CENG 567 Design and Analysis of Algorithms | Fall 2022

# Homework 1

Released: 12/10/2022 | Due:23/10/2022

**Instructions.** You may work with other students, but you must individually write your solutions **in your own words**. If you work with other students or consult outside sources (such as Internet/book), cite your sources.

**Submissions.** Submit a pdf file through odtuclass. LaTeX or Word typed submission is required.

1. **Stable Matching**. Consider the following stable matching instance.

| Men | Preference List |
|---|---|
| X | C, D, A, B |
| Y | D, A, B, C |
| Z | A, B, C, D |
| W | D, C, A, B |

| Women | Preference List |
|---|---|
| A | X, Y, Z, W |
| B | X, Y, Z, W |
| C | Y, Z, X, W |
| D | Z, X, Y, W |

(a) Use the Gale and Shapley algorithm to find a man-optimal stable matching.

(b) Find a woman-optimal stable matching in this instance.

(c) Suppose we are given $n$ men and $n$ women where each man has a preference list of all women and every woman has preference list of all men. On a stable matching instance, prove that if we run the Gale-Shapley algorithm twice, once with men proposing and once with women proposing and we obtain the same matching, then the instance has a unique stable solution.

(d) Show that the Gale-Shapley algorithm terminates within $(n-1)^2+1$ iterations for $n$ boys and $n$ girls and this bound is tight.

2. **Alternative matchings.** Consider the following stable matching instance.

| Men | Preference List |
|---|---|
| X | B, A, D, C |
| Y | C, B, A, D |
| Z | D, A, C, B |
| W | C, B, D, A |

| Women | Preference List |
|---|---|
| A | W, Y, X, Z |
| B | Z, Y, W, X |
| C | X, Z, Y, W |
| D | X, W, Z, Y |

(a) In addition to man-optimal and woman-optimal matchings suppose we have an alternative matching $S = \{(X - A), (Y - B), (Z - C), (W - D)\}$. Is S a stable matching? Explain.

(b) How can we find all stable matchings for a problem where there are $n$ men and $n$ women? Is there a polynomial time algorithm to find all stable matchings? If so, describe it.

(c) Can a woman can end up better off by lying about her preferences? More concerently, suppose a woman $w$ falsely switches the order of $m$ and $m'$ although she prefers $m$ to $m'$. Prove that this is possible by giving an example of a set of preference lists (for $n \geq 3$) for which there is a switch that would improve the partner of a woman who falsely switched preferences.

3. **Order of growth**]. Analyze the growth of $n!$.

(a) Prove that $n! = \mathrm{O}(n^n)$ and $n! = \Omega(2^{n-1})$.

(b) Prove that for any fixed $k > 1$, $n! = \Omega(k^{n-k})$, or find a counter example.

(c) Prove that for any fixed $k > 1$, $n! = \Omega(k^n)$, or find a counter example.

4. **Asymptotics**. Given an array $A$ of $n$ integers, you'd like to output a two-dimensional $n \times n$ array $B$ in which for all $i < j$, $B[i, j] = \max \{A[i], A[i+1], \cdots, A[j]\}$. For $i \geq j$ the value of $B[i, j]$ can be left as is.

---

**for** $i = 1, 2, \cdots, n$ **do**
    **for** $j = i + 1, \cdots, n$ **do**
        compute the maximum of the entries $A[i], A[i+1], \cdots, A[j]$.
        storethe maximum value in $B[i, j]$.
    **end for**
**end for**

---

(a) Find a function $f$ such that the running time of the algorithm is $O(f(n))$, and clearly explain why.

(b) For the same function $f$ argue that the running time of the algorithm is also $\Omega(f(n))$. (This establishes an asymptotically tight bound $\Theta(f(n))$.)

(b) Design and analyze a faster algorithm for this problem. You should give an algorithm with running time $O(g(n))$, where $\lim_{n \to \infty} g(n)/f(n) = 0$.

5. **Big O and $\Omega$.** For each function $f(n)$ below, find (1) the smallest integer constant $H$ such that $f(n) = \mathrm{O}(n^H)$, and (2) the largest positive real constant $L$ such that $f(n) = \Omega(n^L)$. Otherwise, indicate that $H$ or $L$ do not exist. All logarithms are with base 2. Your answer should consist of: (1) the correct value of $H$, (2) a proof that $f(n)$ is $O(n^H)$, (3) the correct value of $L$, (4) a proof that $f(n)$ is $\Omega(n^L)$.

(a) $f(n) = \frac{1}{2}n^2$.

(b) $f(n) = n(\log n)^3$.

(c) $f(n) = \sum_{i=0}^{\lceil \log n \rceil} \frac{n}{2^i}$.

(d) $f(n) = \sum_{i=1}^{n} i^3$.

(e) $f(n) = 2^{(\log n)^2}$.