 8.24677095652446e-13, 9.247843364897147e-13, 1.0700471215565905e-12]
Stopped at Iteration: 13
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 12 : [7.878907019415637e-15, 6.22699886082291e-
14, 1.0825869893378368e-13, 1.4146617834726278e-13, 2.0851775005165014e-13, 2.4373366026208553e-13, 3.427
4342303827794e-13, 3.886077914925512e-13, 5.411254384723218e-13, 5.897241071769783e-13,
7.6216170685371l5e-13, 8.275491669719215e-13, 1.0142761681012917e-12]
Stopped at Iteration: 16
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 15 : [3.160201134108575e-14, 6.057561799205694e-1
4, 1.0919808174978745e-13, 1.230055663784837e-13, 1.902158473657455e-13, 2.1217967933773778e-13,
3.0481023955587383e-13, 3.451536206888943e-13, 4.69213809116731e-13, 5.115545780796051e-13, 6.41764861837
4274e-13, 6.84793712919668e-13, 8.206167158437801e-13, 8.58154929370406e-13, 9.905602538506032e-13, 1.034
250396193814e-12]
Stopped at Iteration: 19
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 18 : [1.3922642070796031e-14, 2.510786366500164e-
14, 4.503642095551398e-14, 5.2068248843792476e-14, 5.653532538357134e-14, 6.250611405059e-14,
8.89552853193372e-14, 1.225672321485509e-13, 1.3968145687723946e-13, 1.7682327368299093e-13, 2.1260812460
319266e-13, 2.692741442185903e-13, 3.078377771856658e-13, 3.637383723615411e-13, 4.274906851417504e-13,
5.010437400576434e-13, 6.039598520517915e-13, 7.61184246940287e-13, 1.0365229969335133e-12]
Stopped at Iteration: 17
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 16 : [2.2222355627704625e-14,
2.4458338321444307e-14, 2.891200436752743e-14, 4.3381267563214576e-14, 9.446059577250723e-14,
1.712073323641009e-13, 2.375021744801872e-13, 3.159630611738403e-13, 3.8550426813497674e-13, 4.55964051655
9947e-13, 5.528060231819074e-13, 6.010358047134707e-13, 7.181636275705519e-13, 7.808108734141027e-13,
9.01929604582814e-13, 9.763456896704859e-13, 1.136888689154689e-12]
Stopped at Iteration: 25
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 24 : [4.785602658835875e-15, 1.1550433283521614e-
14, 2.4841492448564367e-14, 2.731048642031593e-14, 4.7496511725403265e-14, 6.001412786138669e-14, 7.86048
6608807308e-14, 8.929389275083447e-14, 1.1675576363492048e-13, 1.3388037597580122e-13, 1.6738872429183293
e-13, 1.8252264461324143e-13, 2.0938714786993482e-13, 2.2073559943283863e-13, 2.4704890674829243e-13, 2.5
70810536689922e-13, 2.9080964214437477e-13, 3.0910279669910423e-13, 3.550361007417393e-13, 3.970116713989
189e-13, 4.804411776697953e-13, 5.83177334355583e-13, 7.211487822937801e-13, 9.096384169932484e-13, 1.226
4059027547922e-12]
Stopped at Iteration: 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 21 : [4.316186402680142e-15, 1.8421789886071757e-
14, 3.947383558486654e-14, 6.321348132832695e-14, 9.710264526360167e-14, 1.0764141108606308e-13,
1.4458689571042632e-13, 1.4987283929805212e-13, 1.897168981665236e-13, 1.946664224006617e-13,
2.373010897574376e-13, 2.4322960156697e-13, 2.8024186595045856e-13, 2.8587823234013936e-13, 3.29999370072
60207e-13, 3.3937997291187337e-13, 3.9012485252204l3e-13, 4.2947522669013706e-13, 5.198707323649l1e-13, 6

.273620319706589e-13, 8.326248974326028e-13, 1.089932747730716e-12]
Stopped at Iteration: 29
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 28 : [2.448677801118657e-14, 2.875732714122712e-14, 4.7289411212688517e-14, 4.8077457114678596e-14, 6.315841763006064e-14, 7.187036347304416e-14, 9.209490918226361e-14, 1.0060352659551843e-13, 1.338050910142907e-13, 1.437220768424993e-13, 1.776545779676277 8e-13, 1.9107832844502002e-13, 2.2567198423552265e-13, 2.3771507854361787e-13, 2.763083779546784e-13, 2.916177292636073e-13, 3.2116715406278715e-13, 3.3490784368305557e-13, 3.6645578193103215e-13, 3.8020702921046226e-13, 4.1515523082060995e-13, 4.349134643758152e-13, 4.781162384135788e-13, 5.063021466345562e-13, 5.60744092467497e-13, 6.18389608140435e-13, 7.331203371101272e-13, 9.0847857028968 87e-13, 1.2291739643515648e-12]
Stopped at Iteration: 21
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 20 : [1.3550592981784681e-14, 2.4857189068385795e-14, 4.2464393262891103e-14, 4.4981645221016316e-14, 7.427239876518377e-14, 9.495020436009758e-14, 1.4559928311558715e-13, 1.6849731472429604e-13, 2.260097975121098e-13, 2.5722756399377713e-13, 3.3251954504799 74e-13, 3.67537253046563e-13, 4.633086430099578e-13, 5.031404594031965e-13, 6.032594382423251e-13, 6.412712144557081e-13, 7.379000265572126e-13, 7.715413136537267e-13, 8.754460970273991e-13, 9.235975187281094e-13, 1.088542701853974e-12]
Stopped at Iteration: 19
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 18 : [1.8545925517594462e-14, 2.4949355292017368e-14, 4.100640852909322e-14, 4.906492087982701e-14, 6.294096078024899e-14, 6.448110703484206e-14, 7.545860467775078e-14, 8.493257633266104e-14, 1.114806678282708e-13, 1.3077235001669899e-13, 1.5932729626416876e-13, 1.9292436004213108e-13, 2.2399993949452448e-13, 2.783164051370344e-13, 3.4783155198754096e-13, 4.649794188264502e-13, 6.278969180931944e-13, 8.987151897728298e-13, 1.2371730829079578e-12]
Stopped at Iteration: 24
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 23 : [1.0477913892247506e-14, 1.0657077674662631e-14, 2.909072050612647e-14, 6.06617600189783e-14, 1.2119295944939115e-13, 1.652243450008 59e-13, 2.2267128197445775e-13, 2.5132899305225455e-13, 3.1219119146109107e-13, 3.2401387988613257e-13, 3.8460958587950747e-13, 3.9399212069532335e-13, 4.5261896686553394e-13, 4.689878664520434e-13, 5.090531827691309e-13, 5.487769573690217e-13, 5.904786051327184e-13, 6.322034661921119e-13, 6.906005052435608e-13, 7.169574253917194e-13, 8.08493539783706e-13, 8.387471111370436e-13, 9.585113548108858e-13, 1.0548509028650198e-12]
Stopped at Iteration: 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 21 : [2.1681816366733095e-14, 3.272919750503435e-14, 5.616917727761024e-14, 6.196896453377898e-14, 6.93959605950764e-14, 7.742761606369184e-14, 9.351546788364785e-14, 1.0884650690446121e-13, 1.0965016740037323e-13, 1.2328185845672662e-13, 1.3236482531592298e-13, 1.419857763876204e-13, 1.5238651973031455e-13, 1.6491337477178802e-13, 1.78051357247025e-13, 1.978641062836636e-13, 2.2502984600411184e-13, 2.799809742719617e-13, 3.5879846271010733e-13, 4.952971648029993e-13, 7.199973335850294e-13, 1.0965513393662568e-12]
Stopped at Iteration: 19
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 18 : [3.882102315821303e-14, 5.0269489880080356e-14, 9.061051058148872e-14, 9.331374405757408e-14, 1.3465569990268816e-13, 1.389721679428142e-13, 1.8068494084030442e-13, 1.9061572707834986e-13, 2.484661094431163e-13, 2.6047888279970286e-13, 3.2415956552793745e-13, 3.3981742687950277e-13, 4.094623366536287e-13, 4.420094132425472e-13, 5.458553636736403e-13, 6.084035324898859e-13, 7.556958703789465e-13, 8.938628019849899e-13, 1.1622614219931526e-12]
Stopped at Iteration: 18
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 17 : [2.247457472151013e-15, 1.7758339760961575e-14, 2.6279945657122e-14, 3.00972650827724e-14, 5.738909659087958e-14, 9.6403070574184 75e-14, 1.3900803243685204e-13, 1.6719619816719047e-13, 2.124463781672158e-13, 2.435441263030747e-13, 3.18145848812592e-13, 3.6693791845247757e-13, 4.531004845105922e-13, 5.229352368868144e-13, 6.366152133857323e-13, 7.332253356017652e-13, 8.999371460884368e-13, 1.0778416159073307e-12]
Stopped at Iteration: 58
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 57 : [7.443075631101624e-16, 1.3473146184390552e-15, 1.9261737702358067e-15, 2.183059228620027e-15, 2.539921477599111e-15, 3.1560097336593355e-15, 4.460118709563737e-15, 5.778508486795884e-15, 6.904929762901429e-15, 7.812362565231663e-15, 8.75945362636534e-15, 9.855375086968311e-15, 1.1155394630275569e-14, 1.2659633655602195e-14, 1.409048429036317e-14, 1.5528736711938802e-14, 1.7039880771200842e-14, 1.8725796750883263e-14, 2.0504234109543007e-14, 2.193054914659612e-14, 2.3451190839703e-14, 2.474581805277235e-14, 2.6357040802819418e-14, 2.7687962951110892e-14, 2.9544664510440166e-14, 3.1011012496788985e-14, 3.297986375877439e-14, 3.4839654707573584e-14, 3.676006055156216 66e-14, 3.8971144081086236e-14, 4.081133287586131e-14, 4.297619571806806e-14, 4.496318465403129e-14, 4.751957995676641e-14, 4.968321798153765e-14, 5.296867975176288e-14, 5.5192235037396523e-14, 5.865926364194024e-14, 6.154034401609324e-14, 6.577846156225175e-14, 7.03754703452941e-14, 7.650057646427323e-14, 8.351811273075494e-14, 9.33977341067743e-14, 1.0556120348168815e-13, 1.1957172911738755e-13, 1.3504047198585507e-13, 1.5282315895668575e-13, 1.7580181772671963e-13, 2.106105095034392 7e-13, 2.6037201179410574e-13, 3.231623226208407e-13, 4.0729999769278435e-13, 5.052931198019647e-13, 6.189756993835802e-13, 7.525314586991201e-13, 9.13710886514833e-13, 1.1679095948901166e-12]
Stopped at Iteration: 60
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 59 : [1.2175343045793923e-15, 1.770792010256102e-15, 2.407063768118867e-15, 2.852298007783355e-15, 3.3940697690630044e-15, 3.737680072729116e-15, 4.030689163992444e-15, 4.436311891569991e-15, 4.9607194932831854e-15, 5.842648578022002e-15, 7.446855185984976e-15, 9.487695977882283e-15, 1.1441390459687356e-14, 1.3114111122206987e-14, 1.4667295447820095e-14, 1.6284531753767894e-14, 1.8054566765028803e-14, 1.972167093183849e-14, 2.1552455032739886e-14, 2.3161848481334706e-14, 2.4927334598512443e-14, 2.702902950037061e-14, 2.924546977820563e-14, 3.1553924278594624e-14, 3.412742823748601e-14, 3.687764437154469e-14, 3.9484351078374926e-14, 4.2069153428746746e-14, 4.455353551104703e-14, 4.7220830596258365e-14, 4.9632007852070087e-14, 5.269912580477551e-14, 5.4682524328369846e-14, 5.764807709060445e-14, 5.930082088665704e-14, 6.273237259936048e-14, 6.446645476800486e-14, 6.814643242141633e-14, 7.013108555100489e-14, 7.443909283951188e-14, 7.690432823155531e-14, 8.13149247225

5522e-14, 8.42525981890115e-14, 9.023288947827221e-14, 9.51039536322591e-14, 1.0412818789577826e-13, 1.12
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88522e-13, 2.0715790781127253e-13, 2.494698574245597 3e-13, 3.0295425294417583e-13, 3.610823799693413e-13,
4.313679764429699e-13, 5.274569643657377e-13, 6.543043836743623e-13, 8.201368863538896e-13, 1.03632646889
14928e-12]
Stopped at Iteration: 54
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 53 : [1.0590852655056278e-15,
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9.915586688822106e-15, 1.1173762504676583e-14, 1.231495771668507e-14, 1.3453505358550715e-14, 1.508665114
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.3892096082652 67e-14, 2.6124195735913163e-14, 2.8004138684900 98e-14, 2.9517796984610664e-14, 3.0872537810
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3.710497973323358e-14, 3.884486769139514e-14, 4.026006778049524e-14, 4.236897193132462e-14, 4.41699361912
89033e-14, 4.6623593428937435e-14, 4.888375409765902e-14, 5.182154549483597e-14, 5.496680895892998e-14,
5.871433472511415e-14, 6.368517135362698e-14, 6.97314007441193e-14, 7.751063708962453e-14, 8.656561038531
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41855e-13, 6.802307016874907e-13, 8.730455478113772e-13, 1.1108180509251662e-12]
Stopped at Iteration: 60
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 59 : [1.0531330705808813e-15, 1.368579327913553e-
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1.0674688602193224e-14, 1.1547802556009994e-14, 1.225928820022436e-14, 1.3255911514285068e-14, 1.41695918
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1.1587010992614056e-12]
Stopped at Iteration: 60
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 59 : [3.8671495986277653e-16, 6.999560859503146e-
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2952874085e-15, 4.775603400432675e-15, 5.314574374888133e-15, 6.0297348298217705e-15,
7.117772092708774e-15, 8.249985206916e-15, 9.888582911796246e-15, 1.0919197646879624e-14, 1.2738689865777
927e-14, 1.4690949031465293e-14, 1.6769081402845133e-14, 1.9119326862037608e-14, 2.146506639849666e-14, 2
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1.4948045085380528e-13, 1.7863966152745627e-13, 2.2181469098586913e-13, 2.80941090195059e-13, 3.659409487
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1.1890088160862501e-12]
Stopped at Iteration: 57
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 56 : [9.973196677990403e-16, 1.7574452628215224e-
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5600041205e-15, 6.699036868630802e-15, 7.469235585788864e-15, 8.55594352414354e-15, 9.302232134780102e-
15, 1.0541711243814975e-14, 1.1354573085349404e-14, 1.2979706861885633e-14, 1.4357926185788124e-14, 1.640
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94e-14, 2.987657288343347e-14, 3.320806767334157e-14, 3.6928870172431515e-14, 4.090977810880639e-14,
4.5314635628361547e-14, 4.9155913841404664e-14, 5.443309295971044e-14, 5.911433676921027e-14,
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, 2.0626278565054544e-13, 2.1945220872556934e-13, 2.368921638304233e-13, 2.5779929253230736e-13, 2.904798
6479471586e-13, 3.3500776619582235e-13, 4.061089522471640 4e-13, 5.115085247476251e-13, 6.467614216841553e
-13, 8.447714908104346e-13, 1.1246464022322008e-12]
Stopped at Iteration: 55
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 54 : [4.440892098500626e-16, 1.3113712077941596e-
15, 2.614055318227552e-15, 3.85789338306325e-15, 4.7799237888855595e-15, 5.630150512404153e-15, 6.2544910
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2.5658202838575954e-14, 2.775791015889841e-14, 3.0354904879653304e-14, 3.336069451768111e-14,
3.6508332505250646e-14, 3.949963369633575e-14, 4.260272544413187e-14, 4.5148914823715956e-14,
4.742437350698076e-14, 4.975762908989155e-14, 5.2144409735207894e-14, 5.452320827702314e-14, 5.6615224360
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3.6110582497154116e-13, 4.584183246003233e-13, 5.950329612579592e-13, 7.979497012530037e-13,
1.0695500460900877e-12]
Stopped at Iteration:  57
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 56 : [4.77904511291524e-16, 5.732900222770765e-16
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2.501887112812458e-14, 2.6706114346974246e-14, 2.924695960996581e-14, 3.0767889264848474e-14, 3.333946841
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4.299494287262833e-14, 4.581428850181716e-14, 4.749113619444186e-14, 5.026690964332608e-14, 5.21342022863
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9.913575580012051e-13, 1.272593484701509e-12]
Stopped at Iteration:  55
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 54 : [3.6095589860507407e-16, 4.8415880489384e-16
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3.2663764113815634e-14, 3.4473212111375207e-14, 3.6518941569164576e-14, 3.811222370768385e-14, 3.99803332
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4.742114206548012e-14, 5.017487126833645e-14, 5.188614914302828e-14, 5.560681594368523e-14, 5.83512818071
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.23786645343532e-13, 6.39801187968669e-13, 7.961272445216202e-13, 9.849968844686977e-13,
1.2255067134072802e-12]
Stopped at Iteration:  62
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 61 : [2.3392409996599824e-16,
1.8909026986810316e-15, 2.802634892038255e-15, 3.9501518122212455e-15, 4.5448248749482644e-15, 5.40443406
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6.549908373194974e-13, 8.019171634885208e-13, 1.0215581003608212e-12]
Stopped at Iteration:  58
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 57 : [1.4667187010059368e-15, 1.922106489434133e-
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6.783461653761428e-15, 7.587522171154743e-15, 8.09529499923168e-15, 8.897555674916954e-15, 9.536662390814
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6636803e-14, 1.876908253065698e-14, 1.986846425178087e-14, 2.1209790384484815e-14, 2.2308511598012974e-
14, 2.408752902608454e-14, 2.507897201407242e-14, 2.7018764786182097e-14, 2.8017540657570584e-14, 2.98389
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7.625889733278444e-14, 8.434385338751241e-14, 9.361056351213347e-14, 1.0672655960973231e-13, 1.2087179298
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, 2.7818885460104336e-13, 3.479290716469727e-13, 4.4166501471138096e-13, 5.504198243539968e-13, 6.7339297
95077108e-13, 8.117802824169369e-13, 9.685951364489983e-13, 1.2026103113932403e-12]
Stopped at Iteration:  61
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 60 : [3.3010152003457217e-15,
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7.168052695945572e-14, 7.385607730899112e-14, 7.497934139741208e-14, 7.750497226700353e-14, 7.89841810649
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6.365892274908481e-13, 8.574754016502913e-13, 1.1858332946962973e-12]
Stopped at Iteration: 57
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 56 : [8.250611840189458e-16, 1.0113958566134345e-
15, 1.494115627963814e-15, 1.759543712314648e-15, 2.3420049574513617e-15, 2.7338623780092466e-15, 3.03263
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094114e-13, 1.1685408356681628e-12]
Stopped at Iteration: 53
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 52 : [8.838005401797225e-16, 1.7950340879659852e-
15, 2.7608810125840 67e-15, 3.098894074865216e-15, 3.961804080817017e-15, 5.530457940625578e-15,
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2638325228e-13, 2.854513969296105e-13, 3.5770670086163115e-13, 4.620916982839558e-13,
5.950954525293064e-13, 7.429468481675952e-13, 9.21388545046694e-13, 1.1362111615199739e-12]
Stopped at Iteration: 63
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 62 : [9.35542015746369e-16, 1.3406640664724568e-1
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14, 2.664330015211752e-14, 2.8028827168909683e-14, 2.948385246135261e-14, 3.0784559382217133e-14, 3.22657
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5.738317164536524e-13, 7.512777793661267e-13, 9.650556644972612e-13, 1.2578924874592141e-12]
Stopped at Iteration: 60
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 59 : [8.541888312493347e-16, 9.1013321501907 08e-1
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00292921056863 8e-15, 7.273404151216587e-15, 8.743447763872215e-15, 1.0484728915980798e-14, 1.2164029594687
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6.843673236267393e-14, 7.18037647092152e-14, 7.552977273435372e-14, 7.906566863282025e-14, 8.296291364879
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.8949259271619314e-13, 4.884339538778805e-13, 6.246284482175164e-13, 8.132140771610407e-13, 1.09400872003
61715e-12]
Stopped at Iteration: 54
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 53 : [3.345453689609277e-16, 7.739744408424253e-1
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Stopped at Iteration: 57
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 56 : [1.7098571017664614e-15, 2.759582762594743e-
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8.098945741753361e-13, 1.0529665505068607e-12]
Stopped at Iteration: 55
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 54 : [1.945366891031334e-15, 2.0879499568033107e-
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13, 1.0927897178661255e-12]
Stopped at Iteration: 59
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 58 : [1.2424907700676392e-15,
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344991753248e-14, 2.854276127595392e-14, 2.973748507970758e-14, 3.2044799111940545e-14, 3.3099367359419l2
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2970777e-12]
Stopped at Iteration: 59
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 58 : [1.8100291287513035e-15,
2.6699902143533378e-15, 4.7795705513635985e-15, 5.416168265868907e-15, 8.392104988781815e-15,
9.506299209889177e-15, 1.257001095104062e-14, 1.3110765836446934e-14, 1.568575932782663e-14, 1.6443011663
481888e-14, 1.9185778307458334e-14, 1.9697053794957236e-14, 2.2286375453539152e-14, 2.2971933433803164e-14, 2.5501440870451853e-14, 2.6338448196907274e-14, 2.888660145406152e-14, 3.030230414675339e-14,
3.3266949079697387e-14, 3.491880888956211e-14, 3.7950587881023184e-14, 3.9201936298389494e-14, 4.20614471
3840478e-14, 4.347296879395744e-14, 4.595884423105778e-14, 4.725267051223854e-14, 4.987738426375829e-14,
5.1061618807660494e-14, 5.3584266056407934e-14, 5.45978667968337e-14, 5.736306334505642e-14, 5.8265644685
55373e-14, 6.105012595027396e-14, 6.217396329713797e-14, 6.52155251840296e-14, 6.619066760668255e-14,
6.982221903706747e-14, 7.08373521264263e-14, 7.477983694164333e-14, 7.629071248038751e-14, 8.129448340593
955e-14, 8.334348518352623e-14, 8.914853410538788e-14, 9.228155419714277e-14, 9.92353136148868e-14, 1.045
7145776478266e-13, 1.1504822409257652e-13, 1.2700942546593177e-13, 1.4650402391373874e-13, 1.712409101664
938e-13, 2.1151873131082907e-13, 2.710334355128112e-13, 3.487729313905663e-13, 4.473216935221956e-13,
5.61565599753717e-13, 6.717947780826826e-13, 7.866455866907575e-13, 9.098714129204814e-13,
1.054830963688429e-12]
Stopped at Iteration: 52
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 51 : [1.2173062775389947e-15,
2.4275904808484952e-15, 4.196038908138198e-15, 5.7278661494425124e-15, 7.516771660532446e-15,
8.570420999018233e-15, 1.0636701664833941e-14, 1.1771468844109878e-14, 1.414217501324765e-14, 1.628471398
722999e-14, 2.0114183534965638e-14, 2.267423879159772e-14, 2.7160155265352432e-14, 3.024837235284033e-14,
3.489070405240297e-14, 3.773057722178256e-14, 4.2631072721648675e-14, 4.6077787879228726e-14, 5.172446015
598458e-14, 5.5377214369493893e-14, 6.16853737822642e-14, 6.415494604392645e-14, 6.991206924219174e-14, 7
.247476990521203e-14, 7.708033375011039e-14, 7.98462583224726e-14, 8.487615811464079e-14, 8.7336175854329
e-14, 9.188000263688879e-14, 9.418175911676603e-14, 9.874784674060181e-14, 1.0061626333716904e-13,
1.0515280805764265e-13, 1.0714090370805042e-13, 1.1194387179476646e-13, 1.1439813095993898e-13, 1.2031128
2329392e-13, 1.250060439288597e-13, 1.3219853501224913e-13, 1.3917110155278712e-13, 1.4949778385317654e-1
3, 1.6375151069164177e-13, 1.839013603259151e-13, 2.1089038820102978e-13, 2.425265905939644e-13,

