Multi-Agent Systems

HW6: Final Homework Assignment

MSc AI, VU

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IMPORTANT

- This project is an **individual assignment.** So everyone should hand in their own copy.
- The questions below require programming. In addition to the report (addressing the questions and discussing the results of your experiments), also provide the code on which your results are based (preferably as Python notebooks). However, make sure that the **report is self-contained**, i.e. that it can be read and understood without the need to refer to the code. **Only the report will be graded, so it's NOT sufficient to submit only a notebook**.
- Store the pdf-report and the code in a zipped folder that you upload to canvas. **Use your name and VU ID as name for the zipped folder**, e.g.

  `Jane_Doe_1234567.zip`

- This assignment will be **graded.** The max score is 4 and will count towards your final grade.
- Your **final grade (on 10)** will be computed as follows:

  Assignments 1 thru 5 (max 1) + Individual assignment (max 4) + Final exam (max 5)

- Good luck!

1 Monte Carlo Simulation (30%)

The COVID scare was a wake-up call, highlighting the vulnerability of our hyper-connected modern world to the threat of pandemics. Now that the COVID urgency has passed, governments across the globe are rushing to update their emergency action plans and disaster scripts.

As part of this effort, you've been hired by a medical team that has been tasked with developing fast procedures to detect a blood-borne virus. Since these tests need to be administered to large groups in the population, and testing resources are limited, the medics have come up with the following procedure. They started from the assumption that they need to test N blood samples (of as many different individuals) and that N is large (e.g. $N = 10^6$). Furthermore, the probability that an individual is infected is p, where p is relatively small, e.g. $p < 0.1$.

Based on these assumptions they propose the following procedure to **minimise the number of tests** they have to run: Rather than testing each sample individually, take a batch of k samples and mix them. Then this mixed sample is tested for the presence of the viral antigen:

- If the mixed sample tests **negative** (i.e. no viral antigen is detected), then all the individual samples were clear, and you therefore have the result for all k individual samples that went into the batch.
- If the mixed sample tests **positive** (i.e. the viral antigen is present indicating infection), one needs to retest all individual samples that went into the batch, in order to find out which individual(s) are actually infected.

Questions

1. Use Monte Carlo simulation to estimate the optimal batch size k (i.e. the one that minimises the expected number of tests) for a given value of p where p can take values between 10^{-1} and 10^{-4}.
2. In order to convince your superiors that this a good investment, quantify the expected reduction in workload (compared to testing all samples individually).

2 Thompson Sampling for Multi-Armed Bandits (30%)

2.1 Beta distributions to model uncertainty about probabilities

Consider a bandit that for each pull of an arm, produces a binary reward: $r = 1$ (with probability p) or $r = 0$ (with probability $1 - p$). Assuming the bandit has K arms, this means that there are K unknown probabilities $p_1, p_2, \ldots, p_K$ and we need to identify the arm that has the highest probability

$$p^* = \max\{p_1, p_2, \ldots, p_K\},$$

as pulling this arm will result in the highest cumulative reward.

Using the beta-distribution to model the uncertainty on a probability Initially we have no information (total uncertainty) about the values of the probabilities, but each pull of an arm yields a binary outcome (reward), providing some information about the underlying probability, and thus reducing the corresponding uncertainty.

We model our uncertainty about the actual (but unknown) value p using a beta-distribution (cf. `https://en.wikipedia.org/wiki/Beta_distribution`). This is a (unimodal) probability distribution on the interval $[0, 1]$ which depends on two parameters: $\alpha, \beta \geq 1$. The explicit distribution is given by (for α, β integers!):

$$B(x; \alpha, \beta) = \frac{(\alpha + \beta - 1)!}{(\alpha - 1)!\,(\beta - 1)!} x^{\alpha-1}(1 - x)^{\beta-1} \qquad \text{(for } 0 \leq x \leq 1\text{)}.$$

The parameters α and β determine the shape of the distribution:

