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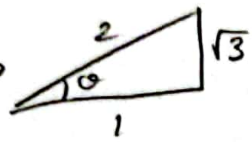
Q1

$$\mathrm{Re}\{(1-j)e^{j\theta}\} = 1 - \mathrm{Re}\{(1-j)e^{-j\theta}\}$$

$$\mathrm{Re}\{(1-j)(\cos\theta + j\sin\theta)\} = 1 - \mathrm{Re}\{(1-j)(\cos\theta - j\sin\theta)\}$$

$$\mathrm{Re}\{\cos\theta + j\sin\theta - j\cos\theta - \underbrace{j^2\sin\theta}_{+\sin\theta}\} = 1 - \mathrm{Re}\{\cos\theta - j\sin\theta - j\cos\theta + \underbrace{j^2\sin\theta}_{-\sin\theta}\}$$

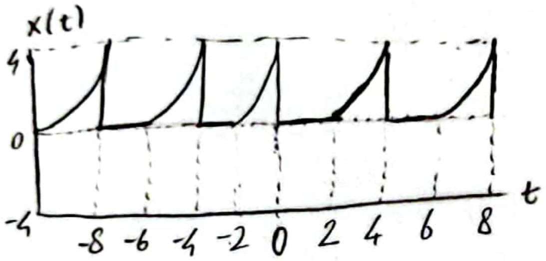
$$\cos\theta + \sin\theta = 1 - \cos\theta + \sin\theta \Rightarrow 2\cos\theta = 1$$

$\cos\theta = \frac{1}{2} \rightarrow$ 

$\cos\theta = \frac{\pi}{3} \pm 2\pi k \quad k = 0,1,2,\ldots$

Q2

Q2.a)



Q2.b) $a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j(2\pi/T_0)kt}\, dt \Rightarrow a_k = \frac{1}{4}\left[\overbrace{\int_0^2 0\, dt}^{0} + \int_2^4 (t-2)^2 e^{-j(\pi/2)kt}\, dt\right]$

Let $M = \frac{-j\pi k}{2} \rightarrow a_k = \frac{1}{4}\cdot\frac{1}{M}\int_2^4 (t-2)^2 e^{Mt} M\, dt$

$$a_k = \frac{1}{4M}\left(\left[(t-2)^2 e^{Mt}\right]_2^4 - \int_2^4 2(t-2)e^{Mt}\, dt\right)$$

$$a_k = \frac{1}{4M}\left(\left[4e^{4M} - 0\right] - 2\cdot\frac{1}{M}\underbrace{\int_2^4 (t-2)e^{Mt} M\, dt}_{\left[(t-2)e^{Mt}\right]_2^4 - \int_2^4 e^{Mt}\, dt \,=\, 2e^{4M} - \frac{(e^{4M} - e^{2M})}{M}}\right)$$

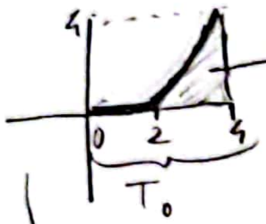
$$a_k = \frac{1}{4M}\left(4e^{4M} - \frac{4e^{4M}}{M} + \frac{2e^{4M} - 2e^{2M}}{M^2}\right) = \frac{4e^{4M}\cdot M^2 - 4e^{4M}\cdot M + 2e^{4M} - 2e^{2M}}{4M^3}$$

$$a_k = \frac{e^{4M}(4M^2 - 4M + 2) - e^{2M}\cdot 2}{4M^3} = \frac{e^{-2j\pi k}(j^2\pi^2k^2 - 2j\pi k + 2) - e^{-j\pi k}\cdot 2}{\frac{-j^3\pi^3k^3}{2}} = \frac{4e^{-j\pi k} + 2e^{-2j\pi k}(\pi^2k^2 + 2j\pi k - 2)}{-j\pi^3k^3}$$

for $k \neq 0$

Q2.c)

$$a_0 = \frac{1}{4}\int_2^4 (t-2)^2\,dt = \frac{1}{4}\left[\frac{(t-2)^3}{3}\right]_2^4 = \frac{1}{4}\left(\frac{8}{3}-0\right) = \frac{2}{3}$$



$$\rightarrow \text{Area} = \int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$$