2.859372147817777e-13, 3.468102310241457e-13, 4.332184478227 88e-13, 5.425782124112082e-13, 6.816685907318289e-13, 8.880941624738738e-13, 1.2502444432495825e-12]
Stopped at Iteration: 52
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 51 : [1.7145146298043535e-15, 2.371338756234451e-15, 4.2611930004413775e-15, 5.023488662092346e-15, 7.064504101588755e-15, 7.64213800272692e-15, 9.761325906827006e-15, 1.0427269905702881e-14, 1.237406437243905e-14, 1.3311774320211246e-14, 1.579937639406498e-14, 1.6655134194362043e-14, 1.9334812342375496e-14, 2.039624603767524e-14, 2.3001813556113246e-14, 2.4578349827212207e-14, 2.7352416407487125e-14, 2.9681719284940843e-14, 3.2838250349568925e-14, 3.6039568605178505e-14, 3.937129428966503e-14, 4.191380696504094e-14, 4.502498731937997e-14, 4.7258594658838976e-14, 5.0655611464397414e-14, 5.325425120597473e-14, 5.717504967972199e-14, 5.919338414064708e-14, 6.296642134317098e-14, 6.444670352223196e-14, 6.828000630426528e-14, 6.958616440597311e-14, 7.340410206267325e-14, 7.506834711547625e-14, 7.981572402004424e-14, 8.224825419026498e-14, 8.8961097714959e-14, 9.349243200678055e-14, 1.0439160041233065e-13, 1.1602097668530876e-13, 1.352323225025493e-13, 1.574825286801782e-13, 1.9091739673548665e-13, 2.3541187506481046e-13, 2.988687821915158e-13, 3.755457668188464e-13, 4.679555599141429975e-13, 5.719475344187978e-13, 6.848656083246297e-13, 8.047927208885329e-13, 9.54823427390727e-13, 1.1897758713835038e-12]
Stopped at Iteration: 58
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 57 : [5.430321800869545e-16, 8.203499797943473e-16, 9.148623080246752e-16, 1.2626070150305604e-15, 1.801184361054347e-15, 3.227035636133032e-15, 4.197551405843797e-15, 4.944981645680402e-15, 5.985881305791062e-15, 7.045620840962 39e-15, 8.332806529750523e-15, 9.558650566134694e-15, 1.1129676552322576e-14, 1.3013037572159 56e-14, 1.4477312828600183e-14, 1.6748113532608 15e-14, 1.8560125283884482e-14, 2.0780161115271 58e-14, 2.2672374042040652e-14, 2.5066952732166328e-14, 2.7353739042937772e-14, 2.988801399409308e-14, 3.220901317356609e-14, 3.4600503308650686e-14, 3.7147537075867505e-14, 3.951551551221055e-14, 4.2572115717691925e-14, 4.45750568640596e-14, 4.778367611109038e-14, 4.942436508645555e-14, 5.2787861074400945e-14, 5.41254572740921e-14, 5.768585226403108e-14, 5.878886636782258e-14, 6.327719796982602e-14, 6.433262638327405e-14, 7.002304928926147e-14, 7.162709005903839e-14, 7.746591565088947e-14, 8.021838950193333e-14, 8.725612624516777e-14, 9.135564420184317e-14, 1.0044309556635937e-13, 1.0819956352948644e-13, 1.211650162911052e-13, 1.340372901669891e-13, 1.5114246652480595e-13, 1.726201503292607e-13, 2.034955961523357e-13, 2.3492610776545876e-13, 2.709323651362472 7e-13, 3.175363381287697e-13, 3.83838446029553e-13, 4.732776062533344e-13, 5.87065291143576e-13, 7.212466024905167e-13, 8.921055714343333e-13, 1.1312950468994884e-12]
Stopped at Iteration: 54
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 53 : [5.385757295718627e-16, 6.566340514819576e-16, 1.0427177888793433e-15, 1.155750629180476e-15, 1.4737129231413661e-15, 1.9437893089108183e-15, 2.145612433336036e-15, 2.790490503358371e-15, 3.62431403686365e-15, 4.8376376598211036e-15, 5.439096088958086e-15, 6.56955671512923e-15, 7.852460559616167e-15, 9.542830966002425e-15, 1.1833317355025559e-14, 1.5039302999940551e-14, 1.8550217224276515e-14, 2.2864946610012405e-14, 2.761402927218934e-14, 3.263247417031566e-14, 3.8066447198856484e-14, 4.380491818355174e-14, 4.957517980021514e-14, 5.546057528751542e-14, 6.232529993227501e-14, 6.80333762329267e-14, 7.649296017745741e-14, 8.241286846937314e-14, 9.066159734427982e-14, 9.673574836342256e-14, 1.054909542630721e-13, 1.1183667245791448e-13, 1.2170727885386283e-13, 1.283533747043612e-13, 1.3908345134112813e-13, 1.456486326495718e-13, 1.5740845386195126e-13, 1.6340132792724494e-13, 1.7637326005393567e-13, 1.828826323601851e-13, 1.9821383670363993e-13, 2.0568930174101183e-13, 2.2425748777470753e-13, 2.3542177571166637e-13, 2.576748263040489e-13, 2.7584517682026367e-13, 3.1160015359973694e-13, 3.489643859545374e-13, 4.036042955534887 7e-13, 4.678686173634473e-13, 5.661123262680051e-13, 7.131845180046397e-13, 9.258525552089486e-13, 1.165897329205361e-12]
Stopped at Iteration: 54
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 53 : [8.087644749205983e-16, 1.3249335957407482e-15, 2.4290629234540925e-15, 3.0405135460863446e-15, 4.146061247937928e-15, 4.782420372440305e-15, 6.3904119928084175e-15, 7.108671992501227e-15, 9.088421272244223e-15, 9.915136686224245e-15, 1.2258679179904538e-14, 1.2942910321668516e-14, 1.508185309478991e-14, 1.5684189726117082e-14, 1.7899301549729745e-14, 1.8792494096103488e-14, 2.0584090110334836e-14, 2.1643051012288528e-14, 2.308857900853461e-14, 2.4158022855406126e-14, 2.5372549583559718e-14, 2.6804845948364638e-14, 2.7855424033867627e-14, 2.948145988502471e-14, 3.0379332638137145e-14, 3.205070771583931e-14, 3.3074545581172604e-14, 3.5142345650158404e-14, 3.653115216745286e-14, 3.89960466415651e-14, 4.047193091528154e-14, 4.283344535378625e-14, 4.4251059514354406e-14, 4.6575058164431766e-14, 4.819015303188787e-14, 5.0894614698781573e-14, 5.324826916463052e-14, 5.788632957178733e-14, 6.288991322368012e-14, 6.915515271151125e-14, 7.671698208671595e-14, 8.756371648853053e-14, 1.006052967530715e-13, 1.171773373807876e-13, 1.360724889148 14e-13, 1.6146107331828975e-13, 1.9928567580952763e-13, 2.4994350178476763e-13, 3.1287039942245525e-13, 4.011792725947173e-13, 5.141197790109 08e-13, 6.518118192949442e-13, 8.506617053932924e-13, 1.10609454740824e-12]
Stopped at Iteration: 55
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 54 : [1.3055450971845835e-16, 7.199081524990568e-16, 1.1449490341370995e-15, 2.2952381253666686e-15, 3.4912823763749137e-15, 4.9801824494951534e-15, 6.105171571572453446e-15, 7.052044325564744e-15, 8.21670011091468e-15, 8.836769013362705e-15, 1.0100968083535381e-14, 1.0675776676891411e-14, 1.2162203926596361e-14, 1.284823184132185e-14, 1.4115500957374393e-14, 1.5139613890731723e-14, 1.591005565854425e-14, 1.7124099409200082e-14, 1.7892912284290156e-14, 1.9114048995555192e-14, 1.9838905412545456e-14, 2.11170772014465e-14, 2.1873754767176724e-14, 2.299795450292679e-14, 2.3800593461131054e-14, 2.4719018363774202e-14, 2.563937968758608e-14, 2.6490610373069175e-14, 2.7542596483319926e-14, 2.8309258094296708e-14, 2.966207153548617e-14, 3.057941101970341e-14, 3.196917336041359e-14, 3.310372036583504e-14, 3.487186607361648e-14, 3.6516726372185536e-14, 3.898788195096 88e-14, 4.1515627317385104e-14, 4.4784054809781e-14, 4.785998540719724e-14, 5.1455532974769696e-14, 5.589700730012004e-14, 6.202078472189624e-14, 7.018977109239462e-14, 8.02997344785579e-14, 9.45347538721443e-14, 1.1731683060311872e-13, 1.4984545930640098e-13, 1.9110146743393275e-13, 2.414454382692851e-13, 3.138380203746517e-13, 4.284822968093614e-13, 6.095623139114715e-13, 8.542871294329566e-13, 1.1377772196546915e-12]
Stopped at Iteration: 65
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 64 : [4.734810070718259e-16, 1.350165493242293e-15, 2.2930688839128702e-15, 3.767690401498907e-15, 5.362085263283062e-15, 6.199435606097226e-15,