- If $\alpha = \beta = 1$ then we have the uniform distribution;
- If $\alpha = \beta$ the distribution is symmetric about $x = 1/2$.
- If $\alpha > \beta$ the density is right-leaning (i.e. concentrated in the neighbourhood of 1). In fact, one can compute the mean explicitly:

$$X \sim B(x;\,\alpha, \beta) \quad \Longrightarrow \quad EX = \frac{\alpha}{\alpha+\beta} = \frac{1}{1+(\beta/\alpha)},$$

  indicating that the ratio β/α determines the position of the mean.
- Larger values of α and β produce a more peaked distribution. This follows from the formula for the variance:

$$X \sim B(x;\,\alpha, \beta) \quad \Longrightarrow \quad Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

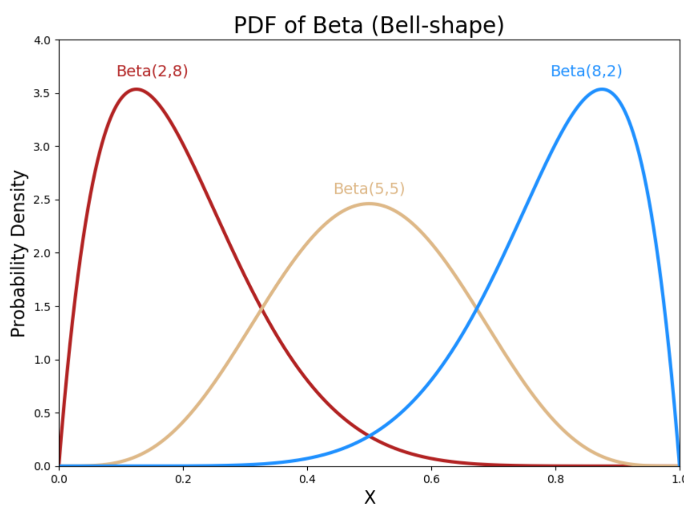


Figure 1: Some probability densities for the Beta-distribution with different parameters.

2.2 Thompson's Bayesian update rule

Although the beta-distribution seems like a reasonable model to quantify the uncertainty on a probability, there is a deeper reason for its use. Updating a (prior) beta-density with binary observations, results in a new beta-density with updated parameters (this is an example of what is known as **conjugated priors**). Specifically, if the **prior** is modeled as $B(x, \alpha, \beta)$, and we observe s successes (1) and f failures (0) then the **posterior** would be the beta distribution $B(x; \alpha{+}s, \beta{+}f)$. This observation yields the rationale for Thompson's Bayesian update rule:

- Initialise $\alpha = \beta = 1$ (resulting in a uniform distribution, indicating that all possible values for p are equally likely). Now repeat the following loop:

  1. Sample from the bandit and get reward r (either $r = 1$ or $r = 0$);

2. Update the values for α and β as follows:
   - if $r = 1$, then $\alpha \leftarrow \alpha + 1$
   - if $r = 0$, then $\beta \leftarrow \beta + 1$

   This update rule can be summarized as:

$$\alpha \leftarrow \alpha + r \qquad \beta \leftarrow \beta + (1 - r)$$

Questions

1. Implement the Thompson update rule for single arm bandit (i.e. $k = 1$) and show experimentally that the Beta-density increasingly peaks at the correct value for p. To this end, plot both the evolution of the mean and variance over (iteration)time.