DC coefficient $(a_0) = \frac{2}{3}$ Area of curve in a period $= \frac{8}{3}$

Q2.d)

$$y(t) = \begin{cases} 2(t-2) & 2 \le t \le 4 \\ 0 & 0 \le t < 2 \end{cases}$$

$$b_k = \frac{1}{4}\int_2^4 2(t-2)\,e^{\frac{-j\pi k t}{2}}\,dt = \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{(-j\pi k)}\int_2^4 (t-2)\,e^{\frac{-j\pi k t}{2}}\left(\frac{-j\pi k}{2}\right)dt$$

$$b_k = \frac{1}{-j\pi k}\left(\left[(t-2)e^{\frac{-j\pi k t}{2}}\right]_2^4 - \underbrace{\int_2^4 e^{\frac{-j\pi k t}{2}}\,dt}_{2\left[\frac{e^{\frac{-j\pi k t}{2}}}{(-j\pi k)}\right]_2^4}\right) = \frac{1}{-j\pi k}\left(2e^{-2j\pi k} - 2\left(\frac{-e^{-2j\pi k}+e^{-j\pi k}}{j\pi k}\right)\right)$$

$$b_2 = \frac{-1}{2j\pi}\left(2e^{-4j\pi} - \cancel{2}\left(\frac{e^{-2j\pi}-e^{-4j\pi}}{2j\pi}\right)\right) = \frac{-e^{-4\pi j}}{\pi j} + \frac{\left(e^{-2j\pi}-e^{-4\pi j}\right)}{2j^2\pi^2}$$

$$b_2 = -\frac{\overbrace{e^{-4\pi j}}^{1}}{\pi j} - \underbrace{\frac{\overbrace{e^{-2\pi j}}^{1} - \overbrace{e^{-4\pi j}}^{1}}{2\pi^2}}_{0}$$

$$e^{2\pi j \cdot m} = \cos(2\pi \cdot m) + j\sin(2\pi \cdot m) = 0 \text{ for } m \in \mathbb{Z}$$

$$b_2 = \frac{-1}{\pi j} = \frac{-j}{\pi \cdot j^2} = \frac{j}{\pi}$$

Q3

Q3.a) $x[n] = 5\cos(\overbrace{0.26\pi n}^{\omega_0 T_s = \hat{\omega}_0} + \frac{\pi}{6})$ $f_s = \frac{1}{T_s} = 500$

$$\hat{\omega}_0 = \frac{\omega_0}{f_s} = \frac{\omega_0}{500} = 0.26\pi \Rightarrow \omega_0 = (0.26)\pi \cdot 500 = 130\pi$$

$$f_0 = \frac{130\pi}{2\pi} = 65$$

$$x(n) = 5\cos(130\pi t + \frac{\pi}{6})$$

Folded Aliases: $\hat{\omega}_0 + 2\pi l$ and $2\pi l - \hat{\omega}$ for $l \in \mathbb{Z}$

Possible inputs:

$$x_1(n) = 5\cos((2.26)(500)\pi + \frac{\pi}{6}) = 5\cos(1130\pi + \frac{\pi}{6})$$

$$x_2(n) = 5\cos((1.74)500\pi - \frac{\pi}{6}) = 5\cos(870\pi - \frac{\pi}{6})$$

However frequency of both are greater than $\frac{f_s}{2} = 250Hz$
so they are under sampled

Q3.b) $x(t) = 16\cos(\underbrace{400\pi t}_{} + \frac{3\pi}{4}) + 4\cos(\underbrace{1400\pi t}_{} + \frac{\pi}{2})$

$f_0 \leftarrow \frac{200 \cdot 2\pi}{1000 \rightarrow f_s}$ $\Downarrow$ $\hat{\omega}_0 = 0.4\pi$

$f_0 \leftarrow \frac{700 \cdot 2\pi}{1000 \rightarrow f_s}$ $\cancel{f_{max} > \frac{f_s}{2}}$ so $\hat{\omega}_0 = 2\pi - 1.4\pi = 0.6\pi$ for aliasing (folded alias)