7.696836258337797e-15, 8.044587101777613e-15, 9.329561672386784e-15, 9.766108515937627e-15, 1.104843628893006e-14, 1.195118064580047e-14, 1.371222832709691e-14, 1.481760622982972e-14, 1.7019530891164236e-14, 1.800342877478551e-14, 2.018848010938446e-14, 2.1377601027444113e-14, 2.3551421712401058e-14, 2.4802533604303978e-14, 2.7068015018221484e-14, 2.8259647067295908e-14, 3.088255325250162 6e-14, 3.2181753149757103e-14, 3.520306431236693e-14, 3.673625612392846e-14, 4.073841058654553e-14, 4.20725709149024e-14, 4.638959334345968e-14, 4.7529345396575534e-14, 5.1899632884635253e-14, 5.302494508857426e-14, 5.733355157030596e-14, 5.83321986860628e-14, 6.288810337604126e-14, 6.371554378806632e-14, 6.858148578529292e-14, 6.944570754122606e-14, 7.427386566799719e-14, 7.533572209306027e-14, 8.008278797302355e-14, 8.122871855537776e-14, 8.626176926721605e-14, 8.738688156345734e-14, 9.214203713552344e-14, 9.321759819117379e-14, 9.830878190072279e-14, 9.956682931014909e-14, 1.049335712793854 6e-13, 1.0651374101177124e-13, 1.1236940965619749e-13, 1.1494412214813873e-13, 1.2234167756106295e-13, 1.2795478132057188e-13, 1.3825751887096238e-13, 1.4838425049307083e-13, 1.6450374658803868e-13, 1.84512788211599e-13, 2.1505913639338186e-13, 2.620404882381106e-13, 3.384066921066857e-13, 4.531784653501889e-13, 6.186313389478035e-13, 8.367471155759643e-13, 1.1571835658511316e-12]
Stopped at Iteration: 55
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 54 : [2.0621305970929365e-15, 2.8659829020307078e-15, 4.780373748216849e-15, 5.134741820124848e-15, 6.523962268951885e-15, 6.886314621668129e-15, 7.667809012467526e-15, 8.251001169316024e-15, 8.840319753890458e-15, 9.565728090167016e-15, 1.0029962326353219e-14, 1.0822140216879517e-14, 1.1331148596379156e-14, 1.1901267569231518e-14, 1.2504986009674026e-14, 1.3287543580744252e-14, 1.3664517673729385e-14, 1.495423564293192e-14, 1.5578665249190613e-14, 1.6811543806772023e-14, 1.7219538998969094e-14, 1.8545207776834792e-14, 1.9345294087166368e-14, 2.052292904256652e-14, 2.1677407814311172e-14, 2.333814067508607e-14, 2.5448833741442072e-14, 2.8305080971563303e-14, 3.2339920168058 36e-14, 3.8098874848196 01e-14, 4.7272786986207026e-14, 6.023773656012392e-14, 7.362612196364805e-14, 8.66592496717908e-14, 1.0101934915739745e-13, 1.1448293667019528e-13, 1.2507843490315983e-13, 1.329220782220476e-13, 1.3936686792391304e-13, 1.4529796395868372e-13, 1.509836701780314e-13, 1.5716520108842545e-13, 1.640480610045713e-13, 1.7082000631484268e-13, 1.7917561311122232e-13, 1.9195912127138163e-13, 2.1084359859611064e-13, 2.367369087939053e-13, 2.7681620502917614e-13, 3.330144824207913e-13, 4.144873242045515e-13, 5.379059384534661e-13, 7.178769728579107e-13, 9.406741711280944e-13, 1.2220193510401515e-12]
Stopped at Iteration: 59
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 58 : [1.5741324006608197e-15, 1.7661624504329835e-15, 2.9262195725711065e-15, 4.144653157536617e-15, 6.39801944357624e-15, 7.047220752382177e-15, 9.236966497308576e-15, 9.840800403460174e-15, 1.2103816752196695e-14, 1.3281425485414819e-14, 1.632457097594054e-14, 1.787448532801788e-14, 2.1305403521852262e-14, 2.299858431748544e-14, 2.6640152736461235e-14, 2.8070911329631547e-14, 3.163883201170224e-14, 3.32849720040179e-14, 3.728020876859823e-14, 3.9728342735197576e-14, 4.377287482097465e-14, 4.673139416443116e-14, 5.07724372930848e-14, 5.430646879527259e-14, 5.882669582299244e-14, 6.207659348474861e-14, 6.762514992054078e-14, 7.09868551867805e-14, 7.650181610126204e-14, 7.946058845263693e-14, 8.545825482887229e-14, 8.747470955683741e-14, 9.364755666674623e-14, 9.558210702996684e-14, 1.0225462209114115e-13, 1.0409093934277326e-13, 1.1239328727658903e-13, 1.1439743428162884e-13, 1.2239591023265155e-13, 1.251074010223598e-13, 1.342130962253536e-13, 1.3654433176779504e-13, 1.4629401629441699e-13, 1.4991721156176682e-13, 1.6041834319752536e-13, 1.6716985765233557e-13, 1.8024735404476555e-13, 1.8924678599116066e-13, 2.055944918652492 4e-13, 2.200312932854256e-13, 2.4493670658644767e-13, 2.7412550865566927e-13, 3.172274550665538e-13, 3.688466581814657e-13, 4.361464139091947e-13, 5.152328092493273e-13, 6.385123573063321e-13, 8.206439702478114e-13, 1.0821010631553799e-12]
Stopped at Iteration: 25
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 24 : [1.083938053663377e-14, 2.3485644783308e-14, 4.1207192743711 92e-14, 4.1905484113975104e-14, 7.43565659054216e-14, 1.1729742497386191e-13, 1.4082099787305427e-13, 1.916883310049041e-13, 2.230992394157443e-13, 2.770816355596423e-13, 3.3808188599150316e-13, 4.007602612162032e-13, 4.612983024801933e-13, 5.101741371999689e-13, 5.719841461265605e-13, 5.982672763204878e-13, 6.652790814881903e-13, 6.802856915161279e-13, 7.523592334888442e-13, 7.665706889201644e-13, 8.376666007868531e-13, 8.584340083804694e-13, 9.261064604698738e-13, 9.484304214588261e-13, 1.0160920801611445e-12]
Stopped at Iteration: 37
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 36 : [3.192243713874806e-16, 6.043896176395271e-15, 1.9459615189991646e-14, 3.249536955710297e-14, 4.8196788098159935e-14, 7.354224548391651e-14, 9.389876430581803e-14, 1.0902276220616893e-13, 1.3194844470510126e-13, 1.372851667719181e-13, 1.6076280483730877e-13, 1.6266698999749347e-13, 1.8377242713661683e-13, 1.9341770008137483e-13, 2.1516275144886262e-13, 2.3306475812374926e-13, 2.4959864538527295e-13, 2.8007403711075 31e-13, 2.9518963611603007e-13, 3.3220289058676626e-13, 3.491572225136335e-13, 3.9472578825135153e-13, 4.2368109036806226e-13, 4.831273888343632e-13, 5.204233510858828e-13, 5.742084497581469e-13, 6.077921634638454e-13, 6.59555189863022e-13, 6.886265068547694e-13, 7.340699584632079e-13, 7.667137108933062e-13, 8.016779615325734e-13, 8.417420134417628e-13, 8.704349936243514e-13, 9.173837254470259e-13, 9.439927015932069e-13, 1.0072004624697345e-12]
Stopped at Iteration: 26
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 25 : [8.511940756789619e-15, 2.5821959780868622e-14, 4.299888693866193e-14, 4.544366543269243e-14, 6.408851452091234e-14, 6.770090678886423e-14, 7.479075369435633e-14, 8.519788335640958e-14, 9.096986673931026e-14, 1.0344880326654531e-13, 1.0613390907574373e-13, 1.2112881927426069e-13, 1.4202897372703163e-13, 1.6376571379400875e-13, 2.1354507560953343e-13, 2.50933350859048e-13, 3.2086848707918394e-13, 3.758482860010266e-13, 4.592612363837376e-13, 5.282633223589881e-13, 6.120345816116775e-13, 6.881503986857892e-13, 7.834680162419512e-13, 8.758786189892488e-13, 9.70376251233748e-13, 1.0666348281694995e-12]
Stopped at Iteration: 22
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 21 : [7.555653467602035e-15, 7.802170053389034e-15, 1.2669281992528562e-14, 3.125837041429535e-14, 5.370744013074872e-14, 1.0039771802070306e-13, 1.4681974057878386e-13, 1.9899427760265962e-13, 2.559569219678313e-13, 3.0769698502527995e-13, 3.7437142009401487e-13, 4.0949348389852705e-13, 4.812846842451303e-13, 5.062295708707322e-13, 5.888769167110847e-13, 6.164014735255923e-13, 7.077184068829e-13, 7.370035408662524e-13, 8.331454008452542e-13,

8.703217094254087e-13, 9.824013295201 22e-13, 1.0261094578375683e-12]
Stopped at Iteration: 31
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 30 : [4.2328113057308e-15, 1.8729373589390583e-14, 3.085484331849468e-14, 3.429195485588719e-14, 4.937435173115146e-14, 5.721497666431902e-14, 9.296482515650695e-14, 1.463528587421959e-13, 1.7867277602811396e-13, 2.3452777600272853e-13, 2.662562150568983e-13, 3.1375269865368307e-13, 3.4832711434536153e-13, 3.7814158280877577e-13, 4.217564025979927e-13, 4.40509505490364e-13, 4.886085754021387e-13, 5.029810333178395e-13, 5.501734759190888e-13, 5.700243506854352e-13, 6.187548871965023e-13, 6.475561979830635e-13, 7.038368100279113e-13, 7.365975594165191e-13, 7.915131135307021e-13, 8.212487807551546e-13, 8.798728351825661e-13, 8.997758529100656e-13, 9.6100415491158e-13, 9.739866089807153e-13, 1.0440797797821397e-12]
Stopped at Iteration: 35
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 34 : [7.0022443753096326e-15, 2.0221553849659973e-14, 4.095211430087628e-14, 7.400102741748121e-14, 9.954977316751179e-14, 1.1179380868623442e-13, 1.4537581208883627e-13, 1.4951712965621518e-13, 1.783720799233843e-13, 1.8068367676667863e-13, 2.0344252246512766e-13, 2.155715353844548e-13, 2.3324069390995117e-13, 2.5556861293932666e-13, 2.724499246818865e-13, 3.079380118043207e-13, 3.203078279377425e-13, 3.650558303731689e-13, 3.846275341559169e-13, 4.393172753300528e-13, 4.609901862046734e-13, 5.185036681749627e-13, 5.396408070874891e-13, 5.920744165563212e-13, 6.152188840790479e-13, 6.627085357726158e-13, 6.964302185877535e-13, 7.396579308908395e-13, 7.775857039061252e-13, 8.148074835050004e-13, 8.577230899872361e-13, 8.896632787212618e-13, 9.370741256939206e-13, 9.698100571513297e-13, 1.0278055432863736e-12]
Stopped at Iteration: 35
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 34 : [3.8146704039532e-15, 1.6397025259076576e-14, 2.0275584958795854e-14, 2.125880730430964e-14, 3.052778138779087e-14, 5.566624714016172e-14, 9.252069979687393e-14, 1.3380219481197859e-13, 1.6624827965001365e-13, 2.0152565137925e-13, 2.3934749949058216e-13, 2.6820161614693934e-13, 3.066449821954761e-13, 3.2440494115799813e-13, 3.6749148123021494e-13, 3.7636740606473654e-13, 4.1741756520109216e-13, 4.2399676636133737e-13, 4.652677921496 81e-13, 4.741751827134829e-13, 5.084181972607441e-13, 5.219584004504852e-13, 5.590700551918808e-13, 5.779953436058658e-13, 6.162167082710698e-13, 6.381003154578195e-13, 6.795622893917705e-13, 6.981275800916165e-13, 7.464754723046645e-13, 7.639861031503167e-13, 8.189026456413497e-13, 8.421745406656704e-13, 9.090730473013075e-13, 9.4222224721714e-13, 1.01733196772731e-12]
Stopped at Iteration: 27
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 26 : [8.028130258979056e-16, 1.461222056558723e-14, 5.131365943651513e-14, 7.056923061517165e-14, 9.789539557906736e-14, 1.0287314390490755e-13, 1.264118559057904e-13, 1.2870724171286379e-13, 1.56801593154131e-13, 1.768586658581372e-13, 2.1331794812408377e-13, 2.751059511536309e-13, 3.1534993683441806e-13, 3.889806465151941e-13, 4.1995702395164996e-13, 4.968265551308404e-13, 5.288405155673262e-13, 5.997461798395889e-13, 6.379317828046535e-13, 7.004169832157383e-13, 7.465709917747824e-13, 7.905230365577082e-13, 8.44649757156726e-13, 8.752605423650524e-13, 9.356093155959855e-13, 9.631209523959333e-13, 1.0322260742407636e-12]
Stopped at Iteration: 27
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 26 : [8.263673510687958e-15, 8.527017613390235e-15, 2.1444563403065485e-14, 2.810783941805074e-14, 4.208062110166946e-14, 5.273929149615728e-14, 7.674015697832006e-14, 9.450429068383209e-14, 1.3470788475328287e-13, 1.5996841728463359e-13, 2.083014590856951e-13, 2.305902518088102e-13, 2.754265258871243e-13, 2.9564682353929193e-13, 3.534406138442791e-13, 3.807708330617468e-13, 4.534185658597381e-13, 4.862892789401719e-13, 5.619970541936871e-13, 5.901438013480536e-13, 6.703793744867493e-13, 6.986969795953949e-13, 7.8068983101586e-13, 8.073744210498809e-13, 9.004551859856665e-13, 9.275132647233937e-13, 1.025260386921545e-12]
Stopped at Iteration: 21
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 20 : [5.173151281305917e-14, 6.158658083307076e-14, 1.1367275231327672e-13, 1.2882653476957844e-13, 1.9283299659404635e-13, 2.0973578253276327e-13, 2.924386577475577e-13, 3.066899025380560 6e-13, 3.922597181354981e-13, 4.078391403138966e-13, 5.083015916919 85e-13, 5.191787978313096e-13, 6.147144131986326e-13, 6.230327537706212e-13, 7.219003153613269e-13, 7.268520024494899e-13, 8.245513816742889e-13, 8.2897857339999e-13, 9.348035744784481e-13, 9.439618426722348e-13, 1.0519630061808499e-12]
Stopped at Iteration: 31
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 30 : [8.051471972336653e-15, 9.161453273451155e-15, 1.2834572829258995e-14, 3.4509864228578785e-14, 4.5406178410371084e-14, 6.54825913581868e-14, 9.61623626914191e-14, 1.2345108365228243e-13, 1.6425193939072214e-13, 1.9814014772943697e-13, 2.518619855482908e-13, 2.91297607668137e-13, 3.4814153809469323e-13, 3.842439628225569e-13, 4.4684197605828775e-13, 4.758959802194979e-13, 5.377217563537317e-13, 5.554582925269408e-13, 6.179653938203777e-13, 6.286247772527208e-13, 6.915186921317377e-13, 6.978149021605e-13, 7.576075818049191e-13, 7.626272158651126e-13, 8.202958126926126e-13, 8.247749702928535e-13, 8.802157668139886e-13, 8.866215130067222e-13, 9.407390477565102e-13, 9.526751465301209e-13, 1.0057868705734822e-12]
Stopped at Iteration: 30
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 29 : [1.0495026173031815e-14, 1.9216849069079633e-14, 4.1947219810247527e-14, 5.380705089150227e-14, 8.875113256794315e-14, 1.0187890284359302e-13, 1.4025056247885 63e-13, 1.5124089590872244e-13, 1.939637339178885e-13, 2.04864928505049e-13, 2.530317928384502e-13, 2.6594085412275285e-13, 3.289355194034159e-13, 3.530856263444996e-13, 4.281798055392941e-13, 4.516922244401509e-13, 5.266072534665686e-13, 5.431953565821551e-13, 6.176218024636667e-13, 6.288672197318695e-13, 7.016112700848415e-13, 7.089022757762258e-13, 7.778761483277151e-13, 7.842699631022313e-13, 8.489718536675168e-13, 8.545753819227924e-13, 9.179016923339267e-13, 9.255098573055225e-13, 9.86511560674411e-13, 1.0045061085617197e-12]
Stopped at Iteration: 39
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 38 : [2.351019730834903e-14, 3.5896185693908466e-14, 6.549151781615569e-14, 7.292038509135541e-14, 1.06197118561384e-13, 1.0663557704060884e-13, 1.3490095676515486e-13, 1.377873640183562e-13, 1.592463318278062e-13, 1.6491983310330078e-13, 1.852020472221341e-13, 1.9059922351327073e-13, 2.1686318811568346e-13, 2.2227092370101852e-13, 2.567932946950823e-13, 2.6111531713075323e-13, 3.0214622662399306e-13, 3.0874541222193504e-13, 3.5427250888744655e-13, 3.61926320779218

