PHYS-302 CLASSICAL MECHANICS
Assignment

Homework 2

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A Course Homework Assignment

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Problem 1

Show that the expression

$$\begin{aligned}\lambda^2 &= -1 + x_l^2 + x_{l-1}^2 - 2x_l x_{l-1} x_{l-2} \\ &= \frac{1}{8}\mathrm{tr}[M_l, M_{l-1}]^2\end{aligned} \tag{1}$$

is an invariant of the trace-map

$$x_{l+1} = 2x_l x_{l-1} - x_{l-2}. \tag{2}$$

Solution.

We first treat the trace terms as points and rename them.

$$x_{l+1} = x \qquad x_l = y \qquad x_{l-1} = z \tag{3}$$

Then the recursion relation (7) will transform these points into new ones by iteration since

$$\begin{aligned}x_{l+1} &\to x_{l+2} = 2x_{l+1}x_l - x_{l-1} = 2xy - z \\ x_l &\to x_{l+1} = x_{l+1} = x \\ x_{l-1} &\to x_l = x_l = y\end{aligned} \tag{4}$$

which shows us that the recursion relation is actually equivalent to the map below

$$T(x, y, z) = (2xy - z, x, y) \tag{5}$$

where T is called Fibonacci trace map. Now renaming the terms in the invariant λ^2 we get

$$I = \lambda^2 = x^2 + y^2 + z^2 - 2xyz - 1 \tag{6}$$

In order the expression I to be indeed an invariant, it has to stay unaltered under the action of trace map. Composing the map and I we then have

$$\begin{aligned}T \circ I &= (2xy - z)^2 + x^2 + y^2 - 2(2xy - z)xy - 1 \\ &= 4x^2y^2 + z^2 - 4xyz + x^2 + y^2 - 4x^2y^2 + 2xyz - 1 \\ &= x^2 + y^2 + z^2 - 2xyz - 1 = I\end{aligned} \tag{7}$$

Thus we have shown

$$T \circ I = I.$$

Problem 2

Consider a one-dimensional Fibonacci chain of particles m_i connected by identical nearest-neighbor springs

$$H = \sum \left(\frac{p_i^2}{2m_i} + \frac{1}{2}(x_{i+1} - x_i)^2 \right) \tag{8}$$

where x_i is the displacement of the ith particle and m_i gets two values m_A and m_B. Then, construct vibrational spectrum for the Fibonacci chain of oscillators ABA, $BAAABA$,$ABABAABA$,$BAABAABABAABA$ using numerical computations.

Solution.

Firstly, the Lagrangian is obtained by Legendre's transform of the given Hamiltonian (8).

$$\begin{aligned} L &= \sum p_i \dot{x}_i - H \qquad\qquad \text{where } p_i \rightarrow \dot{x}_i = \frac{\partial H}{\partial p_i} = \sum \frac{p_i}{m_i} \\ &= \sum \left(m_i \dot{x}_i^2 - \frac{(m_i \dot{x}_i)^2}{2m_i} - \frac{1}{2}(x_{i+1} - x_i)^2 \right) \\ &= \sum \left(\frac{1}{2} m_i \dot{x}_i^2 - \frac{1}{2}(x_{i+1} - x_i)^2 \right). \end{aligned} \tag{9}$$

Hence,

$$L = \frac{1}{2} \sum m_i \dot{x}_i^2 - \frac{1}{2} \sum (x_{i+1} - x_i)^2. \tag{10}$$

The parentheses are open to get x_i terms:

$$(x_{i+1} - x_i)^2 + (x_i - x_{i-1})^2 = x_{i+1}^2 + x_i^2 - 2x_{i+1}x_i + x_i^2 + x_{i-1}^2 - 2x_i x_{i-1} \tag{11}$$

Then, the Euler-Lagrange equation applies

$$\frac{\partial}{\partial x_i} \frac{1}{2}(x_{i+1} - x_i)^2 - \frac{d}{dt}(m_i \dot{x}_i) = 0. \tag{12}$$

Therefore,

$$x_{i+1} - 2x_i + x_{i-1} = m_i \ddot{x}_i. \tag{13}$$

Thus, $x_n = \mathrm{Re}[e^{i\omega t} x_n]$ is a solution of (13), so plugged and obtained the following:

$$-2x_n + x_{n+1} + x_{n-1} = -m_i \omega^2 x_n \tag{14}$$

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} -\omega^2 m_i + 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} \tag{15}$$

Thus the initial matrices for Fibonacci chain are,

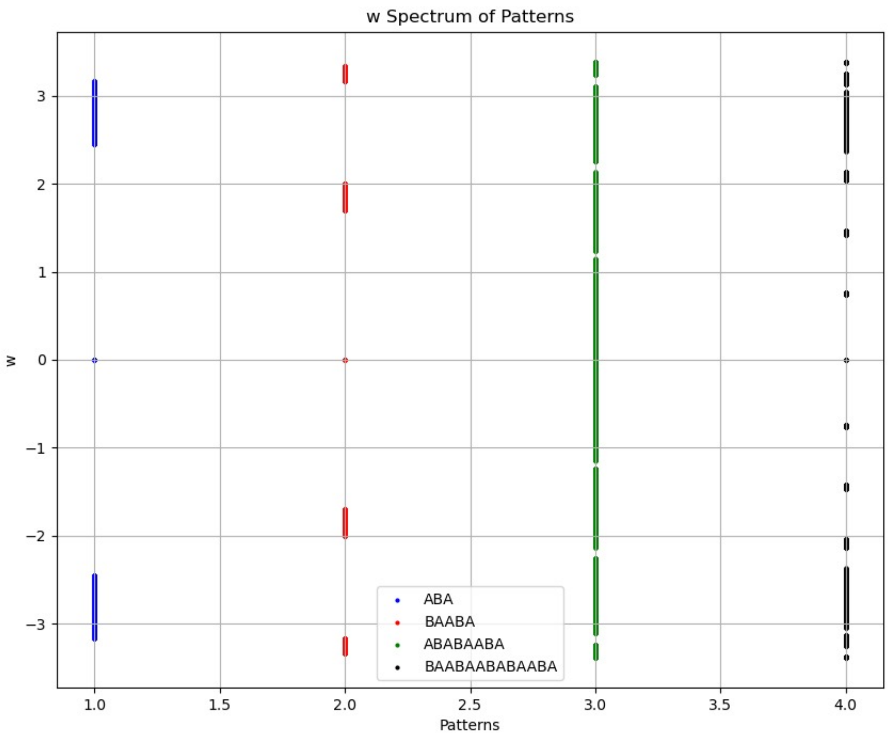
$$M_0 = \begin{pmatrix} -\omega^2 m_B + 2 & -1 \\ 1 & 0 \end{pmatrix} \tag{16}$$