Since $f_{max} \geq \frac{f_s}{2}$, aliasing happens. So discrete signal will interpret it as a different signal, namely:

$$x[n] = 16\cos((0.4\pi)n + \frac{3\pi}{4}) + 4\cos((0.4\pi)n - \frac{\pi}{2})$$

$$t = n \cdot f_s = 1000n \rightarrow y(t) = x[1000n] = 16\cos(400\pi t + \frac{3\pi}{4}) + 4\cos(600\pi t - \frac{\pi}{2})$$

Note that as $f_{max} > \frac{f_s}{2}$ information loss happens and signal can not be fully constructed. Namely $x(t) \neq y(t)$

Q4

Q4.a) $x(t) = [3 + \cos(3\pi(500)t)]\cos(3\pi(1000)t)$

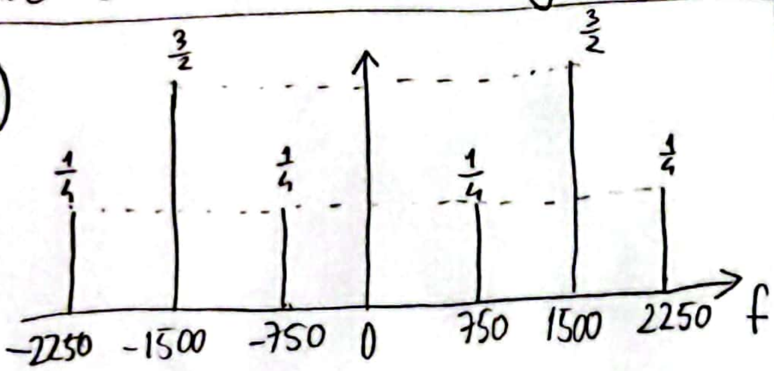
$$x(t) = 3\cos(3000\pi t) + \cos(1500\pi t)\cdot\cos(3000\pi t)$$

Let $\theta = 1500\pi \rightarrow x(t) = 3\left(\frac{e^{2\theta t}+e^{-2\theta t}}{2}\right) + \left(\frac{e^{\theta t}+e^{-\theta t}}{2}\right)\left(\frac{e^{2\theta t}+e^{-2\theta t}}{2}\right)$

$$x(t) = \frac{1}{4}\left(6e^{2\theta t} + 6e^{-2\theta t} + e^{3\theta t} + e^{-3\theta t} + e^{\theta t} + e^{-\theta t}\right)$$

Since $\theta = 1500\pi$ and $\gcd(3\theta, 2\theta, \theta) = \theta \rightarrow T_0 = \frac{1}{\left(\frac{\omega_0}{2\pi}\right)} = 750s$

Q4.b)



Q4.c)

$$f_{max} < \frac{f_s}{2}$$

where $f_{max} = 2250$ Hz in this case

Q5

$$y[n] = \sum_{k=0}^{3} b_k x[n-k] = -2x[n] + 2x[n-1] + 4x[n-2] + 6x[n-3]$$

$n = 3k \rightarrow 2 + 2 + 0 - 6 = -2$

$n = 3k+1 \rightarrow 0 - 2 + 4 + 0 = 2$

$n = 3k+2 \rightarrow -2 + 0 - 4 + 6 = 0$

$$y[n] = \begin{cases} -2 & n = 3k \\ 2 & n = 3k+1 \\ 0 & n = 3k+2 \end{cases}$$

Q6

Q6.a) (1) $w[n] = \alpha 2x_1[n]\cos(\pi n) + \beta 2x_2[n]\cos(\pi n)$
$y[n] = 2(\alpha x_1[n] + \beta x_2[n])\cos(\pi n)$
} $w[n] = y[n]$ ↓ Linear ✓

(2) $w[n] = 2x[n-n_0]\cos(\pi n)$
$y[n-n_0] = 2x[n-n_0]\cos(\pi(n-n_0))$
} $w[n] \neq y[n-n_0]$ ↓ Time Ind. ✗

(3) No dependency on future values of n → Causality ✓

Q6.b) (1) $w[n] = \alpha x_1[n] - \alpha x_1[2n+1] + \beta x_2[n] - \beta x_2[2n+1]$
$y[n] = (\alpha x_1[n] + \beta x_2[n]) - (\alpha x_1[2n+1] + \beta x_2[2n+1])$
} $w[n] = y[n]$ ↓ Linear ✓