33e-13, 4.0553554328310906e-13, 4.1325501533154233e-13, 4.551066390316631e-13, 4.644217652072012e-13,
5.015688399661213e-13, 5.117619124876448e-13, 5.497941846137239e-13, 5.622146392669063e-13, 6.03276744479
9564e-13, 6.204576518053623e-13, 6.669744891604207e-13, 6.956692818182759e-13, 7.445204778826474e-13,
7.791558673223442e-13, 8.276867280435415e-13, 8.648115665806848e-13, 9.142518164137515e-13, 9.53296266857
1558e-13, 1.0053219773589983e-12]
Stopped at Iteration: 31
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 30 : [3.612305846459437e-15, 2.983777288222536e-1
4, 4.6625209961152296e-14, 5.251090209307254e-14, 7.310285893756123e-14, 7.480164490760473e-14,
8.531822954708465e-14, 1.0049692249461423e-13, 1.1585520006022225e-13, 1.2560141493361883e-13, 1.61914160
19482832e-13, 1.8857903337687213e-13, 2.4356725053939837e-13, 2.717861057327915e-13, 3.255592354194006e-1
3, 3.446963707840824e-13, 3.9074707633451875e-13, 4.2400690668509837e-13, 4.817315838615055e-13,
5.154326595800711e-13, 5.860904495779399e-13, 6.093865492132868e-13, 6.746122234017321e-13, 6.98412983361
0579e-13, 7.588046417793703e-13, 7.772380417481541e-13, 8.385188218895983e-13, 8.570348582406981e-13,
9.236960048876314e-13, 9.397219117118932e-13, 1.0249974868121047e-12]
Stopped at Iteration: 21
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 20 : [1.135989110350772e-14, 2.0168366613905533e-
14, 4.582530159780768e-14, 6.181712665143628e-14, 1.0945241001430476e-13, 1.455836991647863e-13,
2.1651671647322178e-13, 2.6161627176811166e-13, 3.418399557334101e-13, 3.7200445819960904e-13, 4.63568982
5094866e-13, 4.897961417528588e-13, 5.886729749712139e-13, 6.129899538509546e-13, 7.159139970255559e-13,
7.362802738555112e-13, 8.354676265903282e-13, 8.532810916736482e-13, 9.561778825725637e-13, 9.723790723338
806e-13, 1.0734197964688647e-12]
Stopped at Iteration: 23
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 22 : [2.8297752404837037e-14,
5.0562534469978045e-14, 9.900473779319924e-14, 1.144057739052223e-13, 1.674259915621551e-13,
1.7521083071277443e-13, 2.3599009026580655e-13, 2.5725641077227684e-13, 3.323358883346694e-13, 3.48100567
5589738e-13, 4.3067660562188865e-13, 4.4121711026330475e-13, 5.20746523017255e-13, 5.379112175596794e-13,
6.237576146973177e-13, 6.369078691317574e-13, 7.256667770912449e-13, 7.365847449231064e-13, 8.30066518187
4295e-13, 8.431727843158424e-13, 9.349833180848474e-13, 9.42243325664041e-13, 1.0373519583444184e-12]
Stopped at Iteration: 45
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 44 : [1.7940056759295784e-14,
1.8239175859272254e-14, 2.8997854131424616e-14, 4.812614088241833e-14, 5.755002985584602e-14,
8.210490876083823e-14, 9.511286543345513e-14, 1.0484064406965013e-13, 1.2769883840518364e-13, 1.304625673
5011142e-13, 1.556608520354364e-13, 1.7503697375246033e-13, 1.9524905164569252e-13, 2.1947474675376517e-1
3, 2.4306286422165706e-13, 2.5580273256408555e-13, 2.8858943310141374e-13, 2.9422411247273614e-13, 3.2003
33384881628e-13, 3.3237102545213394e-13, 3.498558276448132e-13, 3.648119874101469e-13, 3.7919670684552097
e-13, 3.892095408878841e-13, 4.1377483724964436e-13, 4.183598237682118e-13, 4.361093670776842e-13,
4.4847909977483154e-13, 4.606797948180674e-13, 4.783032866261432e-13, 4.891620674676932e-13,
5.061922572169901e-13, 5.239246406303278e-13, 5.427826222118332e-13, 5.576256262155127e-13, 5.81781842481
8885e-13, 5.969852605449618e-13, 6.319343184296324e-13, 6.598294242807124e-13, 7.074586413295042e-13,
7.533272330868304e-13, 8.18727093853041e-13, 8.72971936681457e-13, 9.641061861073155e-13, 1.0251251427668
105e-12]
Stopped at Iteration: 27
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 26 : [2.2737038342689782e-14, 3.233546344957474e-
14, 6.130659843532349e-14, 6.581587493203671e-14, 9.437749618905045e-14, 9.696213190529807e-14,
1.1932604959575622e-13, 1.2696494040423456e-13, 1.573934289855954e-13, 1.6991749936202167e-13, 2.16932380
00988095e-13, 2.373092646989586e-13, 2.960295768025707e-13, 3.1605781091471443e-13, 3.7909169051538945e-13,
4.046019794304539e-13, 4.691277248652276e-13, 4.938524214428732e-13, 5.634380195171858e-13,
5.939468811087731e-13, 6.745728484209778e-13, 6.998200336791433e-13, 7.85644351669146e-13, 8.058951171899
174e-13, 8.909247343085344e-13, 9.153270825085915e-13, 1.0086630166492092e-12]
Stopped at Iteration: 27
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 26 : [9.332995444485081e-15, 3.1330558473590001e-1
4, 5.732022646254657e-14, 7.015531686948781e-14, 1.0718391292466467e-13, 1.2471925641440063e-13,
1.7349258182148063e-13, 1.9622240237523393e-13, 2.5826011273428464e-13, 2.8134416357435597e-13, 3.5347035
48447859e-13, 3.718576042452368e-13, 4.4406968504687035e-13, 4.58756335369317e-13, 5.3484707038623e-13, 5
.429205398082697e-13, 6.202256789860358e-13, 6.250227724002643e-13, 6.978283637558993e-13,
6.993559552014342e-13, 7.7115140988781e-13, 7.774381557300428e-13, 8.4154052239634e-13, 8.605260359465909
e-13, 9.2238987011879e-13, 9.591028377815472e-13, 1.0097022800932173e-12]
Stopped at Iteration: 44
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 43 : [1.1222312233960005e-14,
1.7216960136683193e-14, 2.9574640921163555e-14, 3.4194554739124605e-14, 3.962696427398747e-14, 4.94442056
4113166e-14, 5.545472648566352e-14, 5.664475139981316e-14, 6.45628496088239e-14, 6.991192750500587e-14,
7.826635727023038e-14, 9.32783348291689e-14, 1.0588417273463071e-13, 1.232122846427949e-13, 1.37493845170
8289e-13, 1.5978349169474989e-13, 1.7707398590415946e-13, 2.0096764632494406e-13, 2.1652402493680169e-13,
2.3276948906095286e-13, 2.5001019708833154e-13, 2.6715742676018907e-13, 2.8370897905407173e-13, 3.0020910
79393502e-13, 3.1570068919994456e-13, 3.2582395112151027e-13, 3.433878387469326e-13, 3.531318212649723e-1
3, 3.7091507422553386e-13, 3.806006196669828e-13, 3.9432753646932017e-13, 4.023812153136563e-13,
4.197332969444769e-13, 4.3563797751552783e-13, 4.662823358212476e-13, 5.034556161022361e-13, 5.5038451700
19124e-13, 6.077716189036695e-13, 6.697059763518451e-13, 7.416916465004095e-13, 8.109783803580845e-13,
8.929610023234554e-13, 9.738643132780907e-13, 1.0535093259840186e-12]
Stopped at Iteration: 32
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 31 : [1.3902698293336643e-14,
2.4409461951901408e-14, 5.688986223728873e-14, 7.333350502720463e-14, 1.0478964130469433e-13,
1.0727160234283015e-13, 1.3598001709289503e-13, 1.4974393104686958e-13, 1.673442995263535e-13, 1.98314328
75497345e-13, 2.122594372431057e-13, 2.6061037515288003e-13, 2.751496983003688e-13, 3.3450709903606695e-1
3, 3.6083968217587827e-13, 4.195100525935505e-13, 4.432508606011201e-13, 4.967141802526476e-13,
5.216888502712804e-13, 5.608423417591781e-13, 5.964489299769971e-13, 6.219300294364139e-13, 6.66842710457
2453e-13, 6.895888302469926e-13, 7.346287361578247e-13, 7.589642667879145e-13, 8.09269739330461e-13, 8.40

4072383852699e-13, 8.923632143505454e-13, 9.269960554063575e-13, 9.771429724614963e-13,
1.0125461985645912e-12]
Stopped at Iteration: 21
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 20 : [1.4506808484046563e-14,
1.4592073201766922e-14, 2.338814205351595e-14, 4.524455803461876e-14, 4.863379114915763e-14,
6.192664762929184e-14, 7.87433636675792e-14, 8.564165156233253e-14, 1.0530781737571933e-13, 1.45281342302
9909e-13, 1.992871838008755e-13, 2.7394525667120653e-13, 3.475116696589873e-13, 4.236009302161026e-13, 5.
031959693236302e-13, 5.851091747887488e-13, 6.723764739178593e-13, 7.613813558594592e-13,
8.48939037472742e-13, 9.361975092245734e-13, 1.0346339277195817e-12]
Stopped at Iteration: 47
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 46 : [1.0705321658257254e-14, 3.936507554812229e-
14, 7.14371482785661e-14, 9.012229411832071e-14, 1.3721482692691937e-13, 1.5545355187804425e-13, 2.072419
2366535394e-13, 2.1184211111343968e-13, 2.6164795579051135e-13, 2.6507427418253825e-13, 3.109981844507449
e-13, 3.135015959071317e-13, 3.507976277947597e-13, 3.562961792282704e-13, 3.880265808543383e-13,
3.9659643416424515e-13, 4.2378208217234236e-13, 4.312812605452744e-13, 4.568990482144369e-13,
4.633824937466892e-13, 4.873807895371924e-13, 4.960070071098057e-13, 5.151417686287712e-13, 5.288538651099
9986e-13, 5.424486954275686e-13, 5.6058762258717e-13, 5.719112485876202e-13, 5.902532970915e-13,
6.070373676788856e-13, 6.187246003889647e-13, 6.420378354256062e-13, 6.497551681317011e-13, 6.732342278269
672e-13, 6.805633677040666e-13, 7.012050484458971e-13, 7.141312128996308e-13, 7.319631433556084e-13,
7.507694918846115e-13, 7.778422974534464e-13, 7.927842632543469e-13, 8.294275129552541e-13, 8.440892626548
8532e-13, 8.957050902130105e-13, 9.126462484108711e-13, 9.737424793134596e-13, 9.952319050430163e-13,
1.0646637170377597e-12]
Stopped at Iteration: 24
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 23 : [3.239008818620849e-14, 3.576741962623415e-1
4, 6.095179217355066e-14, 8.056868994520787e-14, 8.110976813570047e-14, 9.552048249338276e-14,
1.1411474450328057e-13, 1.518744963259134e-13, 1.7688360585528804e-13, 2.3286040784976227e-13, 2.773963497
7308143e-13, 3.427993227693191e-13, 4.059297720086644e-13, 4.727693573858181e-13, 5.378958413276226e-13,
5.899919794688681e-13, 6.620301194665341e-13, 7.123705332437662e-13, 7.837679485336142e-13, 8.187841141326
6597e-13, 8.958582282362191e-13, 9.19328693043926e-13, 9.964696692139971e-13, 1.0129993709381418e-12]
Stopped at Iteration: 17
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 16 : [3.124457491778875e-14, 5.919866470881456e-1
4, 1.2098441391700298e-13, 1.344066818467382e-13, 2.0079189401231072e-13, 2.0572506605724282e-13, 2.69008
9409875969e-13, 2.7868060677302097e-13, 3.5665638561595984e-13, 3.8464011790101127e-13, 4.938130847604849
e-13, 5.34902851660454e-13, 6.742433226788818e-13, 7.241940026867156e-13, 8.796722715604261e-13,
9.371012023074948e-13, 1.0924431254126382e-12]
Stopped at Iteration: 21
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 20 : [2.4205432180114366e-15, 3.061228274301417e-
15, 1.7950120938000505e-14, 3.316306956464078e-14, 5.93451716261025e-14, 8.717820834486956e-14, 1.2594659
256584613e-13, 1.7518898404191009e-13, 2.4876183946589085e-13, 3.1437600100705295e-13,
3.951872452834572e-13, 4.4980333149487723e-13, 5.322249166028535e-13, 5.769371981732213e-13, 6.6399096504
72109e-13, 6.988027297285561e-13, 7.927009177957567e-13, 8.213688212757083e-13, 9.158172981590392e-13,
9.412459971627297e-13, 1.0362727608125556e-12]
Stopped at Iteration: 33
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 32 : [2.901654449712905e-14, 3.023472302653717e-1
4, 4.8552808058722817e-14, 5.940934453389551e-14, 6.724029252008006e-14, 8.533641421494192e-14,
9.043616119820479e-14, 1.1001785557853559e-13, 1.2271951687653242e-13, 1.5572828981404797e-13, 1.73208305
38151375e-13, 2.0310932098446302e-13, 2.2703961875200476e-13, 2.5228274950096334e-13, 2.896853557619268e-
13, 3.2723815033191896e-13, 3.7367618948426255e-13, 4.1807186778921485e-13, 4.733945568084068e-13, 5.1832
53714157103e-13, 5.768567973213658e-13, 6.149316352636634e-13, 6.691979029042861e-13,
6.986306619676118e-13, 7.54568550471929e-13, 7.737182117028833e-13, 8.337603085835243e-13, 8.459337740126
428e-13, 9.067772702300798e-13, 9.158257020383973e-13, 9.74781651357853e-13, 9.826891223577782e-13, 1.040
6744022870584e-12]
Stopped at Iteration: 45
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 44 : [2.951862882532993e-14, 3.308856035262136e-1
4, 5.5073275852006825e-14, 6.556692221455179e-14, 7.720572995404862e-14, 8.3354867393732e-14,
1.0996363228531778e-13, 1.1564125016342625e-13, 1.4092559086169547e-13, 1.4754916914639388e-13, 1.6547996
649027149e-13, 1.755257633789934e-13, 1.9269565738346657e-13, 2.037724208927732e-13, 2.1683464900767738e-
13, 2.3130373344822416e-13, 2.403083497376e-13, 2.599348884037238e-13, 2.691475352744733e-13,
2.8891358674294857e-13, 3.050897175560202e-13, 3.207982704878134e-13, 3.4543125773093036e-13,
3.558178471267642e-13, 3.887981186257678e-13, 3.966264067951155e-13, 4.3393731682919724e-13, 4.3947228972
02859e-13, 4.70346561189477e-13, 4.803294679628995e-13, 5.056681626980278e-13, 5.223552567529565e-13,
5.416785264568957e-13, 5.692696706161846e-13, 5.888925211289976e-13, 6.251997194711104e-13, 6.483692066711
661e-13, 6.911538613238904e-13, 7.191067139801696e-13, 7.672899526803359e-13, 8.020848445017784e-13,
8.645014692897835e-13, 9.105138322698254e-13, 9.819459302889873e-13, 1.0500886418374957e-12]
Stopped at Iteration: 43
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 42 : [2.6510186286326324e-15, 5.401402003329181e-
15, 1.680830493303741e-14, 3.536704247474689e-14, 6.446885589173627e-14, 9.005391035371482e-14,
1.2429481426898586e-13, 1.4429267102106677e-13, 1.767550302043694e-13, 1.8208104810797343e-13, 2.178509877
68331228e-13, 2.2211475210331352e-13, 2.525569773888258e-13, 2.6419457765092703e-13, 2.800266380055091e-1
3, 3.013148778110674e-13, 3.089550293270363e-13, 3.392185965319902e-13, 3.5277406858249653e-13,
3.7954748668051155e-13, 4.081982350129755e-13, 4.235228783324997e-13, 4.666824550872422e-13,
4.763791467742159e-13, 5.241497438968224e-13, 5.364882688661675e-13, 5.770006633339758e-13, 5.937616055320
593e-13, 6.277121049600572e-13, 6.480312703880385e-13, 6.793806571139599e-13, 7.015624748593039e-13,
7.271002742141455e-13, 7.510410568453712e-13, 7.742202468166666e-13, 8.000007191173539e-13, 8.28135460650
764e-13, 8.591623080313197e-13, 8.916959453686642e-13, 9.240949211581788e-13, 9.590533979441347e-13,
9.93393279068999e-13, 1.0274983562385876e-12]
Stopped at Iteration: 41

Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 40 : [1.3082104013747172e-14,
1.5062855980436162e-14, 1.9017625219217266e-14, 2.6157499165250625e-14, 4.787644332781838e-14, 6.91069199
3101574e-14, 1.0967959847852206e-13, 1.3418409274630665e-13, 1.6739648411128813e-13, 1.8555873690315097e-
13, 2.2305071813423714e-13, 2.316422823010686e-13, 2.6788998325414167e-13, 2.7206290419668353e-13, 3.1050
530020193084e-13, 3.18734633262242e-13, 3.521760816350688e-13, 3.639282697841517e-13,
3.9355300895065624e-13, 4.052579666705928e-13, 4.354462003587983e-13, 4.460448271753852e-13,
4.794065109268431e-13, 4.965464893134574e-13, 5.280515514126101e-13, 5.564997048362501e-13, 5.81518474249
3184e-13, 6.172096442932295e-13, 6.367871383913701e-13, 6.793403538907481e-13, 6.952481671188906e-13,
7.420340168559405e-13, 7.574370565586745e-13, 8.024103438322893e-13, 8.182197950900503e-13, 8.65556981328
3422e-13, 8.786629134458738e-13, 9.275653521783983e-13, 9.395292066002628e-13, 9.84948544925181e-13, 1.00
0715863290747e-12]
Stopped at Iteration: 76
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 75 : [8.403769026048108e-16, 1.3347344908555387e-
15, 1.7575622502028413e-15, 3.4765488620770738e-15, 5.2047647361301834e-15, 6.549544186475839e-15, 7.1795
6056318393e-15, 7.969431327144995e-15, 8.495016399673604e-15, 8.918102425050313e-15, 9.304146766087577e-
15, 9.73684866413557e-15, 1.0259116806069273e-14, 1.0701434348539339e-14, 1.1253558691027817e-14, 1.15571
02088487532e-14, 1.2444617803733101e-14, 1.2991779498966067e-14, 1.4201282699713874e-14, 1.47854371575326
4e-14, 1.6173654539748666e-14, 1.6975618233482214e-14, 1.897983777256692e-14, 2.0251689331076053e-14, 2.2
728644064829456e-14, 2.496517041167523e-14, 2.7897945879751872e-14, 3.054491334903481e-14,
3.450093075745184e-14, 3.775259978287136e-14, 4.1558392534521826e-14, 4.586414591777082e-14, 5.0644780441
05291e-14, 5.5825511276178714e-14, 6.158115697827774e-14, 6.764591441464013e-14, 7.349320613879083e-14,
7.980644321848557e-14, 8.647063845186987e-14, 9.300200529165952e-14, 1.0000754164019348e-13, 1.0748444026
170447e-13, 1.143688823487106e-13, 1.2188531649811276e-13, 1.292741866024533e-13, 1.3717760071409984e-13,
1.4409235308071814e-13, 1.5290332819289282e-13, 1.6003544862830843e-13, 1.6852896908652508e-13, 1.7656906
203485248e-13, 1.856152728557837e-13, 1.9341070578461196e-13, 2.0300485710907401e-13,
2.1078493732304447e-13, 2.2109353780972097e-13, 2.289337153429667e-13, 2.398131147119637e-13,
2.4827304954312544e-13, 2.597404380321141e-13, 2.6924607367731583e-13, 2.826919925929186e-13,
2.93260438219672e-13, 3.1044203388554894e-13, 3.2395652768200887e-13, 3.4632284843075054e-13, 3.659226036
4008276e-13, 3.9631015761257174e-13, 4.26081191024606e-13, 4.698398986274572e-13, 5.175559536999821e-13,
5.835859922108505e-13, 6.588671025169972e-13, 7.610923699665617e-13, 8.864297708897352e-13, 1.04444402891
75191e-12]
Stopped at Iteration: 80
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 79 : [3.401902337491944e-16, 9.272730124511431e-1
6, 1.9000361658165108e-15, 3.936281123286028e-15, 5.027540739570911e-15, 6.1915055349455034e-15,
6.768384260394841e-15, 8.363339994287661e-15, 9.407790977650455e-15, 1.2195763545799879e-14, 1.3828193058
22667e-14, 1.6661715955572037e-14, 1.8137519513437404e-14, 2.127744138631326e-14, 2.2962038559995156e-14,
2.6443698778817714e-14, 2.8093497750523775e-14, 3.127892206201633e-14, 3.2685317740383354e-14,
3.635445748327364e-14, 3.7694151796102597e-14, 4.1542934285458426e-14, 4.306364442268932e-14, 4.663903806
6441393e-14, 4.792537619207565e-14, 5.1870017910143884e-14, 5.3016961225376465e-14, 5.710584110294215e-
14, 5.813585344043244e-14, 6.239894820563962e-14, 6.315720586961103e-14, 6.786159152520828e-14,
6.854276719874228e-14, 7.280408608507232e-14, 7.356365557584634e-14, 7.825146498209787e-14, 7.89504785910
8806e-14, 8.394956132495126e-14, 8.46608724157508e-14, 8.940914522151433e-14, 9.019449417003675e-14, 9.52
8212528808826e-14, 9.601762099853895e-14, 1.011332720393681e-13, 1.017879611566392e-13,
1.0688371088449556e-13, 1.0747440170270391e-13, 1.1274814961566199e-13, 1.1335572538547118e-13, 1.1830615
881879925e-13, 1.1889810262305048e-13, 1.2395517344053607e-13, 1.2459625182504013e-13,
1.2989574399732657e-13, 1.3062102716170238e-13, 1.3590893493477162e-13, 1.367591960149907e-13, 1.42449275
29864585e-13, 1.435961329111631e-13, 1.4939157542045714e-13, 1.5093800378096005e-13, 1.5676825391682904e-
13, 1.5891176657240136e-13, 1.6559009764562 8e-13, 1.690654744191811e-13, 1.7794007431475247e-13, 1.855885
0299658222e-13, 1.9862899400446945e-13, 2.1021566285231958e-13, 2.2981803916963665e-13, 2.5253335127678 19
e-13, 2.864024502314034e-13, 3.3090356992414625e-13, 3.9187853047374214e-13, 4.649883394400955e-13, 5.537
016831513329e-13, 6.564808286251524e-13, 7.871255064473936e-13, 9.45511128899549e-13,
1.1297866722347969e-12]
Stopped at Iteration: 81
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 80 : [2.302497708361679e-15, 3.640222723225438e-1
5, 4.6120654233613 63e-15, 8.767541595754585e-15, 1.0341964887616452e-14, 1.4021217046082539e-14,
1.4832628081220414e-14, 1.7129531818512607e-14, 1.7628071752821167e-14, 1.968423107915488e-14, 2.04374521
53179522e-14, 2.185010666491269e-14, 2.2867340176195596e-14, 2.3939180314868456e-14, 2.498581723620671e-14, 2.596746157760795e-14, 2.731408220684737e-14, 2.786026551849223e-14, 2.9604128478662145e-14,
2.9991366442203295e-14, 3.155267487606334e-14, 3.2049278474587124e-14, 3.3499911105971774e-14, 3.42292109
3386365e-14, 3.551928882897108e-14, 3.6358794130561085e-14, 3.748308757198652e-14, 3.822071580802482e-
14, 3.941170574979926e-14, 4.0116275971884535e-14, 4.114279415130691e-14, 4.193691833759943e-14,
4.2990971993044e-14, 4.3698737068371515e-14, 4.4983029323224124e-14, 4.5701346651350434e-14, 4.7031836685
64364e-14, 4.7884817607369826e-14, 4.9099791570383845e-14, 5.01156279688782e-14, 5.128661571866619e-14, 5
.2478433010150746e-14, 5.3786848957180273e-14, 5.495477374603252e-14, 5.6356934141334726e-14, 5.748430338
192418e-14, 5.888330547758053e-14, 6.006883295928296e-14, 6.142889390685469e-14, 6.270145952491947e-14,
6.4053562116082 83e-14, 6.567026334689512e-14, 6.740087330642347e-14, 6.948551367351875e-14, 7.164180600305
5398e-14, 7.462197277682261e-14, 7.780728494797586e-14, 8.233568869536636e-14, 8.734335999544028e-14,
9.383916304101685e-14, 1.0100174594261703e-13, 1.1066758082450731e-13, 1.210276229968015e-13, 1.334343898
4515935e-13, 1.4663416803460778e-13, 1.6163516064215862e-13, 1.7708676575742168e-13, 1.9592498948203558e-
13, 2.1654716553734132e-13, 2.3910995162265644e-13, 2.6578469787179706e-13, 2.9842028518334203e-13, 3.364
4613605378404e-13, 3.8079783669181775e-13, 4.3219331396154816e-13, 4.914196792472941e-13, 5.6119343350398
2e-13, 6.4652029956027 01e-13, 7.586931684213187e-13, 9.061571138285843e-13, 1.1011243707875167e-12]
Stopped at Iteration: 90
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 89 : [2.8289946094622708e-15, 4.022432453913721e-
15, 4.907638223892985e-15, 8.352729461002705e-15, 1.024546987896163e-14, 1.444619078798496e-14,
1.5853364942670704e-14, 1.8690722978888274e-14, 1.9340600612682184e-14, 2.3377071003204982e-14, 2.4523178
09087611e-14, 2.8231451954803206e-14, 2.887125431323802e-14, 3.210434596205574e-14, 3.252397189481203e-