2.3 Thompson sampling for K-armed bandit Problem

For binary outcomes, the Thompson update rule offers an alternative for the UCB-based balancing of exploration and exploitation. Specifically, suppose we have a K-armed bandit problem. The k-th arm delivers a reward $r = 1$ with (unknown!) probability p_k (and hence $r = 0$ with probability $1 - p_k$). For each arm ($k = 1, \ldots, K$), the uncertainty about the corresponding p_k is modelled using a Beta-distribution $B(x; \alpha_k, \beta_k)$. Thompson sampling now tries to identify the arem that will deliver the maximal cumulative reward (highest p_k) by proceeding as follows:

Initialise all parameters to 1: $\alpha_k = 1 = \beta_k$; Now repeat the following loop:

- We use the beta-distributions to simulate the pulling of each arm. This means that we sample a value U_k from each of the K Beta-distributions:

$$U_k \sim B(x; \alpha_k, \beta_k) \qquad (k = 1 \ldots K).$$

- For this simulation, determine which arm gave the best result:

$$k_{max} = \arg\max\{U_1, U_2, \ldots, U_K\}$$

- Mindful of the uncertainties on the p_k-values, the above simulation gives us reason to believe that pulling the k_{max} arm is rational (after all, we did a simulation using the available evidence, and this was the result).

- Sample the corresponding arm (i.e. arm k_{max}) and get reward r (either 1 or 0);

- Use the Bayesian update rule for the corresponding parameters:

$$\alpha_{k_{max}} \leftarrow \alpha_{k_{max}} + r \qquad \text{and} \qquad \beta_{k_{max}} \leftarrow \beta_{k_{max}} + (1 - r)$$

Questions

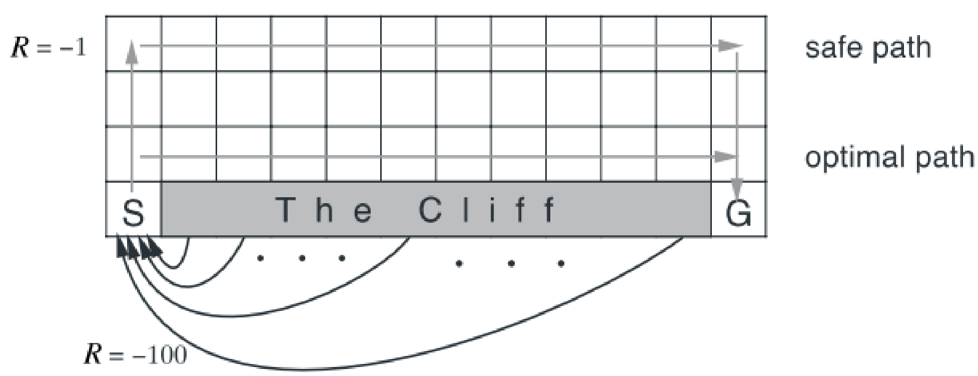
2. Write code to implement Thompson sampling for the above scenario when $K = 3$;

3. Perform numerical experiments in which you compare Thompson sampling with the UCB. Use **total regret** (provide the precise definition that you're using) as your performance criteria. For UCB, experiment with different values of the hyperparameter c. The fact that, for Thompson sampling, you don't need to specify an hyperparameter, is a distinct advantage.

3 Reinforcement Learning: Cliff Walking (40%)

Consider the cliff-walking example (Sutton & Barto, ex. 6.6. p.108). Assume that the grid has 21 columns and 3 rows (above or in addition to the cliff). This is a standard undiscounted, episodic task, with start (S) and goal (G) states, and the usual actions causing movement up, down, right, and left. Reward is -1 on all transitions except:

- the transition to the terminal goal state (G) which has an associated reward of $+20$;
- transitions into the region marked *The Cliff*. Stepping into this region incurs a "reward" of -100 and also terminates the episode.



Questions

1. Use both SARSA and Q-Learning to construct an appropriate policy. Do you observe the difference between the SARSA and Q-learning policies mentioned in the text (safe versus optimal path)? For Q-learning, experiment with a replay-buffer. Discuss.

2. Try different values for ϵ (parameter for ϵ-greedy policy). How does the value of ϵ influence the result? Discuss.

3. Now assume that there is a snake pit in the two bottom cells in the 11th column, i.e. the two cells closest to the cliff). Stumbling in these snake pits also carries a penalty (negative reward) of -100. Does this change the results obtained by SARSA or Q-learning? Discuss.