(2) $w[n] = x[n-n_0] - x[2n-2n_0+1]$
$y[n-n_0] = x[n-n_0] - x[2n-n_0+1]$
} $w[n] \neq y[n-n_0]$ ↓ Time Ind ✗

(3) Not causal since $x[2n+1]$ depends on future values → Causality ✗

Q6.c) (1) $w[n] = -\alpha x_1[n]^2 - \beta x_2[n]^2$
$y[n] = -(\alpha x_1[n] + \beta x_2[n])^2$
} $w[n] \neq y[n]$ ↓ Not Linear

(2) $w[n] = -(x[n-n_0])^2$
$y[n-n_0] = -(x[n-n_0])^2$
} $w[n] = y[n-n_0]$ Time Ind. ✓

(3) Causality ✓

Q6.d)

(1) $w[n] = \alpha(x_1[n] - v[n]) + \beta(x_2[n] - v[n])$

$y[n] = \alpha x_1[n] + \beta x_2[n] - v[n]$

$w[n] \neq y[n]$ Linearity ✓

(2) $w[n] = x[n-n_0] - v[n]$

$y[n-n_0] = x[n-n_0] - v[n-n_0]$

$w[n] \neq y[n-n_0]$ $\Downarrow$ Time Ind. ✗

(3) Causality ✓

Q6.e)

(1) $w[n] = \alpha 2^{x_1[n]} + \beta 2^{x_2[n]}$

$y[n] = 2^{\alpha x_1[n] + \beta x_2[n]}$

$w[n] \neq y[n]$ $\Downarrow$ Linearity ✗

(2) $w[n] = 2^{x[n-n_0]}$

$y[n-n_0] = 2^{x[n-n_0]}$

$w[n] = y[n-n_0]$ $\Downarrow$ Time Ind ✓

(3) Causality ✓

Q6.f)