14, 3.6006822001131044e-14, 3.6629277771601735e-14, 3.9713972876997923e-14, 4.011211935361717e-14, 4.3262
05568666981e-14, 4.3734165846781605e-14, 4.691702668773604e-14, 4.74108473696539e-14,
5.0304554152428896e-14, 5.058825056663905e-14, 5.3406068718151356e-14, 5.373792262339022e-14,
5.634779202712918e-14, 5.6706767688139985e-14, 5.912730158099849e-14, 5.949543314426867e-14, 6.1972100425
10679e-14, 6.23349313948559e-14, 6.452033993609178e-14, 6.485798043987175e-14, 6.69483291299953e-14, 6.73
060536204246e-14, 6.930157682592828e-14, 6.971377373076377e-14, 7.152826490566737e-14, 7.194366591018969e
-14, 7.376327762535327e-14, 7.429483815888232e-14, 7.604127025267823e-14, 7.669441417029402e-14,
7.837528334515103e-14, 7.906805677217619e-14, 8.078931087564688e-14, 8.156760868651311e-14, 8.31531671096
6907e-14, 8.406173382193154e-14, 8.557584509075851e-14, 8.660945530440099e-14, 8.80336684877129e-14, 8.91
4130974663642e-14, 9.049308319237854e-14, 9.174891144883702e-14, 9.29832309987302e-14, 9.442042663767337e
-14, 9.542561977892572e-14, 9.70400540976758e-14, 9.797121720389054e-14, 9.968187453384593e-14, 1.0060758
312268201e-13, 1.0240262500545095e-13, 1.0343035484653845e-13, 1.0529902212137885e-13,
1.0653768771108539e-13, 1.0852091064737946e-13, 1.0999333826787403e-13, 1.1231784097721147e-13, 1.1423688
70943724e-13, 1.171359464461793e-13, 1.198508158666077e-13, 1.2363745061375457e-13, 1.2774973025001306e-
13, 1.3354326838279512e-13, 1.404652882605785e-13, 1.5009621112180862e-13, 1.628487007699844e-13, 1.80583
43186267796e-13, 2.04081654483022e-13, 2.3502745657892804e-13, 2.774558532770567e-13, 3.373994933390102e-
13, 4.1449421600799123e-13, 5.056055725908323e-13, 6.204408720116295e-13, 7.796896522525199e-13,
1.0027453444089024e-12]
Stopped at Iteration: 82
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 81 : [9.176328640993074e-16, 1.5707823640154507e-
15, 2.021016849291205e-15, 3.8244082448341756e-15, 5.224429599290534e-15, 6.4516874892279176e-15, 7.23380
8742572674e-15, 8.419789378067471e-15, 9.34664863443323e-15, 1.0139357968549626e-14, 1.1103311457543777e-
14, 1.1800684367518779e-14, 1.3022919489746104e-14, 1.3531019173699997e-14, 1.432318590281936e-14, 1.4710
27000641106e-14, 1.554795214535824e-14, 1.605528188007315e-14, 1.667885719284578e-14,
1.7436938081903946e-14, 1.806282827883305e-14, 1.8930941243917085e-14, 1.9494876654315516e-14, 2.02772604
86704615e-14, 2.101605491685644e-14, 2.161537848588614e-14, 2.2332130390474406e-14, 2.280984616859257e-
14, 2.3666670010392875e-14, 2.4159938166959277e-14, 2.4925943024452056e-14, 2.549262813917495e-14, 2.6121
764779186275e-14, 2.684710347260706e-14, 2.729510924757582e-14, 2.803777600688322e-14, 2.8431703514246775
e-14, 2.918648896025018e-14, 2.964040858834664e-14, 3.033139643006466e-14, 3.0805949938477357e-14,
3.1520087671796715e-14, 3.208514717692284e-14, 3.279798741602861e-14, 3.342146318762433e-14,
3.4288379756785234e-14, 3.50038041807193e-14, 3.5976790718656354e-14, 3.676140903806626e-14, 3.7731810552
42261e-14, 3.8621637492492205e-14, 3.9489234736729837e-14, 4.033406739715764e-14, 4.1210103689365605e-14,
4.2309012551738883e-14, 4.339039984465555e-14, 4.483905386833364e-14, 4.6380469630741226e-14, 4.8469287550
96324e-14, 5.0927825118992124e-14, 5.3814661110607e-14, 5.706076653551083e-14, 6.079791089786429e-14,
6.5049165474029946e-14, 7.012951108784025e-14, 7.689851363697532e-14, 8.603803176303668e-14, 9.83581379890
8506e-14, 1.1470147066688959e-13, 1.3449482663025762e-13, 1.5870010046293173e-13, 1.8941282580077791e-13,
2.287084511222379e-13, 2.782561226212502e-13, 3.3794191232216805e-13, 4.0770048013668517e-13, 4.868131411
992383e-13, 5.705059492151846e-13, 6.633220507520158e-13, 7.721814604321239e-13, 9.086560523942444e-13,
1.0877561517959951e-12]
Stopped at Iteration: 91
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 90 : [4.449460041899307e-16, 7.554121508808877e-1
6, 8.303201824321225e-16, 1.373777930793697e-15, 1.834329849488096e-15, 2.337395301726563e-15,
2.757760051700895e-15, 3.1179124116983077e-15, 3.4613114479167686e-15, 3.853556717851599e-15, 4.093646096
552696e-15, 4.387856116441954e-15, 4.934606410233493e-15, 5.442554658302665e-15, 6.430301841855291e-15,
7.290477261108524e-15, 8.448848856120477e-15, 9.136503016212019e-15, 1.032998948732425e-14, 1.11007459797
43085e-14, 1.225353237599662e-14, 1.297569967652403e-14, 1.4108509377529027e-14, 1.4821769974685056e-14,
1.583714436916225e-14, 1.6682480181503136e-14, 1.743180361263173e-14, 1.8375454776930294e-14, 1.894356744
3967965e-14, 1.993010112782363e-14, 2.0486590250479703e-14, 2.1559834325444484e-14, 2.2118633388064798e-1
4, 2.3086627468349762e-14, 2.363070269048786e-14, 2.4658563274615392e-14, 2.5061795041068908e-14, 2.61707
6513437384e-14, 2.6659646754590526e-14, 2.7759493524156962e-14, 2.8224189600527756e-14, 2.946458858038058
3e-14, 2.995887639157028e-14, 3.128761061524219e-14, 3.202854377296066e-14, 3.3678307854433595e-14, 3.488
835113225283e-14, 3.705646214688936e-14, 3.882701283109244e-14, 4.133307308469527 6e-14, 4.331197086443223
e-14, 4.609422636174241e-14, 4.80242782106955e-14, 5.088828154613007e-14, 5.2965679795779413e-14, 5.59132
6959282316e-14, 5.786899416993334e-14, 6.096986129447282e-14, 6.292512847343597e-14, 6.632167170263611e-
14, 6.838430687949997e-14, 7.20485566279883e-14, 7.412624870713875e-14, 7.811789415028797e-14, 8.05525076
498137e-14, 8.454759716884702e-14, 8.746703730993237e-14, 9.22648465659293e-14, 9.586372697284416e-14, 1.
0206111030052873e-13, 1.0740999018193467e-13, 1.1585116112864798e-13, 1.2437833203568682e-13, 1.3615620095
599627e-13, 1.4740305446620874e-13, 1.631293366779512e-13, 1.8081163395691674e-13, 2.0301758356962504e-13
, 2.2809148151276847e-13, 2.601725806816776e-13, 2.9533937704236247e-13, 3.3465222749925025e-13, 3.776390
6866027954e-13, 4.270927042240185e-13, 4.816452686353743e-13, 5.451315797877475e-13, 6.230023192893356e-
13, 7.196951047973426e-13, 8.400864244917924e-13, 9.960608672413203e-13, 1.1872403029358642e-12]
Stopped at Iteration: 90
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 89 : [4.029040717284993e-15, 5.6570480594280745e-
15, 6.626822736276781e-15, 1.0440906042819193e-14, 1.2310762615597965e-14, 1.5789420221653126e-14, 1.6686
421853058788e-14, 1.8549016909272414e-14, 1.9149531931413 73e-14, 2.1405810054607972e-14, 2.2380873282082305e-14, 2.4617738346671495e-14, 2.5224437934941263e-14, 2.7306333978111498e-14, 2.7670854009961327e-14, 3
.0040885044635645e-14, 3.063389399487285e-14, 3.286808227081074e-14, 3.339830389955979e-14, 3.57495325237
6521e-14, 3.616293614365307e-14, 3.8596362349201655e-14, 3.904040556251434e-14, 4.105918947283249e-14, 4.
1514972411127 96e-14, 4.376722481882712e-14, 4.429609690485704e-14, 4.649560216285757e-14,
4.6929007136560245e-14, 4.874244003143435e-14, 4.9135619700587016e-14, 5.1099271896883 56e-14,
5.1697546632576494e-14, 5.342766363775814e-14, 5.3952414342249576e-14, 5.558582232828859e-14,
5.600102225093382e-14, 5.750493603174114e-14, 5.816293481295827e-14, 5.952827296532802e-14, 6.04573079649
1678e-14, 6.169192823460339e-14, 6.278015984987898e-14, 6.380662738971452e-14, 6.501952783907005e-14,
6.593890943255732e-14, 6.716551888384582e-14, 6.816694937806334e-14, 6.933734291397299e-14, 7.02635983005
871e-14, 7.150733473168618e-14, 7.235786103319409e-14, 7.366963521408772e-14, 7.445192748564855e-14,
7.60254785098879e-14, 7.685544976939633e-14, 7.860658971070936e-14, 7.953761627311987e-14,
8.146038288323291e-14, 8.25552852801277e-14, 8.493718037400644e-14, 8.616760609903004e-14, 8.890746217090