$$M_1 = \begin{pmatrix} -\omega^2 m_A + 2 & -1 \\ 1 & 0 \end{pmatrix} \tag{17}$$

These has the following recursion relation as known from the lecture,

$$M_{l+1} = M_l M_{l-1} \tag{18}$$

The procedure used to find the wanted chain of oscillators is to multiply the initial matrices M_0 and M_1 by (18). Then the available and forbidden values for traces of these matrices are examined by first getting approximate solutions for roots with using $m_A = 0.5$ and $m_B = 0.25$ and then plotting the corresponding inequalities with Python.



To be clear, the roots found by numerical computations can be listed:

ROOTS WITH USING PYTHON:

ABA ROOTS:

$$x = \{-3.16227766016838, -2.44948974278318, 0, 0, 2.44948974278318, 3.16227766016838\}$$

BAABA ROOTS:

$$x = \{-3.33513202521544, -3.16227766016838, -2.00000000000000, -1.69614102432031, 0, 0, 1.69614102432031, 2.00000000000000, 3.16227766016838, 3.33513202521544\}$$

ABABAABA ROOTS:

$$x = \{-3.38586792641676, -3.23606797749979, -3.10277504909664, -2.24914053812955, -2.12976486609725, -1.23606797749979, -1.14636548903291, 0, 0, 1.14636548903291, 1.23606797749979, 2.12976486609725, 2.24914053812955, 3.10277504909664, 3.23606797749979, 3.38586792641676\}$$

BAABAABABAABA ROOTS:

$$x = \{-3.38586792641676, -3.36687386958356, -3.25282610460807, -3.13152101251502, -3.03975344428299, -2.36899279590520, -2.12976486609725, -2.03900133198064, -1.46308119482860, -1.41421356237310, -0.762528216609821, -0.738274111008558, 0, 0, 0.738274111008558, 0.762528216609821, 1.41421356237310, 1.46308119482860, 2.03900133198064, 2.12976486609725, 2.36899279590520, 3.03975344428299, 3.13152101251502, 3.25282610460807, 3.36687386958356, 3.38586792641676\}$$

ROOTS WITH USING MATLAB:

ABA ROOTS:

$$
\begin{aligned}
x = \{&-3.1622776601683793319988935444327,\\
&-2.4494897427831780981972840747059, 0, 0,\\
&2.4494897427831780981972840747059,\\
&3.1622776601683793319988935444327\}
\end{aligned}
$$

BAABA ROOTS:

$$
\begin{aligned}
x = \{&-3.3351320252154427342010537475 28,\\
&-3.1622776601683793319988935444327, -2.0,\\
&-1.6961410243203067606706250279029, 0, 0,\\
&1.6961410243203067606706250279029,\\
&2.0, 3.1622776601683793319988935444327,\\
&3.3351320252154427342010537475 28\}
\end{aligned}
$$

ABABAABA ROOTS:

$$
\begin{aligned}
x = \{&-3.3858679264167636146125920643112,\\
&-3.2360679774997896964091736687313,\\
&-3.1027750490966407845239050205292,\\
&-2.2491405381295493392619825400 03,\\
&-2.1297648660972519228645331076861,\\
&-1.2360679774997896964091736687313,\\
&-1.1463654890329085547380775194737,\\
&0, 0, 1.1463654890329085547380775194737,\\
&1.2360679774997896964091736687313,\\
&2.1297648660972519228645331076861,\\
&2.2491405381295493392619825400 03,\\
&3.1027750490966407845239050205292,\\
&3.2360679774997896964091736687313,\\
&3.3858679264167636146125920643112\}
\end{aligned}
$$

BAABAABABAABA ROOTS:

$$
\begin{aligned}
x = \{&-3.3858679264167636146125920643112,\\
&-3.3668738695835603503435512740498,\\
&-3.2528261046080714471488550818899,\\
&-3.1315210125150216808931929204 51,\\
&-3.0397534442829913823166127921474,\\
&-2.3689927959052005322093946855211,\\
&-2.1297648660972519228645331076861,\\
&-2.0390013319806432926323476756346,\\
&-1.4630811948285982012481739200426, \ldots
\end{aligned}
$$

-1.4142135623730950488016887242097,
-0.76252821660982114868379823492997,
-0.73827411100855841846032420385428, 0, 0,
0.73827411100855841846032420385428,
0.76252821660982114868379823492997,
1.4142135623730950488016887242097,
1.4630811948285982012481739200426,
2.0390013319806432926323476756346,
2.1297648660972519228645331076861,
2.3689927959052005322093946855211,
3.0397534442829913823166127921474,
3.131521012515021680893192920451,
3.2528261046080714471488550818899,
3.3668738695835603503435512740498,
3.3858679264167636146125920643112}