(1) $w[n] = \alpha(5 + x_1[n]) + \beta(5 + x_2[n])$

$y[n] = 5 + \alpha x_1[n] + \beta x_2[n]$

$w[n] \neq y[n]$ $\Downarrow$ Linearity ✗

(2) $w[n] = 5 + x[n-n_0]$

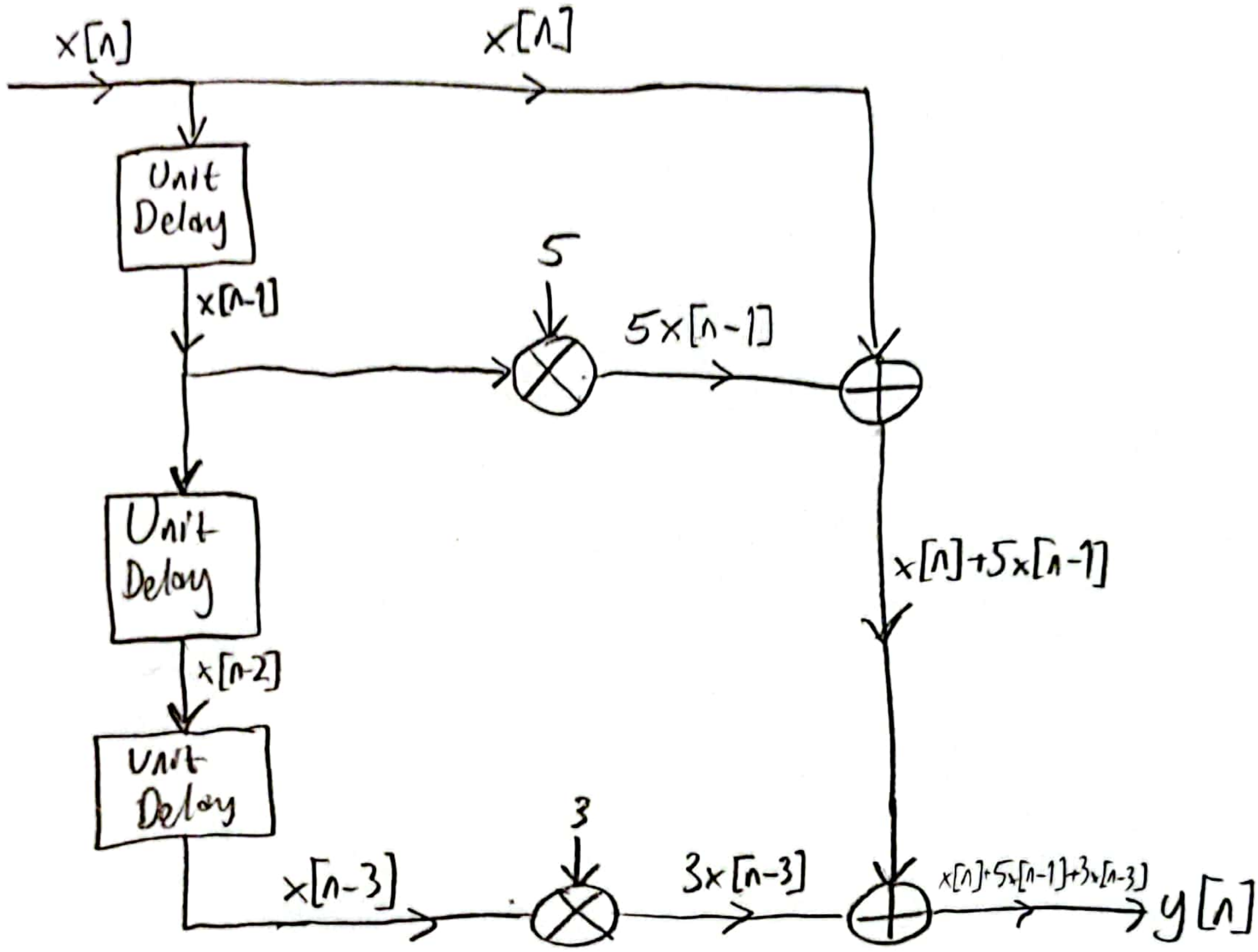
$y[n-n_0] = 5 + x[n-n_0]$

$w[n] \neq y[n-n_0]$ $\Downarrow$ Time Ind. ✓

(3) Causality ✓

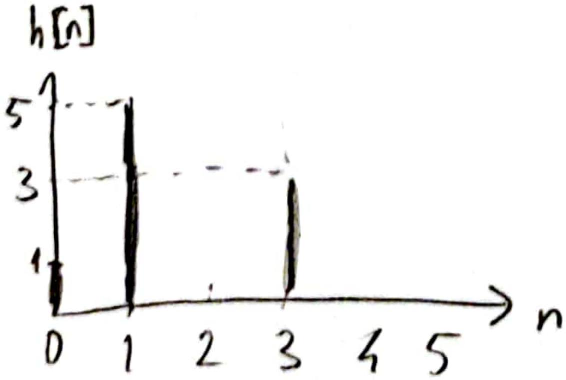
Q7

Q7.a)



Q7.b)

$$h[n] = \begin{cases} 1 & n=0 \\ 5 & n=1 \\ 3 & n=3 \\ 0 & \text{otherwise} \end{cases}$$



Q8

$$x[n] \rightarrow LTI\,1 \rightarrow y_1[n] = 3x[n] + 7x[n-1] + 9x[n-3] + 5x[n-4]$$

$$x[n] = 2u[n] \qquad u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$y_1[n] = \begin{cases} 48 & n \geq 4 \\ 38 & 4 > n \geq 3 \\ 20 & 3 > n \geq 1 \\ 6 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$y_1[n] \longrightarrow LTI\,2 \rightarrow y_2[n] = 3y_1[n] + 7y_1[n-1] + 13y_1[n-2]$$

$$y[n] = y_1[n] + y_2[n] = 4y_1[n] + 7y_1[n-1] + 13y_1[n-2]$$

$$y[n] = \begin{cases} 1152 & n \geq 6 \\ 1022 & n = 5 \\ 718 & n = 4 \\ 552 & n = 3 \\ 298 & n = 2 \\ 122 & n = 1 \\ 24 & n = 0 \\ 0 & n < 0 \end{cases}$$