516e-14, 9.035309397189488e-14, 9.363730350463248e-14, 9.569888248800514e-14, 9.938115889836054e-14,
1.0212706388189358e-13, 1.0668752663845847e-13, 1.1055085055982278e-13, 1.1666395702275104e-13, 1.2226753
084077373e-13, 1.3004308067803523e-13, 1.3774088477044374e-13, 1.4829588902764733e-13,
1.592463798851021e-13, 1.733778564564556e-13, 1.8913967582662377e-13, 2.091196367113362e-13, 2.3325347619
545236e-13, 2.637062913026884e-13, 2.997835738373604e-13, 3.451787370302629 7e-13, 4.0276662800226544e-13,
4.699148837044444e-13, 5.488127714734717e-13, 6.495820924500992e-13, 7.71338666114935e-13, 9.1317715230258
64e-13, 1.0778248675532879e-12]
Stopped at Iteration: 73
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 72 : [8.167292590537281e-16, 8.754851615397648e-1
6, 1.1464942371754414e-15, 1.5848821523214003e-15, 1.9606397386013673e-15, 2.686826226971795e-15, 3.75254
0028376311e-15, 4.388622530793647e-15, 5.476901878623428e-15, 6.074058447678207e-15,
6.7575997102909296e-15, 7.520062577034874e-15, 8.0321917805134e-15, 9.050875363531617e-15,
9.450993405958488e-15, 1.0536696154441295e-14, 1.1134357568087519e-14, 1.173943344573396e-14, 1.269376788
9818044e-14, 1.3027893093216542e-14, 1.398398580579666 1e-14, 1.4285668090005942e-14, 1.5288873404525346e-
14, 1.591870058438575e-14, 1.6922103535940117e-14, 1.7878432023504713e-14, 1.8897027597382937e-14, 2.0246
290265321682e-14, 2.1641963092960827e-14, 2.3222396446769015e-14, 2.4760853204838066e-14, 2.6523774870670
397e-14, 2.808653461561602e-14, 2.967174904810799e-14, 3.162771117568804e-14, 3.3500947125875647e-14,
3.535222934245872e-14, 3.7560069678245775e-14, 3.9450496614000026e-14, 4.1889953515653926e-14, 4.40017695
1591352e-14, 4.6670910674501726e-14, 4.905057870398823e-14, 5.193437375664971e-14, 5.479406741030508e-
14, 5.810287103901347e-14, 6.114823010540787e-14, 6.534338968311658e-14, 6.916453355317419e-14,
7.389882476705029e-14, 7.810771749224015e-14, 8.337841244493691e-14, 8.764432674489948e-14, 9.37626777366
2206e-14, 9.90266040009595e-14, 1.0547441649068537e-13, 1.1160445342230212e-13, 1.2019972906006442e-13, 1
.2814002289816775e-13, 1.3948308096601955e-13, 1.5053840734609332e-13, 1.6523059294010055e-13, 1.80698132
32990386e-13, 2.015341033000559e-13, 2.2508306291649737e-13, 2.586243749541119e-13, 3.023417900530053e-
13, 3.6102895969362315e-13, 4.3829276129047824e-13, 5.429234072373782e-13, 6.763606951271193e-13, 8.39195
4535803383e-13, 1.0291257415359138e-12]
Stopped at Iteration: 75
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 74 : [2.879807922619365e-16, 4.532174994398489e-1
6, 6.057466394642176e-16, 8.142410461299404e-16, 1.1485199078977725e-15, 1.3755688835414749e-15,
1.799986324308495e-15, 2.2227336174841844e-15, 2.5509999227803602e-15, 2.755234183138563e-15, 3.125247337
728437e-15, 3.4879586966699864e-15, 3.8240618986574626e-15, 4.312207563912084e-15, 4.6015361056770085e-15
, 4.763507137334078e-15, 5.113278476526816e-15, 5.44620501943198e-15, 5.77036273259148e-15, 6.20110758493
2955e-15, 6.570437689351426e-15, 7.03550101536051e-15, 7.419784347487515e-15, 7.766172731783773e-15, 8.12
9579036308263e-15, 8.574025907062269e-15, 9.06027827984052e-15, 9.443263298933586e-15, 9.795204434247466e
-15, 1.0187787222441395e-14, 1.0747983352347376e-14, 1.1194279408415239e-14, 1.1917330647348672e-14, 1.26
39982807452103e-14, 1.3486606008375093e-14, 1.4206638705029002e-14, 1.5117258023898 86e-14, 1.6046984858245723e-14, 1.689086747023914e-14, 1.7888763795042472e-14, 1.8778724151772773e-14, 1.987668801200595e-14, 2
.0907716416789923e-14, 2.2161059717467576e-14, 2.3444906543640927e-14, 2.5057570172032143e-14, 2.69884892
20158597e-14, 2.9333344641944426e-14, 3.2212164730078857e-14, 3.5686719106742064e-14, 3.9757321836214e-14
, 4.4465411658908325e-14, 5.0063071406638655e-14, 5.62239204391105e-14, 6.385330701839543e-14, 7.30182792
3794472e-14, 8.329535285812501e-14, 9.521916751777176e-14, 1.0900586766196518e-13, 1.2344186603838898e-
13, 1.3950333787769695e-13, 1.5990734643799818e-13, 1.8358654850430335e-13, 2.110420640497585e-13, 2.4561
091469108504e-13, 2.901066260657986e-13, 3.4316555657598605e-13, 4.0288036691660037e-13, 4.679443133262578e-13, 5.365094530948316e-13, 6.138877754964281e-13, 7.085888932282673e-13, 8.16769820863965e-13, 9.33380
4196783726e-13, 1.070502824095753e-12]
Stopped at Iteration: 74
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 73 : [1.5657746148167142e-15,
2.5961567763318056e-15, 3.3652110766304594e-15, 6.525896129470728e-15, 8.793983077260165e-15,
1.2292835628442002e-14, 1.413914685505732e-14, 1.6961632781180474e-14, 1.811247679991307e-14,
2.2237176814926487e-14, 2.377072960111977e-14, 2.7828546535465842e-14, 2.9260248542856386e-14, 3.34310931
662502e-14, 3.4182771690213564e-14, 3.930862616854036e-14, 4.011644909230836e-14, 4.430875669393703e-14,
4.484172477383671e-14, 4.9092898753267884e-14, 4.9468328772010446e-14, 5.394278092968511e-14, 5.430519467
2516813e-14, 5.806513763932318e-14, 5.8315064111042e-14, 6.221682508388282e-14, 6.244356926809548e-14,
6.643619611159624e-14, 6.671634645879474e-14, 7.06377789078151e-14, 7.095305536832177e-14, 7.515721866734
163e-14, 7.55374455054035e-14, 7.90542500192204e-14, 7.950188758678836e-14, 8.283468714968323e-14, 8.3296
62489958855e-14, 8.679458045871084e-14, 8.73653548475737e-14, 9.05890701296517e-14, 9.122218931845178e-14
, 9.437610345215524e-14, 9.508508166836765e-14, 9.845728960696114e-14, 9.944119240171152e-14,
1.029616354595956e-13, 1.0417690261033186e-13, 1.0813807161854042e-13, 1.0962560564467445e-13, 1.13710411
37724996e-13, 1.1539134692188232e-13, 1.199791069508107e-13, 1.2198706780826383e-13, 1.2762148715144906e-
13, 1.302808331465613e-13, 1.368568728333925e-13, 1.403921441036177e-13, 1.48956214048812e-13, 1.54893539
7598836e-13, 1.6632980795242295e-13, 1.755739803539405e-13, 1.9037915092245467e-13, 2.0371627253493678e-13, 2.2444674953239106e-13, 2.461054293034055e-13, 2.7923911615896556e-13, 3.182942328323825e-13,
3.742044786345759e-13, 4.4407818184028593e-13, 5.373167432260874e-13, 6.520352540815077e-13, 7.9497242915
04601e-13, 9.702683494312539e-13, 1.1932132048612711e-12]
Stopped at Iteration: 76
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 75 : [4.0693410850359307e-16, 1.018141061648773e-
15, 1.3460559814952612e-15, 2.6343763253046304e-15, 3.849281039203536e-15, 5.125175079545757e-15, 6.19279
81116113954e-15, 8.012866549120554e-15, 9.116355704218323e-15, 1.1308603604623031e-14, 1.2512602900555877
e-14, 1.4649355853080212e-14, 1.6068447891455425e-14, 1.90583137474049e-14, 2.0514959529082433e-14, 2.407
6440780789994054e-14, 2.5845675569705786e-14, 2.9213007358651083e-14, 3.1097673708818146e-14, 3.533741863769
2116e-14, 3.749002267428774e-14, 4.210121775955182e-14, 4.441205093752622 4e-14, 4.8750745141393673e-14, 5
.076037061482483e-14, 5.608484577326152e-14, 5.781852183433087e-14, 6.315948054221728e-14,
6.504953553873103e-14, 7.04520615187216e-14, 7.1978117824523e-14, 7.78980050394805e-14,
7.940513670960178e-14, 8.534777675060667e-14, 8.668709151767202e-14, 9.283711632561022e-14, 9.40857358711
0369e-14, 1.003756305623525e-13, 1.0164314266833466e-13, 1.0762193460440058e-13, 1.0873019743640068e-13,
1.1512372828072566e-13, 1.161835074217773e-13, 1.2254142703638343e-13, 1.236794840366049e-13,
1.299412770348124e-13, 1.3098343366095567e-13, 1.3784754970611956e-13, 1.389585809236143e-13, 1.452225367

9522476e-13, 1.4640089702734706e-13, 1.5267457683128948e-13, 1.538574506254665e-13, 1.6038534432909787e-13, 1.6170299923485498e-13, 1.686029261711867e-13, 1.7027633480152375e-13, 1.776898257360621e-13, 1.8030037795954728e-13, 1.8867525505410168e-13, 1.9302894090255909e-13, 2.0412432239772894e-13, 2.12391379110533e-13, 2.290571500818534e-13, 2.441650241971295e-13, 2.692272452409297 7e-13, 2.946573698765511e-13, 3.3050422369419097e-13, 3.660442250211303e-13, 4.151535984313712e-13, 4.72531894802311e-13, 5.504432758216817e-13, 6.528712902959418e-13, 7.826462348982903e-13, 9.360670688790416e-13, 1.1310623916933115e-12]
Stopped at Iteration: 82
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 81 : [2.557831061312827e-15, 3.869228840443033e-15, 4.9609700222568536e-15, 8.92333567686127e-15, 1.092048326685764 3e-14, 1.4732314956829575e-14, 1.592559038508583e-14, 1.8973037513478773e-14, 1.966856180013099e-14, 2.421203301254173 8e-14, 2.5104972239011806e-14, 2.888251764616094e-14, 2.938782351953865e-14, 3.282753095413337e-14, 3.330418175145768e-14, 3.7204813185444917e-14, 3.792037296311376e-14, 4.142412386202686e-14, 4.191272977726019e-14, 4.568285696496358e-14, 4.6197838238479246e-14, 5.00528927920353e-14, 5.0537702907775655e-14, 5.406790673911739e-14, 5.435198514823315e-14, 5.812993708305947e-14, 5.854009013691791e-14, 6.235780991780352e-14, 6.273881939527347e-14, 6.643703147306159e-14, 6.664410890058378e-14, 7.055085423872833e-14, 7.083518415314771e-14, 7.417237408957458e-14, 7.452668592443478e-14, 7.791481338304058e-14, 7.825481907999518e-14, 8.191937530419017e-14, 8.229164084202977e-14, 8.580264408370062e-14, 8.616803015071 6e-14, 8.989813158610205e-14, 9.023494629267747e-14, 9.407981844361927e-14, 9.442316811621387e-14, 9.837092915580951e-14, 9.871439627998576e-14, 1.0286607244164073e-13, 1.0335248681076929e-13, 1.0731301890268094e-13, 1.0787910592324214e-13, 1.1211552507400802e-13, 1.1268760026829427e-13, 1.1728009142917516e-13, 1.179655057762033e-13, 1.2281362560391647e-13, 1.2371985278356453e-13, 1.2922887851238549e-13, 1.3068245746475672e-13, 1.3690896004090134e-13, 1.392733189180005e-13, 1.4613116953291948e-13, 1.4923682758438488e-13, 1.5684630214308062e-13, 1.608257041274274e-13, 1.6969358294068306e-13, 1.753127984912706e-13, 1.8617579472139914e-13, 1.9446310179027106e-13, 2.0757654184666696e-13, 2.1962959740040187e-13, 2.392427939951496e-13, 2.610802112944928e-13, 2.94932716493297e-13, 3.364902500345944e-13, 3.922771090539463e-13, 4.610165632951767e-13, 5.464464363444392e-13, 6.408583948404412e-13, 7.552044412013528e-13, 8.947435351871109e-13, 1.0699834377359412e-12]
Stopped at Iteration: 69
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 68 : [1.5780560497560938e-15, 2.1355724138190532e-15, 2.8943649206673975e-15, 4.892854569981296e-15, 5.6404936530190105e-15, 7.364828509394 33e-15, 8.084232281265791e-15, 9.456302354422648e-15, 1.0003920673620596e-14, 1.1521071210464018e-14, 1.2090825929768622e-14, 1.3585020422349039e-14, 1.4039175608911577e-14, 1.5200348784530975e-14, 1.554557430417088 7e-14, 1.661087399750176e-14, 1.7115670306336103e-14, 1.8106282700156805e-14, 1.8828335350214534e-14, 1.9567252676637747e-14, 2.0279518450279118e-14, 2.084363145356447e-14, 2.1695164937103414e-14, 2.2487130629233768e-14, 2.350123247987971e-14, 2.448264227364621e-14, 2.5703355058243064e-14, 2.730905987063914 7e-14, 2.868566481135 58e-14, 3.0834420863888234e-14, 3.2045118272801 26e-14, 3.482943772761576 6e-14, 3.589858835303054e-14, 3.8596859432360246e-14, 3.962592747318468e-14, 4.254057909343494e-14, 4.379711408730171e-14, 4.684766757177597e-14, 4.845273594419917e-14, 5.149657379356031e-14, 5.363582566441179e-14, 5.71146579312833e-14, 6.041598243428546e-14, 6.470846603323795e-14, 6.889355255769023e-14, 7.413342232898601e-14, 7.96788507593253e-14, 8.641780015872937e-14, 9.398881307001097e-14, 1.0249755283394276e-13, 1.1189042458943012e-13, 1.2450435949972746e-13, 1.3973347635462832e-13, 1.5609746357797995e-13, 1.7439247837968335e-13, 1.9706469075237087e-13, 2.2368559340036595e-13, 2.5624491464849525e-13, 2.9470485879428283e-13, 3.3871866098310195e-13, 3.883167390107919e-13, 4.4993924568024577e-13, 5.174940087791247e-13, 5.92220685164256e-13, 6.741186056382922e-13, 7.620415647314953e-13, 8.553569829907581e-13, 9.643728034640188e-13, 1.0870567324358302e-12]
Stopped at Iteration: 86
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 85 : [7.660765951554329e-16, 1.2289977904770095e-15, 1.6963293896454431e-15, 3.3106588697698126e-15, 3.9581096905573744e-15, 5.139341429541603e-15, 5.834728042202035e-15, 6.554119250504493e-15, 7.152000629955582e-15, 7.696633482130035e-15, 8.292026780305142e-15, 8.862244669187733e-15, 9.248050336856425e-15, 9.636440286681028e-15, 1.001780483358207e-14, 1.0389822245676122e-14, 1.0894140453830088e-14, 1.1337510982484354e-14, 1.1740432169323213e-14, 1.2432071703188064e-14, 1.304468125931136e-14, 1.4330132722314498e-14, 1.5117445163970093e-14, 1.691702444289 88e-14, 1.7806168485931285e-14, 1.990030714305364e-14, 2.0908750121765588e-14, 2.2894588955403672e-14, 2.423178016558673 8e-14, 2.6285439360899795e-14, 2.7708806183927637e-14, 2.9839788241908774e-14, 3.1333531083969856e-14, 3.326427839547016e-14, 3.474430609297366e-14, 3.653114783457787e-14, 3.796623797144857e-14, 3.981658984388738e-14, 4.1315264840477994e-14, 4.326360667923852e-14, 4.4878098279356424e-14, 4.744188209438068e-14, 4.895480140189626e-14, 5.164552194158453e-14, 5.3098857774687693e-14, 5.6047795009028845e-14, 5.747144532508428e-14, 6.088241441179628e-14, 6.233323550943311e-14, 6.572778959054011e-14, 6.728483240764875e-14, 7.104219522838478e-14, 7.262790564660054e-14, 7.673185104026965e-14, 7.845582650277808e-14, 8.275298396797713e-14, 8.478186702748907e-14, 8.954669755687595e-14, 9.189855511338531e-14, 9.666050987073421e-14, 9.957017495948295e-14, 1.0486257698282909e-13, 1.0825635114751231e-13, 1.1440531667178 26e-13, 1.187997690931847e-13, 1.2616454101329865e-13, 1.3174327055870992e-13, 1.4055991286943496e-13, 1.4738660976325641e-13, 1.5764707824188373e-13, 1.6660704023324631e-13, 1.7871312952115836e-13, 1.906577665414974e-13, 2.0645419790938417e-13, 2.2298708251279914e-13, 2.44618368895067e-13, 2.6915455834368887e-13, 3.007151035964639e-13, 3.38636799781671e-13, 3.8954887907422237e-13, 4.553011617853758e-13, 5.41070168091194e-13, 6.467624718413088e-13, 7.85457033608338e-13, 9.704879733678106e-13, 1.2016910523103624e-12]
Stopped at Iteration: 84
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 83 : [1.8665628632001357e-15, 3.0934937692973626e-15, 4.0581428502953105e-15, 7.3189455615905e-15, 9.28108192322584e-15, 1.2446895934123065e-14, 1.3602649045995485e-14, 1.594509560833968e-14, 1.6681939096350536e-14, 1.9810329750936572e-14, 2.0621671048737717e-14, 2.3341863386278 73e-14, 2.3804131483441088e-14, 2.6125737830618874e-14, 2.650375447933689e-14, 2.874517119868692e-14, 2.9387622764492835e-14, 3.110595962881803e-14, 3.1907338053443054e-14, 3.338596772493774e-14, 3.452581933235173e-14, 3.572438769568326e-14, 3.7338033455242846e-14, 3.825119639954457e-14, 4.031289436281481e-14, 4.1068973698327477e-14, 4.329199557144092e-14, 4.381010468321898e-14, 4.6293966799780744e-14, 4.6798321280415003e-14, 4.8972975943196467e-14, 4.951527886277229e-14,

5.155825241022329e-14, 5.224517801446094e-14, 5.402681801060641e-14, 5.512806308532888e-14, 5.65301741452
4405e-14, 5.796070526475475e-14, 5.898029639456534e-14, 6.059091218156833e-14, 6.132832444008498e-14,
6.323996409245653e-14, 6.390910858390732e-14, 6.603347727846606e-14, 6.677128032362809e-14, 6.89150741330
1697e-14, 6.980440543521518e-14, 7.206187141933417e-14, 7.326265878069966e-14, 7.538353710098322e-14,
7.684024899921701e-14, 7.876830492829292e-14, 8.04974754521832e-14, 8.216891324115918e-14, 8.422120389754
897e-14, 8.577258138363175e-14, 8.816455303308007e-14, 8.982379499813711e-14, 9.279017197556288e-14,
9.493671244423974e-14, 9.877921453659756e-14, 1.0199385968368594e-13, 1.0699947934237849e-13, 1.113354187
4594705e-13, 1.1759782425317438e-13, 1.234673758773943 8e-13, 1.3163591397149867e-13, 1.4071789059597047e-
13, 1.5275394440112456e-13, 1.654258728492784e-13, 1.8186135767856752e-13, 2.0000288072899678e-13, 2.2364
693281365965e-13, 2.5319967356311986e-13, 2.91313234807388e-13, 3.306843420993332e-13, 3.73206511684292e-
13, 4.206563957529914e-13, 4.751405555377587e-13, 5.370816954685544e-13, 6.180980452681005e-13,
7.23326793116807e-13, 8.598110045427432e-13, 1.0409937131267021e-12]
Stopped at Iteration: 71
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 70 : [1.809978382895367e-15, 3.351665336667182e-1
5, 4.254071450264682e-15, 7.93136290173246e-15, 9.830931193110648e-15, 1.2893927639926783e-14, 1.36821350
352473e-14, 1.5750352179378686e-14, 1.625183019455717e-14, 1.8992979556069762e-14, 1.972625597062115e-14,
2.228996713582099e-14, 2.2814289604712113e-14, 2.5214753457002325e-14, 2.564552006488985e-14, 2.863764301
632087e-14, 2.926507432540617e-14, 3.199723701766577e-14, 3.251831648439808e-14, 3.521970387959229e-14,
3.5825993063066305e-14, 3.8635262486189025e-14, 3.926426836267727e-14, 4.187640297048626e-14,
4.2317209912539446e-14, 4.5313709317908786e-14, 4.603857703715927e-14, 4.9412338724664985e-14, 5.05104038
25693516e-14, 5.4265514734919466e-14, 5.5635288496516917e-14, 6.01400754871482e-14, 6.18898389333668e-14,
6.662084018757162e-14, 6.872828332055478e-14, 7.439098835821858e-14, 7.698791302601406e-14, 8.33547909076
2233e-14, 8.612922335869104e-14, 9.326172424429661e-14, 9.624406484466136e-14, 1.0465020483488807e-13,
1.0829705095045648e-13, 1.1725041871127687e-13, 1.2149117256883645e-13, 1.307213241399501e-13, 1.36059780
34223125e-13, 1.469600526043719e-13, 1.542133448588453e-13, 1.6751917923298595e-13, 1.7691490476585628e-13,
1.9255731768203e-13, 2.0557811429261205e-13, 2.2383771407416722e-13, 2.3779786843135294e-13, 2.5926068
820447797e-13, 2.8049835290106864e-13, 3.1023217036909323e-13, 3.420231947179735e-13,
3.7923871320842125e-13, 4.157423983642138e-13, 4.60692259084224e-13, 5.065413091968574e-13, 5.60574413499
0928e-13, 6.192222677208151e-13, 6.841319602195317e-13, 7.41077588599265e-13, 8.017891987606953e-13, 8.68
8170162220228e-13, 9.463327517043716e-13, 1.0422056503120046e-12]
Stopped at Iteration: 76
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 75 : [1.0043292858369463e-15,
1.2289629746288664e-15, 1.5746329399562488e-15, 2.0181187662226275e-15, 2.2621317040121154e-15, 2.9419575
08835104e-15, 4.115278387698831e-15, 5.539749392011419e-15, 6.734453554957981e-15, 9.27231974238657e-15,
1.0843554551394624e-14, 1.3237187795093716e-14, 1.514022742156758e-14, 1.784970140938381e-14,
1.930577230152261e-14, 2.2580520037437437e-14, 2.4027489191385343e-14, 2.706058456436e-14, 2.834195875490
9935e-14, 3.1742875333956905e-14, 3.276884933985987e-14, 3.6277759614275534e-14, 3.726705864960364e-14, 4
.045596991284881e-14, 4.121333716747393e-14, 4.506085252270907e-14, 4.567114470935076e-14,
4.9500948105954455e-14, 5.001626985944313e-14, 5.328129425610538 6e-14, 5.360357404364507 4e-14, 5.71256399
0985103e-14, 5.736626162086227e-14, 6.04042059597077e-14, 6.059769512012445e-14, 6.366720345711281e-14,
6.389375754425291e-14, 6.684957023827434e-14, 6.716982125687383e-14, 6.999920937519277e-14, 7.03663125168
1508e-14, 7.308109263104271e-14, 7.347523516428422e-14, 7.612078498111246e-14, 7.663264817365063e-14,
7.932328008568284e-14, 7.990195022978367e-14, 8.305064382734537e-14, 8.373716093637717e-14, 8.70387055125
086e-14, 8.790390480732425e-14, 9.146417440746557e-14, 9.235342721807758e-14, 9.622332417603485e-14,
9.726101806053064e-14, 1.0126008675500711e-13, 1.0257324644155921e-13, 1.0737708889017834e-13, 1.09365063
79677724e-13, 1.152675687232828e-13, 1.1912401704371643e-13, 1.275733096356281e-13, 1.3544708763217 16e-
13, 1.4926991973528215e-13, 1.638344462236087e-13, 1.8663945023925567e-13, 2.134620860105694e-13, 2.48784
70142602587e-13, 2.949403834412772e-13, 3.5279062787601385e-13, 4.225523409778292e-13, 5.0526445066641361e
-13, 6.051568818229646e-13, 7.267121544421949e-13, 8.580426467091822e-13, 1.0075648503672031e-12]
Stopped at Iteration: 72
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 71 : [3.840919289678638e-16, 5.467102812413621e-1
6, 8.272903394457262e-16, 1.1822144801626828e-15, 1.4283383909949358e-15, 1.912020999457426e-15,
2.4013676015813642e-15, 2.8525607995553047e-15, 3.322990681458816e-15, 4.061679890686282e-15,
4.665583829526388e-15, 5.238319576420732e-15, 5.927538452195255e-15, 6.702976579254562e-15, 7.37964136696
8083e-15, 8.011138519666301e-15, 8.648311517353196e-15, 9.250085780242531e-15, 9.683689920591454e-15,
1.0129385707525613e-14, 1.0638365935969836e-14, 1.1042043566425421e-14, 1.1577356985126301e-14, 1.2156426
104876637e-14, 1.2808819369306562e-14, 1.3467474936043 58e-14, 1.4184464724470574e-14, 1.483008228692082e-
14, 1.5508532224603376e-14, 1.6124584872614687e-14, 1.6922218293509477e-14, 1.759718043882513e-14, 1.8405
075367417597e-14, 1.913621844903305e-14, 1.9827616126985356e-14, 2.0571797760532266e-14, 2.12311284645322
93e-14, 2.1954766849789386e-14, 2.290006689637406e-14, 2.383401174176369e-14, 2.48765066173627e-14, 2.603
3134250784736e-14, 2.7504047205370502e-14, 2.90666988516764e-14, 3.089717869999285e-14, 3.29479999714125e
-14, 3.520841545554426e-14, 3.747168995528733e-14, 4.0385541761498287e-14, 4.380675363711946e-14, 4.80813
71934074157e-14, 5.329203320548 1074e-14, 5.938850739819806e-14, 6.572134209526351e-14, 7.31450211826792e-
14, 8.224089450201575e-14, 9.330676489189422e-14, 1.0774005377684423e-13, 1.2666687255375323e-13, 1.49102
772190575e-13, 1.751332499859752 8e-13, 2.0398568950214034e-13, 2.353258989924431e-13, 2.711444858216734e-
13, 3.1297357602301277e-13, 3.632426354162711e-13, 4.268944335627315e-13, 5.075303789085304e-13,
6.12872525295822e-13, 7.51291512633449e-13, 9.297610843228614e-13, 1.1496379376020148e-12]
Stopped at Iteration: 76
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 75 : [1.1964587687908361e-15,
1.4940577413975516e-15, 1.835127761834675e-15, 2.348275328842811e-15, 2.7700746890925847e-15,
3.16615395576295e-15, 3.834234051935987e-15, 5.21087267430732e-15, 6.071276034393076e-15, 7.7402568441643
56e-15, 8.619565636290809e-15, 9.770122490687223e-15, 1.0706995725628689e-14, 1.1688967719154227e-14,
1.2223771249435131e-14, 1.3358790080985613e-14, 1.3728688104771419e-14, 1.51327888952219e-14, 1.574940035
7131383e-14, 1.7840484776482707e-14, 1.9006343388457152e-14, 2.170941623082223e-14, 2.3483649362891422e-14,
2.5876113444068918e-14, 2.7627605385445958e-14, 3.0323330075488357e-14, 3.174847102497645e-14, 3.43889
6205951185e-14, 3.5603048305617275e-14, 3.827344535916323e-14, 3.933207726703451e-14,
4.212406818553573e-14, 4.328828036713022e-14, 4.585978267935322e-14, 4.7152895409875354e-14, 4.9719672172

34849e-14, 5.0952816898711135e-14, 5.365237757532289e-14, 5.478311634368548e-14, 5.743112816552253e-14,
5.837419238038459e-14, 6.142126673694696e-14, 6.22223572903315e-14, 6.544986668081902e-14, 6.642273920127
6e-14, 6.992792423526854e-14, 7.125037011003276e-14, 7.512871960011044e-14, 7.686831244287785e-14,
8.109825393124494e-14, 8.344937579722983e-14, 8.866446530718791e-14, 9.177857096639829e-14, 9.81713064342
7219e-14, 1.0257758015715318e-13, 1.1027722259368507e-13, 1.1674970581351893e-13, 1.26679950498102e-13, 1
.3584467920456007e-13, 1.479657940274183e-13, 1.5941841217398934e-13, 1.7608390920019115e-13, 1.938284861
7562644e-13, 2.167307507681812e-13, 2.4371217731471653e-13, 2.8104937710972525e-13, 3.2480741011232547e-1
3, 3.7587755838070176e-13, 4.305432214607586e-13, 4.89895035766995e-13, 5.562571314032193e-13, 6.38875542
2667497e-13, 7.32812333507614e-13, 8.346706604023645e-13, 9.536338881818025e-13, 1.0988160972043484e-12]
Stopped at Iteration: 80
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 79 : [5.541702536222811e-16, 7.354437517627383e-1
6, 8.3220979088641155e-16, 1.2618368523755171e-15, 1.955364682326266e-15, 2.835037342443892e-15,
4.437350111435365e-15, 5.501326916332957e-15, 6.817451140281528e-15, 8.370353047088404e-15, 9.25253137148
3089e-15, 1.052270736336127e-14, 1.0999195625636432e-14, 1.2344586731961818e-14, 1.2546465270273948e-14,
1.4210020184574093e-14, 1.4444308465627134e-14, 1.5631254676349287e-14, 1.5878261793155346e-14, 1.6908577
408777455e-14, 1.7164459172139918e-14, 1.8242976238018893e-14, 1.8710665382522118e-14,
1.954348174666689e-14, 2.0044292048105356e-14, 2.0963526778609742e-14, 2.1387188564009028e-14, 2.23658234
8624895e-14, 2.263525533251839e-14, 2.375627784251208e-14, 2.393755285991575e-14, 2.5135745162349316e-
14, 2.538006820692782e-14, 2.635688626035392e-14, 2.6731399473180923e-14, 2.7626456706954726e-14, 2.79160
30796921212e-14, 2.891673320587719e-14, 2.9191784739661595e-14, 3.0087915923187425e-14, 3.037575845490007
e-14, 3.1264563517541275e-14, 3.167967377802732e-14, 3.2510527957896744e-14, 3.303922085246613e-14, 3.384
728359726087e-14, 3.440592504044599e-14, 3.5326036890424336e-14, 3.6007601607131863e-14, 3.69970228132299
3e-14, 3.7738067603791366e-14, 3.8827282445346597e-14, 3.9531773586797516e-14, 4.085039998777318e-14, 4.1
65043838758594e-14, 4.3193970164388l3e-14, 4.4255710814020477e-14, 4.626098267449271e-14, 4.8218045094472
935e-14, 5.1038345709344765e-14, 5.4207585586381095e-14, 5.812303274549364e-14, 6.270956558745668e-14, 6.
848778759514803e-14, 7.581921346330973e-14, 8.565944864784225e-14, 9.869683092222338e-14,
1.1608701353202879e-13, 1.3903398279792389e-13, 1.6867714067175513e-13, 2.0600432665478557e-13, 2.5176007
61480947e-13, 3.079531838762697e-13, 3.7865491401934743e-13, 4.654318426269804e-13, 5.722536028821913e-
13, 6.924417672719878e-13, 8.111677095050561e-13, 9.26823890924799e-13, 1.0498338782272718e-12]
Stopped at Iteration: 89
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 88 : [4.514118158641475e-16, 7.627094349024205e-1
6, 1.4171242163577135e-15, 3.0336053131628416e-15, 4.1564295337659885e-15, 5.388381373205224e-15, 6.76078
1187701976e-15, 7.672488832325589e-15, 8.919248801132073e-15, 9.741876630710257e-15,
1.0909110186320119e-14, 1.1685478802744435e-14, 1.2844838825243011e-14, 1.3573734819946965e-14, 1.4778296
697020875e-14, 1.5601832956899665e-14, 1.7080716830273293e-14, 1.837213595112168e-14,
2.0386665742892434e-14, 2.231417126023005e-14, 2.5252653616685132e-14, 2.8138322159296863e-14, 3.22846993
6819878e-14, 3.5651403653785065e-14, 4.019706137789217e-14, 4.370030104322614e-14, 4.81822973396062e-14,
5.184715344955358e-14, 5.6504626504919673e-14, 6.065877089492396e-14, 6.538683213501846e-14, 6.9755650291
03747e-14, 7.442006617688303e-14, 7.900542188241788e-14, 8.420874287702969e-14, 8.869246156072844e-14,
9.445362410385177e-14, 9.858705766131292e-14, 1.0489620921136316e-13, 1.0905528302045563e-13, 1.154021855
7726347e-13, 1.1956355492231242e-13, 1.2618009669053305e-13, 1.3011034184497592e-13, 1.3695802656433l9e-1
3, 1.407171752392947e-13, 1.4769047869393827e-13, 1.5168080977296433e-13, 1.585584323458106e-13,
1.6226327123274298e-13, 1.6910790461815415e-13, 1.7292637456976218e-13, 1.7956811904343219e-13, 1.8317928
480474585e-13, 1.897257172438424e-13, 1.9358430800954398e-13, 2.0021922958683891e-13, 2.043024275326074e-
13, 2.113916529905052e-13, 2.155770435847674e-13, 2.2284081041949493e-13, 2.2731942012188e-13, 2.34072527
0182536e-13, 2.385254863183188e-13, 2.456145941776837e-13, 2.502650744580969e-13, 2.581121474840504e-13,
2.6235914955278985e-13, 2.710311482326147e-13, 2.754524960278893e-13, 2.835357769118826e-13,
2.889433333567005e-13, 2.971873221446363e-13, 3.0406923477274634e-13, 3.1422320667865003e-13, 3.231482920
6562345e-13, 3.3626315056568357e-13, 3.491132123133338e-13, 3.680807684292464 5e-13, 3.8812073521315124e-1
3, 4.1694549443084293e-13, 4.5106054392302823e-13, 4.97192090802453e-13, 5.509833486854525e-13, 6.1678038
20824964e-13, 6.917733745024375e-13, 7.898207579925971e-13, 9.046411703861844e-13, 1.0582206561439814e-
12]
Stopped at Iteration: 76
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 75 : [6.789051506363024e-16, 8.764660044665199e-1
6, 1.2400207135514718e-15, 1.9867530431149345e-15, 3.019459238396605e-15, 4.6917377975548645e-15, 6.71614
06065540696e-15, 9.240664885219235e-15, 1.1836417635577738e-14, 1.4614322532182112e-14, 1.746634163642569
e-14, 2.019041192048632e-14, 2.3326718850300748e-14, 2.5753427576836963e-14, 2.8808386375014843e-14, 3.11
1362417485836e-14, 3.4293267721642973e-14, 3.641025719746452e-14, 3.957340043121554e-14,
4.1518910838207055e-14, 4.452816639239443e-14, 4.6356233749057256e-14, 4.9529039138636354e-14, 5.13004537
9060786e-14, 5.438264417191045e-14, 5.602854936463763e-14, 5.889023630764188e-14, 6.048226053229991e-14,
6.31820032035286e-14, 6.461629259551774e-14, 6.725041331588535e-14, 6.86937292749273e-14, 7.1123290043080
06e-14, 7.263359095385043e-14, 7.489490714479833e-14, 7.664011867536525e-14, 7.86856383389476e-14,
8.046132336732557e-14, 8.256168028999505e-14, 8.437688727214095e-14, 8.647200611520098e-14, 8.82787644838
316e-14, 9.058199192503087e-14, 9.237829228670734e-14, 9.481700898726128e-14, 9.67788219296859e-14, 9.936
286685565825e-14, 1.0154949578874127e-13, 1.0476904654550532e-13, 1.0755142338739147e-13, 1.1172083193576
357e-13, 1.159150795433021e-13, 1.2225827891824191e-13, 1.295622260593924 6e-13, 1.3978695230031283e-13, 1
.5235611368193626e-13, 1.6940430746368941e-13, 1.8979262724106176e-13, 2.1440423640070062e-13, 2.40351746
10855287e-13, 2.678976876920511e-13, 2.979039550867925e-13, 3.3215732426704814e-13, 3.648573690229084e-
13, 3.958029152096021e-13, 4.291861874848538e-13, 4.679290613667416e-13, 5.073257084767791e-13,
5.463295030239813e-13, 5.874512555026195e-13, 6.334280708786838e-13, 6.882031147211754e-13, 7.598615355783
854e-13, 8.560178106862427e-13, 9.76122025401072e-13, 1.1197047188139628e-12]
Stopped at Iteration: 75
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 74 : [1.1048462959905072e-15,
2.0742887712469582e-15, 2.7812609475678665e-15, 3.5221840093398927e-15, 4.589801033396273e-15, 5.64706780
8678071e-15, 6.9043416693739886e-15, 7.871760222622368e-15, 9.36259831090743e-15, 1.062837628023214e-14,
1.2108156823953828e-14, 1.325640686261967e-14, 1.4622786106719282e-14, 1.5509354637136985e-14, 1.69316740
59648656e-14, 1.7945347457893252e-14, 1.9321775073073056e-14, 2.0279651622515876e-14,

2.1658838660617644e-14, 2.2613239796933993e-14, 2.3942988885138462e-14, 2.4872243313333026e-14, 2.6248091868080332e-14, 2.7083875400952612e-14, 2.8373666175228943e-14, 2.912793077644297e-14,
3.0344747694127095e-14, 3.116880313300549e-14, 3.2273531397418165e-14, 3.3108672344580284e-14, 3.4198080391031056e-14, 3.5117671890564976e-14, 3.6129043833722833e-14, 3.7186767939650505e-14,
3.8114020068179036e-14, 3.925036415333876e-14, 4.027843482574854e-14, 4.1506782564004867e-14,
4.264241466792992e-14, 4.4050233122656203e-14, 4.5411199748440226e-14, 4.7078970286348e-14, 4.877404545393166e-14, 5.082955984361707e-14, 5.301006749500424e-14, 5.5736165387174517e-14, 5.895656076168906e-14, 6.307933217122242e-14, 6.814469259720036e-14, 7.418969091649923e-14, 8.100101675761752e-14,
8.916107348858617e-14, 9.835942244170554e-14, 1.0878977835787396e-13, 1.1965172800769102e-13, 1.3067660070875725e-13, 1.4207261580467193e-13, 1.5442493261430705e-13, 1.6815603321137995e-13, 1.8348068254186298e-13, 2.0074790356055497e-13, 2.2067364489852492e-13, 2.4262757514514936e-13, 2.652577849533606e-13, 2.8887338051055046e-13, 3.1431222942203523e-13, 3.4577423170118716e-13, 3.84176190038821e-13,
4.296321113797737e-13, 4.860908091723046e-13, 5.544844201178602e-13, 6.344675097681128e-13, 7.377703658918776e-13, 8.61169601624131e-13, 1.011783051029046e-12]
Stopped at Iteration: 79
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 78 : [6.444790351932819e-16, 7.912812058140041e-16, 1.19480064867488e-15, 1.6175439903111207e-15, 2.2564541186094037e-15, 2.9159337708055833e-15, 3.8068566992521515e-15, 4.472097254686991e-15, 5.47649075800227e-15, 6.4470047684329175e-15, 7.754084012144697e-15, 8.880861856852866e-15, 1.0425203116386539e-14, 1.1493263763132886e-14, 1.3028910932725031e-14, 1.4082130459736579e-14, 1.5694006429248393e-14, 1.6676456194977773e-14, 1.7988677520203074e-14, 1.8853346545017764e-14, 2.011883825892293e-14, 2.0929680999756374e-14, 2.2149809388480963e-14, 2.286472573475753e-14,
2.4064496308328265e-14, 2.469534845014693e-14, 2.5755845808321825e-14, 2.6439266885385243e-14, 2.7369187873903032e-14, 2.8152874967814496e-14, 2.902036572524452e-14, 2.990585071956833e-14, 3.0840711005143925e-14, 3.209679524939206e-14, 3.32534923348863e-14, 3.4669933420995847e-14, 3.605444169084926e-14, 3.7759982936823524e-14, 3.9523272008671584e-14, 4.13979534254689e-14, 4.332376996657413e-14, 4.534200463781615e-14,
4.744274363299398e-14, 4.959077599435331e-14, 5.18823194783543e-14, 5.416250447593477e-14, 5.685951992072437e-14, 5.949643580796254e-14, 6.26847977071478e-14, 6.537471957953007e-14, 6.913763409581914e-14, 7.233865083835093e-14, 7.655880960810634e-14, 8.008636359500046e-14, 8.45556488956255e-14, 8.830549806200728e-14, 9.319090701205966e-14, 9.710723737806252e-14, 1.0234640559063149e-13, 1.0708549385160429e-13, 1.1334226456701942e-13, 1.1956140443991597e-13, 1.278069904355854e-13, 1.3707548084737708e-13, 1.4944106998828322e-13, 1.6429126125008162e-13, 1.837877135263045e-13, 2.041304662948113 7e-13, 2.2973314405294103e-13, 2.597097032773809e-13, 2.968082872022249e-13, 3.425154341913721e-13, 4.0054078749205215e-13, 4.723687195166972e-13, 5.576133007220965e-13, 6.558724965765682e-13, 7.784896569151911e-13, 9.278687943857152e-13,
1.1110051937614087e-12]
Stopped at Iteration: 87
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 86 : [1.8439797127908965e-15,
3.1331044312538877e-15, 5.024711410964066e-15, 5.94678692447034e-15, 7.227638118628258e-15, 8.027693594274047e-15, 8.972466283162041e-15, 9.71801089346145e-15, 1.0571639918309458e-14, 1.1224549962721215e-14, 1.1937613657045725e-14, 1.2493725740389108e-14, 1.3561638508373196e-14, 1.4273266414589958e-14, 1.5538264954651 26e-14, 1.6100356114669223e-14, 1.721187342391736e-14, 1.8175853952049973e-14, 1.941487866309557e-14, 2.0405515327528057e-14, 2.1751446366797406e-14, 2.2864847858911768e-14, 2.4589844969924844e-14, 2.610094590596645e-14, 2.842430423481948e-14, 3.00649813302811e-14, 3.2904261467417864e-14, 3.5038868998140014e-14, 3.807269941561211e-14, 4.046745334141286e-14, 4.360399966820513e-14, 4.607591372461491 3e-14, 4.940608190735178e-14, 5.2320709584371766e-14, 5.563727703225879e-14, 5.848258079046076e-14, 6.185327413267 61e-14,
6.470010737355125e-14, 6.794986179028349e-14, 7.07855380726938e-14, 7.397658654592564e-14, 7.692900290294676e-14, 8.017334574390866e-14, 8.320386334773816e-14, 8.653004882703965e-14, 8.962029236823026e-14,
9.302514385424235e-14, 9.619837136821931e-14, 9.953711475750459e-14, 1.0278239598039759e-13, 1.0590944661902945e-13, 1.0900200076160018e-13, 1.123089518333471e-13, 1.154518004951052e-13, 1.185871183913712e-13,
1.2195458778132726e-13, 1.2518305446314516e-13, 1.285105580184816e-13, 1.3173445936278044e-13, 1.3508954576008965e-13, 1.3844063027183257e-13, 1.417511695059714e-13, 1.4511176065127805e-13, 1.4884152569834463e-13, 1.5221891345107152e-13, 1.5660385113132295e-13, 1.602991188339694e-13, 1.6513287829010343e-13, 1.6949286967932554e-13, 1.7533030148903104e-13, 1.8161870143622422e-13, 1.9017737683230518e-13, 1.9931187496889945e-13, 2.1186257608099787e-13, 2.2603690661676662e-13, 2.448608033838869e-13, 2.6671759483988167e-13, 2.9373466141945975e-13, 3.2533689048346056e-13, 3.6374684378578474e-13, 4.084066657784662e-13, 4.623854587315456e-13, 5.294086251842762e-13, 6.106089290551118e-13, 7.153473813732766e-13, 8.523803516699452e-13,
1.0316940250842588e-12]
Stopped at Iteration: 77
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 76 : [2.236279702081636e-16, 4.541165705115257e-16, 4.800189274167074e-16, 1.247794183085716e-15, 2.093293888147918e-15, 3.390204761337809e-15,
4.504454801425928e-15, 5.853233165427009e-15, 7.16395763297298e-15, 8.712041034327554e-15, 1.0331838851612476e-14, 1.212356333030181e-14, 1.3881378099297445e-14, 1.5590505408678168e-14, 1.7235710153481254e-14,
1.8608930088837292e-14, 2.0172581105209694e-14, 2.133267278177420 2e-14, 2.2973926519767266e-14, 2.4043932498810505e-14, 2.559336466470194e-14, 2.6470603347076387e-14, 2.811527728457512e-14, 2.8984952793895874e-14, 3.058479788469566e-14, 3.1440127919021243e-14, 3.2980039526797514e-14, 3.389662026423515e-14, 3.5300172159864075e-14, 3.6266579700685766e-14, 3.7563542861989876e-14, 3.86245125168236e-14,
3.976039887755141e-14, 4.094925934173825e-14, 4.1903435386447076e-14, 4.3227251037898095e-14, 4.413301135227777e-14, 4.547168403495051e-14, 4.63865831453967e-14, 4.776622286085836e-14, 4.8871888047612e-14, 5.026020416617606e-14, 5.148334403600875e-14, 5.293910611726667e-14, 5.444448502779892e-14,
5.60350101621385e-14, 5.789109796195522e-14, 5.992864638593212e-14, 6.251701279966718e-14,
6.54497670401522e-14, 6.90115680175711e-14, 7.29960308321237e-14, 7.779018778909399e-14, 8.341209019994687e-14, 8.984949280546335e-14, 9.7236466107627e-14, 1.0646517279035927e-13, 1.1731130046242034e-13, 1.2989905006251127e-13, 1.4327385051217233e-13, 1.5765211143967532e-13, 1.734796328618008e-13, 1.912290091133876e-13, 2.1027217554803327e-13, 2.315156063121596e-13, 2.5479574043544967e-13, 2.8399148603727966e-13, 3.200110215540807e-13, 3.6212523383374127e-13, 4.160516819336076 7e-13, 4.822437305776846e-13, 5.580832734301409e-13, 6.462051880510255e-13, 7.441811101703839e-13, 8.593620456508664e-13, 9.98796820475833e-13, 1.1601502657010997e-12]
Stopped at Iteration: 85

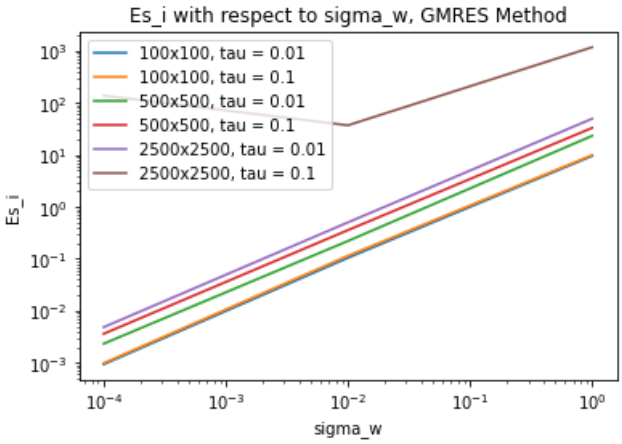
Stopped at Iteration: 85
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 84 : [6.219421421421936e-17, 5.297304958555872e-16, 1.4629560477082436e-15, 2.4999664541014505e-15, 3.2890886462485947e-15, 3.600789761999355e-15, 4.545344573315104e-15, 4.826737063373444e-15, 5.432931813734484e-15, 5.993340783300749e-15, 6.386332130834402e-15, 7.034710695425973e-15, 7.289679618031242e-15, 7.877927533206443e-15, 8.214017090274618e-15, 8.613644585967781e-15, 9.061277287401618e-15, 9.347163681800244e-15, 9.775845118685198e-15, 1.0065778764063823e-14, 1.057115604603965e-14, 1.0872927785109593e-14, 1.1303677218954064e-14, 1.167515090913571e-14, 1.2064605647594512e-14, 1.2465941742839806e-14, 1.2816383297502473e-14, 1.3153968100478906e-14, 1.3464463410238643e-14, 1.3866293713748915e-14, 1.4138029978879876e-14, 1.4506895869079868e-14, 1.4825742511367 23e-14, 1.5310604212372698e-14, 1.6030553110906416e-14, 1.6731232563633074e-14, 1.781342856039487e-14, 1.844384149874651e-14, 1.974661884958091e-14, 2.04973665126993e-14, 2.1614579705397867e-14, 2.2472407592295047e-14, 2.3528288562356783e-14, 2.4402200058743575e-14, 2.5322131083711528e-14, 2.6257123563796728e-14, 2.710025183619161e-14, 2.8112751567375672e-14, 2.899518427869508e-14, 2.999458203143051e-14, 3.0848726214291785e-14, 3.190140131367319e-14, 3.2885686815983596e-14, 3.397819275654947e-14, 3.5067196050391724e-14, 3.6307354527842156e-14, 3.7569992625430395e-14, 3.915782236194446e-14, 4.071879108261739e-14, 4.270048855682173e-14, 4.488421928314164e-14, 4.7515599689809687e-14, 5.0489132365721305e-14, 5.3942641684570357e-14, 5.789395755605141e-14, 6.290394993833545e-14, 6.908468736244586e-14, 7.687275591288319e-14, 8.687182468719175e-14, 1.0009371259864528e-13, 1.1625320121020794e-13, 1.3597525498591 32e-13, 1.5958799598262218e-13, 1.8776461638475819e-13, 2.210689268719907e-13, 2.5867099625133547e-13, 3.02488000957765e-13, 3.534138652560028e-13, 4.125676608686347e-13, 4.836219311036137e-13, 5.673636213649278e-13, 6.629381467748447e-13, 7.758140919680903e-13, 9.10005182605695e-13, 1.0738838739386698e-12]
Stopped at Iteration: 70
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 69 : [1.6126147723554603e-15, 2.0753522589770033e-15, 2.8599840091428238e-15, 3.299633166793977e-15, 3.908519006733878e-15, 4.212139991339904e-15, 5.330707946799549e-15, 6.080452168293759e-15, 7.354683522211765e-15, 8.064713602650081e-15, 8.968444786875201e-15, 9.678185916780792e-15, 1.0845575977139979e-14, 1.1500459733752386e-14, 1.2561759505896219e-14, 1.3317081500452777e-14, 1.4565496956498055e-14, 1.5298107177923363e-14, 1.6697926999972855e-14, 1.7429393340262325e-14, 1.8706696104870157e-14, 1.9494303956747844e-14, 2.0745531468404704e-14, 2.1445328592717995e-14, 2.2913304054092355e-14, 2.3627031845309615e-14, 2.509004367047299e-14, 2.5853604614705982e-14, 2.717575475258325e-14, 2.7943957834258588e-14, 2.918970548505813e-14, 2.995596618524939e-14, 3.1144014059860195e-14, 3.2060476172756203e-14, 3.3382891066410295e-14, 3.4616975569903174e-14, 3.624583156113056e-14, 3.774941833194421e-14, 3.9514987800071015e-14, 4.132272427416085e-14, 4.347749513131158e-14, 4.565765694559069 5e-14, 4.825833976524188e-14, 5.089774469505907e-14, 5.3836753827498793e-14, 5.702336275558319e-14, 6.061442258469258e-14, 6.48812219452504e-14, 7.009863954403472e-14, 7.687735059249973e-14, 8.532777674761756e-14, 9.649987222987657e-14, 1.0996917163945138e-13, 1.2585137563702305e-13, 1.4363963384514997e-13, 1.6368674362215692e-13, 1.885535432523591e-13, 2.1960360469517187e-13, 2.5591575505409566e-13, 2.995352850325217e-13, 3.495515320927508e-13, 4.091409798355518e-13, 4.76243337383061e-13, 5.469691662052562e-13, 6.197214988205411e-13, 6.901218967782469e-13, 7.638538593283168e-13, 8.444313526979373e-13, 9.280093807385387e-13, 1.0234667662592288e-12]
Stopped at Iteration: 74
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 73 : [8.213317772715352e-16, 9.95504236795974e-16, 1.6242602156517248e-15, 1.9101346297598038e-15, 2.222407106971569 6e-15, 2.7783168690339727e-15, 3.05490343370461e-15, 3.4006161591272484e-15, 3.684824888582269e-15, 3.9075010027295545e-15, 4.1494379501632605e-15, 4.49988600744283e-15, 4.89675884945565e-15, 5.174495601966558e-15, 5.588393051780134e-15, 6.056307003443418e-15, 6.741089793209929e-15, 7.258640252602062e-15, 8.18424358774477e-15, 8.916429695357734e-15, 9.957519284340034e-15, 1.078020000118956e-14, 1.1849165093214258e-14, 1.2877219926552103e-14, 1.432460357304715e-14, 1.5508687617700462e-14, 1.727519691231061e-14, 1.8530754124220665e-14, 2.0235313328603578e-14, 2.1693885479134315e-14, 2.35274149298363e-14, 2.5254898631227543e-14, 2.7549199565739483e-14, 2.9632817807684856e-14, 3.194397540429199e-14, 3.42974204397162e-14, 3.671268970729153e-14, 3.900668430912794e-14, 4.1455042669993044e-14, 4.366100302439061e-14, 4.607929014357029e-14, 4.845115419935949e-14, 5.0993438800450135e-14, 5.360267732023405e-14, 5.6619317009868056e-14, 5.949287560108924e-14, 6.301889218749696e-14, 6.677481374786995e-14, 7.119416316628595e-14, 7.609879660053714e-14, 8.215314192284693e-14, 8.891248774565924e-14, 9.75179465512399e-14, 1.0675985258319113e-13, 1.1752936464434706e-13, 1.2982735308335213e-13, 1.4381603846421203e-13, 1.5949175558847736e-13, 1.780541948117035e-13, 1.9946852165568812e-13, 2.23465509142799e-13, 2.481049649410299e-13, 2.750105663707361e-13, 3.0247023241459 08e-13, 3.3147085859724135e-13, 3.6462316547292395e-13, 4.0469161122376e-13, 4.5507096217892e-13, 5.207472555568256e-13, 6.038616263634281e-13, 6.921664787524483e-13, 7.93288043595385e-13, 9.135157031443692e-13, 1.0561522197626904e-12]
Stopped at Iteration: 85
Frobenius_norm(Q_k.T Q_k - I_k) for iterations up to : 84 : [1.864566929993936e-15, 3.1289556318865887e-15, 5.071879216489295e-15, 6.057892347610153e-15, 7.808447224941725e-15, 8.959979736934036e-15, 1.1153394033844395e-14, 1.2514133407889253e-14, 1.4748818625684464e-14, 1.6385622126964723e-14, 1.914234772291564e-14, 2.1209537128913264e-14, 2.4007757340653704e-14, 2.6037064868076046e-14, 2.875919506768627e-14, 3.084562403176071e-14, 3.348552716294936e-14, 3.561864579089017e-14, 3.816536683323052e-14, 4.01640754985281e-14, 4.2768611457659227e-14, 4.474110302144046e-14, 4.7357666175379295e-14, 4.919735187500444e-14, 5.1937137177542975e-14, 5.3526380644027313e-14, 5.6015236016009216e-14, 5.742412823674268e-14, 5.980269093375642e-14, 6.1178406380568e-14, 6.352600462535604e-14, 6.486146530495218e-14, 6.709852942296664e-14, 6.848014368149626e-14, 7.067064897531097e-14, 7.22711206574269e-14, 7.445758873851341e-14, 7.606768344188866e-14, 7.816900370437566e-14, 7.9618078148208e-14, 8.159013264612373e-14, 8.3052528109620 3e-14, 8.498714098488518e-14, 8.64312971176459e-14, 8.83628320802957e-14, 8.987802866746174e-14, 9.169802834393263e-14, 9.315986826408669e-14, 9.500896492660746e-14, 9.660515354009354e-14, 9.831035297996097e-14, 1.0009363746631203e-13, 1.0171029185675108e-13, 1.036069666137568e-13, 1.0514650031182147e-13, 1.0711164865935884e-13, 1.0879663052054021e-13, 1.1094166363436268e-13, 1.1275785086595133e-13, 1.1508001982449687e-13, 1.1708021016031357e-13, 1.195907800475572e-13, 1.2185382666992323e-13, 1.2461385713669785e-13, 1.27257470156428e-13, 1.3066774079344013e-13, 1.3433052046340578e-13, 1.394344151602474e-13, 1.4529518965465047e-13, 1.5342446712551731e-13, 1.6310831680316488e-13, 1.75208645194553e-13, 1.8957558035957201e-13, 2.0783277264423119e-13, 2.3086765400194187e-13, 2.6000104068061987e-13, 2.9552351551765554e-13, 3.372172589589090 5e-13, 3.85733209881226e-13, 4.452957167367626e-13, 5.172962830873506e-13, 6.035113270954881e-13,

7.118581028899633e-13, 8.476652245520667e-13, 1.0184752623699454e-12]

In [106]:

```
plt.figure()
w = [0.0001,0.01,1]
plt.plot(w,gmres_err1_es, label = '100x100, tau = 0.01')
plt.plot(w,gmres_err2_es, label = '100x100, tau = 0.1')
plt.plot(w,gmres_err3_es,label = '500x500, tau = 0.01')
plt.plot(w,gmres_err4_es, label = '500x500, tau = 0.1')
plt.plot(w,gmres_err5_es, label = '2500x2500, tau = 0.01')
plt.plot(w,gmres_err6_es, label = '2500x2500, tau = 0.1')

plt.title("Es_i with respect to sigma_w, GMRES Method")
plt.yscale('log')
plt.xscale('log')
plt.ylabel('Es_i')
plt.xlabel('sigma_w')
plt.legend()
plt.show(block=False)
```



In [ ]: